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Nonlinear Mixed λ-Jordan Triple Derivation on $*$-algebras

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Abstract. Let $\mathcal{A}$ be a $*$-algebra with unit I and P_1 and $P_2 = I - P_1$ includes a non-trivial projections, and let $\lambda \in \mathbb{C} \setminus \{0, -1\}$. In this paper, we aim to study the characterization of nonlinear mixed λ-Jordan triple derivation on $*$-algebras. As an application, we can also apply our results on prime $*$-algebras, factor von-Neumann algebras and standard operator algebras.

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1. Introduction

Consider an $*$-algebra $\mathcal{A}$ defined over the complex field $\mathbb{C}$. Introducing the λ-Jordan product $U\diamondsuit_\lambda V = UV + \lambda VU$ and the skew Lie product $[U, V]_* = UV - VU^*$ for nonzero scalar λ, these algebraic structures have gained significant attention in various research domains, as evidenced by studies such as [1–5, 7, 8, 12]. In the context of additive mappings, an additive derivation is characterized by $\Pi(UV) = \Pi(U)V + U\Pi(V)$ for all $U, V \in \mathcal{A}$. If the additional condition $\Pi(U^*) = \Pi(U)^*$ holds for all $U \in \mathcal{A}$, then Π is termed an additive $*$-derivation. Now, let $\Pi : \mathcal{A} \to \mathcal{A}$ be a map without assuming additivity. The concept of a nonlinear skew Lie derivation is introduced, defined by the relation

$$\Pi([U, V]_*) = [\Pi(U), V]_* + [U, \Pi(V)]_*$$

for all $U, V \in \mathcal{A}$. Notably, Kong and Zhang [3] established the result that every nonlinear skew Lie derivation is, in fact, an additive $*$-derivation.

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Similarly, a map $\Pi : \mathcal{A} \to \mathcal{A}$ is termed a nonlinear skew Lie triple derivation if it satisfies the equation

$$\Pi([[U,V]_*,W]_*) = [[\Pi(U),V]_*,W]_* + [[U,\Pi(V)]_*,W]_* + [[U,V]_*,\Pi(W)]_*$$

for all $U,V,W \in \mathcal{A}$.
Several recent studies have delved into the exploration of derivations and isomorphisms associated with innovative products resulting from the combination of Lie and skew Lie products, as evidenced by works such as [6, 9, 11]. Notably, Zhou et al. [13] established the result that every nonlinear mixed Lie triple derivation on a prime $*$-algebra is, in fact, an additive $*$-derivation. Additionally, Pang et al. [10] demonstrated that every second nonlinear mixed Jordan triple derivable mapping on factor von Neumann algebras also an additive $*$-derivation.

Motivated by these previous works, our paper introduces the λ-Jordan product defined as $U\diamondsuit_\lambda V = UV + \lambda VU$. We specifically focus on the derivation corresponding to the novel product obtained by combining the skew Lie product and the λ-Jordan product. In this context, we define a map $\Pi : \mathcal{A} \to \mathcal{A}$ as a mixed λ-Jordan triple derivation if it satisfies the equation

$$\Pi([U,V]_*\diamondsuit_\lambda W) = [\Pi(U),V]_* \diamond_\Lambda W + [U,\Pi(V)]_*\diamondsuit_\lambda W + [U,V]_*\diamondsuit_\lambda \Pi(W)$$

for all $U,V,W \in \mathcal{A}$. Our main result establishes that Π is a nonlinear mixed λ-Jordan triple derivation on $*$-algebras if and only if Π is an additive $*$-derivation.

2. Main Result

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with unity I containing a non-trivial projection P satisfies*

$$X\mathcal{A}P = 0 \implies X = 0 \tag{$\blacktriangle$}$$

and

$$X\mathcal{A}(I-P) = 0 \implies X = 0. \tag{$\blacktriangledown$}$$

Define a map $\Pi : \mathcal{A} \to \mathcal{A}$ such that

$$\Pi([U,V]_*\diamondsuit_\lambda W) = [\Pi(U),V]_*\diamondsuit_\lambda W) + [U,\Pi(V)]_*\diamondsuit_\lambda W + [U,V]_*\diamondsuit_\lambda \Pi(W).$$

Then Π is an additive $*$-derivation.

Consider a non-trivial projection $P = P_1$ in the algebra $\mathcal{A}$, and let $P_2 = I - P_1$, where I is the unity element of the algebra. Utilizing the Peirce decomposition of $\mathcal{A}$, we express $\mathcal{A}$ as the direct sum $\mathcal{A} = P_1\mathcal{A}P_1 \oplus P_1\mathcal{A}P_2 \oplus P_2\mathcal{A}P_1 \oplus P_2\mathcal{A}P_2$. Denoting the corresponding subspaces as $\mathcal{A}_{11} = P_1\mathcal{A}P_1$, $\mathcal{A}_{12} = P_1\mathcal{A}P_2$, $\mathcal{A}_{21} = P_2\mathcal{A}P_1$, and $\mathcal{A}_{22} = P_2\mathcal{A}P_2$, we can represent any element $U \in \mathcal{A}$ as the sum $U = U_{11} + U_{12} + U_{21} + U_{22}$, where $U_{ij} \in \mathcal{A}_{ij}$ and $U_{ij}^* \in \mathcal{A}_{ji}$ for $i,j = 1,2$. Before proving Theorem 2.1, we need several lemmas and remarks.

Lemma 2.1. $\Pi(0) = 0$.

Proof. It is obvious that

$$\Pi(0) = \Pi([0,0]_*\diamondsuit_\lambda 0) = [\Pi(0),0]_*\diamondsuit_\lambda 0 + [0,\Pi(0)]_*\diamondsuit_\lambda 0 + [0,0]_*\diamondsuit_\lambda \Pi(0) = 0.$$

Lemma 2.2. *Let* $U_{12} \in \mathcal{A}_{12}$ *and* $U_{21} \in \mathcal{A}_{21}$. *Then* $\Pi(U_{12} + U_{21}) = \Pi(U_{12}) + \Pi(U_{21})$.

