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On Lie Homomorphisms of Complex Intuitionistic Fuzzy Lie Algebras

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Abstract. In this paper, we investigate the properties of complex intuitionistic fuzzy Lie subalgebras and ideals under Lie algebra homomorphisms, focusing on both images and inverse images. We provide detailed proofs for several new results concerning the preservation of complex intuitionistic fuzzy structures through homomorphisms, and we introduce additional homomorphism-related properties for these structures. This work extends known results to the context of complex intuitionistic fuzzy Lie algebras, contributing new insights into their behavior under algebraic mappings.

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1. Introduction

Introduced by Lotfi Zadeh in 1965, fuzzy sets provide a mathematical framework for representing and managing imprecise and uncertain information [22]. Unlike classical crisp sets with binary membership values, fuzzy sets allow for gradual degrees of membership, offering a more flexible approach to dealing with uncertainty.

In 1986, Atanassov extended the concept of fuzzy sets by introducing intuitionistic fuzzy sets [6], which account for both membership and non-membership degrees. This concept garnered significant attention, such as in [10, 11], leading to the evolution of intuitionistic fuzzy set theory, which has found utility in diverse domains such as decision-making, control systems, and pattern recognition.

Building on these ideas, complex intuitionistic fuzzy sets (CIFS) were introduced by Alkouri and Salleh in 2012 [5]. CIFS extend intuitionistic fuzzy sets by using complex numbers to represent membership and non-membership values, offering a more expressive

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way to handle uncertainty and ambiguity. This framework has proven useful in decision-making, pattern recognition, and image processing, where complex and conflicting information needs to be systematically managed.

Related work on fuzzy Lie algebras can be found in [3], and studies on bipolar fuzzy soft Lie algebras in [2]. These foundational studies have contributed significantly to the extension of fuzzy algebraic structures and motivate further research in this area.

This work aligns with recent research in the field, such as the study of intuitionistic fuzzy ordered subalgebras in ordered BCI-algebras by Roh et al. [12]. In a recent work [18], we introduced the notion of a complex intuitionistic fuzzy Lie algebra, which characterizes a Lie algebra using elements represented as complex intuitionistic fuzzy sets. The Lie bracket operation is formulated based on the principles of complex intuitionistic fuzzy logic. These complex intuitionistic fuzzy Lie algebras can be perceived as a broader generalization encompassing fuzzy Lie algebras [21], intuitionistic fuzzy Lie algebras [1], and complex fuzzy Lie algebras [13].

2. Some preliminaries on Lie Algebras and Complex Intuitionistic Fuzzy Lie Ideals

2.1. Fundamentals of Lie Algebras

In this section, we delve into the core concepts surrounding Lie algebras, drawing insights from the sources [7] and [9]. We explore the foundational notion of Lie algebras, denoted as $(\mathcal{M}, [.,.])$, where $\mathcal{M}$ signifies a vector space over the field $\mathcal{K}$, and $[.,.] : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ represents a bilinear mapping (called Lie product). The Lie product adheres to two key axioms for any elements $m_1, m_2, m_3 \in \mathcal{M}$:

(i) The condition $[m_1,\ m_2] = 0_{\mathcal{M}}$ holds, leading to the implication of anti-symmetry: $[m_1,\ m_2] = -[m_2,\ m_1]$.

(ii) The Jacobi identity, denoted as

$$[m_1,\ [m_2, m_3]] + [m_2, [m_3, m_1]] + [m_3, [m_1,\ m_2]] = 0_{\mathcal{M}},$$

is satisfied.

It's clear that $[m,\ 0_{\mathcal{M}}] = [0_{\mathcal{M}},\ m] = 0_{\mathcal{M}}$ for any $m \in \mathcal{M}$. Additionally, when $\mathrm{Char}(\mathcal{K}) \neq 2$, the identity $[m_1,\ m_2] = -[m_2,\ m_1]$ for all $m_1,\ m_2 \in \mathcal{M}$ implies $[m,\ m] = 0_{\mathcal{M}}$ for each $m \in \mathcal{M}$.

Various illustrative examples highlight diverse instances of Lie algebras:

(i) Any vector space V can be viewed as an abelian Lie algebra using the Lie bracket $[v_1,\ v_2] = 0_V$ for all elements $v_1,\ v_2 \in V$.

(ii) When a vector space V is endowed with an associative multiplication, it can be transformed into a Lie algebra with the commutator operation $[v_1,\ v_2] = v_1v_2 - v_2v_1$

for all $v_1,\ v_2 \in V$. In the realm of finite-dimensional vector spaces, the set of linear transformations from V to itself, denoted $\mathrm{gl}(V)$, constitutes a Lie algebra, with the bracket defined as $[f_1,\ f_2] = f_1 \circ f_2 - f_2 \circ f_1$ for any $f_1,\ f_2 \in \mathrm{gl}(V)$.

(iii) The Lie algebra structure is evident in the vector space $\mathbb{R}^3$ when the cross product operation is employed as $[\mathbf{a},\ \mathbf{b}] = \mathbf{a} \times \mathbf{b}$.

These examples showcase the wide-ranging applications and properties of Lie algebras. In the context of Lie algebras, a subspace $\mathcal{Q}$ of $\mathcal{M}$ is referred to as a Lie subalgebra if it remains closed under the Lie product operation, implying that $[q_1,\ q_2] \in \mathcal{Q}$ for any $q_1,\ q_2 \in \mathcal{Q}$. Also, $\mathcal{Q}$ is termed a Lie ideal of $\mathcal{M}$ if $[m,\ n] \in \mathcal{N}$ for any $m \in \mathcal{M}$ and $n \in \mathcal{N}$. Notably, the subspaces $0_{\mathcal{M}}$ and $\mathcal{M}$ are considered ideals of $\mathcal{M}$, termed the trivial ideals. Additionally, the center of $\mathcal{M}$, denoted $Z(\mathcal{M})$, consists of elements m in $\mathcal{M}$ for which $[m,\ n] = 0_{\mathcal{M}}$ for all $n \in \mathcal{M}$, making it a Lie ideal of $\mathcal{M}$.

Furthermore, if $\mathcal{Q}$ and $\mathcal{S}$ are ideals of $\mathcal{M}$, then $\mathcal{Q} + \mathcal{S} = \{q + s\ : q \in \mathcal{Q} \text{ and } s \in \mathcal{S}\}$, $[Q, S] = \mathrm{Span}\{[q, s] : q \in \mathcal{Q} \text{ and } s \in \mathcal{S}\}$, and $Q \cap S = \{m : m \in \mathcal{Q} \text{ and } m \in \mathcal{S}\}$ are also considered ideals of $\mathcal{M}$.

