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Employing a Generalization of Open Sets Defined by Ideals to Initiate Novel Rough Approximation Spaces with a Chemical Application

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Abstract. A close similarity and analogy between rough set theory and topology is attributed to the corresponding behavior of lower and upper rough approximations with interior and closure topological operators, respectively. This relation motivates joint studies between topology and this theory. We endeavor by rough set theory to enlarge the knowledge we obtain from the information systems, for this reason, we apply the abstract concept of ideal structures to build new generalized approximation spaces with less vagueness. In the present work, we employ a novel type of nearly open sets in topology so-called "$\mathcal{L}$-$\theta\beta_\lambda$-open" with an ideal structure to introduce novel approximation spaces satisfying the desired properties concerning shrinking the boundary region of uncertainty and expanding the domain of confirmed information. We set up the fundamentals of the proposed rough paradigms and demonstrate their superiority over the preceding paradigms induced by some nearly open sets. Two algorithms are furnished to illustrate the way of specifying the family of $\mathcal{L}$-$\theta\beta_\lambda$-open sets and exploring whether a subset is $\mathcal{L}$-$\theta\beta_\lambda$-definable or $\mathcal{L}$-$\theta\beta_\lambda$-rough. Then, we put forward the concepts of rough membership relations and functions and uncover their core characterizations. Finally, we examine the proposed models to model a real situation in the Chemistry field and clarify how our models improve the outcomes of generalized approximation spaces over the previous models.

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1. Introduction

Nowadays, one can see a rapid growth of interest in the theory of rough sets and its applications, evident from the number of international conferences and workshops dedicated to investigating the progression of rough set theory, as well as the high-quality papers published as a result of this attention. This theory was initiated by Pawlak [40, 41] in the early 1980s as a non-statistical technique to analyze data tables acquired from human experts or measurements. The philosophy of rough set theory in addressing the complex problems individuals face in practical life is based on dividing a set of data containing uncertainty into three regions. The first region includes the confirmed information extracted from this set, terminologically known as the lower approximation. The second region represents the information for which we cannot determine its belonging or non-belonging to the set, known as the upper approximation. The third region, known as the boundary region, is defined as the difference between the upper approximation and the lower approximation.

Rough set theory begins with the concept of an equivalence relationship, which is a strict term when modeling many realistic problems. This strictness prompted many researchers and authors to search for alternative methods to the equivalence classes, leading to the development of the neighborhood idea. Initially defined by Yao, he [50, 51] formulated the concepts of right neighborhoods and left neighborhoods as the equivalents of the equivalence classes derived from Pawlak's original model. Over time, with the desire to increase the confirmed information, other models were proposed to improve the approximation operators and accuracy measures. For instance, rough set paradigms introduced by using minimal neighborhoods [3], containment neighborhoods [5], maximal neighborhoods [8, 16], subset neighborhoods [10, 52], adhesion neighborhoods [36], etcetera.

Attention was paid early by [48] to the similarity between the behaviors of lower and upper rough approximations and interior and closure topological operators. Therefore, topological structures have been proposed to study information systems and apply topological operators as alternative tools for these approximations; see, for instance [4, 17, 22, 34, 43, 46, 49, 53]. Diverse techniques have been introduced to create topological spaces utilizing neighborhood systems. For example, one can take the neighborhood of each point as a subbase of a topology [33] or initiate the topology using the following formula: $\vartheta_\lambda = \{V \subseteq X : \forall \mathsf{y} \in V, \mathbb{G}_\lambda(\mathsf{y}) \subseteq V\}$ [45]. To develop decision-making methods for information systems from a topological standpoint, several authors have employed abstract topological principles and their generalizations, such as nearly open sets [1, 2, 6, 7], supra topology [9], minimal structures [18], infra topology [13], and bitopology [44].

The authors of [32, 47] put forward the notion of ideal over a set X as a nonempty subcollection of the power set of X which is closed under finite union and hereditary property. In [28], new topologies are derived from an old one using ideals. With a strong desire to increase the amount of confirmed information, which gives the decision-maker a greater opportunity to make more accurate decisions, the ideal structure was integrated into generalized approximation spaces. The concept was first employed by Kandil et al. [29]. This concept was later exploited by researchers in the study of information systems, explaining the advantages of this tool in various ways, including topological approaches, as illustrated in many published manuscripts [11, 12, 14, 21, 23, 24, 27, 39]. Researchers have the freedom to choose the tool that is most efficient for addressing the problem and achieving the greatest possible amount of desired characteristics of Pawlak paradigms. Michael [38] came up with a brilliant idea to enlarge a family of semi-open sets using ideals, then some authors [25, 26] followed this technique to aggrandize the classes of α-open, β-open, and pre-open sets.

This work deals with generalized approximation spaces using a topological approach and enhances the prominence of using ideals via rough set theory studies, as a tool to demystify the data. We suggest a broader general framework of topological approximation spaces via ideals, satisfying the desirable characteristics of original models and enhancing decision reliability. The presentation of this article is organized as follows: Section 2 covers the fundamentals required to make the paper self-contained. Then, in Section 3, we define a new class of nearly open sets, namely, $\mathcal{L}$-$\theta\beta_\lambda$-open sets, which is strictly stronger than the class of $\mathcal{L}$-β_λ-open sets. We draw the main properties of this class and articulate its relationships with the preceding ones with the aid of examples. Section 4 is devoted to constructing rough set models utilizing the class of $\mathcal{L}$-$\theta\beta_\lambda$-open sets. We compare the approximation operators, boundary regions, and accuracy values of the proposed paradigms with those presented in other studies. In Section 5, we display a new type of rough membership functions and apply to describe the main concepts of the proposed rough set models. We provide a practical example in Section 6 to illustrate the superiority of the current models over the former models and their applicability in addressing realistic problems. Lastly, we draw conclusions from the present study and summarize its most important findings in Section 7. The presentation of this article is organized as follows: Section 2 covers the fundamentals required to make the paper self-contained. Then, in Section 3, we define a new class of nearly open sets, namely, $\mathcal{L}$-$\theta\beta_\lambda$-open sets, which is strictly stronger than the class of $\mathcal{L}$-β_λ-open sets. We draw the main properties of this class and articulate its relationships with the preceding ones with the aid of examples. Section 4 focuses on developing rough set models using the $\mathcal{L}$-$\theta\beta_\lambda$-open sets. We compare the approximation operators, boundary regions, and accuracy values of the proposed models with those in existing studies. In Section 5, we introduce a new type of rough membership functions and apply them to explain the central concepts of the proposed rough set models. Section 6 provides a practical example that demonstrates the advantages of the current models over previous ones and their effectiveness in solving real-world problems. Finally, in Section 7, we conclude the study by summarizing its key findings.

2. Preliminaries

In this segment, we cover the main contributions via topological (generalized) approximation spaces that are required to understand the main contributions and significance of this manuscript.

Definition 1. *[32, 47] An ideal $\mathcal{L}$ over the universe $X \neq \emptyset$ is a subfamily of the power set of X satisfying the below terms.*

(i) *$V \in \mathcal{L}$ and $Z \in \mathcal{L} \Rightarrow V \cup Z \in \mathcal{L}$.*

(ii) *$V \in \mathcal{L}$ and $Z \subseteq V \Rightarrow Z \in \mathcal{L}$.*

Through this content, X indicates for a finite nonempty set.

Definition 2. *[9] Let $\mathcal{R}$ be a binary relation on X. Then, the λ-neighborhood of an element y in X, symbolized by $\mathbb{G}_\lambda(\mathsf{y}), \lambda \in \{\mathbf{a}, \mathbf{b}, \hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{i}, \mathbf{u}, \hat{\mathbf{i}}, \hat{\mathbf{u}}\}$, is given by:*

(i) $\mathbb{G}_{\mathbf{a}}(\mathsf{y}) = \{\mathsf{x} \in X : \mathsf{y}\mathcal{R}\mathsf{x}\}$.

(ii) $\mathbb{G}_{\mathbf{b}}(\mathsf{y}) = \{\mathsf{x} \in X : \mathsf{x}\mathcal{R}\mathsf{y}\}$.

(iii) $\mathbb{G}_{\hat{\mathbf{a}}}(\mathsf{y}) = \cap_{\mathsf{y} \in \mathbb{G}_{\mathbf{a}}(\mathsf{x})} \mathbb{G}_{\mathbf{a}}(\mathsf{x})$, *or* $\mathbb{G}_{\hat{\mathbf{a}}}(\mathsf{y}) = \emptyset$ *when there does not exists* $\mathbb{G}_{\mathbf{a}}(\mathsf{x})$ *containing* y.

(iv) $\mathbb{G}_{\hat{\mathbf{b}}}(\mathsf{y}) = \cap_{\mathsf{y} \in \mathbb{G}_{\mathbf{b}}(\mathsf{x})} \mathbb{G}_{\mathbf{b}}(\mathsf{x})$, *or* $\mathbb{G}_{\hat{\mathbf{b}}}(\mathsf{y}) = \emptyset$ *when there does not exists* $\mathbb{G}_{\mathbf{b}}(\mathsf{x})$ *containing* y.

(v) $\mathbb{G}_{\mathbf{i}}(\mathsf{y}) = \mathbb{G}_{\mathbf{a}}(\mathsf{y}) \cap \mathbb{G}_{\mathbf{b}}(\mathsf{y})$.

(vi) $\mathbb{G}_{\mathbf{u}}(\mathsf{y}) = \mathbb{G}_{\mathbf{a}}(\mathsf{y}) \cup \mathbb{G}_{\mathbf{b}}(\mathsf{y})$.

(vii) $\mathbb{G}_{\hat{\mathbf{i}}}(\mathsf{y}) = \mathbb{G}_{\hat{\mathbf{a}}}(\mathsf{y}) \cap \mathbb{G}_{\hat{\mathbf{b}}}(\mathsf{y})$.

(viii) $\mathbb{G}_{\hat{\mathbf{u}}}(\mathsf{y}) = \mathbb{G}_{\hat{\mathbf{a}}}(\mathsf{y}) \cup \mathbb{G}_{\hat{\mathbf{b}}}(\mathsf{y})$.

