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Common Fixed Point of Generalized Berinde Type Contraction and an Application

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Abstract. In this paper, we introduce $\lambda_{(s,\varphi,\psi,L)}$-generalized Berinde type contraction and obtain some common fixed point results for such class of contractions the setting of triangular α-admissible mappings with respect to η in the framework of b-metric spaces. Our results generalize and extend some theorems in the literature. An example is given to support our result.

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1. Introduction and preliminaries

The most important tools in fixed point theory is Banach contraction principle. A lot of authors have extended or generalized this contraction and proved the existence of fixed and common fixed point theorems for single valued and multi-valued mappings and some application (see [3–6, 11, 14–18, 21–23]). The concept of the b-metric space was introduced by Czerwik [12] and he also obtained some fixed-point theorems of contractive mappings in b-metric space. Since then, this notion has been used by many authors to obtain various fixed point theorems. Roshan et al. in [18] used the notion of almost generalized contractive mappings in ordered complete b-metric spaces and established some fixed and common fixed point results.

The main goal of this section is to present some definitions and properties of b-metric spaces:

Definition 1.1. ([12]) Let $\digamma$ be a nonempty set. A mapping $\Lambda_b : \digamma \times \digamma \to [0, +\infty)$ is said to be a b-metric if the following three conditions hold for all $u, v \in \digamma$:
(Λ_1) $\Lambda(u, v) = 0 \Rightarrow u = v$;
(Λ_2) $\Lambda(u, v) = \Lambda(v, u)$;
(Λ_3) $\Lambda(u, v) \leq s[\Lambda(u, w) + \Lambda(w, v)]$.
In this case, the pair $(\digamma, \Lambda_b)$ is called a b-metric space.

Example 1.2. *Let $(\digamma, \Lambda_b)$ be a metric space and let $\beta > 1, \varrho \geq 0$ and $\mu > 0$. For $u, v \in \digamma$, set $\Lambda_b(u, v) = \varrho\Lambda_b(u, v)+\mu\Lambda_b(u, v)^\beta$. Then $(\digamma, \Lambda_b)$ is a b-metric space with the parameter $s = 2\beta-1$ and not a metric space on $\digamma$.*

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Example 1.3. *Let $\digamma$ be the set of Lebesgue measurable functions on [0,1] such that $\int_0^1 \mid p(u) \mid^2 < +\infty$. Define*

$$\Lambda_b(u,v) = \int_0^1 \mid p(u) - q(u) \mid^2 d(u).$$

Then Λ_b satisfies the following properties:

(i) $\Lambda_b(u,v) = 0 \Leftrightarrow u = v$

(ii) $\Lambda_b(u,v) = \Lambda_b(v,u)$, *for all* $u, v \in \digamma$

(iii) $\Lambda_b(u,v) \leq 2[\Lambda_b(u,w) + \Lambda_b(w,v)]$, *for all* $u, w, v \in \digamma$.

Definition 1.4. ([20]) Let $(\digamma, \Lambda_b)$ be a b-metric space. Then a sequence $\{u_n\}$ in $\digamma$ is called:
(i) b-convergent if and only if there exists $v \in \digamma$ such that $\Lambda_b(u_n, u) \to 0$, as $n \to +\infty$. In this case, we write $\lim_{n\to+\infty} u_n = u$.
(2) b-Cauchy if and only if $\Lambda_b(u_n, u_m) = 0$ as $n, m \to \infty$.

Proposition 1.5. *([11]) In b-metric space $(\digamma, \Lambda_b)$ the following assertions holds:*
(1) A b-convergent sequence has a unique limit,
(2) Each b-convergent is b-Cauchy,
(3) In general, a b-metric is not continuous.

Proposition 1.6. *([11]) The b-metric space $(\digamma, \Lambda_b)$ is complete if every Cauchy sequence in $\digamma$ b-converges.*

Qawagneh et al. [19] introduced the notion of triangular α-admissible with respect to η for p and q on a set $\digamma$ as the following:

Definition 1.7. ([20])Let $p, q : \digamma \to \digamma$ be two mappings and $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ be two functions such that the following assertions hold:

(i) if $\alpha(u,v) \geq \eta(u,v)$, then $\alpha(pu, qv) \geq \eta(pu, qv)$, and $\alpha(pqu, qpv) \geq \eta(pqu, qpv)$,

(ii) if $\alpha(u,h) \geq \eta(u,h)$, and $\alpha(h,v) \geq \eta(h,v)$, then $\alpha(u,v) \geq \eta(u,v)$,

Lemma 1.8. *([22]) Let $p, q : \digamma \to \digamma$ be two mappings and $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ be two functions such that the pair (p,q) is triangular α-admissible with respect to η. Assume that there exist $u_0 \in \digamma$ such that $\alpha(u_0, pu_0) \geq \eta(u_0, pu_0)$. Define a sequence $\{u_n\}$ in $\digamma$ by $pu_{2n} = u_{2n+1}$ and $qu_{2n+1} = u_{2n+2}$. Then $\alpha(u_n, u_m) \geq \eta(u_n, u_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Berinde [[6],[7],[8],[9],[10]] presented many interesting fixed-point results for various types of contraction mappings. In [8] and [9], he defined the almost contraction map as follows.

Definition 1.9. Let $(\digamma, \Lambda)$ be a metric space. A map $p : \digamma \to \digamma$ is called an almost contraction if there exist a constant $\lambda \in [0,1)$ and some $L \geq 0$ such that:

$$\Lambda(pu, pv) \leq \lambda\Lambda(u,v) + L\Lambda(v, pu)$$

for all $u, v \in \digamma$.