Considering two Lie algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{K}$, a linear transformation $T : \mathcal{M}_1 \to \mathcal{M}_2$ is considered a Lie algebra homomorphism if it satisfies $T([m, n]) = [T(m), T(n)]$ for all $m,\ n \in \mathcal{M}_1$. Furthermore, if T is both a Lie algebra homomorphism and a bijection (one-to-one and onto), it's termed a Lie algebra isomorphism.

An illustrative example showcases the adjoint representation of a Lie algebra. For $m \in \mathcal{M}$, the function $\mathrm{ad}m : \mathcal{M} \to \mathcal{M};\ n \mapsto [m, n]$ is defined, and the set $\mathrm{ad}\mathcal{M} = \{\mathrm{ad}m :\ m \in \mathcal{M}\}$ is shown to be a Lie subalgebra of $\mathrm{gl}(\mathcal{M})$. The function

$$\mathrm{ad} : \mathcal{M} \to \mathrm{ad}\mathcal{M}; m \mapsto \mathrm{ad}m$$

is a Lie algebra homomorphism and is known as the adjoint representation of $\mathcal{M}$.

Moreover, for a Lie algebra homomorphism $f : \mathcal{M}_1 \to \mathcal{M}_2$, the image of f is characterized as $\mathrm{im}(f) = \{f(q) : q \in M_1\}$, while the kernel of f is denoted as $\ker(f) = \{s \in \mathcal{M}_1 : f(s) = 0_{\mathcal{M}_2}\}$. As a consequence, $\mathrm{im}(f)$ emerges as a Lie subalgebra within $\mathcal{M}_2$, and $\ker(f)$ establishes itself as an ideal within $\mathcal{M}_1$.

2.2. Fundamentals of Complex Intuitionistic Fuzzy Lie Ideals

A complex intuitionistic fuzzy set defined on a non-empty set X can be denoted as $\mathcal{E} = (\phi_{\mathcal{E}}, \psi_{\mathcal{E}}) = \{(x,\ \phi_{\mathcal{E}}(x),\ \psi_{\mathcal{E}}(x)) : x \in X\}$. In this representation, $\phi_{\mathcal{E}}(x)$ and $\psi_{\mathcal{E}}(x)$ are complex numbers situated within the unit circle, ensuring that their absolute values satisfy the condition $\mid \phi_{\mathcal{E}}(x) \mid + \mid \psi_{\mathcal{E}}(x) \mid \leq 1$. In this context, $i = \sqrt{-1}$, and the expressions for $\phi_{\mathcal{E}}(x)$ and $\psi_{\mathcal{E}}(x)$ take the form $\rho_{\mathcal{E}}(x)e^{i\zeta_{\mathcal{E}}(x)}$ and $\hat{\rho}_{\mathcal{E}}(x)e^{i\hat{\zeta}_{\mathcal{E}}(x)}$ respectively. Here, $\rho_{\mathcal{E}}(x)$ and $\hat{\rho}_A(x)$ are real numbers ranging from 0 to 1, while $\zeta_{\mathcal{M}}(x)$ and $\hat{\zeta}_{\mathcal{M}}(x)$ are real numbers within the interval of $[0, 2\pi]$.

The notion of complex intuitionistic fuzzy sets (CIFS) can be seen as an expansion of intuitionistic fuzzy sets. When both $\zeta_{\mathcal{E}}(x)$ and $\hat{\zeta}_{\mathcal{E}}(x)$ are set to 0, it reverts to the classical fuzzy set. Furthermore, when $\psi_{\mathcal{E}}(x) = (1 - \rho_{\mathcal{E}}(x))e^{i(2\pi - \zeta_{\mathcal{E}}(x))}$, it produces a complex fuzzy set.

A homogeneous complex intuitionistic fuzzy set is defined as a CIFS that fulfills two conditions for any x and y belonging to the set X: (i) $\rho_{\mathcal{E}}(x) \leq \rho_{\mathcal{E}}(y)$ if and only if $\zeta_{\mathcal{E}}(x) \leq \zeta_{\mathcal{E}}(y)$, and (ii) $\hat{\rho}_{\mathcal{E}}(x) \leq \hat{\rho}_{\mathcal{E}}(y)$ if and only if $\hat{\zeta}_{\mathcal{E}}(x) \leq \hat{\zeta}_{\mathcal{E}}(y)$. In the article, it is assumed that all complex intuitionistic fuzzy sets are of this homogeneous type. Also, for $z_1 = \rho_1 e^{i\zeta_1}$ and $z_2 \rho_2 e^{i\zeta_2}$ ($\rho_1, \rho_2 \in [0, 1]$ and $\zeta_1, \zeta_2 \in [0, 2\pi]$) are two complex numbers, we say $z_1 \leq z_2$ if and only if $\rho_1 \leq \rho_2$ and $\zeta_1 \leq \zeta_2$.

Definition 1. *([18]) A complex intuitionistic fuzzy set $E = (\phi_{\mathcal{E}},\ \psi_A)$ defined on a Lie algebra $\mathcal{M}$ is categorized as a complex intuitionistic fuzzy Lie subalgebra when it meets three conditions for all $m_1, m_2 \in \mathcal{M}$, and $k \in \mathcal{K}$:*

(i) $\phi_{\mathcal{E}}(m_1 + m_2) \geq \phi_{\mathcal{E}}(m_1) \wedge \phi_{\mathcal{E}}(m_2)$ *and* $\psi_{\mathcal{M}}(m_1 + m_2) \leq \psi_{\mathcal{E}}(m_1) \vee \psi_{\mathcal{E}}(m_2)$,

(ii) $\phi_{\mathcal{E}}(km_1) \geq \phi_{\mathcal{E}}(x)$ *and* $\psi_{\mathcal{E}}(km_1) \leq \psi_{\mathcal{E}}(m_1)$*, and*

(iii) $\phi_{\mathcal{E}}([m_1, m_2]) \geq \phi_{\mathcal{E}}(m_1) \wedge \phi_{\mathcal{E}}(m_2)$ *and* $\psi_{\mathcal{E}}([m_1, m_2]) \leq \psi_{\mathcal{E}}(m_1) \vee \psi_{\mathcal{E}}(m_2)$.

