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On Spectrum and Energy of Identity Graph for Group of Integers Modulo n, $\mathbb{Z}_n$

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Abstract. Groups and graphs are two concepts of algebraic mathematics. This paper focuses on group structures that can be expressed in graphs known as identity graphs. We investigate the energy of the identity graph for a group of integers modulo n, $\mathbb{Z}_n$, for odd and even n corresponding to adjacency, Laplacian, and signless Laplacian matrices. It can be seen that the Laplacian and signless Laplacian energies are always equal and are always an even integer. Meanwhile, the adjacency energy is never an odd integer for n is odd.

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Key Words and Phrases: Energy of a graph, Identity graph of a group, $\mathbb{Z}_n$

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1. Introduction

Groups and graphs are two concepts of algebraic mathematics. A group is an algebraic structure from a non-empty set with a binary operation and satisfies associative property, there is an identity element and each element has an inverse. Furthermore, graph theory is a discrete mathematics study that discusses vertices and edges. In this paper, we discuss group structures that can be expressed in graphs, the name is identity graph.

Kandasamy and Smarandache in 2009 [5] described finite groups as graphs. They call this the identity graph because the main key in constructing the graph is determined by the group's identity elements. The discussion on labeling of the identity graph can be found in [10].

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The graph energy concept was pioneered by Gutman in 1978 [4]. It should be noted that the graph energy is never an odd integer [1, 9]. Moreover, several results on the energy of a graph defined on groups can be found in [11, 13, 15]. They worked on non-commuting graphs with Wiener-hosoya, closeness and degree subtraction matrices. Meanwhile, for Sombor energy can be seen in [12]. Shi et al [17] found the energy of picture fuzzy graphs, in line with the signless Laplacian energy [11] and Cayley of interval-valued fuzzy graphs [2]. Kumari et al. [6] presented the quotient energy of the identity graph for $\mathbb{Z}_p$, for prime number p and Romdhini et al. [14] showed the spectral properties of power graph for dihedral groups. Meanwhile, the spectral discussion of the square power graph can be seen in [18]. In addition, Shanthakumari et al. [16] described the Euclidean degree energy and Lokesha et al. [8] investigated the skew energy of a graph.

Inspired by this, we work on the identity matrix. Our focus in this paper is a group of integers modulo n, $\mathbb{Z}_n = \left\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\right\}$. We construct the identity graph based on the group elements as vertices. We develop some graph matrices corresponding to this graph concerning the adjacency, Laplacian, and signless Laplacian matrices. We formulate the graph's characteristic polynomial, spectrum, and energy, and analyze the relationship between those energies. We also observe the energy values to draw interesting conclusions.

2. Preliminaries

In this part, we recall the fundamental definition and theorem that are useful for our main results. We start with the definition of the identity graph.

Definition 1. *[5] The identity graph of a group G, denoted by Γ_G, is a graph whose vertex set is the elements of the group and two distinct vertices u and v will be connected by an edge if $uv = e$ with every member of $G\backslash\{e\}$ is adjacent to e, where e is the identity element of G.*

Throughout this paper, we denote the identity graph for $\mathbb{Z}_n$ as $\Gamma_{\mathbb{Z}_n}$. The next two theorems are the description of $\Gamma_{\mathbb{Z}_n}$, for n is odd and even.

Theorem 1. *[5] If $\mathbb{Z}_n = \left\{\overline{0}, \overline{1}, \overline{2}, ..., \overline{n-1}\right\}$ is a group of order n, $n \geq 3$ with odd n, then the identity graph of $\mathbb{Z}_n$ contains $\frac{n-1}{2}$ of K_3.*

Theorem 2. *[5] If $\mathbb{Z}_n = \left\{\overline{0}, \overline{1}, \overline{2}, ..., \overline{n-1}\right\}$ is a group of order n, $n \geq 2$ with even n, then the identity graph of $\mathbb{Z}_n$ contains $\frac{n-2}{2}$ of K_3 and a K_2.*

The construction of the graph matrices of $\Gamma_{\mathbb{Z}_n}$ is based on the definition of the adjacency, Laplacian, and signless Laplacian matrices. We refer these definition to ([3]) as presented below:

Definition 2. *([3]) The adjacency matrix of order $n \times n$ associated with $\Gamma_{\mathbb{Z}_n}$ is given by $A(\Gamma_{\mathbb{Z}_n}) = [a_{ij}]$ whose (i,j)-th entry*

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Definition 3. *([3]) The $n \times n$ diagonal degree matrix of $\Gamma_{\mathbb{Z}_n}$ is given by $D(\Gamma_{\mathbb{Z}_n}) = [d_{ij}]$ whose (i, j)-th entry*

$$d_{ij} = \begin{cases} d_{v_i}, & \text{if } v_i = v_j \\ 0, & \text{otherwise} \end{cases}$$

where d_{v_i} is the vertex degree of v_i.