Moving forward, we utilize this symbol λ throughout this manuscript to refer to the types of neighbourhoods of $\{\mathbf{a}, \mathbf{b}, \hat{\mathbf{a}}, \hat{\mathbf{b}}, \mathbf{i}, \mathbf{u}, \hat{\mathbf{i}}, \hat{\mathbf{u}}\}$.

Definition 3. *[45] If $\Xi_\lambda : X \to P(X)$ is a mapping that assigns for each y in X a $\mathbb{G}_\lambda$ in $P(X)$, then we called a 3-tuple $(X, \mathcal{R}, \Xi_\lambda)$ a $\mathbb{G}_\lambda$-space.*

Theorem 1. *[30, 31, 45] It may generate a topology ϑ_λ on X using $\mathbb{G}_\lambda$-neighbourhoods by the next formula*

$$\vartheta_\lambda = \{V \subseteq X : \forall \mathsf{y} \in V, \mathbb{G}_\lambda(\mathsf{y}) \subseteq V\}$$

Every member of ϑ_λ is named a λ-open set and we call a subset a λ-closed set if its complement is a λ-open set.

The class of Γ_λ is given by $\Gamma_\lambda = \{F \subseteq X : F' \in \vartheta_\lambda\}$, where F' is the complement of F.

Definition 4. *[45] The λ-lower and λ-upper approximations, λ-boundary region and λ-accuracy of $V \subseteq X$, inspired by the topological space (X, ϑ_λ) given in above theorem, are respectively formulated by the subsequent formulas:*

$\underline{\mathcal{R}}_\lambda(V)$ is the union of all λ-open sets which are contained in V; that is $V = int_\lambda(V)$, where int_λ is the topological λ-interior operator.

$\overline{\mathcal{R}}_\lambda(V)$ is the intersection of all λ-closed sets containing V; that is, $V = cl_\lambda(V)$, where cl_λ is the topological λ-closure operator.

$\mathcal{BND}_\lambda(V) = \overline{\mathcal{R}}_\lambda(V) - \underline{\mathcal{R}}_\lambda(V)$.

$ACC_\lambda(V) = \frac{|\underline{\mathcal{R}}_\lambda(V)|}{|\overline{\mathcal{R}}_\lambda(V)|}$, for each subset $V \neq \emptyset$.

Remember that a subset V is named λ-exact if $\overline{\mathcal{R}}_\lambda(V) = \underline{\mathcal{R}}_\lambda(V)$. Otherwise, V is λ-rough.

In what follows, we recall some definitions of λ-nearly open sets.

Definition 5. *[15, 20] Let $(X, \mathcal{R}, \Xi_\lambda)$ be a $\mathbb{G}_\lambda$-space. $V \subseteq X$ is said to be*

(i) *λ-preopen (P_λ-open), if $int_\lambda(cl_\lambda(V)) \supseteq V$.*

(ii) *λ-semiopen (S_λ-open), if $cl_\lambda(int_\lambda(V)) \supseteq V$.*

(iii) *α_λ-open, if $V \subseteq int_\lambda[cl_\lambda(int_\lambda(V))]$.*

(iv) *β_λ-open (semi preopen), if $V \subseteq cl_\lambda[int_\lambda(cl_\lambda(V))]$.*

(v) *$\delta\beta_\lambda$-open, if $V \subseteq cl_\lambda[int_\lambda(cl^\delta_\lambda(V))]$, where $cl^\delta_\lambda(V) = \{\mathsf{y} \in X : V \cap int_\lambda(cl_\lambda(G)) \neq \emptyset, G \in \vartheta_\lambda$ and $\mathsf{y} \in G\}$.*

(vi) *$\bigwedge_{\beta_\lambda}$-set if $V = \bigwedge_{\beta_\lambda}(V)$, where $\bigwedge_{\beta_\lambda}(V) = \cap\{G : V \subseteq G, G \in \beta_{\lambda O}(X)\}$.*

The families of λ-nearly open subsets of X are assigned by $\eta_{\lambda O}(X)$, where $\eta \in \{\alpha, P, S, \beta, \delta\beta, \bigwedge_\beta\}$. The complements of the λ-nearly open sets are known as λ-nearly closed sets and denoted by $\eta_{\lambda C}(X)$.

Henceforth, we mean by η the elements of the set $\{P, S, \alpha, \beta, \delta\beta, \bigwedge_\beta\}$, unless otherwise stated.

Definition 6. *[15, 20] Let $(X, \mathcal{R}, \Xi_\lambda)$ be a $\mathbb{G}_\lambda$-space and $V \subseteq X$. The η_λ-lower and η_λ-upper approximations, η_λ-boundary regions and η_λ-accuracy of V are respectively given by:*

$\underline{\mathcal{R}}^\eta_\lambda(V) = \cup\{G \in \eta_{\lambda O}(X) : G \subseteq V\} = \eta_\lambda$*-interior of* V.

$\overline{\mathcal{R}}^\eta_\lambda(V) \cap \{H \in \eta_{\lambda C}(X) : V \subseteq H\} = \eta_\lambda$*-closure of* V.

$\mathcal{BND}^\eta_\lambda(V) = \overline{\mathcal{R}}^\eta_\lambda(V) - \underline{\mathcal{R}}^\eta_\lambda(V)$.

$ACC^\eta_\lambda(V) = \frac{|\underline{\mathcal{R}}^\eta_\lambda(V)|}{|\overline{\mathcal{R}}^\eta_\lambda(V)|}$*, where* $|\overline{\mathcal{R}}^\eta_\lambda(V)| \neq 0, |\overline{\mathcal{R}}^\eta_\lambda(V)|$ *denotes the cardinality of* $\overline{\mathcal{R}}^\eta_\lambda(V)$.

Definition 7. *[20] A subset V of a $\mathbb{G}_\lambda$-space (X, R, Ξ_λ) is called:*

(i) *$\delta\beta_\lambda$-definable ($\delta\beta_\lambda$-exact) if $\overline{\mathcal{R}}^{\delta\beta}_\lambda(V) = \underline{\mathcal{R}}^{\delta\beta}_\lambda(V)$ or $\mathcal{BND}^{\delta\beta}_\lambda(V) = \emptyset$.*

(ii) *$\delta\beta_\lambda$-rough if $\overline{\mathcal{R}}^{\delta\beta}_\lambda(V) \neq \underline{\mathcal{R}}^{\delta\beta}_\lambda(V)$ or $\mathcal{BND}^{\delta\beta}_\lambda(V) \neq \emptyset$.*

(iii) *$\bigwedge_{\beta_\lambda}$-definable ($\bigwedge_{\beta_\lambda}$-exact) if $\overline{\mathcal{R}}^{\bigwedge_\beta}_\lambda(V) = \underline{\mathcal{R}}^{\bigwedge_\beta}_\lambda(V)$ or $\mathcal{BND}^{\bigwedge_\beta}_\lambda(V) = \emptyset$.*

(iv) *$\bigwedge_{\beta_\lambda}$-rough if $\overline{\mathcal{R}}^{\bigwedge_\beta}_\lambda(V) \neq \underline{\mathcal{R}}^{\bigwedge_\beta}_\lambda(V)$ or $\mathcal{BND}^{\bigwedge_\beta}_\lambda(V) \neq \emptyset$.*

Definition 8. *[22, 23] Let $\mathcal{L}$ be an ideal on X. We call a subset V of a $\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda)$:*

(i) $\mathcal{L}$-α_λ-*open providing that there exists* $G \in \vartheta_\lambda$ *s.t.* $(V - int_\lambda(cl_\lambda((G)) \in \mathcal{L}$ *and* $(G - V) \in \mathcal{L}$.

(ii) $\mathcal{L}$-P_λ-*open providing that there exists* $G \in \vartheta_\lambda$ *s.t.* $(V - G) \in \mathcal{L}$ *and* $(G - cl_\lambda(V)) \in \mathcal{L}$.

(iii) $\mathcal{L}$-S_λ-*open providing that there exists* $G \in \vartheta_\lambda$ *s.t.* $(V - cl_\lambda(G)) \in \mathcal{L}$ *and* $(G - V) \in \mathcal{L}$.

(iv) $\mathcal{L}$-β_λ-*open providing that there exists* $G \in \vartheta_\lambda$ *s.t.* $(V - cl_\lambda(G)) \in \mathcal{L}$ *and* $(G - cl_\lambda(V)) \in \mathcal{L}$.

(v) $\mathcal{L}$-$\delta\beta_\lambda$-*open providing that there exists* $G \in \vartheta_\lambda$ *s.t.* $(V - cl_\lambda(G)) \in \mathcal{L}$ *and* $(G - cl^\delta_\lambda(V)) \in \mathcal{L}$.

(vi) $\mathcal{L}$-$\bigwedge_{\beta_\lambda}$-*set, if* $V = \mathcal{L} - \bigwedge_{\beta_\lambda}(V)$, *where* $\mathcal{L}$-$\bigwedge_{\beta_\lambda}(V) = \cap\{G : V \subseteq G, G \in \mathcal{L}\text{-}\beta_{\lambda O}(X)\}$.

These sets are called $\mathcal{L}$-λ-*nearly open sets, the complement of the* $\mathcal{L}$-λ-*nearly open sets is called* $\mathcal{L}$-λ-*nearly closed sets, the families of* $\mathcal{L}$-λ-*nearly open sets of* X *denoted by* $\mathcal{L}$-$\eta_{\lambda O}(X)$ *and the families of* $\mathcal{L}$-λ-*nearly closed sets of* X *denoted by* $\mathcal{L}$-$\eta_{\lambda C}(X)$.