Let Φ the set of all increasing and continuous functions $\varphi : [0,+\infty) \to [0,+\infty)$ and let Δ be the set of all lower semi-continuous functions $\psi : [0,+\infty) \to [0,+\infty)$ with $\psi(b) = b$ if and only if $b = 0$.

2. An $\lambda_{(s,\varphi,\psi,L)}$- generalized Berinde type contraction mapping

Now, we will present $\lambda_{(s,\varphi,\phi,L)}$- generalized Berinde type contraction mapping prove our main result for such class of contractions in the framework of b-metric spaces.

Definition 2.1. Let $(\digamma, \Lambda_b)$ be a b-metric space with parameter $s \geq 1$ and $p, q : \digamma \to \digamma$ be a two mappings. Then we consider that the pair (p, q) is $\lambda_{(s,\varphi,\phi,L)}$-generalized Berinde type contraction mapping if there exists $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ be two mappings, $\varphi \in \Omega, \phi \in \Phi, \lambda \in [0, 1), L \geq 0$ such that

$$\varphi\left(s^2\Lambda_b(pu, qv)\right) \leq \lambda\left[\varphi\left(M_{\Lambda_b}(u, v)\right) - \phi\left(M_{\Lambda_b}(u, v)\right) + LN_{\Lambda_b}(u, v)\right], \tag{2.1}$$

holds for all $u, v \in \digamma$, where

$$M_{\Lambda_b}(u, v) = \max\left\{\Lambda_b(u, v), \Lambda_b(u, pu), \Lambda_b(v, qv), \frac{\Lambda_b(u, qv) + \Lambda_b(pu, v)}{2s[1 + \Lambda_b(pu, v)]}\right\},$$

and

$$N_{\Lambda_b}(u, v) = \min\left\{\Lambda_b(u, v), \Lambda_b(u, pu), \Lambda_b(v, qv), \Lambda_b(v, pu)\right\}.$$

Now we begin with our first result.

Theorem 2.2. *Let $(\digamma, \Lambda_b)$ be a complete b-metric space with the constant $s \geq 1$, and (p, q) be two self-mappings on $\digamma$. Suppose that $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ are two functions. Assume that the following conditions hold:*
(i) $\lambda_{(s,\varphi,\phi,L)}$-Berinde type contraction mapping;
(ii) the pair (p, q) is triangular α-admissible with respect to η;
(iii) there exists $u_0 \in \digamma$ such that $\alpha(u_0, pu_0) \geq \eta(u_0, pu_0)$,
(iv) p and q are continuous mappings.
Then, p and q have a common fixed point in $\digamma$.

Proof. Let $u_0 \in \digamma$ such that $\alpha(u_0, pu_0) \geq \eta(u_0, pu_0)$. We define a sequence $\{u_n\} \subset \digamma$ such that $u_{2n+1} = pu_{2n}$ and $u_{2n+2} = qu_{2n+1}$ for all $n \in \mathbb{N}$. If $\exists$ an n_* such that $u_{n_*+1} = u_{n_*}$ for some $n_* \in \mathbb{N}$, then it is very easy to show that p and q have a common fixed point, which completes the proof. Since the pair (p, q) is triangular α-admissible with respect to η, then

$$\alpha(u_1, u_2) = \alpha(pu_0, qu_1) \geq \eta(pu_0, qu_1) = \eta(u_1, u_2)$$

and

$$\alpha(u_2, u_1) = \alpha(pu_1, qu_0) \geq \eta(pu_1, qu_0) = \eta(u_2, u_1).$$

One more time by using triangular α-admissible with respect to η, we get

$$\alpha(u_2, u_3) = \alpha(pu_1, qu_2) \geq \eta(pu_1, qu_2) = \eta(u_2, u_3)$$

and

$$\alpha(u_3, u_2) = \alpha(pu_2, qu_1) \geq \eta(pu_2, qu_1) = \eta(u_3, u_2).$$

By repeating the above steps for $n-$times, we obtain the following $\alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1})$ and $\alpha(u_{n+1}, u_n) \geq \eta(u_{n+1}, u_n)$. By Lemma 1.8, we have $\alpha(u_{2n}, u_{2n+1}) \geq \eta(u_{2n}, u_{2n+1})$ for all $n \in \mathbb{N}$ and since (p, q) is $\lambda_{(s,\varphi,\psi,L)}$-generalized Berinde type contraction mapping, we get

$$\begin{aligned}\varphi(\Lambda_b(u_{2n+1}, u_{2n+2})) &\leq \varphi(s^2\Lambda_b(pu_{2n}, qu_{2n+1}) \\ &\leq \lambda[\varphi(M_{\Lambda_b}(u_{2n}, u_{2n+1})) - \phi(M_{\Lambda_b}(u_{2n}, u_{2n+1})) + LN_{\Lambda_b}(u_{2n}, u_{2n+1})]\end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M_{\Lambda_b}(u_{2n}, u_{2n+1}) &= \max\Big\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n}, pu_{2n}), \Lambda_b(u_{2n+1}, qu_{2n+1}), \\ &\quad \frac{\Lambda_b(u_{2n}, qu_{2n+1}) + \Lambda_b(pu_{2n}, u_{2n+1})}{2s(1+\Lambda_b(pu_{2n}, u_{2n+1}))}\Big\} \\ &= \max\Big\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n+1}, u_{2n+2}), \\ &\quad \frac{\Lambda_b(u_{2n}, u_{2n+2}) + \Lambda_b(u_{2n+1}, u_{2n+1})}{2s(1+\Lambda_b(u_{2n+1}, u_{2n+1}))}\Big\} \\ &= \max\left\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n+1}, u_{2n+2}), \frac{\Lambda_b(u_{2n}, u_{2n+2})}{2s}\right\} \end{aligned}$$