Definition 4. *([3]) The $n \times n$ Laplacian matrix of $\Gamma_{D_{2n}}$ is given by $L(\Gamma_{\mathbb{Z}_n}) = D(\Gamma_{\mathbb{Z}_n}) - A(\Gamma_{\mathbb{Z}_n})$.*

Definition 5. *([3]) The $n \times n$ signless Laplacian matrix of $\Gamma_{\mathbb{Z}_n}$ is given by $SL(\Gamma_{\mathbb{Z}_n}) = D(\Gamma_{\mathbb{Z}_n}) + A(\Gamma_{\mathbb{Z}_n})$.*

The characteristic polynomial of $A(\Gamma_{\mathbb{Z}_n})$ is defined by

$$P_{A(\Gamma_{\mathbb{Z}_n})}(\lambda) = |\lambda I_n - A(\Gamma_{\mathbb{Z}_n})|, \tag{1}$$

where I_n is an $n \times n$ identity matrix. Similarly, notation for other matrices can be used in the same manner.

To formulate the determinant in Equation 1, we need row and column operations to simplify the process. Let R_i be the i-th row and C_i be the i-th column of $P_{A(\Gamma_{\mathbb{Z}_n})}(\lambda)$.

Furthermore, the roots of $P_{A(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ are the eigenvalues of $\Gamma_{\mathbb{Z}_n}$. The graph energy definition is based on the eigenvalues of $\Gamma_{\mathbb{Z}_n}$ as presented below.

Definition 6. *[4] The adjacency energy of $\Gamma_{\mathbb{Z}_n}$ can be written by*

$$E_A(\Gamma_{\mathbb{Z}_n}) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A(\Gamma_{\mathbb{Z}_n})$.

The spectrum of $\Gamma_{\mathbb{Z}_n}$ in accordance with the adjacency matrix is

$$Spec_A(\Gamma_{\mathbb{Z}_n}) = \{(\lambda_1)^{k_1}, (\lambda_2)^{k_2}, \ldots, (\lambda_n)^{k_n}\},$$

where $k_1, k_2, \ldots, k_n$ are the respective multiplicities of eigenvalus. The spectral radius of $\Gamma_{\mathbb{Z}_n}$ corresponding with the adjacency matrix is

$$\rho_A(\Gamma_{\mathbb{Z}_n}) = max\{|\lambda| : \lambda \in Spec_A(\Gamma_{\mathbb{Z}_n})\}.$$

The energy value of $\Gamma_{\mathbb{Z}_n}$ is classified as hyperenergetic if the energy of $\Gamma_{\mathbb{Z}_n}$ is greater than $2(n-1)$ [7].

3. Main Results

In this section, we begin with the analysis of the degree of every vertex in $\Gamma_{\mathbb{Z}_n}$. We need this property for constructing the matrices of $\Gamma_{\mathbb{Z}_n}$.

Theorem 3. *Let $\Gamma_{\mathbb{Z}_n}$ be the identity graph on $\mathbb{Z}_n$. For n is odd, then*

(i) the degree of $\overline{0}$ on $\Gamma_{\mathbb{Z}_n}$ is $deg(\overline{0}) = n-1$, and

(ii) the degree of $\overline{a}$ on $\Gamma_{\mathbb{Z}_n}$ is $deg(\overline{a}) = 2$, for $a \neq 0$.

Proof. From Theorem 1 for odd n, the identity graph of $\mathbb{Z}_n$ contains $\frac{n-1}{2}$ of K_3. Since the identity of $\mathbb{Z}_n$ is $\overline{0}$, then $\overline{0}$ is adjacent to all other vertices in $\mathbb{Z}_n$. This means that the degree of $\overline{0}$ is equal to $n-1$. Meanwhile, for $\overline{a}$, where $a \neq 0$, we know that the inverse of $\overline{a}$ is $\overline{n-a}$, since $\overline{a+(n-a)} = \overline{0}$. This implies that $\overline{a}$ and $\overline{n-a}$ are always adjacent. Therefore, the degree of $\overline{a}$ is 2.

Theorem 4. *Let $\Gamma_{\mathbb{Z}_n}$ be the identity graph on $\mathbb{Z}_n$. For n is even, then*

(i) the degree of $\overline{0}$ on $\Gamma_{\mathbb{Z}_n}$ is $deg(\overline{0}) = n-1$,

(ii) the degree of $\overline{\frac{n}{2}}$ on $\Gamma_{\mathbb{Z}_n}$ is $deg(\overline{\frac{n}{2}}) = 1$, and

(iii) the degree of $\overline{a}$ on $\Gamma_{\mathbb{Z}_n}$ is $deg(\overline{a}) = 2$, for $a \neq 0, \frac{n}{2}$.

Proof. Recall the fact from Theorem 2 that the identity graph of $\mathbb{Z}_n$ contains $\frac{n-2}{2}$ of K_3 and a K_2 for even n. By the same argument with the proofing part of Theorem 3 that $\overline{0}$ is the identity of $\mathbb{Z}_n$, then the degree of $\overline{0}$ is $n-1$. Now, we concern with $\overline{\frac{n}{2}} \in \mathbb{Z}_n$. Since the inverse of $\overline{\frac{n}{2}}$ is itself, then $\overline{\frac{n}{2}}$ is only adjacent to $\overline{0}$ which means the degree of $\overline{\frac{n}{2}}$ is 1. Meanwhile, for $a \neq 0, \frac{n}{2}$, we have $\overline{a+(n-a)} = \overline{0}$. Consequently, $\overline{a}$ is adjacent to $\overline{n-a}$ and also to $\overline{0}$ which we mentioned earlier. Therefore, the degree of $\overline{a}$ is 2.