Proposition 1. *[23]*

(i) Every $\delta\beta_\lambda$-*open is* $\mathcal{L}$-$\delta\beta_\lambda$-*open.*

(ii) Every $\bigwedge_{\beta_\lambda}$-*set is* $\mathcal{L}$-$\bigwedge_{\beta_\lambda}$-*set.*

Proposition 2. *[23] The next implications hold true:*

$$\begin{array}{ccccc} \vartheta_\lambda(\Gamma_\lambda) \Rightarrow & \mathcal{L}\text{-}\alpha_{\lambda O}(\mathcal{L}\text{-}\alpha_{\lambda C}) & & \mathcal{L}\text{-}P_{\lambda O}(\mathcal{L}\text{-}P_{\lambda C}) & \\ & \Downarrow & & \Downarrow & \\ \mathcal{L}\text{-}S_{\lambda O}(\mathcal{L}\text{-}S_{\lambda C}) \Rightarrow & \mathcal{L}\text{-}\beta_{\lambda O}(\mathcal{L}\text{-}\beta_{\lambda C}) & \Rightarrow & \mathcal{L}\text{-}\delta\beta_{\lambda O}(\mathcal{L}\text{-}\delta\beta_{\lambda C}). & \end{array}$$

$$\begin{array}{ccccc} \vartheta_\lambda(\Gamma_\lambda) \Rightarrow & \mathcal{L}\text{-}\alpha_{\lambda O}(\mathcal{L}\text{-}\alpha_{\lambda C}) & & \mathcal{L}\text{-}P_{\lambda O}(\mathcal{L}\text{-}P_{\lambda C}) & \\ & \Downarrow & & \Downarrow & \\ \mathcal{L}\text{-}S_{\lambda O}(\mathcal{L}\text{-}S_{\lambda C}) \Rightarrow & \mathcal{L}\text{-}\beta_{\lambda O}(\mathcal{L}\text{-}\beta_{\lambda C}) & \Rightarrow & \mathcal{L}\text{-}\bigwedge_{\beta_{\lambda O}}(\mathcal{L}\text{-}\bigwedge_{\beta_{\lambda C}}). & \end{array}$$

$$\begin{array}{ccccc} \vartheta_\lambda(\Gamma_\lambda) \Rightarrow & \alpha_{\lambda O}(\alpha_{\lambda C}) & & P_{\lambda O}(P_{\lambda C}) & \\ & \Downarrow & & \Downarrow & \\ & S_{\lambda O}(S_{\lambda C}) & \Rightarrow & \beta_{\lambda O}(\beta_{\lambda C}) & \Rightarrow \delta\beta_{\lambda O}(\delta\beta_{\lambda C}). \end{array}$$

$$\begin{array}{ccccc} \vartheta_\lambda(\Gamma_\lambda) \Rightarrow & \alpha_{\lambda O}(\alpha_{\lambda C}) & & P_{\lambda O}(P_{\lambda C}) & \\ & \Downarrow & & \Downarrow & \\ & S_{\lambda O}(S_{\lambda C}) & \Rightarrow & \beta_{\lambda O}(\beta_{\lambda C}) & \Rightarrow \bigwedge_{\beta_{\lambda O}}(\bigwedge_{\beta_{\lambda C}}). \end{array}$$

Definition 9. *[22, 23] The* $\mathcal{L}$-η_λ-*lower and* $\mathcal{L}$-η_λ-*upper approximations,* $\mathcal{L}$-η_λ-*boundary regions and* $\mathcal{L}$-η_λ-*accuracy of* V *are respectively given by:*

$\underline{\mathcal{R}}^{\mathcal{L}-\eta}_\lambda(V) = \cup\{G \in \mathcal{L}\text{-}\eta_{\lambda O}(X) : G \subseteq V\} = \mathcal{L}$-$\eta_\lambda$-*interior of* V.

$\overline{\mathcal{R}}^{\mathcal{L}-\eta}_\lambda(V) = \cap\{H \in \mathcal{L}\text{-}\eta_{\lambda C}(X) : V \subseteq H\} = \mathcal{L}$-$\eta_\lambda$-*closure of* V.

$\mathcal{BND}^{\mathcal{L}-\eta}_\lambda(V) = \overline{\mathcal{R}}^{\mathcal{L}-\eta}_\lambda(V) - \underline{\mathcal{R}}^{\mathcal{L}-\eta}_\lambda(V)$.

$ACC^{\mathcal{L}-\eta}_\lambda(V) = \frac{|\underline{\mathcal{R}}^{\mathcal{L}-\eta}_\lambda(V)|}{|\overline{\mathcal{R}}^{\mathcal{L}-\eta}_\lambda(V)|}$, *where* $|\overline{\mathcal{R}}^{\mathcal{L}-\eta}_\lambda(V)| \neq 0$.

Remember that a subset V is called an $\mathcal{L}$-η_λ-definable ($\mathcal{L}$-η_λ-exact) set if $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\eta}(V) = \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\eta}(V)$. Otherwise, V is an $\mathcal{L}$-η_λ-rough set.

Theorem 2. *[23] For a subset V of a $\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda)$, we have:*

(i) $\underline{\mathcal{R}}_\lambda^{\alpha}(V) \subseteq \underline{\mathcal{R}}_\lambda^{p}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\gamma}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\delta\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha\beta}(V)$.

(ii) $\underline{\mathcal{R}}_\lambda^{\alpha}(V) \subseteq \underline{\mathcal{R}}_\lambda^{s}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\gamma}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\delta\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha\beta}(V)$.

(iii) $\underline{\mathcal{R}}_\lambda(V) \subseteq \underline{\mathcal{R}}^{\delta\beta}{}_\lambda(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha\beta}(V)$.

(iv) $\underline{\mathcal{R}}_\lambda^{\alpha}(V) \subseteq \underline{\mathcal{R}}_\lambda^{p}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\gamma}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\bigwedge_\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V)$.

(v) $\underline{\mathcal{R}}_\lambda^{\alpha}(V) \subseteq \underline{\mathcal{R}}_\lambda^{s}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\gamma}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\bigwedge_\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V)$.

(vi) $\underline{\mathcal{R}}_\lambda(V) \subseteq \underline{\mathcal{R}}_\lambda^{\bigwedge_\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V)$.

(vii) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\gamma}(V) \subseteq \overline{\mathcal{R}}_\lambda^{p}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\alpha}(V)$.

(viii) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\gamma}(V) \subseteq \overline{\mathcal{R}}_\lambda^{s}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\alpha}(V)$.

(ix) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \overline{\mathcal{R}}^{\delta\beta}{}_\lambda(V) \subseteq \overline{\mathcal{R}}_\lambda(V)$.

(x) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\bigwedge_\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\bigwedge_\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\gamma}(V) \subseteq \overline{\mathcal{R}}_\lambda^{p}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\alpha}(V)$.

(xi) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\bigwedge_\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\bigwedge_\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\gamma}(V) \subseteq \overline{\mathcal{R}}_\lambda^{s}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\alpha}(V)$.

(xii) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\bigwedge_\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\bigwedge_\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda(V)$.

When we combine an ideal $\mathcal{L}$ with a $\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda)$, we write the quadruple $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$; this quadruple is symbolized by $\mathcal{L} - \mathbb{G}_\lambda$-space.

Proposition 3. *[23] For a subset V of an $\mathcal{L} - \mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$, we have:*

(i) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-P}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\delta\beta_\lambda}(V)$.

(ii) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-S}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V)$.

(iii) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-P}(V)$.

(iv) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-S}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha}(V)$.

(v) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-P}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V)$.

(vi) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-S}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V)$.

(vii) $\overline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-P}(V)$.

(viii) $\overline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-S}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha}(V)$.

Definition 10. *Let* $(X, \mathcal{R}, \Xi_\lambda)$ *be a* $\mathbb{G}_\lambda$*-space and* $V \subseteq X$*. The* θ_λ*-closure is given by* $cl_\lambda^\theta(V) = \{\mathsf{y} \in X : V \cap cl_\lambda(G) \neq \emptyset, G \in \vartheta_\lambda$ *and* $\mathsf{y} \in G\}$.

Definition 11. *[42] The rough membership function of a subset* V *of* X *is defined, under an equivalence relation* $\mathcal{R}$ *on* X*, as* $\mu^V : X \to [0,1]$*, where*

$$\mu^V(\mathsf{y}) = \frac{|[\mathsf{y}]_\mathcal{R} \cap V|}{|[\mathsf{y}]_\mathcal{R}|}, \mathsf{y} \in X.$$

$[\mathsf{y}]_\mathcal{R}$ *denotes to an equivalence classes.*

Definition 12. *[23] The* λ*-rough membership functions of a subset* V *of* X *is given by* $\mu_V^\lambda \to [0,1]$*, where*

$$\mu_V^\lambda(\mathsf{y}) = \frac{|\{\cap \mathbb{G}_\lambda(\mathsf{y})\} \cap V|}{|\cap \mathbb{G}_\lambda(\mathsf{y})|}.$$

Definition 13. *[35] The* λ*-rough nearly membership function of a subset* V *of* X *is defined by* $\mu_V^{\eta_\lambda} \to [0,1]$ *as follows*

$$\mu_V^{\eta_\lambda}(\mathsf{y}) = \begin{cases} 1 & : \ 1 \in \psi_V^{\eta_\lambda}(\mathsf{y}) \\ min(\psi_V^{\eta_\lambda}(\mathsf{y})) & : \ otherwise \end{cases}$$

where $\psi_V^{\eta_\lambda}(\mathsf{y}) = \{\frac{|\eta_\lambda(\mathsf{y}) \cap V|}{|\eta_\lambda(\mathsf{y})|} : \mathsf{y} \in \eta_\lambda(\mathsf{y})$ *and* $\eta_\lambda(\mathsf{y}) \in \eta_{\lambda O}(X)\}, \eta \in \{\alpha, P, S, \beta\}$.

Definition 14. *[22, 23] The* $\mathcal{L} - \lambda$*-nearly rough membership functions of a subset* V *of* X *is defined by* $\mu_V^{\mathcal{L}-\eta_\lambda} \to [0,1]$*, as follows*

$$\mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = \begin{cases} 1 & : \ 1 \in \psi_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) \\ min(\psi_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}))) & : \ otherwise \end{cases}$$

where $\psi_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = \{\frac{|\mathcal{L}-\eta_\lambda(\mathsf{y}) \cap V|}{|\mathcal{L}-\eta_\lambda(\mathsf{y})|} : \mathsf{y} \in \mathcal{L} - \eta_\lambda(\mathsf{y})$ *and* $\mathcal{L} - \eta_\lambda(\mathsf{y}) \in \mathcal{L}\text{-}\eta_{\lambda O}(X)\}$.