If condition (iii) is substituted with $\phi_{\mathcal{E}}([m_1, m_2]) \geq \phi_{\mathcal{E}}(m_1) \vee \phi_{\mathcal{E}}(m_2)$, and $\psi_{\mathcal{E}}([m_1, m_2]) \leq \psi_{\mathcal{E}}(m_1) \wedge \psi_{\mathcal{E}}(m_2)$, then $\mathcal{E}$ is denoted as a complex intuitionistic fuzzy Lie ideal within the context of $\mathcal{M}$.

3. Mappings and Inverse Mappings of Complex Intuitionistic Fuzzy Lie Subalgebras (Ideals) Through Lie Morphisms.

Consider Lie algebras $\mathcal{M}_1$ and $\mathcal{M}_2$, a complex intuitionistic subset $E = (\phi_{\mathcal{E}}, \psi_{\mathcal{E}})$ of $\mathcal{M}_1$, and a function $f : \mathcal{M}_1 \to \mathcal{M}_2$. The complex intuitionistic fuzzy subset $f(E)$ of $f(\mathcal{M}_1)$ is defined as $f(E) = (\phi_{f(\mathcal{E})}, \psi_{f(\mathcal{E})})$. Here $(m_2 \in f(\mathcal{M}_1))$,

$$\phi_{f(\mathcal{E})}(m_2) = \sup_{m_1 \in f^{-1}(\{m_2\})}\{\phi_{\mathcal{E}}(m_1)\},$$

and

$$\psi_{f(\mathcal{E})}(m_2) = \inf_{m_1 \in f^{-1}(\{m_2\})}\{\psi_{\mathcal{E}}(m_1)\}.$$

This subset is termed as the mapping of $\mathcal{E}$ through f.

In a similar manner, for a complex intuitionistic fuzzy subset $\mathcal{P} = (\phi_{\mathcal{P}}, \psi_{\mathcal{P}})$ of $\mathcal{M}_2$, the inverse mapping of $\mathcal{P}$ under f, $f^{-1}(\mathcal{P})$, is defined as $(\phi_{f^{-1}(\mathcal{P})}, \psi_{f^{-1}(\mathcal{P})})$, where $(m_1 \in \mathcal{M}_1)$

$$\phi_{f^{-1}(\mathcal{P})}(m_1) = \phi_{\mathcal{P}}(f(m_1)) \text{ and } \psi_{f^{-1}(\mathcal{P})}(m_1) = \psi_{\mathcal{P}}(f(m_1)).$$

Theorem 1. *Assume $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a homomorphism between Lie algebras. If $\mathcal{P} = (\phi_{\mathcal{P}},\ \psi_{\mathcal{P}})$ is a complex intuitionistic fuzzy Lie subalgebra of $\mathcal{M}_2$, then the complex intuitionistic fuzzy set $f^{-1}(\mathcal{P})$ is a complex intuitionistic fuzzy Lie subalgebra of $\mathcal{M}_1$.*

Proof. To show that $f^{-1}(\mathcal{P})$ is a complex intuitionistic fuzzy Lie subalgebra in $\mathcal{M}_1$, we need to verify that it satisfies the properties of a complex intuitionistic fuzzy set and maintains the closure properties under Lie operations within $\mathcal{M}_1$.

We start by checking the homogeneity property. For any $m_1 \in \mathcal{M}_1$, by definition of the inverse image under the homomorphism f, we have:

$$\phi_{f^{-1}(\mathcal{P})}(m_1) = \phi_{\mathcal{P}}(f(m_1)) = \rho_{\mathcal{P}}(f(m_1))e^{i\zeta_{\mathcal{P}}(f(m_1))},$$

and similarly,

$$\psi_{f^{-1}(\mathcal{P})}(m_1) = \psi_{\mathcal{P}}(f(m_1)) = \hat{\rho}_{\mathcal{P}}(f(m_1))e^{i\hat{\zeta}_{\mathcal{P}}(f(m_1))}.$$

Now, consider two elements $m_1, m_2 \in \mathcal{M}_1$. If $\rho_{\mathcal{P}}(f(m_1)) \le \rho_{\mathcal{P}}(f(m_2))$, then by the homogeneity property of $\mathcal{P}$, we know that $\zeta_{\mathcal{P}}(f(m_1)) \le \zeta_{\mathcal{P}}(f(m_2))$. Similarly, if $\hat{\rho}_{\mathcal{P}}(f(m_1)) \le \hat{\rho}_{\mathcal{P}}(f(m_2))$, then $\hat{\zeta}_{\mathcal{P}}(f(m_1)) \le \hat{\zeta}_{\mathcal{P}}(f(m_2))$. Therefore, the homogeneity property holds for $f^{-1}(\mathcal{P})$ in $\mathcal{M}_1$.

Next, we verify the algebraic properties under addition, scalar multiplication, and the Lie bracket. For $m_1, m_2 \in \mathcal{M}_1$ and $k \in \mathcal{K}$, we need to show the following:

1. **Addition**:

$$\phi_{f^{-1}(\mathcal{P})}(m_1 + m_2) = \phi_{\mathcal{P}}(f(m_1 + m_2)) = \phi_{\mathcal{P}}(f(m_1) + f(m_2)),$$

since f is linear. By Definition 1 (the property of complex intuitionistic fuzzy Lie subalgebras), we have:

$$\phi_{\mathcal{P}}(f(m_1) + f(m_2)) \ge \phi_{\mathcal{P}}(f(m_1)) \wedge \phi_{\mathcal{P}}(f(m_2)).$$

Thus,

$$\phi_{f^{-1}(\mathcal{P})}(m_1 + m_2) \ge \phi_{f^{-1}(\mathcal{P})}(m_1) \wedge \phi_{f^{-1}(\mathcal{P})}(m_2).$$

For ψ, we proceed similarly:

$$\psi_{f^{-1}(\mathcal{P})}(m_1 + m_2) = \psi_{\mathcal{P}}(f(m_1 + m_2)) = \psi_{\mathcal{P}}(f(m_1) + f(m_2)),$$

and by Definition 1:

$$\psi_{\mathcal{P}}(f(m_1) + f(m_2)) \le \psi_{\mathcal{P}}(f(m_1)) \vee \psi_{\mathcal{P}}(f(m_2)).$$

Hence,

$$\psi_{f^{-1}(\mathcal{P})}(m_1 + m_2) \le \psi_{f^{-1}(\mathcal{P})}(m_1) \vee \psi_{f^{-1}(\mathcal{P})}(m_2).$$