Theorem 5. *Let $M_{n\times n}$ be the matrix as follows:*

$$M = \begin{bmatrix} a & c & c & \dots & c & c \\ c & b & 0 & \dots & 0 & c \\ c & 0 & b & \dots & c & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c & 0 & c & \dots & b & 0 \\ c & c & 0 & \dots & 0 & b \end{bmatrix},$$

where n is odd, and real numbers a, b, c. The characteristic polynomial of M is

$$P_M(\lambda) = \left(\lambda^2 - (a+b+c)\lambda + a(b+c) - c^2(n-1)\right)(\lambda - b - c)^{\frac{n-3}{2}}(\lambda - b + c)^{\frac{n-1}{2}}.$$

Proof. Let n be an odd number. The characteristic polynomial of M is given by

$$P_M(\lambda) = \begin{vmatrix} \lambda - a & -c & -c & \dots & -c & -c \\ -c & \lambda - b & 0 & \dots & 0 & -c \\ -c & 0 & \lambda - b & \dots & -c & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c & 0 & -c & \dots & \lambda - b & 0 \\ -c & -c & 0 & \dots & 0 & \lambda - b \end{vmatrix},$$

where the numbers a, b, c are real. We need to simplify the above determinant by applying row and column operations.

(i) $R_{\frac{n+1}{2}+i} \longrightarrow R_{\frac{n+1}{2}+i} - R_{\frac{n+3}{2}-i}$, for $i = 1, 2, \dots, \frac{n-1}{2}$.
Then we have

$$P_M(\lambda) = \begin{vmatrix} \lambda - a & -c & -c & \dots & -c & -c \\ -c & \lambda - b & 0 & \dots & 0 & -c \\ -c & 0 & \lambda - b & \dots & -c & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -\lambda + b - c & \dots & \lambda - b + c & 0 \\ 0 & -\lambda + b - c & 0 & \dots & 0 & \lambda - b + c \end{vmatrix}.$$

(ii) $C_{\frac{n+3}{2}-i} \longrightarrow C_{\frac{n+3}{2}-i} + C_{\frac{n+1}{2}+i}$, for $i = 1, 2, \dots, \frac{n-1}{2}$.
Consequently,

$$P_M(\lambda) = \begin{vmatrix} \lambda - a & -2c & -2c & \dots & -c & -c \\ -c & \lambda - b - c & 0 & \dots & 0 & -c \\ -c & 0 & \lambda - b - c & \dots & -c & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda - b + c & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda - b + c \end{vmatrix}.$$

(iii) $C_1 \longrightarrow C_1 + \frac{c}{\lambda - b - c} C_2 + \frac{c}{\lambda - b - c} C_3 + \dots + \frac{c}{\lambda - b - c} C_{\frac{n+1}{2}}$.
Hence we can write

$$P_M(\lambda) = \begin{vmatrix} \frac{(\lambda - a)(\lambda - b - c) - c^2(n-1)}{\lambda - b - c} & -2c & -2c & \dots & -c & -c \\ 0 & \lambda - b - c & 0 & \dots & 0 & -c \\ 0 & 0 & \lambda - b - c & \dots & -c & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda - b + c & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda - b + c \end{vmatrix}.$$

It follows that

$$P_M(\lambda) = \left(\lambda^2 - (a+b+c)\lambda + a(b+c) - c^2(n-1)\right)(\lambda - b - c)^{\frac{n-3}{2}}(\lambda - b + c)^{\frac{n-1}{2}},$$

due to it is an upper triangular matrix.

Theorem 6. *Let n is even and $M_{n\times n}$ be the matrix as follows:*

$$M = \begin{bmatrix} a & d & d & \dots & d & d & d & \dots & d & d \\ d & b & 0 & \dots & 0 & 0 & 0 & \dots & 0 & d \\ d & 0 & b & \dots & 0 & 0 & 0 & \dots & d & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d & 0 & 0 & \dots & b & 0 & d & \dots & 0 & 0 \\ d & 0 & 0 & \dots & 0 & c & 0 & \dots & 0 & 0 \\ d & 0 & 0 & \dots & d & 0 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d & 0 & d & \dots & 0 & 0 & 0 & \dots & b & 0 \\ d & d & 0 & \dots & 0 & 0 & 0 & \dots & 0 & b \end{bmatrix},$$

where a, b, c, d are real numbers. The characteristic polynomial of M is

$$P_M(\lambda) = \left(\lambda^3 - (a+b+c+d)\lambda^2 + ((a+c)(b+d) + ac - d^2(n-1))\lambda + (b+d)(d^2 - ac) + cd^2(n-2)\right)$$
$$(\lambda - b - d)^{\frac{n}{2}-2}(\lambda - b + d)^{\frac{n}{2}-1}..$$

Proof. Let n is even, and a, b, c, d are real numbers. The characteristic polynomial of M is given by

$$P_M(\lambda) = \begin{vmatrix} \lambda - a & -d & -d & \dots & -d & -d & -d & \dots & -d & -d \\ -d & \lambda - b & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -d \\ -d & 0 & \lambda - b & \dots & 0 & 0 & 0 & \dots & -d & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d & 0 & 0 & \dots & \lambda - b & 0 & -d & \dots & 0 & 0 \\ -d & 0 & 0 & \dots & 0 & \lambda - c & 0 & \dots & 0 & 0 \\ -d & 0 & 0 & \dots & -d & 0 & \lambda - b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d & 0 & -d & \dots & 0 & 0 & 0 & \dots & \lambda - b & 0 \\ -d & -d & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda - b \end{vmatrix}.$$

We need to simplify the above determinant by applying row and column operations.