Proof. Let $T = \Pi(U_{12} + U_{21}) - \Pi(U_{12}) - \Pi(U_{21})$. Since $[U_{12}, P_1]_*\diamondsuit_\lambda P_2 = 0$ and by using Lemma 2.1, we have

$$\begin{aligned}\Pi([U_{12}+V_{21},P_1]_*\diamondsuit_\lambda P_2) &= \Pi([U_{12},P_1]_*\diamondsuit_\lambda P_2) + \Pi([V_{21},P_1]_*\diamondsuit_\lambda P_2)\\ &= [\Pi(U_{12}),P_1]_*\diamondsuit_\lambda P_2 + [U_{12},\Pi(P_1)]_*\diamondsuit_\lambda P_2 + [U_{12},P_1]_*\diamondsuit_\lambda \Pi(P_2)\\ &\quad +[\Pi(V_{21}),P_1]_*\diamondsuit_\lambda P_2 + [V_{21},\Pi(P_1)]_*\diamondsuit_\lambda P_2 + [V_{21},P_1]_*\diamondsuit_\lambda \Pi(P_2).\end{aligned}$$

On the other hand, we find

$$\begin{aligned}\Pi([U_{12}+V_{21},P_1]_*\diamondsuit_\lambda P_2 &= [\Pi(U_{12}+V_{21}),P_1]_*\diamondsuit_\lambda P_2 + [U_{12}+V_{21},\Pi(P_1)]_*\diamondsuit_\lambda P_2\\ &\quad +[U_{12}+V_{21},P_1]_*\diamondsuit_\lambda \Pi(P_2).\end{aligned}$$

From the above two equations, we get $[T, P_1]_*\diamondsuit_\lambda P_2 = 0$. That means $-P_1T^*P_2+\lambda P_2TP_1 = 0$. Multiplying by P_2 from the left and since $\lambda \neq 0$, we get $P_2TP_1 = 0$. Similarly, one can show that $P_1TP_2 = 0$.
Now, for every $X_{21} \in \mathcal{A}_{21}$, it follows from $[X_{21}, U_{12}]_*\diamondsuit_\lambda P_1 = 0$ and using Lemma 2.1 that

$$\begin{aligned}\Pi([X_{21},U_{12}+V_{21}]_*\diamondsuit_\lambda P_1) &= \Pi([X_{21},U_{12}]_*\diamondsuit_\lambda P_1) + \Pi([X_{21},V_{21}]_*\diamondsuit_\lambda P_1)\\ &= [\Pi(X_{21}),U_{12}]_*\diamondsuit_\lambda P_1 + [X_{21},\Pi(U_{12})]_*\diamondsuit_\lambda P_1\\ &\quad +[X_{21},U_{12}]_*\diamondsuit_\lambda \Pi(P_1) + [\Pi(X_{21}),V_{21}]_*\diamondsuit_\lambda P_1\\ &\quad +[X_{21},\Pi(V_{21})]_*\diamondsuit_\lambda P_1 + [X_{21},V_{21}]_*\diamondsuit_\lambda \Pi(P_1).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Pi([X_{21},U_{12}+V_{21}]_*\diamondsuit_\lambda P_1) &= [\Pi(X_{21}),U_{12}+V_{21}]_*\diamondsuit_\lambda P_1 + [X_{21},\Pi(U_{12}+V_{21})]_*\diamondsuit_\lambda P_1\\ &\quad +[X_{21},U_{12}+V_{21}]_*\diamondsuit_\lambda \Pi(P_1).\end{aligned}$$

From the above expressions, we find that $[X_{21}, T]_*\diamondsuit_\lambda P_1 = 0$. That means $X_{21}TP_1 - \lambda P_1TX_{21}^* = 0$. Multiplying both sides by P_1 from right, we get $X_{21}TP_1 = 0$. By using (▲) and (▼), we have $P_1TP_1 = 0$. Similarly, we can show that $P_2TP_2 = 0$. Hence, $T = 0$ i.e., $\Pi(U_{12} + U_{21}) = \Pi(U_{12}) + \Pi(U_{21})$.

Lemma 2.3. *For any* $U_{ij} \in \mathcal{A}_{ij}, 1 \leq i,j \leq 2$, *we have*

$$\Pi(\sum_{i,j=1}^{2} U_{ij}) = \sum_{i,j=1}^{2} \Pi(U_{ij}).$$

Proof. Let $T = \Pi(U_{11}+U_{12}+U_{21}+U_{22}) - \Pi(U_{11}) - \Pi(U_{12}) - \Pi(U_{21}) - \Pi(U_{22})$. For every $X_{12} \in \mathcal{A}_{12}$, also $[P_1, U_{11}]_* \diamondsuit_\lambda X_{12} = [P_1, U_{22}]_* \diamondsuit_\lambda X_{12} = 0$ and using Lemmas 2.1 and 2.2, we get

$$\begin{aligned}
\Pi([P_1, U_{11}+U_{12}+U_{21}+U_{22}]_* \diamondsuit_\lambda X_{12}) &= \Pi([P_1, U_{11}]_* \diamondsuit_\lambda X_{12}) + \Pi([P_1, U_{12}]_* \diamondsuit_\lambda X_{12}) \\
&\quad + \Pi([P_1, U_{21}]_* \diamondsuit_\lambda X_{12}) + \Pi([P_1, U_{22}]_* \diamondsuit_\lambda X_{12}) \\
&= [\Pi(P_1), U_{11}]_* \diamondsuit_\lambda X_{12} + [P_1, \Pi(U_{11})]_* \diamondsuit_\lambda X_{12} \\
&\quad + [P_1, U_{11}]_* \diamondsuit_\lambda \Pi(X_{12}) + [\Pi(P_1), U_{12}]_* \diamondsuit_\lambda X_{12} \\
&\quad + [P_1, \Pi(U_{12})]_* \diamondsuit_\lambda X_{12} + [P_1, U_{12}]_* \diamondsuit_\lambda \Pi(X_{12}) \\
&\quad + [\Pi(P_1), U_{21}]_* \diamondsuit_\lambda X_{12} + [P_1, \Pi(U_{21})]_* \diamondsuit_\lambda X_{12} \\
&\quad + [P_1, U_{21}]_* \diamondsuit_\lambda \Pi(X_{12}) + [\Pi(P_1), U_{22}]_* \diamondsuit_\lambda X_{12} \\
&\quad + [P_1, \Pi(U_{22})]_* \diamondsuit_\lambda X_{12} + [P_1, U_{22}]_* \diamondsuit_\lambda \Pi(X_{12}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Pi([P_1, U_{11}+U_{12}+U_{21}+U_{22}]_* \diamondsuit_\lambda X_{12}) &= [\Pi(P_1), U_{11}+U_{12}+U_{21}+U_{22}]_* \diamondsuit_\lambda X_{12} \\
&\quad + [P_1, \Pi(U_{11}+U_{12}+U_{21}+U_{22})]_* \diamondsuit_\lambda X_{12} \\
&\quad + [P_1, U_{11}+U_{12}+U_{21}+U_{22}]_* \diamondsuit_\lambda \Pi(X_{12}).
\end{aligned}$$