Lemma 1. *[23] Let* V *be a subset of an* $\mathcal{L} - \mathbb{G}_\lambda$*-space* $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$*. Then*

(i) $\mu_V^\lambda(\mathsf{y}) = 1 \Rightarrow \mu_V^{\eta_\lambda}(\mathsf{y}) = 1 \Rightarrow \mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = 1, \forall\, \mathsf{y} \in X$.

(ii) $\mu_V^\lambda(\mathsf{y}) = 0 \Rightarrow \mu_V^{\eta_\lambda}(\mathsf{y}) = 0 \Rightarrow \mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = 0, \forall\, \mathsf{y} \in X$.

Definition 15. *[45] Let* $(X, \mathcal{R}, \Xi_\lambda)$ *be a* $\mathbb{G}_\lambda$*-space,* $\mathsf{y} \in X$ *and* $V \subseteq X$*:*

(i) *If* $\mathsf{y} \in \underline{\mathcal{R}}_\lambda(V)$*, then* y λ*-certainly belongs to* V*, denoted by* $\mathsf{y}\, \underline{\in}_\lambda V$.

(ii) *If* $\mathsf{y} \in \overline{\mathcal{R}}_\lambda(V)$*, then* y λ*-probably belongs to* V*, denoted by* $\mathsf{y}\, \overline{\in}_\lambda V$.

(iii) *If* $\mathsf{y} \in \underline{\mathcal{R}}_\lambda^\eta(V)$*, then* y λ*-nearly certainly (*η_λ*-certainly) belongs to* V*, denoted by* $\mathsf{y}\, \underline{\in}_\lambda^\eta V, \eta \in \{\alpha, P, S, \beta\}$.

(iv) *If* $\mathsf{y} \in \overline{\mathcal{R}}_{\lambda}^{\eta}(V)$, *then* y λ*-nearly probably (*η_{λ}*-probably) belongs to* V, *denoted by* $\mathsf{y} \,\overline{\in}_{\lambda}^{\eta} V, \eta \in \{\alpha, P, S, \beta\}$.

Definition 16. *[23] Let* $(X, \mathcal{R}, \Xi_{\lambda})$ *be a* $\mathbb{G}_{\lambda}$*-space,* $\mathsf{y} \in X$ *and* $V \subseteq X$*:*

(i) *If* $\mathsf{y} \in \underline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\eta}(V)$, *then* y *is* λ*-nearly certainly with respect to* $\mathcal{L}$ *(* $\mathcal{L} - \eta_{\lambda}$*-certainly) belongs to* V, *denoted by* $\mathsf{y} \,\underline{\in}_{\lambda}^{\mathcal{L}-\eta} A$.

(ii) *If* $\mathsf{y} \in \overline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\eta}(V)$, *then* y *is* λ*-nearly probably with respect to* $\mathcal{L}$ *(briefly* $\mathcal{L} - \eta_{\lambda}$*-probably) belongs to* V, *denoted by* $\mathsf{y} \,\overline{\in}_{\lambda}^{\mathcal{L}-\eta} A$.

Proposition 4. *[23] The subsequent properties hold true for each subset* V.

(i) *if* $\mathsf{y} \,\underline{\in}_{\lambda} A \Rightarrow \mathsf{y} \,\underline{\in}_{\lambda}^{\eta} A \Rightarrow \mathsf{y} \,\underline{\in}_{\lambda}^{\mathcal{L}-\eta} A$.

(ii) *if* $\mathsf{y} \,\overline{\in}_{\lambda}^{\mathcal{L}-\eta} A \Rightarrow \mathsf{y} \,\overline{\in}_{\lambda}^{\eta} A \Rightarrow \mathsf{y} \,\overline{\in}_{\lambda} A$.

3. $\mathcal{L}$-$\theta\beta_{\lambda}$-open sets

This section aims to adopt a fresh class of nearly open sets called $\mathcal{L}$-$\theta\beta_{\lambda}$-open sets, serving as an introduction to building rough set paradigms. This type of nearly open sets is established by replacing the empty difference of $\theta\beta$-open sets with the belonging of difference to the ideal, which enlarges the class of $\theta\beta$-open sets. We conclude the core characterizations of this class and elucidate its relationship with the forgoing classes.

Definition 17. *A subset* V *of an* $\mathcal{L}-\mathbb{G}_{\lambda}$*-space* $(X, \mathcal{R}, \Xi_{\lambda}, \mathcal{L})$ *is called* $\mathcal{L}$-$\theta\beta_{\lambda}$*-open providing that* $\exists\, G \in \vartheta_{\lambda}$ *s.t.* $(V - cl_{\lambda}(G)) \in \mathcal{L}$ *and* $(G - cl_{\lambda}^{\theta}(V)) \in \mathcal{L}$. *We call a complement of a* $\mathcal{L}$-$\theta\beta_{\lambda}$*-open set an* $\mathcal{L}$-$\theta\beta_{\lambda}$*-closed set. The classes of all* $\mathcal{L}$-$\theta\beta_{\lambda}$*-open and* $\mathcal{L}$-$\theta\beta_{\lambda}$*-closed are respectively symbolized by* $\mathcal{L}$-$\theta\beta_{\lambda O}(X)$ *and* $\mathcal{L}$-$\theta\beta_{\lambda C}(X)$.

Example 1. *Let*

$$X = \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4, \mathsf{y}_5\}, \mathcal{L} = \{\emptyset, \{\mathsf{y}_3\}\},$$

and

$$\mathcal{R} = \{(\mathsf{y}_1, \mathsf{y}_1), (\mathsf{y}_1, \mathsf{y}_2), (\mathsf{y}_2, \mathsf{y}_2), (\mathsf{y}_3, \mathsf{y}_3), (\mathsf{y}_3, \mathsf{y}_4), (\mathsf{y}_4, \mathsf{y}_3), (\mathsf{y}_4, \mathsf{y}_4), (\mathsf{y}_5, \mathsf{y}_2), (\mathsf{y}_5, \mathsf{y}_3), (\mathsf{y}_5, \mathsf{y}_4)\}.$$

Then, the topology generated by a relation $\mathcal{R}$ *in the case of* $\lambda = \mathbf{a}$ *is* $\vartheta_{\mathbf{a}} = \{X, \emptyset, \{\mathsf{y}_2\}, \{\mathsf{y}_1, \mathsf{y}_2\}, \{\mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}, \mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4, \mathsf{y}_5\}\}$ *and* $\mathcal{L}$-$\theta\beta_{\mathbf{a}}O(X)$ *is the power set of* X.

We demonstrate in the next result that the class of $\mathcal{L}$-$\theta\beta_{\lambda}$-open sets is wider than the classes of $\mathcal{L}$-$\delta\beta_{\lambda}$-open sets, $\mathcal{L}$-$\bigwedge_{\beta_{\lambda}}$-sets.

Proposition 5. (i) *Every* $\mathcal{L}$-$\delta\beta_{\lambda}$*-open set is* $\mathcal{L}$-$\theta\beta_{\lambda}$*-open set.*

(ii) *Every* $\mathcal{L}$-$\bigwedge_{\beta_{\lambda}}$*-set is* $\mathcal{L}$-$\theta\beta_{\lambda}$*-open set.*

Proof. It is evident by Definitions 8 [23] and 17.

Remark 1. *Example 1 yields an evidence that the converse of Proposition 5 fails. By this example, we remark that* $\mathcal{L}$-$\theta\beta_{\mathbf{a}}O(X) = P(X)$, $\mathcal{L}$-$\delta\beta_{\mathbf{a}}O(X) = P(X) - \{\{y_5\}\}$, *and* $\mathcal{L}$-$\bigwedge_{\beta_{\mathbf{a}}}O(X) = \{X, \emptyset, \{y_2\}, \{y_3\}, \{y_4\}, \{y_5\}, \{y_1, y_2\}, \{y_2, y_3\}, \{y_2, y_4\}, \{y_2, y_5\}, \{y_3, y_4\}$, $\{y_3, y_5\}$, $\{y_4, y_5\}$, $\{y, y_2, y_3\}$, $\{y_1, y_2, y_5\}, \{y_1, y_2, y_4\}, \{y_2, y_3, y_4\}, \{y_2, y_3, y_5\}, \{y_2, y_4, y_5\}, \{y_3, y_4, y_5\}$ $, \{y_1, y_2, y_3, y_4\}$, $\{y_1, y_2, y_3, y_5\}, \{y_1, y_2, y_4, y_5\}$,$\{y_2, y_3, y_4, y_5\}\}$. *Now,* $\{y_5\}$ *is an* $\mathcal{L}$-$\theta\beta_{\mathbf{a}}O(X)$*-open set, but it is neither an* $\mathcal{L}$-$\delta\beta_{\mathbf{a}}O(X)$*-open set nor an* $\mathcal{L}$-$\bigwedge_{\beta_{\mathbf{a}}}$*-set.*

Also, the next result clarifies that the class of $\mathcal{L}$-$\theta\beta_\lambda$-open sets is wider than the classes of $\delta\beta_\lambda$-open sets and $\bigwedge_{\beta_\lambda}$-sets.

Proposition 6. **(i)** *Every* $\delta\beta_\lambda$*-open set is* $\mathcal{L}$-$\theta\beta_\lambda$*-open set.*

(ii) *Every* $\bigwedge_{\beta_\lambda}$*-set is* $\mathcal{L}$-$\theta\beta_\lambda$*-open set.*

Proof. By using Propositions 1 [23] and 5.

