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On the Symmetric Block Design With Parameters (220, 73, 24) Admitting a Group of Order 73

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Abstract. In this paper, we have demonstrated that for a putative symmetric block design $\mathcal{D}$ with parameters (220,73,24) constructed by group G of order 73, there exists only one orbit structure up to isomorphism. The full automorphism group for this orbit structure is provided.

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Key Words and Phrases: Symmetric block design, Orbit structure, Automorphism group

1. Introduction and preliminaries

A $2-(v,k,\lambda)$ block design $(\mathcal{P},\mathcal{B},I)$ is said to be *symmetric* if the relation $|\mathcal{P}| = |\mathcal{B}| = v$ holds and in that case we often speak of a symmetric design with parameters (v,k,λ). The integer $n = k-\lambda$ is called the order of the symmetric block design. The collection of the parameter sets (v,k,λ) for which a symmetric $2-(v,k,\lambda)$ block design exists is often called the "spectrum". The determination of the spectrum for symmetric block designs is a widely open problem. For example, a finite projective plane of order n is a symmetric design with parameters $(n^2+n+1, n+1, 1)$ and it is still unknown whether finite projective planes of non–prime–power order may exist at all.

The existence/non-existence of a symmetric block design has often required "ad hoc" treatments even for a single parameter set (v,k,λ). The most famous instance of this circumstance is perhaps the non-existence of the projective plane of order 10, see [9].

It is worthwhile to investigate symmetric block designs with supplementary characteristics, frequently entailing the premise that a non-trivial automorphism group acts on the design under scrutiny. See for instance [5].

Investigating symmetric block designs of order 49 within the realm of symmetric block designs of square order holds notable significance. Despite the existence of 15 potential parameters (v,k,λ) for symmetric block designs of order 49, only limited results have been established thus far (see [4], [6]). Given the large number of points (blocks) in symmetric

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block designs of order 49, investigating sporadic cases can be highly challenging, unless one assumes the existence of a collineation group.

Several techniques exist for constructing symmetric block designs, each demonstrating effectiveness in specific situations. In this instance, we opt for the tactical decompositions method, as employed by Z. Janko in [7], also referenced in [8], assuming the action of a particular automorphism group on the design under construction. This paper focuses on analyzing a symmetric block design denoted as $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ with parameters $(220, 73, 24)$. At present, the existence or non-existence of such a design remains uncertain to the best of our knowledge. Additionally, we posit that the specified design admits a particular automorphism group of order 73. We expect that the reader has a grasp of fundamental concepts in design theory, as outlined in references such as [2], [3] and [10]. If g denotes an automorphism of a symmetric design $\mathcal{D}$ characterized by parameters (v, k, λ), it is observed that g fixes an equal number of points and blocks, as detailed in [10, Theorem 3.1, p.78]. The sets of these fixed elements we denote by $F_{\mathcal{P}}(g)$ and $F_{\mathcal{B}}(g)$ each, and their number simply by $|F(g)|$. For number of fixed points shall we use the following upper bound, as delineated in [10, Corollary 3.7, p. 82]:

$$|F(g)| \leq k + \sqrt{k - \lambda}. \tag{1}$$

It's established that an automorphism group G of a symmetric block design exhibits an equal number of orbits on both the set of points $\mathcal{P}$ and the set of blocks $\mathcal{B}$, as outlined in [10, Theorem 3.3, p.79]. This number is denoted by t.

We utilize the notation and terminology introduced in Section 1 of [5], recapitulating certain essential relations for the reader's convenience. Consider $\mathcal{D}$ as a symmetric block design characterized by parameters (v, k, λ), with G representing a subgroup of the automorphism group $Aut(\mathcal{D})$. The point orbits of G on $\mathcal{P}$ are denoted as $\mathcal{P}_1, \mathcal{P}_2, \dots \mathcal{P}_t$, and the line orbits of G on $\mathcal{B}$ as $\mathcal{B}_1, \mathcal{B}_2, \dots \mathcal{B}_t$. Let $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. Clearly,

$$\sum_{r=1}^{t} \omega_r = \sum_{i=1}^{t} \Omega_i = v. \tag{2}$$

Consider γ_{ir} as the number of points from $\mathcal{P}_r$ situated on a block from $\mathcal{B}_i$; evidently, this number remains invariant regardless of the chosen block. Similarly, let Γ_{js} be the number of blocks from $\mathcal{B}_j$ intersecting a point from $\mathcal{P}_s$. It is evident that,

$$\sum_{r=1}^{t} \gamma_{ir} = k \text{ and } \sum_{j=1}^{t} \Gamma_{js} = k. \tag{3}$$