By comparing the above two equations, we get $[P_1, M]_* \diamondsuit_\lambda X_{12} = 0$ from which we obtain $P_1 T X_{12} - T X_{12} - \lambda X_{12} T P_1 = 0$. Multiplying P_2 from left and right, we get $P_2 T X_{12} = 0$. By using (▲) and (▼), we have $P_2 T P_1 = 0$. Similarly, we can show that $P_1 T P_2 = 0$. Again for $X_{12} \in \mathcal{A}_{12}$, it follows from $[X_{12}, U_{11}]_* \diamondsuit_\lambda P_2 = [X_{12}, U_{12}]_* \diamondsuit_\lambda P_2 = [X_{12}, U_{12}]_* \diamondsuit_\lambda P_2 = 0$ that

$$\begin{aligned}
\Pi([X_{12}, U_{11}+U_{12}+U_{21}+U_{22}]_* \diamondsuit_\lambda P_2) &= \Pi([X_{12}, U_{11}]_* \diamondsuit_\lambda P_2) + \Pi([X_{12}, U_{12}]_* \diamondsuit_\lambda P_2) \\
&\quad + \Pi([X_{12}, U_{21}]_* \diamondsuit_\lambda P_2) + \Pi([X_{12}, U_{22}]_* \diamondsuit_\lambda P_2) \\
&= [\Pi(X_{12}), U_{11}]_* \diamondsuit_\lambda P_2 + [X_{12}, \Pi(U_{11})]_* \diamondsuit_\lambda P_2 \\
&\quad + [X_{12}, U_{11}]_* \diamondsuit_\lambda \Pi(P_2) + [\Pi(X_{12}), U_{12}]_* \diamondsuit_\lambda P_2 \\
&\quad + [X_{12}, \Pi(U_{12})]_* \diamondsuit_\lambda P_2 + [X_{12}, U_{12}]_* \diamondsuit_\lambda \Pi(P_2) \\
&\quad + [\Pi(X_{12}), U_{21}]_* \diamondsuit_\lambda P_2 + [X_{12}, \Pi(U_{21})]_* \diamondsuit_\lambda P_2 \\
&\quad + [X_{12}, U_{21}]_* \diamondsuit_\lambda \Pi(P_2) + [\Pi(X_{12}), U_{22}]_* \diamondsuit_\lambda P_2 \\
&\quad + [X_{12}, \Pi(U_{22})]_* \diamondsuit_\lambda P_2 + [X_{12}, U_{22}]_* \diamondsuit_\lambda \Pi(P_2).
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
\Pi([X_{12}, U_{11}+U_{12}+U_{21}+U_{22}]_* \diamondsuit_\lambda P_2) &= [\Pi(X_{12}), U_{11}+U_{12}+U_{21}+U_{22}]_* \diamondsuit_\lambda P_2 \\
&\quad + [X_{12}, \Pi(U_{11}+U_{12}+U_{21}+U_{22})]_* \diamondsuit_\lambda P_2 \\
&\quad + [X_{12}, U_{11}+U_{12}+U_{21}+U_{22}]_* \diamondsuit_\lambda \Pi(P_2).
\end{aligned}$$

By the above two equations, we get $[X_{12}, T]_* \diamondsuit_\lambda P_2 = 0$. That means that $X_{12} T P_2 - \lambda P_2 T X_{12}^* = 0$. When we multiply both sides by P_1 on the left, the result is $X_{12} T P_2 = 0$.

By using (▲) and (▼), we have $P_2TP_2 = 0$. Similarly, $P_1TP_1 = 0$. Hence, $T = 0$ i.e., $\Pi(U_{11} + U_{12} + U_{21} + U_{22}) = \Pi(U_{11}) + \Pi(U_{12}) + \Pi(U_{21}) + \Pi(U_{22})$.

Lemma 2.4. *For any* $U_{ij}, V_{ij} \in \mathcal{A}_{ij}$ *with* $(1 \leq i \neq j \leq 2)$, *we have*

$$\Pi(U_{ij} + V_{ij}) = \Pi(U_{ij}) + \Pi(V_{ij}).$$

Proof. Initially, we establish the result for $i = 1$ and $j = 2$. Let $T = \Pi(U_{12} + V_{12}) - \Pi(U_{12}) - \Pi(V_{12})$. Since $[X_{12}, U_{12}]_* \diamondsuit_\lambda P_2 = 0$, and using Lemma 2.1, we get

$$\begin{aligned}\Pi([X_{12}, U_{12} + V_{12}]_* \diamondsuit_\lambda P_2) &= \Pi([X_{12}, U_{12}]_* \diamondsuit_\lambda P_2) + \Pi([X_{12}, V_{12}]_* \diamondsuit_\lambda P_2) \\ &= [\Pi(X_{12}), U_{12}]_* \diamondsuit_\lambda P_2 + [X_{12}, \Pi(U_{12})]_* \diamondsuit_\lambda P_2 \\ &\quad + [X_{12}, U_{12}]_* \diamondsuit_\lambda \Pi(P_2) + [\Pi(X_{12}), V_{12}]_* \diamondsuit_\lambda P_2 \\ &\quad + [X_{12}, \Pi(V_{12})]_* \diamondsuit_\lambda P_2 + [X_{12}, V_{12}]_* \diamondsuit_\lambda \Pi(P_2).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Pi([X_{12}, U_{12} + V_{12}]_* \diamondsuit_\lambda P_2) &= [\Pi(X_{12}), U_{12} + V_{12}]_* \diamondsuit_\lambda P_2 + [X_{12}, \Pi(U_{12} + V_{12})]_* \diamondsuit_\lambda P_2 \\ &\quad + [X_{12}, U_{12} + V_{12}]_* \diamondsuit_\lambda \Pi(P_2).\end{aligned}$$

By comparing the last two expressions, we get $[X_{12}, T]_* \diamondsuit_\lambda P_2 = 0$. That means $X_{12}TP_2 - P_2TX_{12}^* = 0$. By left-multiplying both sides of the preceding equation by P_1 and utilizing (▲) and (▼), we obtain $P_2TP_2 = 0$. Similarly, we can show that $P_1TP_1 = 0$.
Now, again for any $X_{12} \in \mathcal{A}_{12}$. Since $[P_1, U_{12}]_* \diamondsuit_\lambda X_{12} = 0$ and using Lemma 2.1, we have

$$\begin{aligned}\Pi([P_1, U_{12} + V_{12}]_* \diamondsuit_\lambda X_{12}) &= \Pi([P_1, U_{12}]_* \diamondsuit_\lambda X_{12}) + \Pi([P_1, V_{12}]_* \diamondsuit_\lambda X_{12}) \\ &= [\Pi(P_1), U_{12}]_* \diamondsuit_\lambda X_{12} + [P_1, \Pi(U_{12})]_* \diamondsuit_\lambda X_{12} \\ &\quad + [P_1, U_{12}]_* \diamondsuit_\lambda \Pi(X_{12}) + [\Pi(P_1), V_{12}]_* \diamondsuit_\lambda X_{12} \\ &\quad + [P_1, \Pi(V_{12})]_* \diamondsuit_\lambda X_{12} + [P_1, U_{12}]_* \diamondsuit_\lambda \Pi(X_{12}).\end{aligned}$$

On the other hand, we find

$$\begin{aligned}\Pi([P_1, U_{12} + V_{12}]_* \diamondsuit_\lambda X_{12}) &= [\Pi(P_1), U_{12} + V_{12}]_* \diamondsuit_\lambda X_{12} + [P_1, \Pi(U_{12} + V_{12})]_* \diamondsuit_\lambda X_{12} \\ &\quad + [P_1, U_{12} + V_{12}]_* \diamondsuit_\lambda \Pi(X_{12}).\end{aligned}$$

From the last two expressions, we find $[P_1, T]_* \diamondsuit_\lambda X_{12} = 0$. That means $P_1TX_{12} - TX_{12} - \lambda X_{12}TP_1 = 0$. Multiplying both sides by P_1 from right and since $\lambda \neq 0$, we have $X_{12}TP_1 = 0$. Thus, $P_2TP_1 = 0$ follows from (▲) and (▼). Similarly, we can show that $P_1TP_2 = 0$. Hence, $T = 0$ i.e.,

$$\Pi(U_{12} + V_{12}) = \Pi(U_{12}) + \Pi(V_{12}).$$

By using the same technique as above, one can show that

$$\Pi(U_{21} + V_{21}) = \Pi(U_{21}) + \Pi(V_{21}).$$

Lemma 2.5. *For any $U_{11}, V_{11} \in \mathcal{A}_{11}$ and $U_{22}, V_{22} \in \mathcal{A}_{22}$, we have*

(i) $\Pi(U_{11} + V_{11}) = \Pi(U_{11}) + \Pi(V_{11})$.