Example 2. *Let*

$$X = \{y_1, y_2, y_3, y_4\}, \mathcal{L} = \{\emptyset, \{y_3\}\},$$

and

$$\mathcal{R} = \{(y_1, y_1), (y_1, y_2), (y_2, y_1), (y_2, y_2), (y_3, y_3), (y_4, y_3), (y_4, y_4))\}.$$

Then, the topology generated by a relation $\mathcal{R}$ *in the case of* $\lambda = \mathbf{a}$ *is* $\vartheta_{\mathbf{a}} = \{X, \emptyset, \{y_3\}, \{y_1, y_2\}, \{y_1, y_2, y_3\}\}$. *Now,* $\mathcal{L}$-$\theta\beta_{\mathbf{a}}O(X)$ *is the power set of* X *and* $\delta\beta_{\mathbf{a}}O(X)$ *is* $P(X) \setminus \{\{y_4\}\}$. *One can check that* $\{y_4\}$ *is an* $\mathcal{L}$-$\theta\beta_{\mathbf{a}}O(X)$*-open set, but it is not* $\delta\beta_{\mathbf{a}}O(X)$*-open.*

Example 3. *Let*

$$X = \{y_1, y_2, y_3, y_4\}, \mathcal{L} = \{\emptyset, \{y_3\}\}$$

and

$$\mathcal{R} = \{(y_1, y_1), (y_1, y_3), (y_2, y_1), (y_2, y_3), (y_3, y_3), (y_4, y_4)\}$$

Then, the topology generated by a relation $\mathcal{R}$ *in the case of* $\lambda = \mathbf{a}$ *is* $\vartheta_{\mathbf{a}} = \{X, \emptyset, \{y_3\}, \{y_4\}, \{y_1, y_3\}, \{y_3, y_4\}, \{y_1, y_3, y_4\}\}$. *Note that* $\mathcal{L}$-$\theta\beta_{\mathbf{a}}O(X) = P(X)$ *and* $\bigwedge_{\beta_{\mathbf{a}}}O(X) = \{X, \emptyset, \{y_2\}, \{y_3\}, \{y_4\}, \{y_1, y_3\}, \{y_2, y_3\}, \{y_2, y_4\}, \{y_3, y_4\}, \{y_1, y_2, y_3\}, \{y_1, y_3, y_4\}, \{y_2, y_3, y_4\}\}$.

The next proposition elucidates that the the class of $\mathcal{L}$-$\theta\beta_\lambda$-open sets is proper wider than the class of $\mathcal{L}$-$\delta\beta_\lambda$-open sets. Therefore, the class of $\mathcal{L}$-$\theta\beta_\lambda$-open sets is also wider than the classes of all $\mathcal{L}$-λ-near open sets introduced in Definition 8 [21], i.e., $\mathcal{L}$-β_λ-open, $\mathcal{L}$-P_λ-open, $\mathcal{L}$-S_λ-open and $\mathcal{L}$-α_λ-open sets. Moreover, it is wider than the classes of $\delta\beta_\lambda$-open sets. Hence, it is also wider than all classes of λ-near open sets introduced in Definition 5 [15], i.e., β_λ-open, P_λ-open, S_λ-open and α_λ-open sets.

Proposition 7. *The next implications hold true:*

$$\begin{array}{ccccc} \vartheta_\lambda(\Gamma_\lambda) \Rightarrow \mathcal{L}\text{-}\alpha_{\lambda O}(\mathcal{L}\text{-}\alpha_{\lambda C}) & & \mathcal{L}\text{-}P_{\lambda O}(\mathcal{L}\text{-}P_{\lambda C}) & & \\ \Downarrow & & \Downarrow & & \\ \mathcal{L}\text{-}S_{\lambda O}(\mathcal{L}\text{-}S_{\lambda C}) & \Rightarrow & \mathcal{L}\text{-}\beta_{\lambda O}(\mathcal{L}\text{-}\beta_{\lambda C}) & \Rightarrow & \mathcal{L}\text{-}\delta\beta_{\lambda O}(\mathcal{L}\text{-}\delta\beta_{\lambda C}) \\ & & & & \Downarrow \\ & & & & \mathcal{L}\text{-}\theta\beta_{\lambda O}(\mathcal{L}\text{-}\theta\beta_{\lambda C}). \end{array}$$

$$\begin{array}{ccccc} \vartheta_\lambda(\Gamma_\lambda) \Rightarrow \mathcal{L}\text{-}\alpha_{\lambda O}(\mathcal{L}\text{-}\alpha_{\lambda C}) & & \mathcal{L}\text{-}P_{\lambda O}(\mathcal{L}\text{-}P_{\lambda C}) & & \\ \Downarrow & & \Downarrow & & \\ \mathcal{L}\text{-}S_{\lambda O}(\mathcal{L}\text{-}S_{\lambda C}) & \Rightarrow & \mathcal{L}\text{-}\beta_{\lambda O}(\mathcal{L}\text{-}\beta_{\lambda C}) & \Rightarrow & \mathcal{L}\text{-}\bigwedge_{\beta_{\lambda O}}(\mathcal{L}\text{-}\bigwedge_{\beta_{\lambda C}}) \\ & & & & \Downarrow \\ & & & & \mathcal{L}\text{-}\theta\beta_{\lambda O}(\mathcal{L}\text{-}\theta\beta_{\lambda C}). \end{array}$$

$$\begin{array}{ccccc} \vartheta_\lambda(\Gamma_\lambda) \Rightarrow \alpha_{\lambda O}(\alpha_{\lambda C}) & & P_{\lambda O}(P_{\lambda C}) & & \\ \Downarrow & & \Downarrow & & \\ S_{\lambda O}(S_{\lambda C}) & \Rightarrow & \beta_{\lambda O}(\beta_{\lambda C}) & \Rightarrow & \delta\beta_{\lambda O}(\delta\beta_{\lambda C}) \\ & & & & \Downarrow \\ & & & & \mathcal{L}\text{-}\theta\beta_{\lambda O}(\mathcal{L}\text{-}\theta\beta_{\lambda C}). \end{array}$$

$$\begin{array}{ccccc} \vartheta_\lambda(\Gamma_\lambda) \Rightarrow \alpha_{\lambda O}(\alpha_{\lambda C}) & & P_{\lambda O}(P_{\lambda C}) & & \\ \Downarrow & & \Downarrow & & \\ S_{\lambda O}(S_{\lambda C}) & \Rightarrow & \beta_{\lambda O}(\beta_{\lambda C}) & \Rightarrow & \bigwedge_{\beta_{\lambda O}}(\bigwedge_{\beta_{\lambda C}}) \\ & & & & \Downarrow \\ & & & & \mathcal{L}\text{-}\theta\beta_{\lambda O}(\mathcal{L}\text{-}\theta\beta_{\lambda C}). \end{array}$$

Proof. By Propositions 2 [23], 5, 6 the proof is obvious.

Theorem 3. *The union of two $\mathcal{L}$-$\theta\beta_\lambda$-open subsets is $\mathcal{L}$-$\theta\beta_\lambda$-open. That is, the family of $\mathcal{L}$-$\theta\beta_\lambda$-open subsets is closed under finite union.*

Proof. Take arbitrary two $\mathcal{L}$-$\theta\beta_\lambda$-open subsets V and W. Then, there are open sets G and H s.t. the four sets $(V \setminus cl_\lambda(G)), (G \setminus cl^\theta_\lambda(V)), (W \setminus cl_\lambda(H))$ and $(W - cl^\theta_\lambda(B))$ belong to $\mathcal{L}$. Since $(G \setminus cl^\theta_\lambda(V \cup W)) \subseteq (G \setminus cl^\theta_\lambda(V)) \in \mathcal{L}$, $(H \setminus cl^\theta_\lambda(V \cup W)) \subseteq (H \setminus cl^\theta_\lambda(W)) \in \mathcal{L}$, we have $(G \setminus cl^\theta_\lambda(V \cup W)) \cup (H \setminus cl^\theta_\lambda(V \cup W)) \in \mathcal{L}$. Let $Z = G \cup H$, then $(Z - cl^\theta_\lambda(V \cup W)) \in \mathcal{L}$. Also, $(V \setminus cl_\lambda(Z)) \subseteq (V \setminus cl_\lambda(G)) \in \mathcal{L}$ and $(W \setminus cl_\lambda(Z)) \subseteq (W \setminus cl_\lambda(H)) \in \mathcal{L}$. Then, $(V \setminus cl_\lambda(Z)) \cup (W \setminus cl_\lambda(Z)) \subseteq (V \setminus cl_\lambda(G)) \cup (W \setminus cl_\lambda(H)) \in \mathcal{L}$ and so $((V \cup W) \setminus cl_\lambda(Z)) \subseteq (V \setminus cl_\lambda(G)) \cup (W \setminus cl_\lambda(H)) \in \mathcal{L}$. Hence, $V \cup W$ is an $\mathcal{L}$-$\theta\beta_\lambda$-open subset.

In Algorithm 1, we present the steps to calculate the family of $\mathcal{L}$-$\theta\beta_\lambda$-open subsets.

Input : The universal set X, a relation $\mathcal{R}$, and an ideal $\mathcal{L}$ under consideration.
Output: The family of $\mathcal{L}$-$\theta\beta_\lambda$-open subsets.

Ask the the expert(s) to give a relation $\mathfrak{L}$ over X;
Choose a λ type;
for *every* $y \in X$ **do**
  compute $\mathbb{G}_\lambda(y)$
end
Construct a topology $\vartheta_\lambda = \{V \subseteq X : \forall y \in V, \mathbb{G}_\lambda(y) \subseteq V\}$ on X;
Initiate $\mathcal{C}_1 = \{cl_\lambda(V) : V \in \vartheta_\lambda\}$;
Construct a θ-topology $\vartheta^\theta_\lambda = \{V \in \vartheta_\lambda : int^\theta_\lambda(V) = V\}$;
Define $\mathcal{F} = P(X) \setminus \vartheta_\lambda$;
Initiate $\mathcal{C}_2 = \{cl^\theta_\lambda(W) : W \in \mathcal{F}\}$;
Ask the the expert(s) to give an ideal $\mathcal{L}$ over X;
for *every* $W \in \mathcal{F}$ **do**
  if $\exists\ V \in \vartheta_\lambda$ *s.t.* $(W \setminus cl_\lambda(V)) \in \mathcal{L}$ *and* $(V \setminus cl^\theta_\lambda(W)) \in \mathcal{L}$ **then**
    W is an $\mathcal{L}$-$\theta\beta_\lambda$-open set;
    $W \in \mathcal{F}^*$
  else
    W is not an $\mathcal{L}$-$\theta\beta_\lambda$-open set
  end
end
$\mathcal{L}$-$\theta\beta_{\lambda O}(X) = \vartheta_\lambda \cup \mathcal{F}^*$.