By [3, Lemma 5.3.1. p.221], the division of both the point set $\mathcal{P}$ and the block set $\mathcal{B}$ constitutes a tactical decomposition of design $\mathcal{D}$ as defined in [3, p.210]. Consequently, the ensuing equations are valid:

$$\Omega_i \cdot \gamma_{ir} = \omega_r \cdot \Gamma_{ir}, \tag{4}$$

$$\sum_{r=1}^{t} \gamma_{ir}\Gamma_{jr} = \lambda\Omega_j + \delta_{ij}(k-\lambda), \tag{5}$$

$$\sum_{i=1}^{t} \Gamma_{ir}\gamma_{is} = \lambda\omega_s + \delta_{rs}(k-\lambda), \tag{6}$$

where δ_{ij}, δ_{rs} are the Kronecker symbols.

For verification of these equations, readers are encouraged to consult [3] and [5]. Combining Equation (5) with (4) results in

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r}\gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij}(k-\lambda). \tag{7}$$

Definition 1. *We denote*

$$[L_i, L_j] = \sum_{r=1}^{t} \frac{\Omega_j}{\omega_r}\gamma_{ir}\gamma_{jr}, 1 \leq i, j \leq t$$

and term these expressions as the orbit products. The $(t \times t)$*-matrix* (γ_{ir}) *is called the orbit structure of the block design* $\mathcal{D}$*.*

An automorphism of an orbit structure entails a permutation of rows followed by a permutation of columns, maintaining the matrix unchanged. It's evident that the collection of all such automorphisms forms a group, referred to as the automorphism group of that orbit structure.

The initial step in constructing a design is to identify all potential orbit structures. The subsequent step, typically referred to as indexing, involves specifying which γ_{ir} points of the orbit $\mathcal{P}_r$ lie on the blocks of the block orbit $\mathcal{B}_i$ for each coefficient γ_{ir} of the orbit matrix. Naturally, this process only needs to be performed for a representative of each block orbit, since the other blocks in that orbit can be generated by producing all G-images of the selected representative.

2. Main results

Let $\mathcal{D}$ represent the symmetric block design with parameters $(220, 73, 24)$. Given that $v = 1 + 3 \cdot 73$, to construct the symmetric block design $\mathcal{D}$, we employ the cyclic group $G = \langle \rho | \rho^{73} = 1 \rangle$ of order 73 as a collineation group.

Lemma 1. *Let* ρ *be an element of* G *with* $o(\rho) = 73$*. Then* $\langle \rho \rangle$ *fixes exactly one point and one block.*

Proof. According to [10, Theorem 3.1], the group $\langle \rho \rangle$ fixes the same number of points and blocks. Let this number be denoted by f. Clearly, $f \equiv 220 (\text{mod } 73)$, which means $f \equiv 1 (\text{mod } 73)$. The upper bound (1) for the number of fixed points gives $f \in \{1, 74\}$.

Since $o(\rho) > \lambda$, applying a result from M. Aschbacher [1, Lemma 2.6, p.274] necessitates the fixed structure being a subdesign of $\mathcal{D}$. However, there is no symmetric block design with $v = 74$ and $\lambda = 24$ (no $k \in N$ satisfies $24 \cdot (v-1) = k \cdot (k-1)$). Therefore, f must be equal to 1.

We set $\mathcal{P}_I = \{I_0, I_1, \cdots, I_{72}\}$, $I = 1, 2, 3$, for the non-trivial point orbits of the group G. Consecuently, G uniquely acts as a permutation group on these point orbits. Therefore, the generator of G can be defined as follows.

$$\rho = (\infty)(I_0, I_1, \cdots, I_{72}), I = 1, 2, 3,$$

where ∞ is the fixed point of the collineation, the non-trivial $\langle\rho\rangle$-orbits are the numbers 1, 2, 3, and $\infty, 1_0, 1_1, \cdots, 3_{72}$ represent all points of the symmetric block design $\mathcal{D}$.

In the following steps, we will construct a representative block for each block orbit.