and

$$N_{\Lambda_b}(u_{2n}, u_{2n+1}) = \min\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n}, pu_{2n}), \Lambda_b(u_{2n+1}, qu_{2n+1}), \Lambda_b(u_{2n+1}, pu_{2n})\}$$

i.e.,

$$N_{\Lambda_b}(u_{2n}, u_{2n+1}) = 0 \tag{2.3}$$

Since

$$\begin{aligned} \frac{\Lambda_b(u_{2n}, u_{2n+2})}{2s} &\le \frac{s[\Lambda_b(u_{2n}, u_{2n+1}) + \Lambda_b(u_{2n+1}, u_{2n+2})]}{2s} \\ &\le \frac{\Lambda_b(u_{2n}, u_{2n+1}) + \Lambda_b(u_{2n+1}, u_{2n+2})}{2} \le \max\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n+1}, u_{2n+2})\}, \end{aligned}$$

we get

$$M_{\Lambda_b}(u_{2n}, u_{2n+1})) \le \max\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n+1}, u_{2n+2})\}. \tag{2.4}$$

Taking (2.3) and (2.4) into account,(2.2) yields

$$\begin{aligned} \varphi(\Lambda_b(u_{2n+1}, u_{2n+2})) &\le \lambda\left[\varphi\left(\max\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n+1}, u_{2n+2})\}\right)\right. \\ &\quad \left.-\lambda\phi\left(\max\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n+1}, u_{2n+2})\}\right)\right] \\ &< \varphi\left(\max\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n+1}, u_{2n+2})\}\right) \\ &\quad - \phi\left(\max\{\Lambda_b(u_{2n}, u_{2n+1}), \Lambda_b(u_{2n+1}, u_{2n+2})\}\right). \end{aligned}$$

Now, we will show that $\Lambda_b(u_{2n+1}, u_{2n+2}) \le \Lambda_b(u_{2n}, u_{2n+1})$. Arguing by contradiction, we assume $\Lambda_b(u_{2n+1}, u_{2n+2}) > \Lambda_b(u_{2n}, u_{2n+1})$. Therefore, we have two cases.
Case 1: $M_{\Lambda_b}(u_{2n}, u_{2n+1}) = \Lambda_b(u_{2n}, u_{2n+1})$. Then

$$\varphi(\Lambda_b(u_{2n+1}, u_{2n+2}) < \varphi(\Lambda_b(u_{2n}, u_{2n+1})) - \phi(\Lambda_b(u_{2n}, u_{2n+1})) < \varphi(\Lambda_b(u_{2n}, u_{2n+1}))$$

Since φ is increasing, we have $\Lambda_b(u_{2n+1}, u_{2n+2}) < \Lambda_b(u_{2n}, u_{2n+1})$. which is a contradiction.
Case 2: $M_{\Lambda_b}(u_{2n}, u_{2n+1}) = \Lambda_b(u_{2n+1}, u_{2n+2})$. Then

$$\varphi(\Lambda_b(u_{2n+1}, u_{2n+2}) < \varphi(\Lambda_b(u_{2n+1}, u_{2n+2})) - \phi(\Lambda_b(u_{2n+1}, u_{2n+2})) < \varphi(\Lambda_b(u_{2n+1}, u_{2n+2}))$$

Since φ is increasing, we have $\Lambda_b(u_{2n+1}, u_{2n+2}) < \Lambda_b(u_{2n+1}, u_{2n+2})$. Which is a impossible. Hence from the above we have $\Lambda_b(u_{2n+1}, u_{2n+2}) \le \Lambda_b(u_{2n}, u_{2n+1})$
By similar way, we can prove that $\Lambda_b(u_{2n}, u_{2n+1}) \le \Lambda_b(u_{2n-1}, u_{2n})$. So, we conclude that $\Lambda_b(u_n, u_{n+1}) \le \Lambda_b(u_{n-1}, u_n)$. that is, the sequence $\Lambda_b(u_{n+1}, u_{n+1})$ is a decreasing sequence and bounded below for all $n \in \mathbb{N}$. Therefore there $\exists\ \omega \ge 0$ such that

$$\lim_{n\to\infty} \Lambda_b(u_n, u_{n+1}) = \omega.$$

We want to prove that $\omega = 0$. Now, we have

$$\varphi(\omega) \leq \lambda[\varphi(\omega) - \phi(\omega)] < \varphi(\omega) - \phi(\omega) < \varphi(\omega)$$

which is a contradiction. Hence

$$\lim_{n\to\infty} \Lambda_b(u_n, u_{n+1}) = 0. \tag{2.5}$$

Now, we want to prove that $\{u_n\}$ is a Cauchy sequence by Lemma 1.8, $\exists\ \varepsilon > 0$ and two subsequences $\{u_{m_i}\}$ and $\{u_{n_i}\}$ of $\{u_n\}$ with $m_i > n_i > i$ such that

$$\Lambda_b(u_{n_i}, u_{m_i}) \geq \varepsilon$$

$$\Lambda_b(u_{n_i-1}, u_{m_i}) < \varepsilon.$$

By using the triangular inequality, we have

$$\begin{aligned} \varepsilon &\leq \Lambda_b(u_{n_i}, u_{m_i}) \leq \Lambda_b(u_{n_i}, u_{n_i-1}) + s\Lambda_b(u_{n_i-1}, u_{m_i}) \\ &< s[\Lambda_b(u_{n_i}, u_{n_i-1}) + \varepsilon] \end{aligned} \tag{2.6}$$

