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Local Total Antimagic Chromatic Number for the Disjoint Union of Star Graphs

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Abstract. Let G be a graph with n vertices and m edges without isolated vertices. A local total antimagic labeling of a graph G is defined as there is a bijection f from the set $V(G) \cup E(G) \to \{1, 2, ..., n+m\}$, such that, any two adjacent vertices, any two adjacent edges, a vertex and an edge incident to the vertex does not receive the same weight. The vertex weight is the sum of the edge labels incident to that vertex and the edge weight is the sum of its end vertex labels. The local total antimagic chromatic number is the minimum number of colors taken over all induced by local total antimagic colorings (labelings) of G, which is denoted by $\chi_{lt}(G)$. In this paper, we determine the local total antimagic chromatic number for the disjoint union of star graphs.

2020 Mathematics Subject Classifications: 05C78, 05C15

Key Words and Phrases: Local antimagic graphs, chromatic number, total coloring

1. Introduction

Let G be a graph with no isolated vertices, neither multiple edges nor loops. The set of vertices and edges of $G(V, E)$ are denoted by $V(G)$ and $E(G)$ respectively. For graph theoretic terminology, we refer to Chartrand and Lesniak [9] and [10]. Graph coloring and graph labeling are the significant research areas in graph theory. The coloring of a graph G is known as the assignment of colors to vertices or edges or both is called as vertex coloring or edge coloring or total coloring. In this study, we consider the total coloring of a graph.

Graph coloring has various applications in network optimization and scheduling problems. Total Coloring, being a slight variation and generalization of the usual vertex coloring. The main application of total coloring is scheduling trains in a region which has high rail traffic with several interconnected tracks, and the region between two stations either consisting of dense forests with wild animals frequently found roaming on tracks at a specific time, or unmanned level-crossing gates with peak hour traffic congestion problems.

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In order to schedule the appropriate train timetables, one can use the total coloring of the rail network graph, where the train stations are vertices of the graph and the connecting tracks form the edges. The scheduling involving multi-factors can be devised using total coloring. The advancement in signal processing and control systems using Z and L-transforms are given in [13].

Definition 1. [39] The total coloring of G is defined as a map $f : V(G) \cup E(G) \to K$, where K is a set of colors, satisfying the following three conditions:

(i) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$;

(ii) $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$; and

(iii) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and an incident edge $e = vx$.

The minimum number of colors used in any total coloring of G is called the total chromatic number $\chi_t(G)$. A trivial lower bound for total chromatic number is maximum degree of the graph G plus one. Behzad [8] and Vizing [38] posed the conjecture independently that, for any graph G, $\chi_t(G) \leq \Delta(G) + 2$. This lower and upper bounds leads to every graph G has $\Delta(G) + 1 \leq \chi_t(G) \leq \Delta(G) + 2$.

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. To learn more about graph labeling we refer Gallian's [12] survey of graph labeling. Graph labeling have been motivated by practical problems, labeled graphs serve useful mathematical models for a broad range of applications such as: coding theory, including the design of good types codes, synch-set codes, missile guidance codes and convolutional codes with optimal auto correlation properties.

The idea of an antimagic labeling of a graph were proposed by Hartsfield and Ringel [11].

Definition 2. [11] A bijection $f : E \to \{1, 2, 3, ..., |E|\}$ is called an antimagic labeling of G, if $w(u) \neq w(v)$ for any two distinct vertices u, $v \in V(G)$. The weight of the vertex u is defined as $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of edges incident to u. A graph G is antimagic if G admits an antimagic labeling.

Hartsfield and Ringel [11] proposed the following conjectures.

Conjecture 1. [11] Every connected graph other than K_2 is antimagic.

Conjecture 2. [11] Every tree other than K_2 is antimagic.

Based on the above two conjectures several authors are studied and obtained several results. In [14–17], the authors studies the b-chromatic number of some special and standard graphs. Arumugam and Nalliah [3] found the super (a, d)-edge antimagic total labelings of friendship graphs in 2012. For further results one can refer [4, 5, 24–27]. In 2017, Nalliah and Arumugam [28] determined the super $(a, 3)$-edge antimagic total labeling for union of two stars. For further study see in [31, 33, 34].

In 2017, Arumugam et al.[6] proposed the local antimagic chromatic number using antimagic labeling and vertex coloring concepts, which is given as follows.

Definition 3. [6] A local vertex antimagic labeling is a bijection $f : E \to \{1, 2, ..., |E|\}$ such that $w(u) \neq w(v)$,for all $uv \in E(G)$, where $w(u) = \sum_{e \in E(u)} f(e)$, and $E(u)$ is the set of edges incident to u. If a graph G admits a local antimagic labeling, then G is called local antimagic. The minimum number of colors taken over all colorings induced by local antimagic labelings of G is called local antimagic chromatic number, denoted by $\chi_{la}(G)$.