The $\langle\rho\rangle-$fixed block can be expressed as:

$$L_1 = (1_0 1_1 \cdots 1_{72})$$

or

$$L_1 = 1_{73}.$$

Let L_2, L_3, and L_4 be the representative blocks for the three non-trivial block orbits. The second ρ–orbit block, L_2, of block design $\mathcal{D}$, constructed by the collineation ρ, can be expressed as

$$L_2 = \infty 1_{a_1} 2_{a_2} 3_{a_3},$$

where $a_i, i = 1, 2, 3$, represent the multiplicities of the occurrence of orbit numbers $1, 2$, and 3 in the orbit block L_2. The multiplicities of these orbit numbers must satisfy the following conditions:

$$a_1 + a_2 + a_3 = 72.$$

Because $|L_1 \cap L_2| = 24$, we have $a_1 = 24$. From (7) we have
$[L_2, L_2] = 73/1 \cdot 1 \cdot 1 + 73/73 \cdot a_1^2 + 73/73 \cdot a_2^2 + 73/73 \cdot a_3^2 = 24 \cdot 73 + 73 - 24 = 1801$, i.e.

$$a_1^2 + a_2^2 + a_3^2 = 1728$$

or

$$a_2^2 + a_3^2 = 1152,$$

whence it follows that the multiplicities of appearances in block L_2 yield the constraints $0 \leq a_i \leq 33, i = 2, 3$.

To minimize isomorphic cases in the orbit structures at the final stage, we can assume without loss of generality that $a_2 \geq a_3$ for block L_2.

Using computational methods, we have demonstrated that there exists exactly one orbit type for block L_2 that meets the aforementioned conditions:

	a_1	a_2	a_3
1.	24	24	24

The form of the third orbit block L_3, created using the collineation ρ, is as follows:

$$L_3 = 1_{b_1} 2_{b_2} 3_{b_3}.$$

Here, b_i, where $i = 1, 2, 3$ represent the frequencies of orbit numbers $1, 2,$ and 3 appearing in orbit block L_3.

The occurrences of orbit numbers must adhere to the following criteria:

$$b_1 + b_2 + b_3 = 73.$$

$[L_1 \cap L_3] = 24$ implies $b_1 = 24$. From (7) we have
$[L_3, L_3] = b_1^2 + b_2^2 + b_3^2 = 24 \cdot 73 + 73 - 24 = 1801$
or
$b_2^2 + b_3^2 = 1225.$
Based on the previous relation, we deduce the constraints $0 \leq b_i \leq 35$, where $i = 2, 3$.
$[L_2, L_3] = a_1 b_1 + a_2 b_2 + a_3 b_3 = 24 \cdot 73 = 1752.$

Through computational analysis, we have confirmed the existence of precisely two orbit types for block L_3 that meet the aforementioned criteria:

	b_1	b_2	b_3
1.	24	28	21
2.	24	21	28

It is evident that among the contenders for block L_3 are also blocks L_4. Consequently, we examine pairs of blocks $\{L_3, l_4\}$ that are mutually compatible. Through this approach, we have determined that, up to isomorphism, there exists precisely one orbit structure for the symmetric block design with parameters $(220, 73, 24)$ under the action of the collineation ρ of order 73:

Orbit structure:

SO	1	73	73	73
	0	73	0	0
	1	24	24	24
	0	24	28	21
	0	24	21	28

Full automporphism group of the orbit stucture is:

$$Aut(SO) = \{1, (3\ 4)(\bar{3}\ \bar{4})\}$$

Thus we have

Theorem 1. *There exists precisely one orbit structure, up to isomorphism, for the symmetric block design $\mathcal{D}$ characterized by parameters $(220, 73, 24)$ and accommodating the group G of order 73.*

Remark 1. *The specific indexing of this orbit structure to generate an example remains an unresolved issue.*

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References

[1] M. Aschbacher. On collineation groups of symmetric block designs. *J. Comb. Theory Ser. A*, 11:272–281, 1971.

[2] T. Beth, D Jungnickel, and H. Lenz. *Design Theory.* Bibliographisches Institut, Mannheim-Wien-Zürich, 1999.

[3] A. Beutelspacher. *Einführung in die endliche Geometrie I.* Bibliographisches Institut, Mannheim-Wien-Zürich, 1985.

[4] D. Crnković. Some new menon designs with parameters (196,91,42). *Mathematical Communications*, 10(2):169–175, 2005.

[5] V. Ćepulić. On symmetric block designs $(40, 13, 4)$ with automorphisms of order 5. *Discrete Math.*, 28:45–60, 1994.

[6] M. Gashi. A construction of a symmetric design with parameters (195,97,48) with help of frobenius group of order 4656. *International Mathematical Forum*, 5(8):383 – 388, 2009.

[7] Z. Janko. Coset enumeration in groups and constructions of symmetric designs. *Combinatorics*, '90:275–277, 1992.

[8] Z. Janko and Tran van Trung. Construction of a new symmetric block design for $(78, 22, 6)$ with help of tactical decompositions. *J. Comb. Theory, Ser. A*, 40:451–455, 1995.

[9] C. W. H. Lam. The search for a finite projective plane of order 10. *Amer. Math. Monthly*, 98:305–318, 1992.

[10] E. Lander. *Symmetric designs: An algebraic approach.* Cambridge University Press, 1983.