(i) $R_{\frac{n}{2}+1+i} \longrightarrow R_{\frac{n}{2}+1+i} - R_{\frac{n}{2}+1-i}$, for $i = 1, 2, \dots, \frac{n}{2} - 1$.

Then we obtain

$$P_M(\lambda) = \begin{vmatrix} \lambda-a & -d & -d & \dots & -d & -d & -d & \dots & -d & -d \\ -d & \lambda-b & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -d \\ -d & 0 & \lambda-b & \dots & 0 & 0 & 0 & \dots & -d & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d & 0 & 0 & \dots & \lambda-b & 0 & -d & \dots & 0 & 0 \\ -d & 0 & 0 & \dots & 0 & \lambda-c & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -\lambda+b-d & 0 & \lambda-b+d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -\lambda+b-d & \dots & 0 & 0 & 0 & \dots & \lambda-b+d & 0 \\ 0 & -\lambda+b-d & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda-b+d \end{vmatrix}.$$

(ii) $C_{\frac{n}{2}+1-i} \longrightarrow C_{\frac{n}{2}+1-i} + C_{\frac{n}{2}+1+i}$, for $i = 1, 2, \dots, \frac{n}{2} - 1$.
Consequently, we have

$$P_M(\lambda) = \begin{vmatrix} \lambda-a & -2d & -2d & \dots & -2d & -d & -d & \dots & -d & -d \\ -d & \lambda-b-d & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -d \\ -d & 0 & \lambda-b-d & \dots & 0 & 0 & 0 & \dots & -d & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d & 0 & 0 & \dots & \lambda-b-d & 0 & -d & \dots & 0 & 0 \\ -d & 0 & 0 & \dots & 0 & \lambda-c & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda-b+d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda-b+d & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda-b+d \end{vmatrix}.$$

(iii) $C_1 \longrightarrow C_1 + \frac{d}{\lambda-b-d}C_2 + \frac{d}{\lambda-b-d}C_3 + \dots + \frac{d}{\lambda-b-d}C_{\frac{n}{2}-1} + \frac{d}{\lambda-b-d}C_{\frac{n}{2}}$.
Then, we can state that $P_M(\lambda)$ is

$$\begin{vmatrix} \frac{(\lambda-a)(\lambda-b-d)-d^2(n-2)}{\lambda-b-d} & -2d & -2d & \dots & -2d & 0 & -d & \dots & -d & -d \\ 0 & \lambda-b-d & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -d \\ 0 & 0 & \lambda-b-d & \dots & 0 & 0 & 0 & \dots & -d & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda-b-d & 0 & -d & \dots & 0 & 0 \\ -d & 0 & 0 & \dots & 0 & \lambda-c & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda-b+d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda-b+d & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda-b+d \end{vmatrix}.$$

(iv) $C_1 \longrightarrow C_1 + \frac{d}{\lambda-c}C_{\frac{n}{2}+1}$.
It follows that $P_M(\lambda)$ is

$$\begin{vmatrix} \frac{(\lambda-a)(\lambda-c)(\lambda-b-d)-d^2((n-2)(n-c)+(\lambda-b-d)}{(\lambda-b-d)(\lambda-c)} & -2d & -2d & \dots & -2d & 0 & -d & \dots & -d & -d \\ 0 & \lambda-b-d & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -d \\ 0 & 0 & \lambda-b-d & \dots & 0 & 0 & 0 & \dots & -d & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda-b-d & 0 & -d & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda-c & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda-b+d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & \lambda-b+d & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \lambda-b+d \end{vmatrix}.$$

The matrix form is upper triangular, consequently, we derive the following formula:

$$P_M(\lambda) = \big(\lambda^3 - (a+b+c+d)\lambda^2 + ((a+c)(b+d) + ac - d^2(n-1))\lambda + (b+d)(d^2 - ac) + cd^2(n-2)\big)$$
$$(\lambda - b - d)^{\frac{n}{2}-2}(\lambda - b + d)^{\frac{n}{2}-1}.$$

3.1. Adjacency Matrix

In this part, the goal is to provide the energy formula of $\Gamma_{\mathbb{Z}_n}$ associated with the adjacency matrix. We begin with the case for n is odd.