The local antimagic chromatic number has been studied by several authors for the various families of graphs. In 2022, Lau et.al [22] obtained the local antimagic chromatic number of lexicographic product graph. Also in [23], the authors are given the complete characterization of s-bridge graphs with local antimagic chromatic number 2. Nalliah et.al [29] found the local vertex coloring for generalized friendship graph. Shankar and Nalliah [37] studied the local vertex antimagic chromatic number of some wheel related graph. Also in [36], the authors found the local antimagic chromatic number for the corona product of wheel and null graphs. In 2023, Lau and Nalliah [19] determined the local antimagic chromatic number of a corona product of graph. In [2], authors found the degree based topological indices of corona product graphs. For further study see in [18, 20, 21, 30].

An edge version of local antimagic labeling were introduced by Agustin et al.[1] in 2017, which is given as follows:

Definition 4. [1] A local edge antimagic labeling is defined as there is a bijection $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$, for any two adjacent edges e_1 and e_2 with their weights $w(e_1) \neq w(e_2)$. The edge weight of an edge $e = xy$ is defined by $w(e = xy) = f(x) + f(y)$. If a graph G admits a local edge antimagic labeling, then G is called local edge antimagic. The minimum number of colors taken over all colorings induced by local edge antimagic labelings of G is called local edge antimagic chromatic number, denoted by $\chi'_{lea}(G)$.

Agustin et al.[1] found the local edge antimagic chromatic number for the path graph, star graph, cycle graph and friendship graph. Rajkumar and Nalliah [32] determined the local edge antimagic chromatic number of wheel graph, helm graph, closed helm graph and double star graph.

In [35], Sandhiya and Nalliah introduced the concept of local total antimagic labeling and its related parameter local total antimagic chromatic number.

Definition 5. [35] *Let G be a graph with n vertices, m edges and no isolated vertices. A local total antimagic coloring of a graph G is a bijection $f : V(G) \cup E(G) \to \{1, 2, ..., n+m\}$, if*

(i). $w(u) \neq w(v)$ for any two adjacent vertices $u, v \in V(G)$;

(ii). $w(e) \neq w(e')$ for any two adjacent edges $e = uv, e' = u'v' \in E(G)$; and

(iii). $w(v) \neq w(e)$ for any vertex $v \in V(G)$ and for any edge $e = uv \in E(G)$ that is incident to the same vertex v.

The weight of a vertex u is defined by $w(u) = \sum_{e \in E(u)} f(e)$, where $E(u)$ is the set of all incident edges of a vertex u, and an edge weight is $w(xy) = f(x) + f(y)$, $xy \in E(G)$.

Definition 6. [35] *The local total antimagic chromatic number is the minimum number of colors taken over all colorings induced by local total antimagic colorings of G, which is denoted by* $\chi_{lt}(G)$.

Observation 1. [35] *If G admits a local total antimagic coloring, then* $\chi_{lt}(G) \geq \chi_t(G)$.

Observation 2. [35] *The graph* K_2 *does not admit a local total antimagic labeling.*

Theorem 1. [35] *Let G be a graph with at least three vertices and k pendants. Then* $\chi_{lt}(G) \geq k+1$.

Theorem 2. [35] *For the star graph* $K_{1,n}$*, we have* $\chi_{lt}(K_{1,n}) = n+1$.

Theorem 3. [35] *For the double star graph* $S_{n,n}, n > 5$*, we have* $\chi_{lt}(S_{n,n}) = 2n+2$.

Martin Bača [7], estimated the bounds of the local antimagic chromatic number for the disjoint union of multiple copies of graphs and obtained the local antimagic chromatic number for the disjoint union of stars, cycles and caterpillar graphs. Rajkumar and Nalliah [31] found the local edge antimagic chromatic number for the disjoint union of star graphs, ladder graph and cycle graph. The disjoint union of star graphs is used to model and analyze communication networks, where each star graph represents a central hub connected to several nodes. Many authors found the different types of antimagic chromatic numbers for the union of star graphs. In this paper, we determine the local total antimagic chromatic number of disjoint union of star graph $mK_{1,n}$.

2. Main Results

Let a and b be two positive integers with $a < b$, we denote $[a, b] = \{a, a+1, ..., b-1, b\}$ and $c-set$ denotes the set of corresponding values taken by the formula f or (w).