2. **Scalar Multiplication**: For any $k \in \mathcal{K}$, we have:

$$\phi_{f^{-1}(\mathcal{P})}(km_1) = \phi_{\mathcal{P}}(f(km_1)) = \phi_{\mathcal{P}}(kf(m_1)),$$

and by Definition 1:

$$\phi_{\mathcal{P}}(kf(m_1)) \ge \phi_{\mathcal{P}}(f(m_1)).$$

Thus,

$$\phi_{f^{-1}(\mathcal{P})}(km_1) \ge \phi_{f^{-1}(\mathcal{P})}(m_1).$$

Similarly, for ψ:

$$\psi_{f^{-1}(\mathcal{P})}(km_1) = \psi_{\mathcal{P}}(f(km_1)) = \psi_{\mathcal{P}}(kf(m_1)),$$

and by Definition 1:

$$\psi_{\mathcal{P}}(kf(m_1)) \leq \psi_{\mathcal{P}}(f(m_1)),$$

so

$$\psi_{f^{-1}(\mathcal{P})}(km_1) \leq \psi_{f^{-1}(\mathcal{P})}(m_1).$$

3. **Lie Bracket**: Finally, for the Lie bracket, we have:

$$\phi_{f^{-1}(\mathcal{P})}([m_1, m_2]) = \phi_{\mathcal{P}}(f([m_1, m_2])) = \phi_{\mathcal{P}}([f(m_1), f(m_2)]),$$

and by Definition 1:

$$\phi_{\mathcal{P}}([f(m_1), f(m_2)]) \geq \phi_{\mathcal{P}}(f(m_1)) \wedge \phi_{\mathcal{P}}(f(m_2)),$$

so

$$\phi_{f^{-1}(\mathcal{P})}([m_1, m_2]) \geq \phi_{f^{-1}(\mathcal{P})}(m_1) \wedge \phi_{f^{-1}(\mathcal{P})}(m_2).$$

Similarly, for ψ:

$$\psi_{f^{-1}(\mathcal{P})}([m_1, m_2]) = \psi_{\mathcal{P}}(f([m_1, m_2])) = \psi_{\mathcal{P}}([f(m_1), f(m_2)]),$$

and by Definition 1:

$$\psi_{\mathcal{P}}([f(m_1), f(m_2)]) \leq \psi_{\mathcal{P}}(f(m_1)) \vee \psi_{\mathcal{P}}(f(m_2)),$$

so

$$\psi_{f^{-1}(\mathcal{P})}([m_1, m_2]) \leq \psi_{f^{-1}(\mathcal{P})}(m_1) \vee \psi_{f^{-1}(\mathcal{P})}(m_2).$$

Thus, $f^{-1}(\mathcal{P})$ satisfies the necessary properties and constitutes a complex intuitionistic fuzzy Lie subalgebra within $\mathcal{M}_1$.

Corollary 1. *Assume $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a homomorphism between Lie algebras. If $\mathcal{P} = (\phi_{\mathcal{P}},\ \psi_{\mathcal{P}})$ is a complex intuitionistic fuzzy Lie ideal within $\mathcal{M}_2$, then the complex intuitionistic fuzzy set $f^{-1}(\mathcal{P})$ is also a complex intuitionistic fuzzy Lie ideal in $\mathcal{M}_1$.*

Proof. We will follow the same general approach used in the proof of Theorem 1. The only key difference lies in the third requirement of Definition 1, which pertains to the behavior under the Lie bracket, as the structure involved is now a fuzzy Lie ideal rather than a fuzzy subalgebra.

We begin by considering two elements $m_1, m_2 \in \mathcal{M}_1$. Our goal is to show that the inverse image $f^{-1}(\mathcal{P})$ satisfies the condition for being a complex intuitionistic fuzzy Lie ideal.

For the membership function $\phi_{f^{-1}(\mathcal{P})}$ under the Lie bracket, we have:

$$\phi_{f^{-1}(\mathcal{P})}([m_1, m_2]) = \phi_{\mathcal{P}}(f([m_1, m_2])) = \phi_{\mathcal{P}}([f(m_1), f(m_2)]),$$

where the second equality follows from the fact that f is a homomorphism. Since $\mathcal{P}$ is a fuzzy Lie ideal in $\mathcal{M}_2$, by Definition 1, we know:

$$\phi_{\mathcal{P}}([f(m_1), f(m_2)]) \geq \phi_{\mathcal{P}}(f(m_1)) \vee \phi_{\mathcal{P}}(f(m_2)).$$

Thus, we obtain:

$$\phi_{f^{-1}(\mathcal{P})}([m_1, m_2]) \geq \phi_{f^{-1}(\mathcal{P})}(m_1) \vee \phi_{f^{-1}(\mathcal{P})}(m_2).$$

This demonstrates that $\phi_{f^{-1}(\mathcal{P})}$ satisfies the required condition under the Lie bracket.

For the non-membership function $\psi_{f^{-1}(\mathcal{P})}$, we similarly have:

$$\psi_{f^{-1}(\mathcal{P})}([m_1, m_2]) = \psi_{\mathcal{P}}(f([m_1, m_2])) = \psi_{\mathcal{P}}([f(m_1), f(m_2)]).$$

Since $\mathcal{P}$ is a fuzzy Lie ideal, it follows that:

$$\psi_{\mathcal{P}}([f(m_1), f(m_2)]) \leq \psi_{\mathcal{P}}(f(m_1)) \wedge \psi_{\mathcal{P}}(f(m_2)).$$

Therefore, we have:

$$\psi_{f^{-1}(\mathcal{P})}([m_1, m_2]) \leq \psi_{f^{-1}(\mathcal{P})}(m_1) \wedge \psi_{f^{-1}(\mathcal{P})}(m_2).$$

Thus, the set $f^{-1}(\mathcal{P})$ satisfies the conditions for being a complex intuitionistic fuzzy Lie ideal in $\mathcal{M}_1$, completing the proof.

Lemma 1. *([1]) If $f : \mathcal{M}_1 \to \mathcal{M}_2$ represents a Lie algebra homomorphism, and $\mathcal{E} = (\phi_{\mathcal{E}}, \psi_{\mathcal{E}})$ constitutes an intuitionistic fuzzy Lie subalgebra within $\mathcal{M}_1$, then the intuitionistic fuzzy set $f(\mathcal{E})$ transforms into an intuitionistic fuzzy Lie subalgebra over the domain* $\mathrm{im}(f)$.