Theorem 7. *Let $\Gamma_{\mathbb{Z}_n}$ be the identity graph on $\mathbb{Z}_n$. The adjacency energy of $\Gamma_{\mathbb{Z}_n}$ for odd n is*

$$E_A(\Gamma_{\mathbb{Z}_n}) = n - 2 + \sqrt{4n-3}.$$

Proof. According to Theorems 1 and 3, we can construct an $n \times n$ adjacency matrix of $\Gamma_{\mathbb{Z}_n}$ as follows:

$$A(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array} \tag{2}$$

Following the principle of Theorem 5 with $a = b = 0$ and $c = 1$, then we derive

$$P_{A(\Gamma_{\mathbb{Z}_n})}(\lambda) = \big(\lambda^2 - \lambda - (n-1)\big)(\lambda - 1)^{\frac{n-3}{2}}(\lambda + 1)^{\frac{n-1}{2}}.$$

The roots of $P_{A(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ are $\lambda_1 = 1$ of multiplicity $\frac{n-3}{2}$, $\lambda_2 = -1$ of multiplicity $\frac{n-1}{2}$, and $\lambda_{3,4} = \frac{1}{2} \pm \frac{\sqrt{4n-3}}{2}$ of multiplicity 1, respectively. Consequently, the spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$Spec_A(\Gamma_{\mathbb{Z}_n}) = \left\{ \left(\frac{1}{2} + \frac{\sqrt{4n-3}}{2}\right)^1, (1)^{\frac{n-3}{2}}, (-1)^{\frac{n-1}{2}}, \left(\frac{1}{2} - \frac{\sqrt{4n-3}}{2}\right)^1 \right\}.$$

It is clear that the spectral radius of $\Gamma_{\mathbb{Z}_n}$ is

$$\rho_A(\Gamma_{\mathbb{Z}_n}) = \frac{1}{2} + \frac{\sqrt{4n-3}}{2}.$$

Therefore, the adjacency energy of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$\begin{aligned} E_A(\Gamma_{\mathbb{Z}_n}) &= \left(\frac{n-3}{2}\right)|1| + \left(\frac{n-1}{2}\right)|-1| + \left|\frac{1}{2} \pm \frac{\sqrt{4n-3}}{2}\right| \\ &= n - 2 + \sqrt{4n-3}. \end{aligned}$$

Theorem 8. *Let $\Gamma_{\mathbb{Z}_n}$ be the identity graph on $\mathbb{Z}_n$. The characteristic polynomial of $A(\Gamma_{\mathbb{Z}_n})$ for even n is*

$$P_{A(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - \lambda^2 - (n-1)\lambda + 1\right)(\lambda - 1)^{\frac{n}{2}-2}(\lambda + 1)^{\frac{n}{2}-1}.$$

Proof. According to Theorems 2 and 4, we can construct an $n \times n$ adjacency matrix of $\Gamma_{\mathbb{Z}_n}$ as follows:

$$A(\Gamma_{\mathbb{Z}_n}) = \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{\frac{n}{2}-1} \\ \overline{\frac{n}{2}} \\ \overline{\frac{n}{2}+1} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{\frac{n}{2}-1} & \overline{\frac{n}{2}} & \overline{\frac{n}{2}+1} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}. \quad (3)$$

According to Theorem 6 with $a = b = c = 0$, $d = 1$, consequently, we derive the following formula:

$$P_{A(\Gamma_{\mathbb{Z}_n})}(\lambda) = \left(\lambda^3 - \lambda^2 - (n-1)\lambda + 1\right)(\lambda - 1)^{\frac{n}{2}-2}(\lambda + 1)^{\frac{n}{2}-1}.$$

3.2. Laplacian Matrix

This part focuses on the Laplacian matrix of $\Gamma_{\mathbb{Z}_n}$, for odd and even n, followed by calculating the energy.

Theorem 9. *Let $\Gamma_{\mathbb{Z}_n}$ be the identity graph on $\mathbb{Z}_n$. The Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ for odd n is*

$$E_L(\Gamma_{\mathbb{Z}_n}) = 3(n-1).$$

Proof. Based on Theorem 3, we have $n \times n$ degree matrix of $\Gamma_{\mathbb{Z}_n}$ as $diag(n-1, 2, 2, ..., 2, 2)$. According to Definition 4 and Equation 2, we can construct an $n \times n$ Laplacian matrix of $\Gamma_{\mathbb{Z}_n}$ as follows:

$$L(\Gamma_{\mathbb{Z}_n}) = D(\Gamma_{\mathbb{Z}_n}) - A(\Gamma_{\mathbb{Z}_n}) \quad (4)$$

$$= \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 & -1 \\ -1 & 2 & 0 & \dots & 0 & -1 \\ -1 & 0 & 2 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & -1 & \dots & 2 & 0 \\ -1 & -1 & 0 & \dots & 0 & 2 \end{pmatrix} \end{array}. \quad (5)$$

According to Theorem 5 with $a = n - 1$, $b = 2$, and $c = -1$, then we obtain

$$P_{L(\Gamma_{\mathbb{Z}_n})}(\lambda) = \lambda(\lambda - n)(\lambda - 1)^{\frac{n-3}{2}}(\lambda - 3)^{\frac{n-1}{2}}.$$