Algorithm 1: Determination of the family of $\mathcal{L}$-$\theta\beta_\lambda$-open subsets

4. Approximations spaces by using $\mathcal{L}$-$\theta\beta_\lambda$-open sets

Herein, we establish novel rough paradigms inspired by the family of $\mathcal{L}$-$\theta\beta_\lambda$-open sets. We focus on the role of the proposed rough paradigms in developing decision-making methods through the preservation of most properties of the standard model given by Pawlak and heighten the accuracy measures of extracted knowledge compared to paradigms studied in the literature. Additionally, we make comparisons between the proposed models for all cases of λ with the assistance of counterexamples.

Definition 18. *Let V be a subset of an $\mathcal{L}-\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$. We respectively define the $\mathcal{L}$-$\theta\beta_\lambda$-lower, $\mathcal{L}$-$\theta\beta_\lambda$-upper approximations, $\mathcal{L}$-$\theta\beta_\lambda$-boundary regions and $\mathcal{L}$-$\theta\beta_\lambda$-accuracy of V as follows:*

$\underline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V) = \cup\{G \in \mathcal{L}\text{-}\theta\beta_{\lambda O}(X) : G \subseteq A\} = \mathcal{L}\text{-}\theta\beta_\lambda\text{-interior of } V.$

$\overline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V) = \cap\{H \in \mathcal{L}\text{-}\theta\beta_{\lambda C}(X) : V \subseteq H\} = \mathcal{L}\text{-}\theta\beta_\lambda\text{-closure of } V.$

$\mathcal{BND}^{\mathcal{L}-\theta\beta}_\lambda(V) = \overline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V) - \underline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V).$

$ACC^{\mathcal{L}-\theta\beta}_\lambda(V) = \frac{|\underline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V)|}{|\overline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V)|}$, *where* $|\overline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V)| \neq 0.$

Proposition 8. *Let V, W be subsets of an $\mathcal{L}-\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$. Then,*

(i) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq V \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$ *equality hold if* $V = \emptyset$ *or* X.

(ii) $V \subseteq W \Rightarrow \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(W)$.

(iii) $V \subseteq W \Rightarrow \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(W)$.

(iv) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V \cap W) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \cap \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(W)$.

(v) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V \cup W) \supseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \cup \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(W)$.

(vi) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V \cup W) \supseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \cup \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(W)$.

(vii) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V \cap W) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \cap \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(W)$.

(viii) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) = (\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V'))'$, $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) = (\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V'))'$.

(ix) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)) = \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(x) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)) = \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(xi) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V))$.

(xii) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V))$.

Proof. The proof is warranted by using the properties of $\mathcal{L}$-$\theta\beta_\lambda$-interior and $\mathcal{L}$-$\theta\beta_\lambda$-closure operators.

Definition 19. *A subset V of an $\mathcal{L}-\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$ is named an $\mathcal{L}$-$\theta\beta_\lambda$-definable (an $\mathcal{L}$-$\theta\beta_\lambda$-exact) set if $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) = \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$. Otherwise, V is an $\mathcal{L}$-$\theta\beta_\lambda$-rough set.*

In Example 1 $V = \{\mathsf{y}_3\}$ is $\mathcal{L}$-$\theta\beta_{\mathbf{a}}$-exact.

To articulate the relationships between the present rough paradigms (Definition 18) and those given in Definition 9 [22, 23], we provide the next two results.

Theorem 4. *Let V be a subset of an $\mathcal{L}-\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$. Then:*

(i) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-P}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(ii) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-S}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(iii) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-P}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\Lambda_{\beta\lambda}}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\theta_{\beta\lambda}}(V)$.

(iv) $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-S}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\Lambda_{\beta\lambda}}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\theta_{\beta\lambda}}(V)$.

(v) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-P}(V)$.

(vi) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-S}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha}(V)$.

(vii) $\overline{\mathcal{R}}^{\mathcal{L}-\theta_{\beta_\lambda}}(V) \subseteq \overline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-P}(V)$.

(viii) $\overline{\mathcal{R}}^{\mathcal{L}-\theta_{\beta_\lambda}}(V) \subseteq \overline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-S}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\alpha}(V)$.

Proof. It is warranted by Proposition 3.

Corollary 1. *Let V be a subset of an $\mathcal{L}-\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$. Then:*

(i) $\mathcal{BND}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-P}(V)$.

(ii) $\mathcal{BND}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-\delta\beta}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-S}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-\alpha}(V)$.

(iii) $\mathcal{BND}^{\mathcal{L}-\theta_{\beta_\lambda}}(V) \subseteq \mathcal{BND}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-P}(V)$.

(iv) $\mathcal{BND}^{\mathcal{L}-\theta_{\beta_\lambda}}(V) \subseteq \mathcal{BND}^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-\beta}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-S}(V) \subseteq \mathcal{BND}_\lambda^{\mathcal{L}-\alpha}(V)$.

(v) $ACC_\lambda^{\mathcal{L}-P}(V) \leqslant ACC_\lambda^{\mathcal{L}-\beta}(V) \leqslant ACC_\lambda^{\mathcal{L}-\delta\beta}(V) \leqslant ACC_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(vi) $ACC_\lambda^{\mathcal{L}-\alpha}(V) \leqslant ACC_\lambda^{\mathcal{L}-S}(V) \leqslant ACC_\lambda^{\mathcal{L}-\beta}(V) \leqslant ACC_\lambda^{\mathcal{L}-\delta\beta}(V) \leqslant ACC_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(vii) $ACC_\lambda^{\mathcal{L}-P}(V) \leqslant ACC_\lambda^{\mathcal{L}-\beta}(V) \leqslant ACC^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V) \leqslant ACC^{\mathcal{L}-\theta_{\beta_\lambda}}(V)$.

(viii) $ACC_\lambda^{\mathcal{L}-\alpha}(V) \leqslant ACC_\lambda^{\mathcal{L}-S}(V) \leqslant ACC_\lambda^{\mathcal{L}-\beta}(V) \leqslant ACC^{\mathcal{L}-\bigwedge_{\beta_\lambda}}(V) \leqslant ACC^{\mathcal{L}-\theta_{\beta_\lambda}}(V)$.

Remark 2. *By Example 1, we will illustrate that the converse of the implications in Theorem 4 and Corollary 1 is not always true as follows.*

(i) *If* $V = \{\mathsf{y}_5\}$, *then* $\underline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = A, \overline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = A$, $\mathcal{BND}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = \emptyset, ACC_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = 1$, *and* $\underline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\delta\beta}(V) = \emptyset, \overline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\delta\beta}(V) = A$, $\mathcal{BND}_{\mathbf{a}}^{\mathcal{L}-\delta\beta}(V) = A, ACC_{\mathbf{a}}^{\mathcal{L}-\delta\beta}(V) = 0$.

(ii) *If* $V = \{\mathsf{y}\}$, *then* $\underline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = A, \overline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = A, \mathcal{BND}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = \emptyset, ACC_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = 1$, *and* $\underline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}}(V) = \emptyset, \overline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}}(V) = A, \mathcal{BND}^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}}(V) = A, ACC^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}}(V) = 0$.

Corollary 2. *Let V be a subset of an $\mathcal{L}-\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$. Then:*

(i) *V is $\mathcal{L}$-α_λ-exact $\Rightarrow$ V is $\mathcal{L}$-S_λ-exact $\Rightarrow$ V is $\mathcal{L}$-β_λ-exact $\Rightarrow$ V is $\mathcal{L}$-$\delta\beta_\lambda$-exact $\Rightarrow$ V is $\mathcal{L}$-$\theta\beta_\lambda$-exact.*

(ii) *V is $\mathcal{L}$-P_λ-exact $\Rightarrow$ V is $\mathcal{L}$-β_λ-exact $\Rightarrow$ V is $\mathcal{L}$-$\delta\beta_\lambda$-exact $\Rightarrow$ V is $\mathcal{L}$-$\theta\beta_\lambda$-exact.*

(iii) *V is λ-exact $\Rightarrow$ V is $\mathcal{L}$-α_λ-exact $\Rightarrow$ V is $\mathcal{L}$-S_λ-exact $\Rightarrow$ V is $\mathcal{L}$-β_λ-exact $\Rightarrow$ V is $\mathcal{L}$-$\bigwedge_{\beta_\lambda}$-exact $\Rightarrow$ V is $\mathcal{L}$-θ_{β_λ}-exact.*

(iv) *V is $\mathcal{L}$-P_λ-exact $\Rightarrow$ V is $\mathcal{L}$-β_λ-exact $\Rightarrow$ V is $\mathcal{L}$-$\bigwedge_{\beta_\lambda}$-exact $\Rightarrow$ V is $\mathcal{L}$-θ_{β_λ}-exact.*

(v) *V is $\mathcal{L}$-$\theta\beta_\lambda$-rough $\Rightarrow$ V is $\mathcal{L}$-$\delta\beta_\lambda$-rough $\Rightarrow$ V is $\mathcal{L}$-β_λ-rough $\Rightarrow$ V is $\mathcal{L}$-S_λ-rough $\Rightarrow$ V is $\mathcal{L}$-α_λ-rough.*

(vi) *V is $\mathcal{L}$-$\theta\beta_\lambda$-rough $\Rightarrow$ V is $\mathcal{L}$-$\delta\beta_\lambda$-rough $\Rightarrow$ V is $\mathcal{L}$-β_λ-rough $\Rightarrow$ V is $\mathcal{L}$-P_λ-rough.*

(vii) *V is $\mathcal{L}$-θ_{β_λ}-rough $\Rightarrow$ V is $\mathcal{L}$-$\bigwedge_{\beta_\lambda}$-rough $\Rightarrow$ V is $\mathcal{L}$-β_λ-rough $\Rightarrow$ V is $\mathcal{L}$-S_λ-rough $\Rightarrow$ V is $\mathcal{L}$-α_λ-rough.*

(vii) *V is $\mathcal{L}$-θ_{β_λ}-rough $\Rightarrow$ V is $\mathcal{L}$-$\bigwedge_{\beta_\lambda}$-rough $\Rightarrow$ V is $\mathcal{L}$-β_λ-rough $\Rightarrow$ V is $\mathcal{L}$-P_λ-rough.*

Remark 3. *By Example 1, we will illustrate that the converse of the implications in Corollary 2 fails.*

(i) If $V = \{\mathsf{y}_5\}$, then it is $\mathcal{L}$-$\theta\beta_{\mathbf{a}}$-exact, but it is not $\mathcal{L}$-$\delta\beta_{\mathbf{a}}$-exact and consequently, not $\mathcal{L}$-$\beta_{\mathbf{a}}$-exact, not $\mathcal{L}$-$S_{\mathbf{a}}$-exact, not $\mathcal{L}$-$\alpha_{\mathbf{a}}$-exact and not $\mathcal{L}$-$P_{\mathbf{a}}$-exact.