Proposition 1. *The graph* mK_2 *is not a local total antimagic.*

Proof. Let $V(mK_2) = \{u_i, v_i, 1 \leq i \leq m\}$ and $E(mK_2) = \{u_iv_i, 1 \leq i \leq m\}$ be the vertex and edge sets of mK_2. Then $|V(mK_2)| = 2m$ and $|E(mK_2)| = m$. Suppose the graph $G = mK_2$ admits a local total antimagic labeling. Then there exists a local total antimagic labeling $f : V(mK_2) \cup E(mK_2) \to \{1, 2, 3, ..., 3m\}$. Let $e = uv \in E(mK_2)$ with label $f(e)$. Then $w(u) = w(v) = f(e)$, which is a contradiction that adjacent vertices u and v received the same weight. Thus, mK_2 has no local total antimagic labeling f.

Theorem 4. *Let* $G = mK_{1,2}$ *be the disjoint union of m copies of star* $K_{1,2}$*. Then* $\chi_{lt}(mK_{1,2}) = 2m+1$.

Proof. Let $V(mK_{1,2}) = \{c_j, u_1^j, u_2^j,\ 1 \leq j \leq m\}$ and $E(mK_{1,2}) = \{c_ju_1^j, c_ju_2^j,\ 1 \leq j \leq m\}$ be the vertex and edge sets of $mK_{1,2}$. Then $|V(mK_{1,2})| = 3m$ and $|E(mK_{1,2})| = 2m$. Define a labeling $f : V(mK_{1,2}) \cup E(mK_{1,2}) \to \{1, 2, 3, ..., 5m\}$ by

$$f(u_i^j) = \begin{cases} j, & i = 1,\ 1 \leq j \leq m, \text{c-set is } \{1, 2, 3, ..., m-1, m\} \\ m+j, & i = 2,\ 1 \leq j \leq m, \text{c-set is } \{m+1, m+2, ..., 2m\} \end{cases}$$

$$f(c_j) = 3m + 1 - j,\ 1 \le j \le m, \text{c-set is } \{3m, 3m-1, ..., 2m+1\}$$

$$f(c_j u_i^j) = \begin{cases} 5m+1-j, & i = 1,\ 1 \le j \le m, \text{c-set is } \{5m, 5m-1, ..., 4m+1\} \\ 3m+j, & i = 2,\ 1 \le j \le m, \text{c-set is } \{3m+1, 3m+2, ..., 4m\} \end{cases}$$

Then the vertex and edge weights are

$$\begin{aligned} w(u_i^j) &= f(c_j u_i^j),\ i = 1, 2 \text{ and } 1 \le j \le m \\ w(c_j) &= 8m + 1,\ 1 \le j \le m \\ w(c_j u_1^j) &= 3m + 1,\ 1 \le j \le m \\ w(c_j u_2^j) &= 4m + 1,\ 1 \le j \le m \end{aligned}$$

The union of all distinct weights of the set C is given as follows:

$$\begin{aligned} C =& \{w(u_1^j) \cup w(u_2^j) \cup w(c_j) \cup w(c_j u_1^j) \cup w(c_j u_2^j), 1 \le j \le m\} \\ =& \{3m+1, 3m+2, ..., 4m\} \cup \{4m+1, 4m+2, ..., 5m\} \cup \{8m+1\} \cup \{3m+1\} \cup \{4m+1\} \\ =& \{3m+1, 3m+2, ..., 4m, 4m+1, ..., 5m\} \cup \{8m+1\} \end{aligned}$$

and hence $|C| = 2m + 1$. So that f provides a local total antimagic labeling with $2m+1$ colors. Thus $\chi_{lt}(mK_{1,2}) \le 2m+1$. By Theorem 1 [35], we get $\chi_{lt}(mK_{1,2}) \ge 2m+1$. Thus $\chi_{lt}(mK_{1,2}) = 2m + 1$.

Example 1. *The local total antimagic labeling of* $5K_{1,2}$ *with* 11 *colors are* $\{16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 41\}$ *given in Figure 1.*

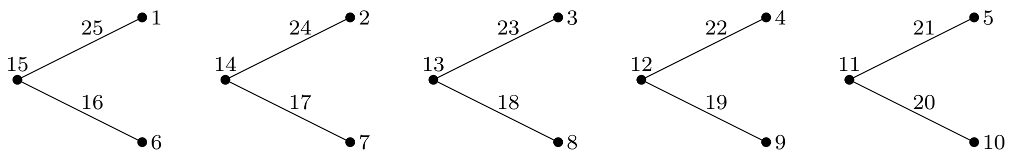

Figure 1: Local total antimagic labeling of $5K_{1,2}$ with 11 colors $\{16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 41\}$

Theorem 5. *Let* $mK_{1,n},\ n \ge 3,\ m \ge 2$ *be the disjoint union of* m *copies of star* $K_{1,n}$*. Then* $\chi_{lt}(mK_{1,n}) = mn + 1$.