Lemma 2. *([14]) If $\mathcal{E} = (\phi_{\mathcal{E}},\ \psi_{\mathcal{E}})$ represents a complex intuitionistic fuzzy set of a Lie algebra $\mathcal{M}$, then $\mathcal{E}$ qualifies as a complex intuitionistic fuzzy Lie ideal (or subalgebra) of $\mathcal{M}$ if and only if the associated intuitionistic fuzzy subset*

$$\overline{\mathcal{E}} = \{(m, \rho_{\mathcal{E}}(m),\ \hat{\rho}_{\mathcal{E}}(m))\ :\ m \in \mathcal{M}\}$$

emerges as an intuitionistic fuzzy Lie ideal (or subalgebra) of $\mathcal{M}$.

Theorem 2. *Assume $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a homomorphism between Lie algebras. If $\mathcal{E} = (\phi_{\mathcal{E}},\ \psi_{\mathcal{E}})$ constitutes a complex intuitionistic fuzzy Lie subalgebra within $\mathcal{M}_1$, then the complex intuitionistic fuzzy set $f(\mathcal{E})$ forms a complex intuitionistic fuzzy Lie subalgebra in the context of the image of f.*

Proof. Initially, we demonstrate the homogeneity of $f(\mathcal{E})$ as follows:

$$\begin{aligned}\phi_{f(\mathcal{E})}(n) &= \sup_{n=f(m)} \{\phi_{\mathcal{E}}(m)\} \\ &= \sup_{n=f(m)} \{\rho_{\mathcal{E}}(m) e^{i\zeta_{\mathcal{E}}(m)}\}\end{aligned}$$

$$= \sup_{n=f(m)} \{\rho_{\mathcal{E}}(m)\} e^{i(\sup_{n=f(m)}\{\zeta_{\mathcal{E}}(m)\})} \qquad \text{(since } \mathcal{E} \text{ is homogeneous)}.$$

In a similar manner, we can derive

$$\psi_{f(\mathcal{E})}(n) = \inf_{n=f(m)} \{\hat{\rho}_{\mathcal{E}}(m)\} e^{i(\inf_{n=f(m)}\{\hat{\zeta}_{\mathcal{E}}(m)\})}.$$

Next, we analyze the case where n_1 and n_2 belong to the image of f, denoted as $\mathrm{im}(f)$, and where $\sup_{n_1=f(m)}\{\rho_{\mathcal{E}}(m)\} \leq \sup_{n_2=f(m)}\{\rho_{\mathcal{E}}(m)\}$. Suppose, for the sake of contradiction, that $\sup_{n_2=f(m)}\{\zeta_{\mathcal{E}}(m)\} < \sup_{n_1=f(m)}\{\zeta_{\mathcal{E}}(m)\}$. This implies the existence of $m_1 \in \mathcal{M}_1$ such that $f(m_1) = n_1$ and $\sup_{n_2=\varphi(m)}\{\zeta_{\mathcal{E}}(m)\} < \zeta_{\mathcal{E}}(m_1)$. Consider the case where $f(m) = n_2$. This leads to $\zeta_{\mathcal{E}}(m) < \zeta_{\mathcal{E}}(m_1)$, and due to the homogeneity of $\mathcal{E}$, we deduce that $\rho_{\mathcal{E}}(m) < \rho_{\mathcal{E}}(m_1)$. Consequently, we find $\sup_{n_2=f(m)}\{\rho_{\mathcal{E}}(m)\} < \rho_{\mathcal{E}}(m_1)$. This directly contradicts our assumption $\sup_{n_1=f(m)}\{\rho_{\mathcal{E}}(m)\} \leq \sup_{n_2=f(m)}\{\rho_{\mathcal{E}}(m)\}$. By employing a similar reasoning, we can establish that if $\inf_{n_1=f(m)}\{\hat{\rho}_{\mathcal{E}}(m)\} \leq \inf_{n_2=f(m)}\{\hat{\rho}_{\mathcal{E}}(m)\}$, then $\inf_{n_1=f(m)}\{\hat{\zeta}_{\mathcal{E}}(m)\} \leq \inf_{n_2=f(m)}\{\hat{\zeta}_{\mathcal{E}}(m)\}$.

As a result of the above analysis, we conclude that $f(\mathcal{E})$ exhibits homogeneity within $\mathrm{im}(f)$.

Given that $\mathcal{E}$ is a complex intuitionistic fuzzy Lie subalgebra, we can deduce from Lemma 2 that $\overline{\mathcal{E}} = \{(m, \rho_{\mathcal{E}}(m), \hat{\rho}_{\mathcal{E}}(m)) :\ m \in \mathcal{M}_1\}$ constitutes an intuitionistic fuzzy Lie subalgebra. According to Lemma 1, when considering the transformation of an intuitionistic fuzzy Lie subalgebra, it maintains its nature as an intuitionistic fuzzy Lie subalgebra. Consequently, the transformed set $f(\overline{\mathcal{E}}) = \{(n, \rho_{\mathcal{E}_{f(\mathcal{E})}}(n), \hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}(n)) :\ n \in \mathrm{im}(f)\}$ represents an intuitionistic fuzzy Lie subalgebra within the domain of $\mathrm{im}(f)$.

For any elements n_1 and n_2 belonging to $\mathrm{im}(f)$, as well as for scalar $k \in \mathcal{K}$, we can ascertain the following assertions concerning $\mathcal{E}_{f(\mathcal{E})}$:

(i) $\rho_{\mathcal{E}_{f(\mathcal{E})}}(n_1 + n_2) \geq \rho_{\mathcal{E}_{f(\mathcal{E})}}(n_1) \wedge \rho_{\mathcal{E}_{f(\mathcal{E})}}(n_2)$ and $\hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}(n_1 + n_2) \leq \hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}(n_1) \vee \hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}(n_2)$,

(ii) $\rho_{\mathcal{E}_{f(\mathcal{E})}}(kn_1) \geq \rho_{\mathcal{E}_{f(\mathcal{E})}}(n_1)$ and $\hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}(kn_1) \leq \hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}(n_1)$,

(iii) $\rho_{\mathcal{E}_{f(\mathcal{E})}}([n_1,\ n_2]) \geq \rho_{\mathcal{E}_{f(\mathcal{E})}}(n_1) \wedge \rho_{\mathcal{E}_{f(\mathcal{E})}}(n_2)$ and $\hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}([n_1,\ n_2]) \leq \hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}(n_1) \vee \hat{\rho}_{\mathcal{E}_{f(\mathcal{E})}}(n_2)$.