Letting $i \to +\infty$ on both sides of (2.6) and using (2.5), we obtain

$$\varepsilon \leq \lim_{n\to+\infty} \Lambda_b(u_{n_i}, u_{m_i}) < s\varepsilon. \tag{2.7}$$

From triangular inequality, we have

$$\Lambda_b(u_{n_i}, u_{m_i}) \leq s[\Lambda_b(u_{n_i}, u_{n_i+1}) + \Lambda_b(u_{n_i+1}, u_{m_i})], \tag{2.8}$$

and

$$\Lambda_b(u_{n_i+1}, u_{m_i}) \leq s[\Lambda_b(u_{n_i+1}, u_{n_i}) + \Lambda_b(u_{n_i}, u_{m_i})]. \tag{2.9}$$

By taking upper limit as $i \to +\infty$ in (2.8) and applying (2.5) , (2.7) , we get

$$\varepsilon \leq \limsup_{i\to+\infty} \Lambda_b(u_{n_i}, u_{m_i}) \leq s\left(\limsup_{+i\to+\infty} \Lambda_b(u_{n_i+1}, u_{m_i})\right).$$

Again, by letting the upper limit as $i \to +\infty$ in (2.9), we have

$$\limsup_{i\to+\infty} \Lambda_b(u_{n_i+1}, u_{m_i}) \leq s\left(\limsup_{i\to+\infty} \Lambda_b(u_{n_i}, u_{m_i})\right) \leq s.s\varepsilon = s^2\varepsilon.$$

Thus

$$\frac{\varepsilon}{s} \leq \limsup_{i\to+\infty} \Lambda_b(u_{n_i+1}, u_{m_i}) \leq s^2\varepsilon. \tag{2.10}$$

Similarly,

$$\frac{\varepsilon}{s} \leq \limsup_{i\to+\infty} \Lambda_b(u_{n_i}, u_{m_i+1}) \leq s^2\varepsilon. \tag{2.11}$$

By using the triangular inequality, we get

$$\Lambda_b(u_{n_i+1}, u_{m_i}) \leq s[\Lambda_b(u_{n_i+1}, u_{m_i+1}) + \Lambda_b(u_{m_i+1}, u_{m_i})]. \tag{2.12}$$

On letting $i \to +\infty$ in (2.12) and using the inequalities (2.5) , (2.10) , we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{i\to+\infty} \Lambda_b(u_{n_i+1}, u_{m_i+1}). \tag{2.13}$$

By following the above methods, we find

$$\limsup_{i\to+\infty}\Lambda_b(u_{n_i+1},u_{m_i+1})\leq s^3\varepsilon. \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$\frac{\varepsilon}{s}\leq\limsup_{i\to+\infty}\Lambda_b(u_{n_i+1},u_{m_i+1})\leq s^3\varepsilon. \tag{2.15}$$

By Lemma 1.8, we have

$$\alpha(u_{n_i+1},u_{m_i+1})\geq\eta(u_{n_i+1},u_{m_i+1}).$$

Thus, we have

$$\begin{aligned}\varphi(\Lambda_b(u_{n_i+1},u_{m_i+1})) &\leq \varphi(s^2\Lambda_b(u_{n_i+1},u_{m_i+1}))\\ &\leq \lambda\left[\varphi\left(M_{\Lambda_b}(u_{n_i},u_{m_i})\right)-\phi\left(M_{\Lambda_b}(u_{n_i},u_{m_i})\right)+L\left(N_{\Lambda_b}(u_{n_i},u_{m_i})\right)\right]\\ &= \left[\lambda\varphi\left(M_{\Lambda_b}(u_{n_i},u_{m_i})\right)-\lambda\phi\left(M_{\Lambda_b}(u_{n_i},u_{m_i})\right)+\lambda L\left(N_{\Lambda_b}(u_{n_i},u_{m_i})\right)\right],\end{aligned}$$

where

$$\begin{aligned}M_{\Lambda_b}(u_{n_i},u_{m_i}) &= \max\{\Lambda_b(u_{n_i},u_{m_i}),\Lambda_b(u_{n_i},pu_{n_i}),\Lambda_b(u_{m_i},qu_{m_i}),\\ &\qquad \frac{\Lambda_b(u_{n_i},qu_{m_i})+\Lambda_b(pu_{n_i},u_{m_i})}{2s(1+\Lambda_b(pu_{n_i},u_{m_i}))}\}.\end{aligned}$$

$$\begin{aligned}N_{\Lambda_b}(u_{n_i},u_{m_i}) &= \min\left\{\Lambda_b(u_{n_i},u_{m_i}),\Lambda_b(u_{n_i},pu_{n_i}),\Lambda_b(u_{m_i},qu_{m_i}),\Lambda_b(u_{m_i},pu_{n_i})\right\}\\ &= \min\left\{\Lambda_b(u_{n_i},u_{m_i}),\Lambda_b(u_{n_i},u_{n_i+1}),\Lambda_b(u_{m_i},u_{m_i+1}),\Lambda_b(u_{m_i},u_{n_i+1})\right\}\end{aligned} \tag{2.16}$$