Proof. Let $V(mK_{1,n}) = \{c_j, u_i^j,\ 1 \le i \le n,\ 1 \le j \le m\}$ and $E(mK_{1,n}) = \{c_j u_i^j,\ 1 \le i \le n,\ 1 \le j \le m\}$ be the vertex and edge sets of $mK_{1,n}$. Then $|V(mK_{1,n})| = m(n+1)$, $|E(mK_{1,n})| = mn$ and $|V(mK_{1,n}) \cup E(mK_{1,n})| = 2mn + m$.

Case 1: When n is even

Define a labeling $f : V(mK_{1,n}) \cup E(mK_{1,n}) \to \{1, 2, 3, ..., 2mn + m\}$ by

$$\begin{aligned} f(u_i^j) &= m(i-1) + j,\ 1 \le i \le n,\ 1 \le j \le m, \text{c-set is given in Table 1} \\ f(c_j) &= m(n+1) + 1 - j,\ 1 \le j \le m \end{aligned}$$

c-set is $\{mn+m,\ mn+m-1,\ ...,\ mn+2,\ mn+1\}$

$$f(c_j u_i^j) = \begin{cases} 2m(n+1)+1-mi-j, \text{ if } i \text{ is odd},\ 1 \le i \le n-1,\ 1 \le j \le m \\ \text{ c- set is given in Table 2} \\ m(2n+1)-mi+j, \text{ if } i \text{ is even},\ 1 \le i \le n,\ 1 \le j \le m \\ \text{ c- set is given in Table 3} \end{cases}$$

The above labeling f is given in Tables 1, 2 and 3 for easy reading.

Table 1: Labels for the pendant vertices u_i^j of $mK_{1,n}$, where n is even

i	u_i^1	u_i^2	u_i^3	...	u_i^{m-1}	u_i^m
1	1	2	3	...	m-1	m
2	m+1	m+2	m+3	...	2m-1	2m
...	...	...	...	...	...	...
...	...	...	...	...	...	...
n-1	m(n-2)+1	m(n-2)+2	m(n-2)+3	...	m(n-1)-1	m(n-1)
n	m(n-1)+1	m(n-1)+2	m(n-1)+3	...	mn-1	mn

Table 2: Labels for the pendant edges $c_j u_i^j$ of $mK_{1,n}$, where n is even and i is odd

i	$c_1 u_i^1$	$c_2 u_i^2$	$c_3 u_i^3$	...	$c_{m-1} u_i^{m-1}$	$c_m u_i^m$
1	2mn+m	2mn+m-1	2mn+m-2	...	2mn+2	2mn+1
3	2mn-m	2mn-m-1	2mn-m-2	...	2mn-2m+2	2mn-2m+1
...	...	...	...	...	...	...
...	...	...	...	...	...	...
n-3	mn+5m	mn+5m-1	mn+5m-2	...	mn+4m+2	mn+4m+1
n-1	mn+3m	mn+3m-1	mn+3m-2	...	mn+2m+2	mn+2m+1

Table 3: Labels for the pendant edges $c_j u_i^j$ of $mK_{1,n}$, where n is even and i is even

i	$c_1 u_i^1$	$c_2 u_i^2$	$c_3 u_i^3$	...	$c_{m-1} u_i^{m-1}$	$c_m u_i^m$
2	2mn-m+1	2mn-m+2	2mn-m+3	...	2mn-1	2mn
4	2mn-3m+1	2mn-3m+2	2mn-3m+3	...	2mn-2m-1	2mn-2m
...	...	...	...	...	...	...
...	...	...	...	...	...	...
n-2	mn+3m+1	mn+3m+2	mn+3m+3	...	mn+4m-1	mn+4m
n	mn+m+1	mn+m+2	mn+m+3	...	mn+2m-1	mn+2m

Then the vertex and edge weights are

$$w(u_i^j) = f(c_j u_i^j),\ 1 \le i \le n,\ 1 \le j \le m$$
$$w(c_j) = \frac{mn(3n+2)+n}{2} = 2mn + \frac{n}{2}[m(3n-2)+1],\quad 1 \le j \le m$$

$$w(c_j u_i^j) = mn + mi + 1,\ 1 \le i \le n,\ 1 \le j \le m$$
$$\text{c-set is}\{mn + m + 1,\ mn + 2m + 1,\ ...,\ 2mn - m + 1,\ 2mn + 1\}$$