(ii) If $V = \{\mathsf{y}\}$, then it is $\mathcal{L}$-$\theta\beta_{\mathbf{a}}$-exact, but it is not $\mathcal{L}$-$\bigwedge\beta_{\mathbf{a}}$-exact and consequently, not $\mathcal{L}$-$\beta_{\mathbf{a}}$-exact, not $\mathcal{L}$-$S_{\mathbf{a}}$-exact, not $\mathcal{L}$-$\alpha_{\mathbf{a}}$-exact and not $\mathcal{L}$-$P_{\mathbf{a}}$-exact.

We elucidate the interrelations between the present rough paradigms (Definition 18) and the those displayed in Definition 4 [45] and Definition 6 [15, 20].

Theorem 5. *Let V be a subset of an $\mathcal{L}-\mathbb{G}_\lambda$-space $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$. Then:*

(i) $\underline{\mathcal{R}}_\lambda^\alpha(V) \subseteq \underline{\mathcal{R}}_\lambda^p(V) \subseteq \underline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \underline{\mathcal{R}}_\lambda^\beta(V) \subseteq \underline{\mathcal{R}}_\lambda^{\delta\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(ii) $\underline{\mathcal{R}}_\lambda^\alpha(V) \subseteq \underline{\mathcal{R}}_\lambda^s(V) \subseteq \underline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \underline{\mathcal{R}}_\lambda^\beta(V) \subseteq \underline{\mathcal{R}}_\lambda^{\delta\beta}(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(iii) $\underline{\mathcal{R}}_\lambda^\alpha(V) \subseteq \underline{\mathcal{R}}_\lambda^p(V) \subseteq \underline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \underline{\mathcal{R}}_\lambda^\beta(V) \subseteq \underline{\mathcal{R}}_\lambda^{\bigwedge\beta}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\theta_{\beta_\lambda}}(V)$.

(iv) $\underline{\mathcal{R}}_\lambda^\alpha(V) \subseteq \underline{\mathcal{R}}_\lambda^s(V) \subseteq \underline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \underline{\mathcal{R}}_\lambda^\beta(V) \subseteq \underline{\mathcal{R}}^{\bigwedge_{\beta_\lambda}}(V) \subseteq \underline{\mathcal{R}}^{\mathcal{L}-\theta_{\beta_\lambda}}(V)$.

(v) $\underline{\mathcal{R}}_\lambda(V) \subseteq \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$.

(vi) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^\beta(V) \subseteq \overline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \overline{\mathcal{R}}_\lambda^p(V) \subseteq \overline{\mathcal{R}}_\lambda^\alpha(V)$.

(vii) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\delta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^\beta(V) \subseteq \overline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \overline{\mathcal{R}}_\lambda^s(V) \subseteq \overline{\mathcal{R}}_\lambda^\alpha(V)$.

(viii) $\overline{\mathcal{R}}^{\mathcal{L}-\theta_{\beta_\lambda}}(V) \subseteq \overline{\mathcal{R}}_\lambda^{\bigwedge\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^\beta(V) \subseteq \overline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \overline{\mathcal{R}}_\lambda^p(V) \subseteq \overline{\mathcal{R}}_\lambda^\alpha(V)$.

(ix) $\overline{\mathcal{R}}^{\mathcal{L}-\theta_{\beta_\lambda}}(V)) \subseteq \overline{\mathcal{R}}_\lambda^{\bigwedge\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda^\beta(V) \subseteq \overline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \overline{\mathcal{R}}_\lambda^s(V) \subseteq \overline{\mathcal{R}}_\lambda^\alpha(V)$.

(x) $\overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \subseteq \overline{\mathcal{R}}_\lambda(V)$.

Proof.

(i) By Theorem 2 [23], $\underline{\mathcal{R}}_\lambda^\alpha(V) \subseteq \underline{\mathcal{R}}_\lambda^p(V) \subseteq \underline{\mathcal{R}}_\lambda^\gamma(V) \subseteq \underline{\mathcal{R}}_\lambda^\beta(V) \subseteq \underline{\mathcal{R}}_\lambda^{\delta\beta}(V)$ and $\underline{\mathcal{R}}_\lambda^{\delta\beta}(V)) = \cup\{G \in \delta\beta_{\lambda O}(X) : G \subseteq A\} \subseteq \cup\{G \in \mathcal{L}\text{-}\theta\beta_{\lambda O}(X) : G \subseteq A\} = \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$ (by Proposition 6).

(ii)–(iv) It is similar to (i).

(v) By Theorem 2 [23], $\underline{\mathcal{R}}_{\lambda}(V) \subseteq \underline{\mathcal{R}}^{\delta\beta_{\lambda}}(V)$, and by (1) $\underline{\mathcal{R}}^{\delta\beta_{\lambda}}(V) \subseteq \underline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(V)$. Hence, $\underline{\mathcal{R}}_{\lambda}(V) \subseteq \underline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(V)$.

(vi)–(x) They are similar to (i)–(v).

The subsequent corollary points out that the greater the size of the boundary region, the lower the accuracy measures.

Corollary 3. *If V is a subset of an $\mathcal{L}-\mathbb{G}_{\lambda}$-space $(X, \mathcal{R}, \Xi_{\lambda}, \mathcal{L})$, then the next properties are satisfied.*

(i) $\mathcal{BND}^{\mathcal{L}-\theta\beta_{\lambda}}(V) \subseteq \mathcal{BND}_{\lambda}^{\delta\beta}(V) \subseteq \mathcal{BND}_{\lambda}^{\beta}(V) \subseteq \mathcal{BND}_{\lambda}^{\gamma}(V) \subseteq \mathcal{BND}_{\lambda}^{p}(V) \subseteq \mathcal{BND}_{\lambda}^{\alpha}(V)$.

(ii) $\mathcal{BND}^{\mathcal{L}-\theta\beta_{\lambda}}(V) \subseteq \mathcal{BND}_{\lambda}^{\delta\beta}(V) \subseteq \mathcal{BND}_{\lambda}^{\beta}(V) \subseteq \mathcal{BND}_{\lambda}^{\gamma}(V) \subseteq \mathcal{BND}_{\lambda}^{s}(V) \subseteq \mathcal{BND}_{\lambda}^{\alpha}(V)$.

(iii) $\mathcal{BND}^{\mathcal{L}-\theta_{\beta_{\lambda}}}(V) \subseteq \mathcal{BND}_{\lambda}^{\bigwedge_{\beta}}(V) \subseteq \mathcal{BND}_{\lambda}^{\beta}(V) \subseteq \mathcal{BND}_{\lambda}^{\gamma}(V) \subseteq \mathcal{BND}_{\lambda}^{p}(V) \subseteq \mathcal{BND}_{\lambda}^{\alpha}(V)$.

(iv) $\mathcal{BND}^{\mathcal{L}-\theta_{\beta_{\lambda}}}(V) \subseteq \mathcal{BND}_{\lambda}^{\bigwedge_{\beta}}(V) \subseteq \mathcal{BND}_{\lambda}^{\beta}(V) \subseteq \mathcal{BND}_{\lambda}^{\gamma}(V) \subseteq \mathcal{BND}_{\lambda}^{s}(V) \subseteq \mathcal{BND}_{\lambda}^{\alpha}(V)$.

(v) $\mathcal{BND}_{\lambda}^{\mathcal{L}-\theta\beta}(V) \subseteq \mathcal{BND}_{\lambda}(V)$.

(vi) $ACC_{\lambda}^{\alpha}(V) \leqslant ACC_{\lambda}^{p}(V) \leqslant ACC_{\lambda}^{\gamma}(V) \leqslant ACC_{\lambda}^{\beta}(V) \leqslant ACC_{\lambda}^{\delta\beta}(V) \leqslant ACC_{\lambda}^{\mathcal{L}-\theta\beta}(V)$.

(vii) $ACC_{\lambda}^{\alpha}(V) \leqslant ACC_{\lambda}^{s}(V) \leqslant ACC_{\lambda}^{\gamma}(V) \leqslant ACC_{\lambda}^{\beta}(V) \leqslant ACC_{\lambda}^{\delta\beta}(V) \leqslant ACC_{\lambda}^{\mathcal{L}-\theta\beta}(V)$.

(viii) $ACC_{\lambda}^{\alpha}(V) \leqslant ACC_{\lambda}^{p}(V) \leqslant ACC_{\lambda}^{\gamma}(V) \leqslant ACC_{\lambda}^{\beta}(V) \leqslant ACC_{\lambda}^{\bigwedge_{\beta}}(V) \leqslant ACC_{\lambda}^{\mathcal{L}-\theta_{\beta}}(V)$.

(ix) $ACC_{\lambda}^{\alpha}(V) \leqslant ACC_{\lambda}^{s}(V) \leqslant ACC_{\lambda}^{\gamma}(V) \leqslant ACC_{\lambda}^{\beta}(V) \leqslant ACC_{\lambda}^{\bigwedge_{\beta}}(V) \leqslant ACC_{\lambda}^{\mathcal{L}-\theta_{\beta}}(V)$.

(x) $ACC_{\lambda}(V) \leqslant ACC_{\lambda}^{\mathcal{L}-\theta\beta}(V)$.

Remark 4. *The converse of the implications in Theorem 5 and Corollary 3 is not true in general as shown in*

(i) Example 2, if $V = \{y_4\}$, *then* $\underline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = A, \overline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = A$, $\mathcal{BND}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = \emptyset, ACC^{\mathcal{L}-\theta\beta}{}_{\mathbf{a}}(V) = 1$, *and* $\underline{\mathcal{R}}^{\delta\beta}{}_{\mathbf{a}}(V) = \emptyset$, $\overline{\mathcal{R}}^{\delta\beta}{}_{\mathbf{a}}(V) = A$ *and* $\mathcal{BND}^{\delta\beta}{}_{\mathbf{a}}(V) = A, ACC^{\delta\beta}{}_{\mathbf{a}}(V) = 0$.