The roots of $P_{L(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ are $\lambda_1 = 0$ of multiplicity 1, $\lambda_2 = n$ of multiplicity 1, $\lambda_3 = 1$ of multiplicity $\frac{n-3}{2}$, and $\lambda_4 = 3$ of multiplicity $\frac{n-1}{2}$. Consequently, the spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$Spec_L(\Gamma_{\mathbb{Z}_n}) = \left\{ (n)^1, (1)^{\frac{n-3}{2}}, (3)^{\frac{n-1}{2}}, (0)^1 \right\}.$$

It is clear that the spectral radius of $\Gamma_{\mathbb{Z}_n}$ is

$$\rho_L(\Gamma_{\mathbb{Z}_n}) = n.$$

Therefore, the Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$\begin{aligned} E_L(\Gamma_{\mathbb{Z}_n}) &= (1)\,|n| + \left(\frac{n-1}{2}\right)|3| + \left(\frac{n-3}{2}\right)|1| + (1)\,|0| \\ &= 3(n-1). \end{aligned}$$

Theorem 10. *Let $\Gamma_{\mathbb{Z}_n}$ be the identity graph on $\mathbb{Z}_n$. The Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ for even n is*

$$E_L(\Gamma_{\mathbb{Z}_n}) = 3n - 4.$$

Proof. From Theorem 2, we have the degree of every vertex in $\Gamma_{\mathbb{Z}_n}$ for even n. Then the diagonal matrix of $\Gamma_{\mathbb{Z}_n}$ is $D(\Gamma_{\mathbb{Z}_n}) = diag(n-1, 2, 2, ..., 2, 1, 2, ..., 2)$. Based on Definition 4 and Equation 3, we can construct an $n \times n$ Laplacian matrix of $\Gamma_{\mathbb{Z}_n}$ as follows:

$$L(\Gamma_{\mathbb{Z}_n}) = D(\Gamma_{\mathbb{Z}_n}) - A(\Gamma_{\mathbb{Z}_n}) \quad (6)$$

$$
= \begin{array}{c} \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{\frac{n}{2}-1} \\ \overline{\frac{n}{2}} \\ \overline{\frac{n}{2}+1} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array}
\overset{\begin{array}{cccccccccc} \overline{0} & \overline{1} & \overline{2} & \ldots & \overline{\frac{n}{2}-1} & \overline{\frac{n}{2}} & \overline{\frac{n}{2}+1} & \ldots & \overline{n-2} & \overline{n-1} \end{array}}{\begin{pmatrix}
n-1 & -1 & -1 & \ldots & -1 & -1 & -1 & \ldots & -1 & -1 \\
-1 & 2 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & -1 \\
-1 & 0 & 2 & \ldots & 0 & 0 & 0 & \ldots & -1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & \ldots & 2 & 0 & -1 & \ldots & 0 & 0 \\
-1 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\
-1 & 0 & 0 & \ldots & -1 & 0 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & -1 & \ldots & 0 & 0 & 0 & \ldots & 2 & 0 \\
-1 & -1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 2
\end{pmatrix}}. \quad (7)
$$

Following the guideline in Theorem 6 with $a = n-1$, $b = 2$, $c = 1$, and $d = -1$, then we can write the following expression:

$$P_{L(\Gamma_{\mathbb{Z}_n})}(\lambda) = \lambda(\lambda - n)(\lambda - 1)^{\frac{n}{2}-1}(\lambda - 3)^{\frac{n}{2}-1}.$$

The roots of $P_{L(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ are $\lambda_1 = 0$ of multiplicity 1, $\lambda_2 = n$ of multiplicity 1, $\lambda_3 = 1$ of multiplicity $\frac{n}{2} - 1$, and $\lambda_4 = 3$ of multiplicity $\frac{n}{2} - 1$. Consequently, the spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$Spec_L(\Gamma_{\mathbb{Z}_n}) = \left\{(n)^1, (3)^{\frac{n}{2}-1}, (1)^{\frac{n}{2}-1}, (0)^1\right\}.$$

It is clear that the spectral radius of $\Gamma_{\mathbb{Z}_n}$ is

$$\rho_L(\Gamma_{\mathbb{Z}_n}) = n.$$

Therefore, the Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$\begin{aligned} E_L(\Gamma_{\mathbb{Z}_n}) &= (1)\,|n| + \left(\frac{n}{2} - 1\right)|3| + \left(\frac{n}{2} - 1\right)|1| + (1)\,|0| \\ &= 3n - 4. \end{aligned}$$

3.3. Signless Laplacian Matrix

Next, we show the energy of $\Gamma_{\mathbb{Z}_n}$ with respect to the signless Laplacian matrix, for odd and even n.