(ii) $\Pi(U_{22} + V_{22}) = \Pi(U_{22}) + \Pi(V_{22})$.

Proof. Let $T = \Pi(U_{11} + V_{11}) - \Pi(U_{11}) - \Pi(V_{11})$. On the one hand, we have

$$\begin{aligned}\Pi([P_2, U_{11} + V_{11}]_* \diamondsuit_\lambda P_1) &= [\Pi(P_2), U_{11} + V_{11}]_* \diamondsuit_\lambda P_1 + [P_2, \Pi(U_{11} + V_{11})]_* \diamondsuit_\lambda P_1 \\ &\quad + [P_2, U_{11} + V_{11}]_* \diamondsuit_\lambda \Pi(P_1).\end{aligned}$$

On the other hand, it follows from $[P_2, U_{11}]_* \diamondsuit_\lambda P_1 = 0$ that

$$\begin{aligned}\Pi([P_2, U_{11} + V_{11}]_* \diamondsuit_\lambda P_1) &= \Pi([P_2, U_{11}]_* \diamondsuit_\lambda P_1) + \Pi([P_2, V_{11}]_* \diamondsuit_\lambda P_1) \\ &= [\Pi(P_2), U_{11}]_* \diamondsuit_\lambda P_1 + [P_2, \Pi(U_{11})]_* \diamondsuit_\lambda P_1 + [P_2, U_{11}]_* \diamondsuit_\lambda \Pi(P_1) \\ &\quad + [\Pi(P_2), V_{11}]_* \diamondsuit_\lambda P_1 + [P_2, \Pi(V_{11})]_* \diamondsuit_\lambda P_1 + [P_2, V_{11}]_* \diamondsuit_\lambda \Pi(P_1).\end{aligned}$$

By comparing the last two equations, we find $[P_2, T]_* \diamondsuit_\lambda P_1 = 0$. This gives $P_2 T P_1 - \lambda P_1 T P_2 = 0$ and hence, $P_2 T P_1 = P_1 T P_2 = 0$.
Again for any $X_{12} \in \mathcal{A}_{12}$ and since $[X_{12}, U_{11}]_* \diamondsuit_\lambda P_2 = 0$, we find

$$\begin{aligned}\Pi([X_{12}, U_{11} + V_{11}]_* \diamondsuit_\lambda P_2) &= \Pi([X_{12}, U_{11}]_* \diamondsuit_\lambda P_2) + \Pi([X_{12}, V_{11}]_* \diamondsuit_\lambda P_2) \\ &= [\Pi(X_{12}), U_{11}]_* \diamondsuit_\lambda P_2 + [X_{12}, \Pi(U_{11})]_* \diamondsuit_\lambda P_2 \\ &\quad + [X_{12}, U_{11}]_* \diamondsuit_\lambda \Pi(P_2) + [\Pi(X_{12}), V_{11}]_* \diamondsuit_\lambda P_2 \\ &\quad + [X_{12}, \Pi(V_{11})]_* \diamondsuit_\lambda P_2 + [X_{12}, V_{11}]_* \diamondsuit_\lambda \Pi(P_2).\end{aligned}$$

From the other side, we get

$$\begin{aligned}\Pi([X_{12}, U_{11} + V_{11}]_* \diamondsuit_\lambda P_2) &= [\Pi(X_{12}), U_{11} + V_{11}]_* \diamondsuit_\lambda P_2 + [X_{12}, \Pi(U_{11} + V_{11})]_* \diamondsuit_\lambda P_2 \\ &\quad + [X_{12}, U_{11} + V_{11}]_* \diamondsuit_\lambda \Pi(P_2).\end{aligned}$$

From the last two equations, we get $[X_{12}, T]_* \diamondsuit_\lambda P_2 = 0$. That means $X_{12} T P_2 - \lambda P_2 T X_{12}^* = 0$. Thus, $X_{12} T P_2 = 0$. By using (▲) and (▼), we get $P_2 T P_2 = 0$. Similarly, we can show that $P_1 T P_1 = 0$. Hence, $T = 0$ i,e.

$$\Pi(U_{11} + V_{11}) = \Pi(U_{11}) + \Pi(V_{11}).$$

(**ii**). By using the same argument as in (i), one can show that

$$\Pi(U_{22} + V_{22}) = \Pi(U_{22}) + \Pi(V_{22}).$$

Lemma 2.6. Π *is an additive map.*

Proof. For any $U, V \in \mathcal{A}$, we write $U = U_{11} + U_{12} + U_{21} + U_{22}$ and $V = V_{11} + V_{12} + V_{21} + V_{22}$. By using Lemmas 2.3 - 2.5, we get

$$\begin{aligned}\Pi(U+V) &= \Pi(U_{11}+U_{12}+U_{21}+U_{22}+V_{11}+V_{12}+V_{21}+V_{22}) \\ &= \Pi(U_{11}+V_{11})+\Pi(U_{12}+V_{12})+\Pi(U_{21}+V_{21})+\Pi(U_{22}+V_{22}) \\ &= \Pi(U_{11})+\Pi(V_{11})+\Pi(U_{12})+\Pi(V_{12})+\Pi(U_{21})+\Pi(V_{21})+\Pi(U_{22})+\Pi(V_{22}) \\ &= \Pi(U_{11}+U_{12}+U_{21}+U_{22})+\Pi(V_{11}+V_{12}+V_{21}+V_{22}) \\ &= \Pi(U)+\Pi(V).\end{aligned}$$

Lemma 2.7. *(i)* $\Pi(P_1)^* = \Pi(P_1)$, $\Pi(P_2)^* = \Pi(P_2)$.

(ii) $P_1\Pi(P_1)P_2 = -P_1\Pi(P_2)P_2$.

(iii) $P_2\Pi(P_2)P_1 = -P_2\Pi(P_1)P_1$.