Given that $f(\mathcal{E})$ is endowed with homogeneity, we can thereby conclude that $f(\mathcal{E})$ qualifies as a complex intuitionistic fuzzy Lie subalgebra situated within the realm of $\mathrm{im}(f)$.

Here, we present the proof of the subsequent outcome, originally established in [8] for Lie superalgebras, adapted to the context of Lie algebra homomorphisms. This outcome demonstrates that the transformation of an intuitionistic fuzzy Lie ideal through a Lie algebra homomorphism maintains its nature as an intuitionistic fuzzy Lie ideal as well.

Corollary 2. *Consider a Lie algebra homomorphism $f : \mathcal{M}_1 \to \mathcal{M}_2$. If $E = (\phi_{\mathcal{E}}, \psi_{\mathcal{E}})$ constitutes an intuitionistic fuzzy Lie ideal within $\mathcal{M}_1$, then the intuitionistic fuzzy set $f(\mathcal{E})$ preserves its character as an intuitionistic fuzzy Lie ideal even within the domain $\mathrm{im}(f)$.*

Proof. It suffices to demonstrate that for any n_1 and n_2 in $\mathrm{im}(f)$, the conditions $\phi_{f(\mathcal{E})}([n_1,\ n_2]) \geq \phi_{f(\mathcal{E})}(n_1) \vee \phi_{f(\mathcal{E})}(n_2)$ and $\psi_{f(\mathcal{E})}([n_1,\ n_2]) \leq \psi_{f(\mathcal{E})}(n_1) \wedge \psi_{f(\mathcal{E})}(n_2)$ hold. Let n_1 and n_2 be elements of $\mathrm{im}(f)$.

Suppose, for the sake of contradiction, that $\phi_{\psi(\mathcal{E})}([n_1,\ n_2]) < \phi_{f(\mathcal{E})}(n_1) \vee \phi_{f(\mathcal{E})}(n_2)$. Let t be chosen from the interval $[0,\ 1]$ such that $\phi_{f(\mathcal{E})}([n_1,\ n_2]) < t < \phi_{f(\mathcal{E})}(n_1) \vee \phi_{f(\mathcal{E})}(n_2)$. Without any loss of generality, we can assume $\phi_{f(\mathcal{E})}(n_1) \geq \phi_{f(\mathcal{E})}(n_2)$. Consequently, it follows that $\phi_{f(\mathcal{E})}([n_1,\ n_2]) < t < \sup_{n_1=f(m)}\{\phi_{\mathcal{E}}(m)\}$. Hence, there exists an q within $\mathcal{M}_1$ such that $f(q) = n_1$ and $\phi_{f(\mathcal{E})}([n_1,\ n_2]) < t < \phi_{\mathcal{E}}(q)$.

For any s in $\mathcal{M}_1$ with $f(s) = n_2$, it holds that $f([q,s]) = [f(q), f(s)] = [n_1,\ n_2]$. As a result, we find that

$$\begin{aligned}
\phi_{f(\mathcal{E})}([n_1,\ n_2]) &= \sup_{[n_1,\ n_2]=f([m_1,\ m_2])} \phi_{\mathcal{E}}([m_1,\ m_2]) \\
&\geq \phi_{\mathcal{E}}([q,s]) \\
&\geq \phi_{\mathcal{E}}(q) \vee \phi_{\mathcal{E}}(s) > t > \phi_{f(\mathcal{E})}([n_1,\ n_2]),
\end{aligned}$$

leading to a contradiction.

Additionally, if $\psi_{f(\mathcal{E})}([n_1,\ n_2]) > \psi_{f(\mathcal{E})}(n_1) \wedge \psi_{f(\mathcal{E})}(n_2)$, then a value r within the interval $[0,\ 1]$ can be selected such that $\psi_{f(\mathcal{E})}([n_1,\ n_2]) > r > \psi_{f(\mathcal{E})}(n_1) \wedge \psi_{f(\mathcal{E})}(n_2)$. Without loss of generality, let's assume $\psi_{f(\mathcal{E})}(n_1) \leq \psi_{f(\mathcal{E})}(n_2)$. This leads to $\psi_{f(\mathcal{E})}([n_1,\ n_2]) > r > \inf_{n_1=f(m)} \psi_{\mathcal{E}}(m)$, which enables the identification of an q in $\mathcal{M}_1$ such that $f(q) = n_1$ and $\psi_{f(\mathcal{E})}([n_1,\ n_2]) > r > \psi_{\mathcal{E}}(q)$.

By choosing $s \in \mathcal{M}_1$ with $f(s) = n_2$, it becomes evident that $f([q,s]) = [f(q), f(s)] = [n_1,\ n_2]$. Consequently,

$$\begin{aligned}
\psi_{f(\mathcal{E})}([n_1,\ n_2]) &= \inf_{[n_1,\ n_2]=f([m_1,\ m_2])} \psi_{\mathcal{E}}([m_1,\ m_2]) \\
&\leq \psi_{\mathcal{E}}([q,s]) \\
&\leq \psi_{\mathcal{E}}(q) \wedge \psi_{\mathcal{E}}(s) < r < \psi_{f(\mathcal{E})}([n_1,\ n_2]),
\end{aligned}$$

leading to a contradiction. Therefore, it can be concluded that $f(\mathcal{E})$ indeed constitutes an intuitionistic fuzzy Lie ideal within $\mathrm{im}(f)$.

In Corollary 2, we establish that if $f : \mathcal{M}_1 \to \mathcal{M}_2$ serves as a Lie algebra homomorphism, and $\mathcal{E} = (\phi_{\mathcal{E}}, \psi_{\mathcal{E}})$ constitutes an intuitionistic fuzzy Lie ideal within $\mathcal{M}_1$, then the intuitionistic fuzzy set $f(\mathcal{E})$ transforms into an intuitionistic fuzzy Lie ideal within $\mathrm{im}(f)$. This result, combined with the insights from Theorem 2, allows us to expand this understanding into the realm of complex intuitionistic fuzzy Lie algebras.

Corollary 3. *Let $f : \mathcal{M}_1 \to \mathcal{M}_2$ represent a Lie algebra homomorphism. If $\mathcal{E} = (\phi_{\mathcal{E}}, \psi_{\mathcal{E}})$ denotes a complex intuitionistic fuzzy Lie ideal of $\mathcal{M}_1$, then the complex intuitionistic fuzzy set $f(\mathcal{E})$ evolves into a complex intuitionistic fuzzy Lie ideal within the domain* $\mathrm{im}(f)$.