Theorem 11. *Let $\Gamma_{\mathbb{Z}_n}$ be the identity graph on $\mathbb{Z}_n$. The signless Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ for odd n is*

$$E_{SL}(\Gamma_{\mathbb{Z}_n}) = 3(n-1).$$

Proof. Based on Theorem 3, we have $n \times n$ degree matrix of $\Gamma_{\mathbb{Z}_n}$ as $diag(n-1, 2, 2, ..., 2, 2)$. According to Definition 5 and Equation 2, we can construct an $n \times n$ signless Laplacian matrix of $\Gamma_{\mathbb{Z}_n}$ as follows:

$$SL(\Gamma_{\mathbb{Z}_n}) = D(\Gamma_{\mathbb{Z}_n}) + A(\Gamma_{\mathbb{Z}_n}) \tag{8}$$

$$= \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} n-1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 0 & \dots & 0 & 1 \\ 1 & 0 & 2 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 1 & \dots & 2 & 0 \\ 1 & 1 & 0 & \dots & 0 & 2 \end{pmatrix} \end{array} \tag{9}$$

From Theorem 5 with $a = n-1$, $b = 2$ and $c = 1$, we can simplify $P_{SL(\Gamma_{\mathbb{Z}_n})}(\lambda)$ as follows:

$$P_{SL(\Gamma_{\mathbb{Z}_n})}(\lambda) = (\lambda^2 - (2+n)\lambda + 2(n-1))(\lambda - 3)^{\frac{n-3}{2}}(\lambda - 1)^{\frac{n-1}{2}}.$$

The roots of $P_{SL(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ are $\lambda_1 = 3$ of multiplicity $\frac{n-3}{2}$, $\lambda_2 = 1$ of multiplicity $\frac{n-1}{2}$, $\lambda_{3,4} = \frac{2+n}{2} \pm \frac{\sqrt{n^2-4n+12}}{2}$ of multiplicity 1, respectively. Consequently, the spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$Spec_{SL}(\Gamma_{\mathbb{Z}_n}) = \left\{ \left(\frac{2+n}{2} + \frac{\sqrt{n^2-4n+12}}{2} \right)^1, (3)^{\frac{n-3}{2}}, (1)^{\frac{n-1}{2}}, \left(\frac{2+n}{2} - \frac{\sqrt{n^2-4n+12}}{2} \right)^1 \right\}.$$

It is clear that the spectral radius of $\Gamma_{\mathbb{Z}_n}$ is

$$\rho_{SL}(\Gamma_{\mathbb{Z}_n}) = \frac{2+n}{2} + \frac{\sqrt{n^2-4n+12}}{2}.$$

Therefore, the signless Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$\begin{aligned} E_{SL}(\Gamma_{\mathbb{Z}_n}) &= \left(\frac{n-3}{2} \right) |3| + \left(\frac{n-1}{2} \right) |1| + \left| \frac{2+n}{2} \pm \frac{\sqrt{n^2-4n+12}}{2} \right| \\ &= 3(n-1). \end{aligned}$$

Theorem 12. *Let $\Gamma_{\mathbb{Z}_n}$ be the identity graph on $\mathbb{Z}_n$. The signless Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ for even n is*

$$E_{SL}(\Gamma_{\mathbb{Z}_n}) = 3n - 4.$$

Proof. Since the diagonal matrix of $\Gamma_{\mathbb{Z}_n}$ is $D(\Gamma_{\mathbb{Z}_n}) = diag(n-1, 2, 2, ..., 2, 1, 2, ..., 2)$. Based on Definition 5 and Equation 3, we can construct an $n \times n$ signless Laplacian matrix of $\Gamma_{\mathbb{Z}_n}$ as follows:

$$SL(\Gamma_{\mathbb{Z}_n}) = D(\Gamma_{\mathbb{Z}_n}) + A(\Gamma_{\mathbb{Z}_n}) \tag{10}$$

$$= \begin{array}{c} \\ \overline{0} \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{\frac{n}{2}-1} \\ \overline{\frac{n}{2}} \\ \overline{\frac{n}{2}+1} \\ \vdots \\ \overline{n-2} \\ \overline{n-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \overline{0} & \overline{1} & \overline{2} & \dots & \overline{\frac{n}{2}-1} & \overline{\frac{n}{2}} & \overline{\frac{n}{2}+1} & \dots & \overline{n-2} & \overline{n-1} \end{array} \\ \begin{pmatrix} n-1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 2 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 2 & 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 1 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 2 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 2 \end{pmatrix} \end{array}. \tag{11}$$

Again, by Theorem 6 with $a = n-1$, $b = 2$, $c = 1$, and $d = 1$, we have

$$\begin{aligned} P_{SL(\Gamma_{\mathbb{Z}_n})}(\lambda) =& (\lambda^3 - (n+3)\lambda^2 + 3n\lambda - 2(n-2))(\lambda-3)^{\frac{n}{2}-2}(\lambda-1)^{\frac{n}{2}-1} \\ =& (\lambda-2)(\lambda^2 - (n+1)\lambda + n - 2)(\lambda-3)^{\frac{n}{2}-2}(\lambda-1)^{\frac{n}{2}-1}. \end{aligned}$$

The roots of $P_{SL(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ are $\lambda_1 = 2$ of multiplicity 1, $\lambda_2 = 3$ of multiplicity $\frac{n}{2} - 2$, $\lambda_3 = 1$ of multiplicity $\frac{n}{2} - 1$, and $\lambda_{4,5} = \frac{n+1}{2} \pm \frac{\sqrt{n^2-2n+9}}{2}$ of multiplicity 1, respectively. Consequently, the spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$Spec_{SL}(\Gamma_{\mathbb{Z}_n}) = \left\{ \left(\frac{n+1}{2} + \frac{\sqrt{n^2-2n+9}}{2} \right)^1, (3)^{\frac{n}{2}-2}, (2)^1, (1)^{\frac{n}{2}-1}, \left(\frac{n+1}{2} - \frac{\sqrt{n^2-2n+9}}{2} \right)^1 \right\}.$$