Proof. (i) In the view of $[P_1, P_1]_* \diamondsuit_\lambda I = 0$ and using Lemma 2.1, we have

$$\begin{aligned}0 &= \Pi([P_1, P_1]_* \diamondsuit_\lambda I) \\ &= [\Pi(P_1), P_1]_* \diamondsuit_\lambda I + [P_1, \Pi(P_1)]_* \diamondsuit_\lambda I + [P_1, P_1]_* \diamondsuit_\lambda \Pi(I) \\ &= (1+\lambda)(P_1\Pi(P_1)^* - P_1\Pi(P_1)).\end{aligned}$$

Since $\lambda \neq -1$, we have $P_1\Pi(P_1)^* = P_1\Pi(P_1)$. That means

$$P_1\Pi(P_1)^*P_1 = P_1\Pi(P_1)P_1. \tag{2.1}$$

$$P_1\Pi(P_1)^*P_2 = P_1\Pi(P_1)P_2 \tag{2.2}$$

$$P_2\Pi(P_1)^*P_1 = P_2\Pi(P_1)P_1 \tag{2.3}$$

Also, $[P_1, P_2]_* \diamondsuit_\lambda I = 0$ and $\Pi(0) = 0$, we have

$$\begin{aligned}0 &= \Pi([P_1, P_2]_* \diamondsuit_\lambda I = 0) \\ &= [\Pi(P_1), P_2]_* \diamondsuit_\lambda I + [P_1, \Pi(P_2)]_* \diamondsuit_\lambda I + [P_1, P_2]_* \diamondsuit_\lambda \Pi(I) \\ &= (1+\lambda)(P_2\Pi(P_1)P_2 - P_2\Pi(P_1)^*P_2).\end{aligned}$$

That is,

$$P_2\Pi(P_1)^*P_2 = P_2\Pi(P_1)P_2. \tag{2.4}$$

From equations (2.1)-(2.4), we conclude $\Pi(P_1)^* = \Pi(P_1)$.
Similarly, by using the same technique as above, we get $\Pi(P_2)^* = \Pi(P_2)$.
(ii) Again $[P_1, P_2]_* \diamondsuit_\lambda P_2 = 0$ and $\Pi(0) = 0$, we find

$$0 = \Pi([P_1, P_2]_* \diamondsuit_\lambda P_2)$$

$$
\begin{aligned}
&= [\Pi(P_1), P_2]_* \diamondsuit_\lambda P_2 + [P_1, \Pi(P_2)]_* \diamondsuit_\lambda P_2 + [P_1, P_2]_* \diamondsuit_\lambda \Pi(P_2) \\
&= \Pi(P_1)P_2 - P_2\Pi(P_1)^* P_2 + \lambda P_2\Pi(P_1)P_2 - \lambda P_2\Pi(P_1)^* + P_1\Pi(P_2)P_2 - \lambda P_2\Pi(P_2)P_1.
\end{aligned}
$$

Multiplying above equation by P_1 from left, we get $P_1\Pi(P_1)P_2 = -P_1\Pi(P_2)P_2$.
(iii) By using the same technique as in (ii), we can show that $P_2\Pi(P_2)P_1 = -P_2\Pi(P_1)P_1$.

Lemma 2.8. *For every $U_{ij} \in \mathcal{A}_{ij}(1 \leq i \neq j \leq 2)$, we have*

$$P_j\Pi(\lambda U_{ij})P_i = 0.$$

Proof. To begin, we establish the result for $i = 1$ and $j = 2$. For any $U_{12} \in \mathcal{A}_{12}$, we get

$$
\begin{aligned}
\Pi(\lambda U_{12}) &= \Pi([P_1, \lambda U_{12}]_* \diamondsuit_\lambda P_2) \\
&= [\Pi(P_1), \lambda U_{12}]_* \diamondsuit_\lambda P_2 + [P_1, \Pi(\lambda U_{12})]_* \diamondsuit_\lambda P_2 + [P_1, \lambda U_{12}]_* \diamondsuit_\lambda \Pi(P_2) \\
&= \lambda\Pi(P_1)U_{12} - \lambda U_{12}\Pi(P_1)P_2 + \lambda^2 P_2\Pi(P_1)U_{12} + P_1\Pi(\lambda U_{12})P_2 \\
&\quad -\lambda P_2\Pi(\lambda U_{12})P_1 + \lambda U_{12}\Pi(P_2) + \lambda^2\Pi(P_2)U_{12}.
\end{aligned}
$$

By left-multiplying the above equation with P_2 and right-multiplying with P_1, we obtain

$$(1 + \lambda)P_2\Pi(\lambda U_{12})P_1 = 0. \tag{2.5}$$

Since $\lambda \neq -1$, we find $P_2\Pi(\lambda U_{12})P_1 = 0$. Similarly, by using the same technique for $i = 2, j = 1$, one can show that $P_1\Pi(\lambda U_{21})P_2 = 0$.

Lemma 2.9. *(i)* $P_1\Pi(P_2)P_1 = P_2\Pi(P_1)P_2 = 0$.

(ii) $P_1\Pi(P_1)P_1 = P_2\Pi(P_2)P_2 = 0$.

Proof. (i) For every $X_{12} \in \mathcal{A}_{12}$, it follows from $[X_{12}, P_1]_* \diamondsuit_\lambda P_1 = 0$ that

$$
\begin{aligned}
0 &= \Pi([X_{12}, P_1]_* \diamondsuit_\lambda P_1) \\
&= [\Pi(X_{12}), P_1]_* \diamondsuit_\lambda P_1 + [X_{12}, \Pi(P_1)]_* \diamondsuit_\lambda P_1 + [X_{12}, P_1]_* \diamondsuit_\lambda \Pi(P_1) \\
&= \Pi(X_{12})P_1 - P_1\Pi(X_{12})^* P_1 + \lambda P_1\Pi(X_{12})P_1 - \lambda P_1\Pi(X_{12})^* \\
&\quad +X_{12}\Pi(P_1)P_1 - \Pi(P_1)X_{12}^* + \lambda X_{12}\Pi(P_1) - \lambda P_1\Pi(P_1)X_{12}^*.
\end{aligned}
$$

By left-multiplying the preceding equation by P_1 and right-multiplying by P_2, and considering the fact that $\lambda \neq 0$, we obtain

$$X_{12}\Pi(P_1)P_2 = P_1\Pi(X_{12})^* P_2. \tag{2.6}$$

That means,

$$P_2\Pi(P_1)X_{12}^* = P_2\Pi(X_{12})P_1.$$

Also, $[P_1, X_{12}]_*\diamondsuit_\lambda X_{12} = 0$ and using Lemma 2.1, we get

$$\begin{aligned} 0 &= \Pi([P_1, X_{12}]_*\diamondsuit_\lambda X_{12}) \\ &= [\Pi(P_1), X_{12}]_*\diamondsuit_\lambda X_{12} + [P_1, \Pi(X_{12})]_*\diamondsuit_\lambda X_{12} + [P_1, X_{12}]_*\diamondsuit_\lambda \Pi(X_{12}) \\ &= -X_{12}\Pi(P_1)^* X_{12} + \lambda X_{12}\Pi(P_1)X_{12} + P_1\Pi(X_{12})X_{12} - \Pi(X_{12})X_{12} - \lambda X_{12}\Pi(X_{12})P_1 \\ &\quad + X_{12}\Pi(X_{12}) + \lambda X_{12}\Pi(X_{12}). \end{aligned}$$