The union of all distinct weights of a set C is given as follows:

$$\begin{aligned} C =& w(u_i^j) \cup w(c_j) \cup w(c_j u_i^j) \\ =& [mn + m + 1,\ 2mn + m] \cup \left\{2mn + \frac{n}{2}[m(3n-2)+1]\right\} \\ & \cup \{mn + m + 1, mn + 2m + 1, ..., 2mn - m + 1, 2mn + 1\} \\ =& [mn + m + 1,\ 2mn + m] \cup \left\{2mn + \frac{n}{2}[m(3n-2)+1]\right\} \end{aligned}$$

and hence $|C| = mn + 1$. So that f admits local total antimagic labeling with $mn + 1$ colors. Thus $\chi_{lt}(mK_{1,n}) \le mn + 1$. By Theorem 1 [35], we get $\chi_{lt}(mK_{1,n}) \ge mn + 1$. Thus $\chi_{lt}(mK_{1,n}) = mn + 1$.

Case 2: When n is odd:

Subcase 2(a): $n = 3$ is odd.

Define a labeling $f : V(mK_{1,3}) \cup E(mK_{1,3}) \to \{1, 2, 3, ..., 7m\}$ by

$$f(u_i^j) = \begin{cases} 3m + 3 - 3j,\ i = 1,\ 1 \le j \le m, \text{ c-set is } \{3m, 3m-3, ..., 9, 6, 3\} \\ 3m + 1 - 3j,\ i = 2,\ 1 \le j \le m, \text{ c-set is } \{3m-2, 3m-5, ..., 7, 4, 1\} \\ 3m + 2 - 3j,\ i = 3,\ 1 \le j \le m, \text{ c-set is } \{3m-1, 3m-4, ..., 8, 5, 2\} \end{cases}$$

$$f(c_j) = 3m - 1 + 2j,\ 1 \le j \le m;$$
$$\text{c-set is } \{3m+1,\ 3m+3,\ 3m+5, ...,\ 5m-3,\ 5m-1\}$$

$$f(c_j u_i^j) = \begin{cases} 7m + 1 - j,\ i = 1,\ 1 \le j \le m, \\ \quad \text{c-set is } \{7m, 7m-1, 7m-2, ..., 6m+2, 6m+1\} \\ 6m + 1 - j,\ i = 2,\ 1 \le j \le m, \\ \quad \text{c-set is } \{6m, 6m-1, 6m-2, ..., 5m+2, 5m+1\} \\ 3m + 2j, \quad i = 3,\ 1 \le j \le m, \\ \quad \text{c-set is } \{3m+2, 3m+4, 3m+6, ..., 5m-2, 5m\} \end{cases}$$

Then the vertex and edge weights are

$$\begin{aligned} w(u_i^j) &= f(c_j u_i^j),\ 1 \le i \le 3,\ 1 \le j \le m \\ w(c_j) &= 16m + 2,\ 1 \le j \le m \\ w(c_j u_1^j) &= 6m + 2 - j, \quad 1 \le j \le m, \\ & \quad \text{c-set is } \{6m+1, 6m, 6m-1, ..., 5m+3, 5m+2\} \\ w(c_j u_2^j) &= 6m - j, \quad 1 \le j \le m, \\ & \quad \text{c-set is } \{6m-1, 6m-2, 6m-3, ..., 5m+1, 5m\} \\ w(c_j u_3^j) &= 6m + 1 - j, \quad 1 \le j \le m, \end{aligned}$$

$$\text{c-set is } \{6m, 6m-1, 6m-2..., 5m+2, 5m+1\}$$

The union of all distinct weights of a set C is given as follows:

$$\begin{aligned} C =& w(u_i^j) \cup w(c_j) \cup w(c_j u_1^j) \cup w(c_j u_2^j) \cup w(c_j u_3^j) \\ =& \{[5m+1,\ 7m] \cup \{3m+2, 3m+4, 3m+6, ..., 5m-2, 5m\}\} \cup \{16m+2\} \\ & \cup [5m+2,\ 6m+1] \cup [5m,\ 6m-1] \cup [5m+1,\ 6m] \\ =& [5m+1,\ 7m] \cup \{3m+2, 3m+4, 3m+6, ..., 5m-2, 5m\} \cup \{16m+2\} \end{aligned}$$

and hence $|C| = 3m+1$. So that f provides a local total antimagic labeling with $3m+1$ colors. Thus $\chi_{lt}(mK_{1,3}) \leq 3m+1$. By Theorem 1 [35], we get $\chi_{lt}(mK_{1,3}) \geq 3m+1$. Thus $\chi_{lt}(mK_{1,3}) = 3m+1$.