Taking the limit as $i\to+\infty$ in the above two expressions and using (2.5),(2.7) ,(2.10) and (2.11), we obtain

$$\varepsilon = \max\{\varepsilon,\frac{\frac{\varepsilon}{s}+\frac{\varepsilon}{s}}{2s}\}\leq\limsup_{i\to+\infty}\Lambda_b(u_{n_i},u_{m_i})\leq\max\{s\varepsilon,\frac{s^2\varepsilon+s^2\varepsilon}{2s}\}=s\varepsilon.$$

$$\limsup_{i\to+\infty}N_{\Lambda_b}(u_{n_i},u_{m_i})=0.$$

From (2.13), we obtain

$$\begin{aligned}\varphi(s\varepsilon) &\leq \varphi(s^2\frac{\varepsilon}{s^2})\leq\varphi(s^2\limsup_{i\to+\infty}\varphi(\Lambda_b(u_{n_i+1},u_{m_i+1}))\\ &\leq \lambda[\varphi(\limsup_{i\to+\infty}M_{\Lambda_b}(u_{n_i},u_{m_i})-\phi(\liminf_{i\to+\infty}M_{\Lambda_b}(u_{n_i},u_{m_i})\\ &\leq \lambda[\varphi(s\varepsilon)-\phi(s\varepsilon)]\\ &\leq \lambda(\varphi(s\varepsilon))-\lambda(\phi(s\varepsilon))\\ &< \lambda\varphi(s\varepsilon)\end{aligned}$$

which leads to a contradiction. Thus $\{u_n\}$ is a Cauchy sequence. Since $\digamma$ is an complete b-metric space and $\alpha(u_{n_i+1},u_{m_i+1})\geq\eta(u_{n_i+1},u_{m_i+1})$ for all $n\in\mathbb{N}_0$, there exists θ such that $\lim_{n\to+\infty}u_n=\theta$. If p is continuous, we have $p\theta=\lim_{n\to+\infty}pu_{2n}=\lim_{n\to+\infty}u_{2n+1}=\theta$. From Condition (2.2), we have:

$$\begin{aligned}\varphi(\Lambda_b(\theta,q\theta)) &\leq \varphi(s^2\Lambda_b(\theta,\theta))\\ &\leq \lambda[(\varphi(M_{\Lambda_b}(\theta,\theta))-\phi(M_{\Lambda_b}(\theta,\theta))+LN_{\Lambda_b}(\theta,\theta)]\end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M_{\Lambda_b}(\theta, \theta) &= \max\{\Lambda_b(\theta, \theta), \Lambda_b(\theta, p\theta), \Lambda_b(\theta, q\theta), \frac{\Lambda_b(\theta, q\theta) + \Lambda_b(p\theta, \theta)}{2s(1 + \Lambda_b(p\theta, \theta))}\} \\ &= \Lambda_b(\theta, q\theta) \end{aligned}$$

and

$$N_{\Lambda_b}(\theta, \theta) = \min\{\Lambda_b(\theta, \theta), \Lambda_b(\theta, p\theta), \Lambda_b(\theta, q\theta), \Lambda_b(\theta, q\theta)\} = 0.$$

By using the properties of φ and ϕ, we have

$$\begin{aligned} \varphi(\Lambda_b(\theta, q\theta)) &= \varphi(s^2\Lambda_b(p\theta, q\theta)) \\ &\leq \lambda[(\varphi(M_{\Lambda_b}(\theta, q\theta)) - \phi(M_{\Lambda_b}(\theta, \theta))] \\ &= \lambda[(\varphi(\Lambda_b(\theta, q\theta)) - \phi(\Lambda_b(\theta, q\theta))] \\ &< \lambda(\varphi(\Lambda_b(\theta, q\theta))). \end{aligned}$$

Hence, $\theta = q\theta$ is θ is the common fixed of p and q. If q is continuous, then, by a similar way of the above, we can prove that p and q have a common fixed point.

Theorem 2.3. *Let $(\digamma, \Lambda_b)$ be a complete b-metric space with the constant $s \geq 1$, and (p, q) be two self-mappings on $\digamma$. Suppose that $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ are two functions. Assume that the following conditions hold:*
(i) $\lambda_{(s,\varphi,\phi,L)}$-Berinde type contraction mapping;
(ii) the pair (p, q) is triangular α-admissible with respect to η;
(iii) If $\exists\ u_0 \in \digamma$ such that $\alpha(u_0, pu_0) \geq \eta(u_0, pu_0)$,
(iv) if $\{u_n\}$ is a sequence in $\digamma$ such that $\alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1})$, for all $n \in \mathbb{N}$ and $u_n \to \theta$ as $n \to \infty$, then $\exists$ a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $\alpha(u_{n_i}, u_*) \geq \eta(u_{n_i}, u_*)$, for all $i \in \mathbb{N}$. Then, p and q have a common fixed point in $\digamma$.