Multiplying the above equation by P_1 from left and right, we find $X_{12}\Pi(X_{12})P_1 = 0$. By using (▲) and (▼), we get $P_2\Pi(X_{12})P_1 = 0$. That means

$$P_1\Pi(X_{12})^* P_2 = 0.$$

From Equation (2.6),(▲) and (▼), we have $P_2\Pi(P_1)P_2 = 0$.
Similarly, we can show that $P_1\Pi(P_2)P_1 = 0$.
(ii) For any $X_{21} \in \mathcal{A}_{21}$ and using Lemma 2.6, we have

$$\Pi([X_{21}, P_1]_*\diamondsuit_\lambda P_1) = \Pi(X_{21}) - \Pi(\lambda X_{21}^*).$$

On the other hand, we have

$$\begin{aligned} \Pi([X_{21}, P_1]_*\diamondsuit_\lambda P_1) &= [\Pi(X_{21}), P_1]_*\diamondsuit_\lambda P_1 + [X_{21}, \Pi(P_1)]_*\diamondsuit_\lambda P_1 + [X_{21}, P_1]_*\diamondsuit_\lambda \Pi(P_1) \\ &= \Pi(X_{21})P_1 - P_1\Pi(X_{21})^* P_1 + \lambda P_1\Pi(X_{21})P_1 - \lambda P_1\Pi(X_{21})^* \\ &\quad + X_{21}\Pi(P_1)P_1 - \lambda P_1\Pi(P_1)X_{21}^* + X_{21}\Pi(P_1) + \lambda\Pi(P_1)X_{21}. \end{aligned}$$

By comparing the aforementioned two equations and subsequently left-multiplying by P_2 and right-multiplying by P_1, we obtain

$$2X_{21}\Pi(P_1)P_1 + \lambda P_2\Pi(P_1)X_{21} + P_2\Pi(\lambda X_{21}^*)P_1 = 0$$

Now, by using Lemma 2.8 and Lemma 2.9(i), we get $X_{21}\Pi(P_1)P_1 = 0$. Hence, $P_1\Pi(P_1)P_1 = 0$. Similarly, we can show that $P_2\Pi(P_2)P_2 = 0$.

Now, let $M = P_1\Pi(P_1)P_2 - P_2\Pi(P_1)P_1$. Then $M = -M^*$. We define a mapping $\Delta : \mathcal{A} \to \mathcal{A}$ as $\Delta(U) = \Pi(U) - (UM - MU)$ for all $U \in \mathcal{A}$. It can be easily verified that for all $U, V, W \in \mathcal{A}$, $\Delta([U, V]_*\diamondsuit_\lambda W) = [\Delta(U), V]_*\diamondsuit_\lambda W) + [U, \Delta(V)]_*\diamondsuit_\lambda W + [U, V]_*\diamondsuit_\lambda \Delta(W)$.

Remark 2.1. *The mapping* Δ *possesses the following properties:*

(i) Δ *is additive.*

(ii) $\Delta(P_1) = \Delta(P_2) = 0$.

(iii) $\Delta(I) = 0$

(iv) For every $U_{ij} \in \mathcal{A}_{ij} (1 \le i \ne j \le 2)$, *we have*

$$P_j\Delta(\lambda U_{ij})P_i = 0$$

(v) Δ *is a* $*$*-derivation if and only if* Π *is an* $*$ *-derivation.*

Proof. (i) Since $[U, M]$ is additive and also using Lemma 2.6, it is clear that Δ is additve.
(ii) By using Lemma 2.9, we have

$$\begin{aligned}\Delta(P_1) &= \Pi(P_1) - P_1\Pi(P_1)P_2 - P_2\Pi(P_1)P_1 \\ &= P_1\Pi(P_1)P_2 + P_2\Pi(P_1)P_1 - P_1\Pi(P_1)P_2 - P_2\Pi(P_1)P_1 \\ &= 0.\end{aligned}$$

Similarly, we can show that $\Delta(P_2) = 0$.
(iii) By using additivity of Δ, we have

$$\Delta(I) = \Delta(P_1 + P_2) = \Delta(P_1) + \Delta(P_2) = 0.$$

(iv) For $i = 1$ and $j = 2$, it can be inferred from Lemma 2.8 that

$$\begin{aligned}P_2\Delta(\lambda U_{12})P_1 &= P_2(\Pi(\lambda U_{12}) - \lambda U_{12}M + M\lambda U_{12})P_1 \\ &= 0.\end{aligned}$$

In the same way, one can show for $i = 2, j = 1$, i.e., $P_1\Delta(\lambda U_{21})P_2 = 0$.
(v) Since $[U, M] = UM - MU$ is an additive $*$-derivation. Therefore, Δ qualifies as a $*$-derivation if and only if Π is a $*$-derivation.

Lemma 2.10. $\Delta(U_{ij}) \subseteq U_{ij}, i, j = 1, 2$.

Proof. First, we prove for $i = 1, j = 1$. For every $U_{11} \in \mathcal{A}_{11}$, it follows from Remark 2.1 that

$$\begin{aligned}0 &= \Delta([P_1, U_{11}]_* \diamondsuit_\lambda P_1) \\ &= [P_1, \Delta(U_{11})]_* \diamondsuit_\lambda P_1 \\ &= P_1\Delta(U_{11})P_1 - \Delta(U_{11})P_1 + \lambda P_1\Delta(U_{11}) - \lambda P_1\Delta(U_{11})P_1.\end{aligned}$$

Left multiplying the above equation by P_2, we find

$$P_2\Delta(U_{11})P_1 = 0. \tag{2.7}$$

Similarly, again by using Remark 2.1, we have

$$\begin{aligned}0 &= \Delta([P_2, U_{11}]_* \diamondsuit_\lambda P_1) \\ &= [P_2, \Delta(U_{11})]_* \diamondsuit_\lambda P_1 \\ &= P_2\Delta(U_{11})P_1 - \lambda P_1\Delta(U_{11})P_2.\end{aligned}$$

Multiplying P_2 on the right and since $\lambda \neq 0$, we get

$$P_1\Delta(U_{11})P_2 = 0 \tag{2.8}$$

Now, for every $X_{12} \in \mathcal{A}_{12}$ and $\Delta(P_2) = 0$, it follows that

$$\begin{aligned} 0 &= \Delta([X_{12}, U_{11}]_* \diamondsuit_\lambda P_2) \\ &= [\Delta(X_{12}), U_{11}]_* \diamondsuit_\lambda P_2 + [X_{12}, \Delta(U_{11})]_* \diamondsuit_\lambda P_2 \\ &= -U_{11}\Delta(X_{12})^* P_2 + \lambda P_2 \Delta(X_{12}) U_{11} + X_{12}\Delta(U_{11})P_2 \\ &\quad -\lambda P_2 \Delta(U_{11}) X_{12}^*. \end{aligned}$$

Multiplying P_1 from left and P_2 from right and using Lemma 2.8, we get

$$X_{12}\Delta(U_{11})P_2 = U_{11}\Delta(X_{12})^* P_2 = U_{11}(P_2\Delta(X_{12})P_1)^* = 0.$$