4. More Homomorphism Properties of Complex Intuitionistic Fuzzy Lie Algebras

Let $\mathcal{E} = (\phi_{\mathcal{E}},\ \psi_{\mathcal{E}})$ and $\mathcal{P} = (\phi_{\mathcal{P}},\ \psi_{\mathcal{P}})$ be two complex intuitionistic fuzzy sets on a Lie algebra $\mathcal{M}$ over a field $\mathcal{K}$. The sum of $\mathcal{E}$ and $\mathcal{P}$, which was defined by Chen and Zhang [8] in the case of intuitionistic fuzzy Lie superalgebras, is defined as follows:

$$\mathcal{E} + \mathcal{P} = (\phi_{\mathcal{E}+\mathcal{P}},\ \psi_{\mathcal{E}+\mathcal{P}}),$$

where

$$\phi_{\mathcal{E}+\mathcal{P}}(m) = \sup_{m_1+m_2} \{\phi_{\mathcal{E}}(m_1) \wedge \phi_{\mathcal{P}}(m_2)\},$$

and

$$\psi_{\mathcal{E}+\mathcal{P}}(m) = \inf_{m=m_1+m_2} \{\psi_{\mathcal{E}}(m_1) \vee \psi_{\mathcal{P}}(m_2)\}.$$

Let $\mathcal{E} = (\phi_{\mathcal{E}}, \psi_{\mathcal{E}})$ and $\mathcal{Q} = (\phi_{\mathcal{Q}}, \psi_{\mathcal{Q}})$ be two complex intuitionistic fuzzy sets on the same set M, where $\phi_{\mathcal{E}} = \rho_{\mathcal{E}} e^{i\zeta_{\mathcal{E}}}, \psi_{\mathcal{E}} = \hat{\rho}_{\mathcal{E}} e^{i\hat{\zeta}_{\mathcal{E}}}, \phi_{\mathcal{Q}} = \rho_{\mathcal{Q}} e^{i\zeta_{\mathcal{Q}}}$, and $\psi_{\mathcal{Q}} = \hat{\rho}_{\mathcal{Q}} e^{i\hat{\zeta}_{\mathcal{Q}}}$. We say that $\mathcal{E}$ is homogeneous with $\mathcal{Q}$ if the following hold for all $m,\ n \in M$:

(i) $\rho_{\mathcal{E}}(m) \leq \rho_{\mathcal{Q}}(n)$ if and only if $\zeta_{\mathcal{E}}(m) \leq \zeta_{\mathcal{Q}}(n)$ (in this case $\phi_{\mathcal{E}}(m) \leq \phi_{\mathcal{Q}}(n)$),

(ii) $\hat{\rho}_{\mathcal{E}}(m) \leq \hat{\rho}_{\mathcal{Q}}(n)$ if and only if $\hat{\zeta}_{\mathcal{E}}(m) \leq \hat{\zeta}_{\mathcal{Q}}(n)$ (in this case $\psi_{\mathcal{E}}(m) \leq \psi_{\mathcal{Q}}(n)$).

Theorem 3. *([14]) Let $\mathcal{E} = (\phi_{\mathcal{E}},\ \psi_{\mathcal{E}})$ and $\mathcal{P} = (\phi_{\mathcal{P}},\ \psi_{\mathcal{P}})$ be two complex intuitionistic fuzzy Lie ideals on $\mathcal{M}$ such that $\mathcal{E}$ is homogeneous with $\mathcal{P}$ and $\mathcal{E} + \mathcal{P}$ is homogeneous. Then $\mathcal{E} + \mathcal{P}$ is a complex intuitionistic fuzzy Lie ideal of $\mathcal{M}$.*

Consider a surjective Lie algebra homomorphism $f : \mathcal{M}_1 \to \mathcal{M}_2$. Suppose we have two complex intuitionistic fuzzy Lie ideals, denoted as $\mathcal{E}$ and $\mathcal{P}$, on $\mathcal{M}_1$. Assume that $\mathcal{E}$ is homogeneous with respect to $\mathcal{P}$, and the sum $\mathcal{E} + \mathcal{P}$ is also homogeneous. In accordance with Corollary 3, we can conclude that $f(\mathcal{E} + \mathcal{P})$ constitutes a complex intuitionistic fuzzy Lie ideal of the image of f, denoted as $\mathrm{im}(f)$. The subsequent theorem establishes the relationship between the sets $f(\mathcal{E} + \mathcal{P})$, $f(\mathcal{E})$, and $f(\mathcal{P})$.

Theorem 4. *Consider a surjective Lie algebra homomorphism $f : \mathcal{M}_1 \to \mathcal{M}_2$. Let $\mathcal{E} = (\phi_{\mathcal{E}}, \psi_{\mathcal{E}})$ and $\mathcal{P} = (\phi_{\mathcal{P}}, \psi_{\mathcal{P}})$ be complex intuitionistic fuzzy Lie ideals on $\mathcal{M}_1$ such that $\mathcal{E}$ is homogeneous with respect to $\mathcal{P}$. Then, $f(\mathcal{E} + \mathcal{P}) = f(\mathcal{E}) + f(\mathcal{P})$.*

Proof. Let $n \in \mathcal{M}_2$. Then

$$\begin{aligned}
\phi_{f(\mathcal{E}+\mathcal{P})}(n) &= \sup_{n=f(m)} \{\phi_{\mathcal{E}+\mathcal{P}}(m)\} \\
&= \sup_{n=f(m)} \{ \sup_{m=m_1+m_2} \{\phi_{\mathcal{E}}(m_1) \wedge \phi_{\mathcal{P}}(m_2)\}\} \\
&= \sup_{n=q+s} \{ \sup_{q=f(m_1)} \{\phi_{\mathcal{E}}(m_1)\} \wedge \sup_{s=f(m_2)} \{\phi_{\mathcal{P}}(m_2)\}\} \\
&= \phi_{f(\mathcal{E})+f(\mathcal{P})}(n).
\end{aligned}$$

In addition,

$$\begin{aligned}\psi_{f(\mathcal{E}+\mathcal{P})}(n) &= \inf_{n=f(m)}\{\psi_{\mathcal{E}+\mathcal{P}}(m)\}\\ &= \inf_{n=f(m)}\{\inf_{m=m_1+m_2}\{\psi_{\mathcal{E}}(m_1)\vee\psi_{\mathcal{P}}(m_2)\}\}\\ &= \inf_{n=q+s}\{\inf_{q=f(m_1)}\{\psi_{\mathcal{E}}(m_1)\}\vee\inf_{s=f(m_2)}\{\psi_{\mathcal{P}}(m_2)\}\}\\ &=\psi_{f(\mathcal{E})+f(\mathcal{P})}(n).\end{aligned}$$

Therefore, $f(\mathcal{E}+\mathcal{P}) = f(\mathcal{E}) + f(\mathcal{P})$.