Proof. Following similar arguments as in the proof of Theorem 2.2, we obtain a sequence $\{u_n\}$ is defined by $u_{2n+1} = pu_{2n}$ and $u_{2n+2} = pu_{2n+1}$ for all $n \in \mathbb{N}$ converging to $u_* \in \digamma$ such that $\alpha(u_{2n}, u_{2n+1}) \geq \eta(u_{2n}, u_{2n+1})$ for all $n \in \mathbb{N}$. By (iv), there exist a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $\alpha(u_{n_i}, u_*) \geq \eta(u_{n_i}, u_*)$, for all $i \in \mathbb{N}$. Therefore

$$\begin{aligned} \varphi(\Lambda_b(u_{2n_i+1}, qu_*)) &\leq \varphi(s^2\Lambda_b(pu_{2n_i}, qu_*) \\ &\leq \lambda[(\varphi(M_{\Lambda_b}(u_{2n_i}, u_*)) - \phi(M_{\Lambda_b}(u_{2n_i}, u_*)) + LN_{\Lambda_b}(u_{2n_i}, u_*)] \end{aligned} \tag{2.17}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M_{\Lambda_b}(u_{2n_i}, u_*)) &= \max\{\Lambda_b(u_{2n_i}, u_*), \Lambda_b(u_{2n_i}, pu_{2n_i}), \Lambda_b(u_*, qu_*), \frac{\Lambda_b(u_{2n_i}, qu_*) + \Lambda_b(pu_{2n_i}, u_*)}{2s(1 + \Lambda_b(pu_{2n_i}, u_*))}\} \\ &= \max\{\Lambda_b(u_{2n}, u_*), \Lambda_b(u_{2n}, u_{2n_i+1}), \Lambda_b(u_*, qu_*), \frac{\Lambda_b(u_{2n_i}, qu_*) + \Lambda_b(u_{2n_i+1}, u_*)}{2s(1 + \Lambda_b(u_{2n_i+1}, u_*))}\} \end{aligned}$$

and

$$\begin{aligned} N_{\Lambda_b}(u_{2n_i}, u_*) &= \min\{\Lambda_b(u_{2n_i}, u_*), \Lambda_b(u_{2n_i}, pu_{2n_i}), \Lambda_b(u_*, qu_*), \Lambda_b(u_*, pu_{2n_i})\} \\ &= \min\{\Lambda_b(u_{2n_i}, u_*), \Lambda_b(u_{2n_i}, u_{2n_i+1}), \Lambda_b(u_*, qu_*), \Lambda_b(u_*, u_{2n_i+1})\}. \end{aligned}$$

Since

$$\limsup_{i\to\infty} \frac{\Lambda_b(u_{2n_i}, qu_*) + \Lambda_b(u_{2n_i+1}, u_*)}{2s(1 + \Lambda_b(u_{2n_i+1}, u_*))} \leq \frac{\Lambda_b(u_*, qu_*)}{2}.$$

By taking $i \to \infty$ in (2.18) and (2.18) using (2.5), we deduce that

$$\limsup_{i\to\infty} M_{\Lambda_b}(u_{2n_i}, u_*)) = \Lambda_b(u_*, qu_*)$$

and

$$\limsup_{i\to\infty} N_{\Lambda_b}(u_{2n_i}, u_*)) = 0.$$

From (2.17)and taking in account () and (), we have

$$\begin{aligned} \varphi(\Lambda_b(u_*, qu_*)) &\leq \lambda[\varphi(\Lambda_b(u_*, qu_*)) - \phi(\Lambda_b(u_*, qu_*)))] \quad (2.18) \\ &< \lambda\varphi(\Lambda_b(u_*, qu_*)) - \lambda\phi(\Lambda_b(u_*, qu_*)). \quad (2.19) \end{aligned}$$

By definition of φ and ϕ, we have a contradiction. Hence $\Lambda_b(u_*, qu_*) = 0$, i.e.,

$$qu_* = u_*.$$

By the same way we can prove that $pu_* = u_*$.

Definition 2.4. Let $(\digamma, \Lambda_b)$ be a b-metric space with parameter $s \geq 1$, $p, q : \digamma \to \digamma$ and $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ be two functions. Let $\varphi \in \Omega$, $\phi \in \Phi$ and $\lambda \in [0, 1)$. Then the pair (p, q) is called $\lambda_(s, \varphi, \phi)$-contraction mapping of type (B) if $\alpha(u, v) \geq \eta(u, v)$, then

$$\varphi\left(s^2\Lambda_b(pu, qv)\right) \leq \lambda\left[\varphi\left(M_{\Lambda_b}(u, v)\right) - \phi\left(M_{\Lambda_b}(u, v)\right)\right], \quad (2.20)$$

where $\lambda \in [0, 1)$ $\varphi \in \Omega$, $\phi \in \Phi$ and

$$M_{\Lambda_b}(u, v) = \max\left\{\Lambda_b(u, v), \Lambda_b(u, pu), \Lambda_b(v, qv), \frac{\Lambda_b(u, qv) + \Lambda_b(pu, v)}{2s[1 + \Lambda_b(pu, v)]}\right\}.$$

.

The proof of the followings two theorems follows from Theorem 2.2 and Theorem 2.3 by putting $L = 0$.

Theorem 2.5. *Let $(\digamma, \Lambda_b)$ be a complete b-metric space with the constant $s \geq 1$, and (p, q) be two self-mappings on $\digamma$. Suppose that $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ are two functions. Assume that the following conditions hold:*
(i) $\lambda_(s, \varphi, \phi)$- contraction type (B) mapping;
(ii) the pair (p, q) is triangular α-admissible with respect to η;
(iii) There exists $u_0 \in \digamma$ such that $\alpha(u_0, pu_0) \geq \eta(u_0, pu_0)$,
(iv) p and q are continuous mappings.
Then, p and q have a common fixed point in $\digamma$.