Subcase 2(b): $n \geq 5$ is odd

Define a labeling $f : V(mK_{1,n}) \cup E(mK_{1,n}) \rightarrow \{1, 2, 3, ..., 2mn+m\}$ by

$$f(u_i^j) = m(n-j)+i,\ 1 \leq i \leq n,\ 1 \leq j \leq m; \text{c-set is given in Table 4}$$

$$f(c_j) = \begin{cases} mn+2j, \text{ if } m \text{ is odd and } 1 \leq j \leq m \\ \quad \text{c-set is } \{mn+2,\ mn+4,\ ...,\ mn+2m\} \\ mn-1+2j, \text{if } m \text{ is even and } 1 \leq j \leq m \\ \quad \text{c-set is } \{mn+1,\ mn+3,\ ...,\ mn+2m-1\} \end{cases}$$

$$f(c_j u_i^j) = \begin{cases} 2m(n+1)+1-mi-j, \text{ if } i \text{ is odd, } 1 \leq i \leq n-2 \text{ and } 1 \leq j \leq m \\ \quad \text{c-set is given in Table 5} \\ m(2n+1)-mi+j, \text{ if } i \text{ is even, } 1 \leq i \leq n-3, \text{ and } 1 \leq j \leq m \\ \quad \text{c-set is given in Table 6} \\ m(n+3)+1-j, \text{ if } i = n-1 \text{ is even and } 1 \leq j \leq m \\ mn-1+2j, \text{ if } i = n \text{ and } m \text{ is odd, } 1 \leq j \leq m \\ \quad \text{c-set is } \{mn+1,\ mn+3,\ ...,\ mn+2m-1\} \\ mn+2j, \text{ if } i = n \text{ and } m \text{ is even, } 1 \leq j \leq m \\ \quad \text{c-set is } \{mn+2,\ mn+4,\ ...,\ mn+2m\} \end{cases}$$

The above labeling f is given in Tables 4, 5 and 6 for easy reading.

Then the vertex and edge weights are

$$w(u_i^j) = f(c_j u_i^j),\ 1 \leq i \leq n \text{ and } 1 \leq j \leq m$$

$$w(c_j) = \begin{cases} \frac{3mn^2+2mn-m+n-1}{2} = 2mn+m+\frac{m[(3n-2)n-3]+(n-1)}{2}, \text{ if } m \text{ is odd, } 1 \leq j \leq m \\ \frac{3mn^2+2mn-m+n+1}{2} = 2mn+m+\frac{m[(3n-2)n-3]+(n+1)}{2}, \text{ if } m \text{ is even, } 1 \leq j \leq m \end{cases}$$

$$w(c_j u_i^j) = \begin{cases} 2mn+i+2j-nj, \text{ if } m \text{ is odd, } 1 \leq j \leq m \text{ and } 1 \leq i \leq n \\ \text{c-set is given in Table 7} \\ 2mn-1+i+2j-nj, \text{ if } m \text{ is even, } 1 \leq j \leq m \text{ and } 1 \leq i \leq n \\ \text{c-set is given in Table 8} \end{cases}$$

Table 4: Labels for the pendant vertices u_i^j of $mK_{1,n}$, where n is odd and $m > 1$

i	u_i^1	u_i^2	u_i^3	...	u_i^{m-1}	u_i^m
1	(m-1)n+1	(m-2)n+1	(m-3)n+1	...	n+1	1
2	(m-1)n+2	(m-2)n+2	(m-3)n+2	...	n+2	2
...	...	...	...	...	...	...
...	...	...	...	...	...	...
n-1	mn-1	(m-1)n-1	(m-2)n-1	...	2n-1	n-1
n	mn	(m-1)n	(m-2)n	...	2n	n

Table 5: Labels for the pendant edges $c_ju_i^j$ of $mK_{1,n}$, where n is odd and i is odd

i	$c_1u_i^1$	$c_2u_i^2$	$c_3u_i^3$	...	$c_{m-1}u_i^{m-1}$	$c_mu_i^m$
1	2mn+m	2mn+m-1	2mn+m-2	...	2mn+2	2mn+1
3	2mn-m	2mn-m-1	2mn-m-2	...	2mn-2m+2	2mn-2m+1
...	...	...	...	...	...	...
...	...	...	...	...	...	...
n-4	mn+6m	mn+6m-1	mn+6m-2	...	mn+5m+2	mn+5m+1
n-2	mn+4m	mn+4m-1	mn+4m-2	...	mn+3m+2	mn+3m+1

The above weights are given in Tables 7 and 8 for easy reading.