It is clear that the spectral radius of $\Gamma_{\mathbb{Z}_n}$ is

$$\rho_{SL}(\Gamma_{\mathbb{Z}_n}) = \frac{n+1}{2} + \frac{\sqrt{n^2-2n+9}}{2}.$$

Therefore, the signless Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ is as follows:

$$\begin{aligned} E_{SL}(\Gamma_{\mathbb{Z}_n}) &= \left(\frac{n}{2} - 2 \right) |3| + (1)\,|2| + \left(\frac{n}{2} - 1 \right) |1| + \left| \frac{n+1}{2} \pm \frac{\sqrt{n^2-2n+9}}{2} \right| \\ &= 3n - 4. \end{aligned}$$

4. Discussion

From the results of the previous section, we can conclude several interesting statements.

Corollary 1. *The Laplacian energy of* $\Gamma_{\mathbb{Z}_n}$ *is always similar to the signless Laplacian energy of* $\Gamma_{\mathbb{Z}_n}$*.*

Corollary 2. *The energy of* $\Gamma_{\mathbb{Z}_n}$ *is always an even integer associated with the Laplacian and signless Laplacian matrices.*

Corollary 3. *The energy of* $\Gamma_{\mathbb{Z}_n}$ *for odd* n *is never an odd integer associated with the adjacency matrix.*

Corollary 4. $\Gamma_{\mathbb{Z}_n}$ *is hyperenergetic associated with the Laplacian and signless Laplacian matrices.*

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References

[1] R B Bapat and S Pati. Energy of a Graph Is Never an Odd Integer. *Bulletin of Kerala Mathematics Association*, 1:129–132, 2004.

[2] R A Borzooei and H Rashmanlou. Cayley interval-valued fuzzy graphs. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 78(3):83–94, 2016.

[3] A E Brouwer and W H Haemers. *Spectra of graphs.* Springer, New York, 2011.

[4] I Gutman. The Energy of Graph. *Ber. Math.-Stat. Sekt. Forschungsz. Graz*, 103:1–2, 1978.

[5] W B V Kandasamy and F Smarandache. *Groups as Graphs.* Editura CuArt, Romania, 2009.

[6] L M Kumari, L Pandiselvi, and K Palani. Quotient energy of zero divisor graphs and identity graph. *Baghdad Sci. J.*, 20(1(SI)):277–282, 2023.

[7] X Li, Y Shi, and I Gutman. *Graph Energy.* Springer, New York, 2012.

[8] V Lokesha, , Y Shanthakumari, and Y Zeba. Energy and Skew Energy of a Modified Graph. *Creat. Math. Inform.*, 30(1):41–48, 2021.

[9] S Pirzada and I Gutman. Energy of a Graph Is Never the Square Root of an Odd Integer. *Applicable Analysis and Discrete Mathematics*, 2:118–121, 2008.

[10] N M Rilwan and A S Hussain. Different kinds of cordial labeling on identity graphs. *Eur. Chem. Bull.*, 12(10):12374–12381, 2023.

[11] M U Romdhini and A Nawawi. Degree subtraction energy of commuting and non-commuting graphs for dihedral groups. *International Journal of Mathematics and Computer Science*, 18(3):497–508, 2023.

[12] M U Romdhini and A. Nawawi. On the spectral radius and Sombor energy of the non-commuting graph for dihedral groups. *Mal. J. Fund. Appl. Sci.*, 20(1):65–73, 2024.

[13] M U Romdhini, A. Nawawi, F Al-Sharqi, and A Al-Quran. Closeness energy of non-commuting graph for dihedral groups. *Eur. J. Pure Appl. Math.*, 17(1):212–221, 2024.

[14] M U Romdhini, A. Nawawi, F Al-Sharqi, and A Al-Quran. Spectral Properties of Power Graph of Dihedral Groups. *Eur. J. Pure Appl. Math.*, 17(2):591–603, 2024.

[15] M U Romdhini, A. Nawawi, F Al-Sharqi, A Al-Quran, and S R Kamali. Wiener-Hosoya energy of non-commuting graph for dihedral groups. *Asia Pac. J. Math.*, 11(9):1–9, 2024.

[16] Y Shanthakumari, M Smitha, and V Lokesha. Euclidean Degree Energy Graphs. *Montes Taurus J. Pure Appl. Math.*, 3(1):89–105, 2021.

[17] X Shi, S Kosari, A A Talebi, S H Sadati, and H Rashmanlou. Investigation of the main energies of picture fuzzy graph and its applications. *Int. J. Comput. Intell. Syst.*, 15(1):31, 2022.

[18] A Siwach, V Bhatia, A Sehgal, and P Rana. Characteristic polynomial of maximum and minimum matrix of square power graph of dihedral group of order $2n$ with odd natural number n. *Int. J. Stat. Appl. Math.*, 9(3):57–64, 2024.