Theorem 2.6. *Let $(\digamma, \Lambda_b)$ be a complete b-metric space with the constant $s \geq 1$, and (p, q) be two self-mappings on $\digamma$. Suppose that $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ are two functions. Assume that the following conditions hold:*
(i) $\lambda_(s, \varphi, \phi)$-contraction mapping type (B);
(ii) the pair (p, q) is triangular α-admissible with respect to η;
(iii) If $\exists$ $u_0 \in \digamma$ such that $\alpha(u_0, pu_0) \geq \eta(u_0, pu_0)$,
(iv) if $\{u_n\}$ is a sequence in $\digamma$ such that $\alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1})$, for all $n \in \mathbb{N}$ and $u_n \to \theta$ as $n \to \infty$, then there exist a subsequence $\{u_{n_i}$ of $\{u_n\}$ such that $\alpha(u_{n_i}, u_*) \geq \eta(u_{n_i}, u_*)$, for all $i \in \mathbb{N}$.
Then, p and q have a common fixed point in $\digamma$.

The following corollaries are consequences of Theorem 2.2 and Theorem 2.3.

Corollary 2.7. *Let $(\digamma, \Lambda_b)$ be a complete b-metric space with the constant $s \geq 1$, and p be a self-mapping on $\digamma$. Suppose that $\alpha, \eta : \digamma \times \digamma \to \mathbb{R}$ are two functions. Suppose that the following conditions hold:*
(i)

$$If \;\; \alpha(u,v) \geq \eta(u,v) \Rightarrow \varphi\left(s^2\Lambda_b(pu,pv)\right) \leq \lambda\left[\varphi\left(M_{\Lambda_b}(u,v)\right) - \phi\left(M_{\Lambda_b}(u,v)\right) + \left(N_{\Lambda_b}(u,v)\right)\right], \tag{2.21}$$

(ii)p is triangular α-admissible with respect to η;
(iii) If $\exists\, u_0 \in \digamma$ such that $\alpha(u_0, pu_0) \geq \eta(u_0, pu_0)$,
(iv) p is a continuous mappings.
Then, p has a fixed point in $\digamma$.

Proof. The conclusion follows from Theorem 2.2 by taking $q = p$.

Corollary 2.8. *Let $(\digamma, \Lambda_b)$ be a complete b-metric space with the constant $s \geq 1$, and p be a self-mapping on $\digamma$. Suppose that $\alpha : \digamma \times \digamma \to \mathbb{R}$ are two functions. Assume that the following conditions hold:*
(i)

$$If \;\; \alpha(u,v) \geq 1 \Rightarrow \varphi\left(s^2\Lambda_b(pu,pv)\right) \leq \lambda\left[\varphi\left(M_{\Lambda_b}(u,v)\right) - \phi\left(M_{\Lambda_b}(u,v)\right) + \left(N_{\Lambda_b}(u,v)\right)\right], \tag{2.22}$$

(ii)p is triangular α-admissible with respect to η;
(iii) There exists $u_0 \in \digamma$ such that $\alpha(u_0, pu_0) \geq 1$,
(iv) p is a continuous mappings.
Then, p has a fixed point in $\digamma$.

Proof. The proof follows Corollary 2.7 by defining $\eta : \digamma \times \digamma \to \mathbb{R}$ via $\eta(u,v) = 1$.

Remark 2.9. Since a b-metric space is a metric space when $s = 1$, so our Theorems can be seen as a generalizations and extensions of several comparable results in metric spaces and b-metric spaces.

The following example illustrates the above result.

Example 2.10. *Let $\digamma = \{1, 2, 3, 4\}$. Define $\Lambda_b : \digamma \times \digamma \to [0, +\infty)$ as follows:*

$$\Lambda_b(u,v) = \Lambda_b(v,u) = 0 \;\; if \;\; u \neq v, \; u = v$$

$$\Lambda_b(u,v) = \Lambda_b(v,u) = 2 \;\; if \;\; u = 1, \; v = 2$$

$$\Lambda_b(u,v) = \Lambda_b(v,u) = 1 \;\; if \;\; u = 1, \; v = 3$$

$$\Lambda_b(u,v) = \Lambda_b(v,u) = 10 \;\; if \;\; u, v = 1, 2, 3, \; v = 4$$

Define $\varphi(t) = e^t, \phi(t) = \frac{e^t}{2+e^t}, \lambda = \frac{1}{2}, L = 2$ and define the mappings $p, q : \digamma \to \digamma$ by

$$p1 = p2 = p3 = 1, p4 = 3$$

$$q1 = 2, q2 = q3 = q4 = 1.$$

It is obvious that $(\digamma, \Lambda_b)$ is a complete b-metric space with the constant $s = 2$. We show that the condition (2.1) is true. We put

$$\varphi\left(s^2\Lambda_b(pu,qv)\right) = A, \varphi(M_{\Lambda_b}(u,v)) = B, \phi(M_{\Lambda_b}(u,v)) = C \;\; and \;\; N_{\Lambda_b}(u,v) = D.$$

Then we have the following cases:

Table 1: The possible values of u, v

$\Lambda_b(u,v)$	A	$\lambda[B-C+D]$	$A \leq \lambda[B-C+D]$	✓
$\Lambda_b(1,1)$	≈ 2.72	≈ 3.30	$2.72 < 3.30$	✓
$\Lambda_b(2,2)$	1	≈ 3.30	$1 < 3.30$	✓
$\Lambda_b(3,3)$	1	≈ 1.07	$1 < 1.07$	✓
$\Lambda_b(4,4)$	≈ 54.60	≈ 11012.73	$54.60 < 11012.73$	✓
$\Lambda_b(1,2)$	1	≈ 3.30	$1 < 3.30$	✓
$\Lambda_b(1,3)$	1	≈ 1.07	$1 < 1.07$	✓
$\Lambda_b(1,4)$	1	≈ 11012.73	$1 < 11012.73$	✓
$\Lambda_b(2,3)$	1	≈ 3.30	$1 < 3.30$	✓
$\Lambda_b(2,4)$	1	≈ 11012.73	$1 < 11012.73$	✓
$\Lambda_b(3,4)$	1	≈ 11012.73	$1 < 11012.73$	✓

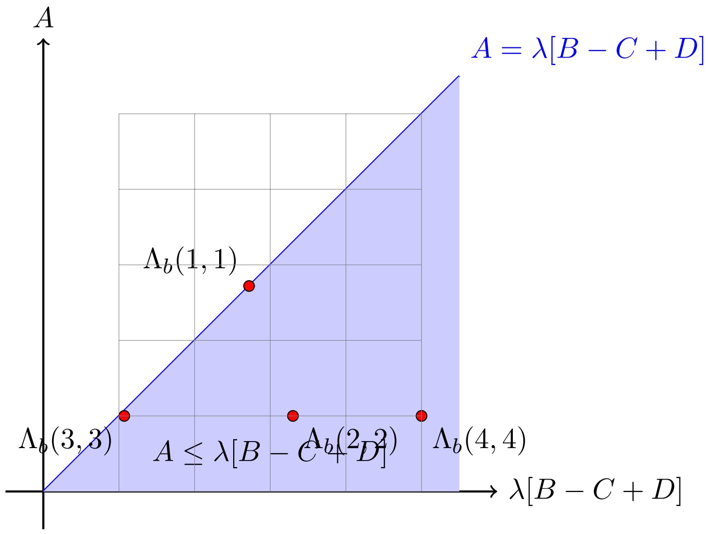

Figure 1. Satisfing the enquality $A \leq \lambda[B-C+D]$

Thus, all the conditions of Theorem 2.1 are satisfied and hence p and q have a common fixed point. Indeed, 1 is a common fixed point of p and q.

3. Application

Fixed point theorem has numerous applications, such as fractional differential equations ([1], [2], [13]), the significance of these types of equations is their utilization in modeling in many subjects. In this section, we utilize our results to demonstrate the existence and uniqueness of the Fredholm type integral equation.
Now, Consider the set $\digamma = C([0,1],(-\infty,\infty))$ and the following Fredholm type integral equation:

$$\acute{p}(t) = \int_0^1 S(t,s,\acute{p}(t))\,ds, \; for\; t,s \in [0,1], \tag{3.1}$$

where $S(t,s,\acute{p}(t))$ is a continuous function on $[0,1]\times[0,1] \to (-\infty,\infty)$.
Now, define $\Lambda_b : \digamma \times \digamma \to C$ and $(p,q) \mapsto \mid \acute{p}(t) - q(t) \mid$.

Note that $(\digamma, \Lambda_b)$ is a complete b-metric space, where the parameter $s = 2$.

Theorem 3.1. *Suppose that for all $p, q \in \digamma$*
(1) $\mid S(t,s,\acute{p}(t)) - S(t,s,q(t)) \mid \leq \frac{|\acute{p}(t)-q(t)|}{2}$.

(2) $\mid S(t,s,\int_0^1 S(t,s,\acute{p}(t))\,ds) - S(t,s,q\int_0^1 S(t,s,q(t))\,ds) \mid \leq \mid S(t,s,\acute{p}(t)) - S(t,s,q(t)) \mid$ *for all* t, s.
Then the integral equation 3.1 has a unique solution.

Proof. Let $\acute{p}(t) : F \to F$ defined by $\acute{p}(t) = \int_0^1 S(t,s,\acute{p}(t))\,ds$, then
$\Lambda_b(\acute{p}, q) = \mid \acute{p}(t) - q(t) \mid$. Now we have

$$\begin{aligned}
\Lambda_b(\acute{p}(t), q(t)) &= \mid \acute{p}(t) - q(t) \mid \\
&= \mid S(t,s,\int_0^1 S(t,s,\acute{p}(t))\,ds) - S(t,s,q\int_0^1 S(t,s,q(t))\,ds) \mid \\
&\leq \mid S(t,s,\acute{p}(t)) - S(t,s,q(t)) \mid \\
&\leq \frac{\mid \acute{p}(t) - q(t) \mid}{2} \\
&\leq \frac{1}{2}\Lambda_b(\acute{p}(t), q(t)) \\
&= \lambda\left[\varphi\left(M_{\Lambda_b}(\acute{p}(t), q(t))\right) - \phi\left(M_{\Lambda_b}(\acute{p}(t), q(t))\right)\right],
\end{aligned}$$

where $\varphi(t) = t$ and $\phi(t) = \frac{t}{2}$. Also the parameter $s < 3$.
Hence, all the hypotheses of Theorem 2.2, are fulfilled and then the equation 3.1 has a unique solution.

4. Conclusion

We have demonstrated the existence and uniqueness of a fixed point for self-mapping in b-metric spaces under diverse nonlinear mappings with continuous control functions. Also, we show an application of our results to Fredholm-type integral equations. Additionally, we would like to bring the researchers consideration to the following question.

4.1. Question

Under what conditions we will get the same results for self-mapping in partial b-metric spaces?

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