Thus, $X_{12}\Delta(U_{11})P_2 = 0$. By using (▲) and (▼), we have

$$P_2\Delta(U_{11})P_2 = 0. \tag{2.9}$$

From Equations (2.7)-(2.9), we have $\Delta(U_{11}) \subseteq U_{11}$. Similarly, we can show that $\Delta(U_{22}) \subseteq U_{22}$.
Next, we establish the result for $i = 1, j = 2$. Additionally, for any $U_{12} \in \mathcal{A}_{12}$ and $\Delta(P_2) = 0$, we have

$$\begin{aligned} \Delta(U_{12}) &= \Delta([P_1, U_{12}]_* \diamondsuit_\lambda P_2) \\ &= [P_1, \Delta(U_{12})]_* \diamondsuit_\lambda P_2 \\ &= P_1\Delta(U_{12})P_2 - \lambda P_2\Delta(U_{12})P_1. \end{aligned}$$

By left and right multiplying the above equation by P_1, we obtain

$$P_1\Delta(U_{12})P_1 = 0. \tag{2.10}$$

Additionally, by left and right-multiplying by P_2, we find

$$P_2\Delta(U_{12})P_2 = 0. \tag{2.11}$$

Similarly, multiplying P_2 from left and P_1 from right and since $\lambda \neq -1$, we get

$$P_2\Delta(U_{12})P_1 = 0 \tag{2.12}$$

From equations (2.10)- (2.12), we get $\Delta(U_{12}) \subseteq U_{12}$.
Similarly, by using the same technique as above, one can prove that $\Delta(U_{21}) \subseteq U_{21}$.

Lemma 2.11. *For any $U_{i,j}, V_{i,j} \in \mathcal{A}_{ij}, 1 \leq i, j \leq 2$, we have*

(i) $\Delta(U_{11}V_{12}) = \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12})$ *and* $\Delta(U_{22}V_{21}) = \Delta(U_{22})V_{21} + U_{22}\Delta(V_{21})$.

(ii) $\Delta(U_{12}V_{21}) = \Delta(U_{12})V_{21} + U_{12}\Delta(V_{21})$ *and* $\Delta(U_{21}V_{12}) = \Delta(U_{21})V_{12} + U_{21}\Delta(V_{12})$.

(iii) $\Delta(U_{11}V_{11}) = \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11})$ *and* $\Delta(U_{22}V_{22}) = \Delta(U_{22})V_{22} + U_{22}\Delta(V_{22})$.

(iv) $\Delta(U_{12}V_{22}) = \Delta(U_{12})V_{22} + U_{12}\Delta(V_{22})$ *and* $\Delta(U_{21}V_{11}) = \Delta(U_{21})V_{11} + U_{21}\Delta(V_{11})$.

Proof. (i) Using Lemma 2.10 and $\Delta(P_2) = 0$, we get

$$\begin{aligned}\Delta([U_{11}, V_{12}]_* \diamondsuit_\lambda P_2) &= [\Delta(U_{11}), V_{12}]_* \diamondsuit_\lambda P_2 + [U_{11}, \Delta(V_{12})]_* \diamondsuit_\lambda P_2 \\ &= \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12}).\end{aligned}$$

On the other side, we get

$$\Delta([U_{11}, V_{12}]_* \diamondsuit_\lambda P_2) = \Delta(U_{11}V_{12}).$$

By comparing the above two equations, we get $\Delta(U_{11}V_{12}) = \Delta(U_{11})V_{12} + U_{11}\Delta(V_{12})$.
Similarly, we can show that $\Delta(U_{22}V_{21}) = \Delta(U_{22})V_{21} + U_{22}\Delta(V_{21})$.
(ii) For any $X_{12} \in \mathcal{A}_{12}$ and by using Lemma 2.11 (1), we have

$$\begin{aligned}\Delta([U_{12}, V_{21}]_* \diamondsuit_\lambda X_{12}) &= \Delta(U_{12}V_{21}X_{21}) \\ &= \Delta(U_{12}V_{21})X_{12} + U_{12}V_{21}\Delta(X_{12}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Delta([U_{12}, V_{21}]_* \diamondsuit_\lambda X_{12}) &= [\Delta(U_{12}), V_{21}]_* \diamondsuit_\lambda X_{12} + [U_{12}, \Delta(V_{21})]_* \diamondsuit_\lambda X_{12} \\ &\quad + [U_{12}, V_{21}]_* \diamondsuit_\lambda \Delta(X_{12}) \\ &= \Delta(U_{12})V_{21}X_{12} + U_{12}\Delta(V_{21})X_{12} + U_{12}V_{21}\Delta(X_{12}).\end{aligned}$$

From the above two expressions, we get

$$(\Delta(U_{12}V_{21}) - \Delta(U_{12})V_{21} - U_{12}\Delta(V_{21}))X_{12} = 0.$$

Thus, by (▲) and (▼), $\Delta(U_{12}V_{21}) = \Delta(U_{12})V_{21} + U_{12}\Delta(V_{21})$. Similarly, we can show that $\Delta(U_{21}V_{12}) = \Delta(U_{21})V_{12} + U_{21}\Delta(V_{12})$.
(iii) For any $X_{12} \in \mathcal{A}_{12}$, it follows from Lemma 2.11(i) that

$$\Delta(U_{11}V_{11}X_{12}) = \Delta(U_{11}V_{11})X_{12} + U_{11}V_{11}\Delta(X_{12}).$$

Again using Lemma 2.11 from the other side, we have

$$\begin{aligned}\Delta(U_{11}V_{11}X_{12}) &= \Delta(U_{11})V_{11}X_{12} + U_{11}\Delta(V_{11}X_{12}) \\ &= \Delta(U_{11})V_{11}X_{12} + U_{11}\Delta(V_{11})X_{12} + U_{11}V_{11}\Delta(X_{12}).\end{aligned}$$

By comparing the aforementioned two equations, we obtain $(\Delta(U_{11}V_{11}) - \Delta(U_{11})V_{11} - U_{11}\Delta(V_{11}))X_{12} = 0$. Therefore, utilizing (▲) and (▼), we conclude that $\Delta(U_{11}V_{11}) = \Delta(U_{11})V_{11} + U_{11}\Delta(V_{11})$.
Similarly, one can show that $\Delta(U_{22}V_{22}) = \Delta(U_{22})V_{22} + U_{22}\Delta(V_{22})$.
(iv) For any $X_{21} \in \mathcal{A}_{21}$. It follows from Lemma 2.11(2) that

$$\Delta(U_{12}V_{22}X_{21}) = \Delta(U_{12}V_{22})X_{21} + U_{12}V_{22}\Delta(X_{21}).$$

Again on the other side, it follows from Lemma 2.11(i) and Lemma 2.11(ii) that

$$\begin{aligned}\Delta(U_{12}V_{22}X_{21}) &= \Delta(U_{12})V_{22}X_{21} + U_{12}\Delta(V_{22}X_{21}) \\ &= \Delta(U_{12})V_{22}X_{21} + U_{12}\Delta(V_{22})X_{21} + U_{12}V_{22}\Delta(X_{21}).\end{aligned}$$

From the above two equations and using (▲) and (▼), we find $\Delta(U_{12}V_{22}) = \Delta(U_{12})V_{22} + U_{12}\Delta(V_{22})$. Similarly, one can show that $\Delta(U_{21}V_{11}) = \Delta(U_{21})V_{11} + U_{21}\Delta(V_{11})$.