For m is odd, the union of all distinct weights of a set C_{odd} is given as follows:

$$\begin{aligned}
C_{odd} =&\{w(u_i^j) \cup w(c_j) \cup w(c_ju_i^j),\ 1 \leq j \leq m \text{ and } 1 \leq i \leq n\}\\
=&[mn+2m+1,\ 2mn+m] \cup \{mn+1, mn+3, ..., mn+2m-1\}\\
&\cup \left\{2mn+m+\frac{m[(3n-2)n-3]+(n-1)}{2}\right\} \cup [mn+2m+1,\ 2mn+2]\\
=&[mn+2m+1,\ 2mn+m] \cup \{mn+1, mn+3, ..., mn+2m-1\}\\
&\cup \left\{2mn+m+\frac{m[(3n-2)n-3]+(n-1)}{2}\right\}\\
|C_{odd}| =&m(n-1)+m+1 = mn+1
\end{aligned}$$

Table 6: Labels for the pendant edges $c_ju_i^j$ of $mK_{1,n}$, where n is odd and i is even

i	$c_1u_i^1$	$c_2u_i^2$	$c_3u_i^3$	...	$c_{m-1}u_i^{m-1}$	$c_mu_i^m$
2	2mn-m+1	2mn-m+2	2mn-m+3	...	2mn-1	2mn
4	2mn-3m+1	2mn-3m+2	2mn-3m+3	...	2mn-2m-1	2mn-2m
...	...	...	...	...	...	...
...	...	...	...	...	...	...
n-3	mn+4m+1	mn+4m+2	mn+4m+3	...	mn+5m-1	mn+5m
n-1	mn+3m	mn+3m-1	mn+3m-2	...	mn+2m+2	mn+2m+1

Table 7: Weights for the pendant edges $c_j u_i^j$ of $mK_{1,n}$, where n is odd and m is odd

i	$c_1u_i^1$	$c_2u_i^2$	$c_3u_i^3$	...	$c_{m-1}u_i^{m-1}$	$c_mu_i^m$
1	mn+(m-1)n+3	mn+(m-2)n+5	mn+(m-3)n+7	...	mn+2m+(n-1)	mn+2m+1
2	mn+(m-1)n+4	mn+(m-2)n+6	mn+(m-3)n+8	...	mn+2m+n	mn+2m+2
...	...	...	...	...	...	...
...	...	...	...	...	...	...
n-1	2mn+1	mn+(m-1)n+3	mn+(m-2)n+5	...	mn+2m+2n-3	mn+2m+n-1
n	2mn+2	mn+(m-1)n+4	mn+(m-2)n+6	...	mn+2m+2n-2	mn+2m+n

Table 8: Weights for the pendant edges $c_j u_i^j$ of $mK_{1,n}$, where n is odd and m is even

i	$c_1u_i^1$	$c_2u_i^2$	$c_3u_i^3$	...	$c_{m-1}u_i^{m-1}$	$c_mu_i^m$
1	mn+(m-1)n+2	mn+(m-2)n+4	mn+(m-3)n+6	...	mn+2m+(n-2)	mn+2m
2	mn+(m-1)n+3	mn+(m-2)n+5	mn+(m-3)n+7	...	mn+2m+(n-1)	mn+2m+1
...	...	...	...	...	...	...
...	...	...	...	...	...	...
n-1	2mn	mn+(m-1)n+2	mn+(m-2)n+4	...	mn+2m+2n-4	mn+2m+n-2
n	2mn+1	mn+(m-1)n+3	mn+(m-2)n+5	...	mn+2m+2n-3	mn+2m+n-1

For m is even, the union of all distinct weights of a set C_{even} is given as follows:

$$\begin{aligned}
C_{even} =&[mn+2m+1,\ 2mn+m] \cup \{mn+2, mn+4, ..., mn+2m\} \\
&\cup \left\{2mn+m+\frac{m[(3n-2)n-3]+(n+1)}{2}\right\} \cup [mn+2m,\ 2mn+1] \\
&[mn+2m+1,\ 2mn+m] \cup \{mn+2, mn+4, ..., mn+2m\} \\
&\cup \left\{2mn+m+\frac{m[(3n-2)n-3]+(n+1)}{2}\right\} \\
|C_{even}| =&m(n-1)+m+1 = mn+1
\end{aligned}$$

So that f provides a local total antimagic labeling for $mK_{1,n}$ with $mn+1$ colors, and hence $\chi_{lt}(mK_{1,n}) \leq mn+1$. By Theorem 1 [35], we get $\chi_{lt}(mK_{1,n}) \geq mn+1$. Thus $\chi_{lt}(mK_{1,n}) = mn+1$.