Let X be a nonempty set. Let $\phi_{\mathcal{E}}(x) = \rho_{\mathcal{E}}(x)e^{i\zeta_{\mathcal{E}}(x)}$, $\psi_{\mathcal{E}}(x) = \hat{\rho}_{\mathcal{E}}(x)e^{i\hat{\zeta}_{\mathcal{E}}(x)}$, and $\mathcal{E} = \{(x,\ \phi_{\mathcal{E}}(x),\ \psi_{\mathcal{E}}(x)) :\ x\in X\}$ be a complex intuitionistic fuzzy set within X. For $\alpha,\ \hat{\alpha}\in [0,\ 1]$ and $\beta,\ \hat{\beta}\in[0,\ 2\pi]$, the set

$$\mathcal{E}_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})} = \{x : \rho_{\mathcal{E}}(x)\geq\alpha,\ \zeta_{\mathcal{E}}(x)\geq\beta,\ \hat{\rho}_{\mathcal{E}}(x)\leq\hat{\alpha},\ \hat{\zeta}_{\mathcal{E}}(x)\leq\hat{\beta}\}$$

is called the upper level subset of the complex intuitionistic fuzzy subset $\mathcal{E}$. The subsets

$$\mathcal{E}_{(\alpha^{>},\beta)}^{(\hat{\alpha}^{<},\hat{\beta})} = \{x : \rho_{\mathcal{E}}(x)>\alpha,\ \zeta_{\mathcal{E}}(x)\geq\beta,\ \hat{\rho}_{\mathcal{E}}(x)<\hat{\alpha},\ \hat{\zeta}_{\mathcal{E}}(x)\leq\hat{\beta}\},$$

$$\mathcal{E}_{(\alpha,\beta^{>})}^{(\hat{\alpha},\hat{\beta}^{<})} = \{x : \rho_{\mathcal{E}}(x)\geq\alpha,\ \zeta_{\mathcal{E}}(x)>\beta,\ \hat{\rho}_{\mathcal{E}}(x)\leq\hat{\alpha},\ \hat{\zeta}_{\mathcal{E}}(x)<\hat{\beta}\},$$

and

$$\mathcal{E}_{(\alpha^{>},\beta^{>})}^{(\hat{\alpha}^{<},\hat{\beta}^{<})} = \{x : \rho_{\mathcal{E}}(x)>\alpha,\ \zeta_{\mathcal{E}}(x)>\beta,\ \hat{\rho}_{\mathcal{E}}(x)<\hat{\alpha},\ \hat{\zeta}_{\mathcal{E}}(x)<\hat{\beta}\}$$

are called strong upper level subsets of the complex intuitionistic fuzzy subset A. The following theorem was obtained by S. Shaqaqha in the setting of complex fuzzy Lie subalgebras [13]. We extend it to the case of complex intuitionistic fuzzy Lie subalgebras.

Theorem 5. *Let $f : \mathcal{M}_1\to\mathcal{M}_2$ be a Lie algebra homomorphism. If $\mathcal{P} = (\phi_{\mathcal{P}},\ \psi_{\mathcal{P}})$ is a complex intuitionistic fuzzy set of $\mathcal{M}_2$. For $\alpha,\ \hat{\alpha}\in[0,\ 1]$ and $\beta,\ \hat{\beta}\in[0,\ 2\pi]$, we have*

(i) $f^{-1}\left(\mathcal{P}_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}\right) = \left(f^{-1}(\mathcal{P})\right)_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$,

(ii) $f^{-1}\left(\mathcal{P}_{(\alpha^{>},\beta)}^{(\hat{\alpha}^{<},\hat{\beta})}\right) = \left(f^{-1}(\mathcal{P})\right)_{(\alpha^{>},\beta)}^{(\hat{\alpha}^{<},\hat{\beta})}$,

(iii) $f^{-1}\left(\mathcal{P}_{(\alpha,\beta^{>})}^{(\hat{\alpha},\hat{\beta}^{<})}\right) = \left(f^{-1}(\mathcal{P})\right)_{(\alpha,\beta^{>})}^{(\hat{\alpha},\hat{\beta}^{<})}$,

(iv) $f^{-1}\left(\mathcal{P}_{(\alpha^{>},\beta^{>})}^{(\hat{\alpha}^{<},\hat{\beta}^{<})}\right) = \left(f^{-1}(\mathcal{P})\right)_{(\alpha^{>},\beta^{>})}^{(\hat{\alpha}^{<},\hat{\beta}^{<})}$.

Proof.

(i) $m\in f^{-1}\left(\mathcal{P}_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}\right)$ if and only if $f(m)\in\mathcal{P}_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$ if and only if $\phi_{\mathcal{P}}(f(m)) = \phi_{f^{-1}(\mathcal{P})}(m)\geq\alpha e^{i\beta}$ and $\psi_{\mathcal{P}}(f(m)) = \psi_{f^{-1}(\mathcal{P})}(m)\leq\hat{\alpha}e^{i\hat{\beta}}$ if and only if $m\in\left(f^{-1}(\mathcal{P})\right)_{(\alpha,\beta)}^{(\hat{\alpha},\hat{\beta})}$. The proofs of (ii), (iii) and (iv) are same.

5. Conclusions

In this paper, we explored the interaction between complex intuitionistic fuzzy sets and Lie algebra homomorphisms, deriving several significant results. We showed that the images and preimages of complex intuitionistic fuzzy Lie subalgebras and ideals under homomorphisms maintain their structural properties, extending known results in fuzzy and intuitionistic fuzzy algebra to the complex intuitionistic fuzzy setting.

The method used focused on analyzing the effects of homomorphisms on membership and non-membership functions, providing a deeper understanding of the preservation of these fuzzy structures. Our results generalize existing findings on fuzzy Lie algebras [21], intuitionistic fuzzy Lie algebras [1], and complex fuzzy Lie algebras [13], offering new insights into their behavior in more expressive contexts.

These findings pave the way for further research, including extensions to gamma rings [14, 15], n-Lie algebras [16], and Hom-Lie algebras [17]. Additionally, investigating the application of these methods to complex Pythagorean Lie algebras [19, 20] could reveal new structural insights.

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