Lemma 2.12. $\Delta(U^*) = \Delta(U)^*$ *for all* $U \in \mathcal{A}$.

Proof. For any $X_{12} \in \mathcal{A}_{12}$, it follows from Remark 2.1 and Lemma 2.11(i) that

$$\begin{aligned}\Delta([U_{11}, P_1]_* \diamondsuit_\lambda X_{12}) &= \Delta(U_{11}X_{12}) - \Delta(U_{11}^*X_{12}) \\ &= \Delta(U_{11})X_{12} + U_{11}\Delta(X_{12}) - \Delta(U_{11}^*)X_{12} - U_{11}^*\Delta(X_{12}).\end{aligned}$$

Alternatively, it can be deduced from $\Delta(P_1) = 0$ that

$$\begin{aligned}\Delta([U_{11}, P_1]_* \diamondsuit_\lambda X_{12}) &= [\Delta(U_{11}), P_1]_* \diamondsuit_\lambda X_{12} + [U_{11}, P_1]_* \diamondsuit_\lambda \Delta(X_{12}) \\ &= \Delta(U_{11})X_{12} - \Delta(U_{11})^*X_{12} + U_{11}\Delta(X_{12}) - U_{11}^*\Delta(X_{12}).\end{aligned}$$

From the above two equations, we have $(\Delta(U_{11})^* - \Delta(U_{11}^*))X_{12} = 0$. Now, by using (▲) and (▼), we get

$$\Delta(U_{11})^* = \Delta(U_{11}^*). \tag{2.13}$$

Similarly, by using the same technique, one can show that

$$\Delta(U_{22})^* = \Delta(U_{22}^*). \tag{2.14}$$

Again, it follows from Lemma 2.10 and $\Delta(P_2) = 0$ that

$$\begin{aligned}\Delta([U_{12}, P_2]_* \diamondsuit_\lambda X_{12}) &= [\Delta(U_{12}), P_2]_* \diamondsuit_\lambda X_{12} + [U_{12}, P_2]_* \diamondsuit_\lambda \Delta(X_{12}) \\ &= -\Delta(U_{12})^*X_{12} - \lambda X_{12}\Delta(U_{12})^* - U_{12}^*\Delta(X_{12}) - \lambda\Delta(X_{12})U_{12}^*.\end{aligned}$$

On the other hand, using Lemma 2.11, we have

$$\begin{aligned}\Delta([U_{12}, P_2]_* \diamondsuit_\lambda X_{12}) &= \Delta(-U_{12}^*X_{12} - \lambda X_{12}U_{12}^*) \\ &= -\Delta(U_{12}^*X_{12}) - \Delta(X_{12}\lambda U_{12}^*) \\ &= -\Delta(U_{12}^*)X_{12} - U_{12}^*\Delta(X_{12}) - \lambda\Delta(X_{12})U_{12}^* - X_{12}\Delta(\lambda U_{12}^*).\end{aligned}$$

By comparing the above two equations, we get

$$(\Delta(U_{12})^* - \Delta(U_{12}^*))X_{12} + \lambda X_{12}\Delta(U_{12})^* - X_{12}\Delta(\lambda U_{12}^*) = 0.$$

By left-multiplying both sides by P_2 and utilizing (▲) and (▼), we obtain

$$\Delta(U_{12})^* = \Delta(U_{12}^*). \tag{2.15}$$

Similarly, we can show that

$$\Delta(U_{21})^* = \Delta(U_{21}^*). \tag{2.16}$$

From equations (2.13)-(2.16) and using additivity of Δ, we get $\Delta(U^*) = \Delta(U)^*$.

Proof of Theorem 2.1 For every $U, V \in \mathcal{A}$, we can write $U = U_{11} + U_{12} + U_{21} + U_{22}$ and $V = V_{11} + V_{12} + V_{21} + V_{22}$. Since, Δ is additive and using Lemma 2.11, we get

$$\begin{aligned}
\Delta(UV) &= \Delta(U_{11}V_{11} + U_{11}V_{12} + U_{12}V_{21} + U_{12}V_{22} \\
&\quad + U_{21}V_{11} + U_{21}V_{12} + U_{22}V_{21} + U_{22}V_{22}) \\
&= \Delta(U_{11}V_{11}) + \Delta(U_{11}V_{12}) + \Delta(U_{12}V_{21}) + \Delta(U_{12}V_{22}) \\
&\quad + \Delta(U_{21}V_{11}) + \Delta(U_{21}V_{12}) + \Delta(U_{22}V_{21}) + \Delta(U_{22}V_{22}) \\
&= \Delta(U_{11} + U_{12} + U_{21} + U_{22})(V_{11} + V_{12} + V_{21} + V_{22}) \\
&\quad + (U_{11} + U_{12} + U_{21} + U_{22})\ \Delta(V_{11} + V_{12} + V_{21} + V_{22}) \\
&= \Delta(U)V + U\Delta(V).
\end{aligned}$$

So, Δ is a derivation. By using Lemma 2.12, Δ is an additive $*$-derivation. Hence, by Remark 2.1, Π is an additive $*$-derivation. This completes the proof of Theorem 2.1.

The corollaries following directly from Theorem 2.1 are as follows:

Corollary 2.1. *Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$ containing identity operator I. Suppose that $\mathcal{A}$ is closed under adjoint operation. Define $\Pi : \mathcal{A} \to \mathcal{A}$ such that*

$$\Pi([U, V]_* \diamondsuit_\lambda W) = [\Pi(U), V]_* \diamondsuit_\lambda W) + [U, \Pi(V)]_* \diamondsuit_\lambda W + [U, V]_* \diamondsuit_\lambda \Pi(W)$$

for all $U, V, W \in \mathcal{A}$. Then Π is an additive $*$-derivation.

Corollary 2.2. *Let $\mathcal{A}$ ba a factor von Neumann algebra with dim$\mathcal{M} \geq 2$. Define $\Pi : \mathcal{M} \to \mathcal{M}$ such that*

$$\Pi([U, V]_* \diamondsuit_\lambda W) = [\Pi(U), V]_* \diamondsuit_\lambda W) + [U, \Pi(V)]_* \diamondsuit_\lambda W + [U, V]_* \diamondsuit_\lambda \Pi(W)$$

for all $U, V, W \in \mathcal{A}$. Then Π is an additive $*$-derivation.

Corollary 2.3. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I containing non-trivial projection P. A map $\Pi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Pi([U, V]_* \diamondsuit_\lambda W) = [\Pi(U), V]_* \diamondsuit_\lambda W) + [U, \Pi(V)]_* \diamondsuit_\lambda W + [U, V]_* \diamondsuit_\lambda \Pi(W)$$

for all $U, V, W \in \mathcal{A}$. Then Π is an additive $*$-derivation.

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