Example 2. *The local total antimagic labeling of* $5K_{1,3}$ *with* 16 *colors are* $\{17, 19, 21, 23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 82\}$ *given in Figure 2.*

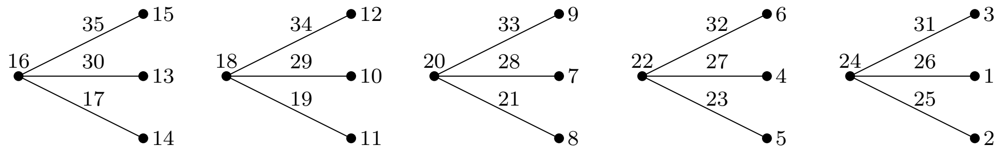

Figure 2: Local total antimagic labeling of $5K_{1,3}$ with 16 colors $\{17, 19, 21, 23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 82\}$

Example 3. *The local total antimagic labeling of* $4K_{1,6}$ *with* 25 *colors are* $[29 \quad 52] \cup \{243\}$ *given in Figure 3.*

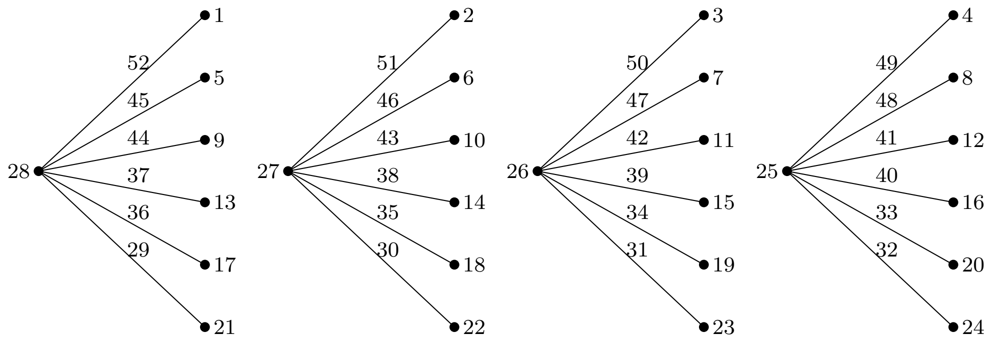

Figure 3: Local total antimagic labeling of $4K_{1,6}$ with 25 colors $[29 \quad 52] \cup \{243\}$

Example 4. *The local total antimagic labeling of* $5K_{1,7}$ *with* 36 *colors are* $\{36, 38, 40, 42, 44\} \cup [47 \quad 75] \cup \{403\}$ *are given in Figure 4.*

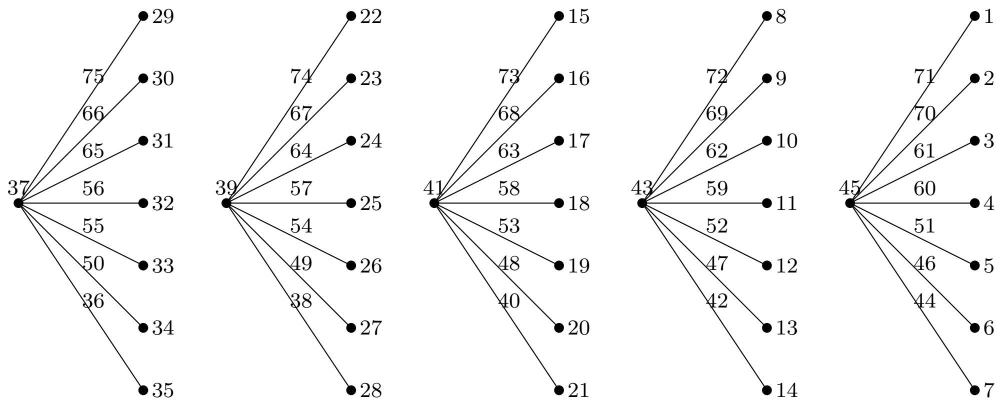

Figure 4: Local total antimagic labeling of $5K_{1,7}$ with 36 colors $\{36, 38, 40, 42, 44\} \cup [47 \quad 75] \cup \{403\}$

3. Conclusions

In this paper, we investigated the local total antimagic labeling and local total antimagic chromatic number for the disjoint union of star graphs with some copies of star graphs. The disjoint union of star graphs, equipped with an antimagic labeling, has various applications in network optimization, coding theory, graph decomposition, scheduling, resource allocation, data storage and transportation. These applications leverage the structural properties of the union of star graphs, combined with the anti-magic labeling, to solve the complex problems efficiently. The local total antimagic labeling for the other families of disconnected graphs are still open. In future we will estimate the real time applications of local total antimagic labeling of both connected and disconnected graphs.

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