(ii) Example 3, if $V = \{y\}$, *then* $\underline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = A, \overline{\mathcal{R}}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = A, \mathcal{BND}_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = \emptyset, ACC_{\mathbf{a}}^{\mathcal{L}-\theta\beta}(V) = 1$, *and* $\underline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}}(V) = \emptyset, \overline{\mathcal{R}}^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}}(V) = A, \mathcal{BND}^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}}(V) = A$, $ACC^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}}(V) = 0$.

Corollary 4. *For a subset V of an $\mathcal{L}-\mathbb{G}_{\lambda}$-space $(X, \mathcal{R}, \Xi_{\lambda}, \mathcal{L})$, we have the next results.*

(i) *V is α_λ-exact $\Rightarrow$ V is S_λ-exact $\Rightarrow$ V is β_λ-exact $\Rightarrow$ $\delta\beta_\lambda$-exact $\Rightarrow$ V is $\mathcal{L}$-$\theta\beta_\lambda$-exact.*

(ii) *V is P_λ-exact $\Rightarrow$ V is β_λ-exact $\Rightarrow$ V is $\delta\beta_\lambda$-exact $\Rightarrow$ V is $\mathcal{L}$-$\theta\beta_\lambda$-exact.*

(iii) *V is α_λ-exact $\Rightarrow$ V is S_λ-exact $\Rightarrow$ V is β_λ-exact $\Rightarrow$ V is $\bigwedge_{\beta_\lambda}$-exact $\Rightarrow$ V is $\mathcal{L}$-θ_{β_λ}-exact.*

(iv) *V is P_λ-exact $\Rightarrow$ V is β_λ-exact $\Rightarrow$ V is $\bigwedge_{\beta_\lambda}$-exact $\Rightarrow$ V is $\mathcal{L}$-θ_{β_λ}-exact.*

(v) *V is λ-exact $\Rightarrow$ V is $\mathcal{L}$-$\theta\beta_\lambda$-exact.*

(vi) *V is $\mathcal{L}$-$\theta\beta_\lambda$-rough $\Rightarrow$ V is $\delta\beta_\lambda$-rough $\Rightarrow$ V is β_λ-rough $\Rightarrow$ V is S_λ-rough $\Rightarrow$ V is α_λ-rough.*

(vii) *V is $\mathcal{L}$-$\theta\beta_\lambda$-rough $\Rightarrow$ V is $\delta\beta_\lambda$-rough $\Rightarrow$ V is β_λ-rough $\Rightarrow$ V is P_λ-rough.*

(viii) *V is $\mathcal{L}$-θ_{β_λ}-rough $\Rightarrow$ V is $\bigwedge_{\beta_\lambda}$-rough $\Rightarrow$ V is β_λ-rough $\Rightarrow$ V is S_λ-rough $\Rightarrow$ V is α_λ-rough.*

(ix) *V is $\mathcal{L}$-θ_{β_λ}-rough $\Rightarrow$ V is $\bigwedge_{\beta_\lambda}$-rough $\Rightarrow$ V is β_λ-rough $\Rightarrow$ V is P_λ-rough.*

(x) *V is $\mathcal{L}$-$\theta\beta_\lambda$-rough $\Rightarrow$ V is λ-rough.*

Remark 5. *The converse of Corollary 4 is wrong in general. We demonstrate this claim in the following.*

(i) Example 2, if $V = \{\mathsf{y}_4\}$, then it is $\mathcal{L}$-$\theta\beta_{\mathbf{a}}$-exact, but it is neither $\delta\beta_{\mathbf{a}}$-exact nor R-exact.

(ii) Example 3, if $V = \{\mathsf{y}\}$, then it is $\mathcal{L}$-$\theta\beta_{\mathbf{a}}$-exact, but it is neither $\bigwedge_{\beta_{\mathbf{a}}}$-exact nor **a**-*exact.*

Remark 6. *We can say that the present rough set models (Definition 18), with the comparison of Abd El-Monsef et al.'s method 4 [45], Amer et al.'s method [15] and Hosny's method 6 [20] and Hosny's method 9 [22, 23], enlarge the confirmed knowledge by maximizing the $\mathcal{L}$-$\theta\beta_\lambda$-lower approximations and minimizing the $\mathcal{L}$-$\theta\beta_\lambda$-upper approximations as illustrated in Theorems 4 and 5. That is, the present approach successfully shrinks the boundary region, which refer to size of ambiguity. Furthemore, Corollaries 3 and 1 confirm that the our accuracy introduced in Definition 18 is greater than the previous ones in Definitions 4 [15], 6 [15, 20] and 9 [22, 23].*

In Algorithm 2, we present the steps to calculate a subset's boundary region and accuracy measure and determine whether an $\mathcal{L}$-$\theta\beta_\lambda$-definable set or an $\mathcal{L}$-$\theta\beta_\lambda$-rough set.

Input : The universal set X, a relation $\mathcal{R}$, and an ideal $\mathcal{L}$ under consideration.
Output: Boundary region $\mathcal{BND}_{\lambda}^{\mathcal{L}-\theta\beta}$ and accuracy measure $ACC_{\lambda}^{\mathcal{L}-\theta\beta}$ of a subset.

Carry out steps 1–20 given in Algorithm 1;
Build $\mathcal{L}\text{-}\theta\beta_{\lambda C}(X) = \{H \subseteq X : H^c \in \mathcal{L}\text{-}\theta\beta_{\lambda O}(X)\}$;
for *a subset* $E \subseteq X$ **do**
  compute $\underline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(E) = \cup\{G \in \mathcal{L}\text{-}\theta\beta_{\lambda O}(X) : G \subseteq E\}$;
  compute $\overline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(E) = \cap\{H \in \mathcal{L}\text{-}\theta\beta_{\lambda C}(X) : E \subseteq H\}$;
  compute $\mathcal{BND}_{\lambda}^{\mathcal{L}-\theta\beta}(E) = \overline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(E) - \underline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(E)$;
  compute $ACC_{\lambda}^{\mathcal{L}-\theta\beta}(E) = \frac{|\underline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(E)|}{|\overline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(E)|}$
end
Print $\mathcal{BND}_{\lambda}^{\mathcal{L}-\theta\beta}(E)$;
Print $ACC_{\lambda}^{\mathcal{L}-\theta\beta}(E)$;
if $ACC_{\lambda}^{\mathcal{L}-\theta\beta}(E) = 1$ **then**
  Print E is an $\mathcal{L}\text{-}\theta\beta_{\lambda}$-definable set
else
  Print E is an $\mathcal{L}\text{-}\theta\beta_{\lambda}$-rough set
end

Algorithm 2: Calculate the boundary region and accuracy measure of a subset

5. $\mathcal{L}$-$\theta\beta_{\lambda}$-rough membership functions

In this segment, we introduce the notion of $\mathcal{L}$-$\theta\beta_{\lambda}$-rough membership functions as a generalization of classical rough membership functions. We exploit this notion to describe the approximation operators given in the preceding section.

Definition 20. *Let* $(X, \mathcal{R}, \Xi_{\lambda}, \mathcal{L})$ *be an* $\mathcal{L} - \mathbb{G}_{\lambda}$*-space,* $\mathsf{y} \in X$*, and* $V \subseteq X$*.*

(i) *if* $\mathsf{y} \in \underline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(V)$*, then* y *is* λ*-*$\theta\beta$*-certainly with respect to* $\mathcal{L}$ *(* $\mathcal{L}-\theta\beta_{\lambda}$*-certainly) belongs to* V*, denoted by* $\mathsf{y}\ \underline{\in}^{\mathcal{L}-\theta\beta_{\lambda}} V$*.*

(ii) *if* $\mathsf{y} \in \overline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(V)$*, then* y *is* λ*-*$\theta\beta$*-probably with respect to* $\mathcal{L}$ *(briefly* $\mathcal{L} - \theta\beta_{\lambda}$*-probably) belongs to* V*, denoted by* $\mathsf{y}\ \overline{\in}^{\mathcal{L}-\theta\beta_{\lambda}} V$*.*

It is called λ*-*$\theta\beta$*-strong and* λ*-*$\theta\beta$*-weak membership relations with respect to* $\mathcal{L}$ *respectively.*

Remark 7. *According to Definition 18, the* $\mathcal{L}$*-*$\theta\beta_{\lambda}$*-lower and* $\mathcal{L}$*-*$\theta\beta_{\lambda}$*-upper approximations for any* $V \subseteq X$ *can be written as:*

(i) $\underline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(V) = \{\mathsf{y} \in X : \mathsf{y}\ \underline{\in}^{\mathcal{L}-\theta\beta_{\lambda}} V\}$.

(ii) $\overline{\mathcal{R}}_{\lambda}^{\mathcal{L}-\theta\beta}(V) = \{\mathsf{y} \in X : \mathsf{y}\ \overline{\in}^{\mathcal{L}-\theta\beta_{\lambda}} V\}$.

Lemma 2. *Let* $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$ *be an* $\mathcal{L}-\mathbb{G}_\lambda$*-space and* $V \subseteq X$*. Then*

(i) *if* $\mathsf{y} \underline{\in}^{\mathcal{L}-\theta\beta_\lambda} V$*, then* $\mathsf{y} \in V$*.*

(ii) *if* $\mathsf{y} \in V$*, then* $\mathsf{y} \overline{\in}^{\mathcal{L}-\theta\beta_\lambda} V$*.*

Proof. Straightforward.

Proposition 9. *Let* $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$ *be an* $\mathcal{L}-\mathbb{G}_\lambda$*-space and* $V \subseteq X$*. Then*

(i) *if* $\mathsf{y} \underline{\in}_\lambda A \Rightarrow \mathsf{y} \underline{\in}^\eta_\lambda A \Rightarrow \mathsf{y} \underline{\in}^{\mathcal{L}-\eta}_\lambda V \Rightarrow \mathsf{y} \underline{\in}^{\mathcal{L}-\theta\beta_\lambda} V$*.*

(ii) *if* $\mathsf{y} \overline{\in}^{\mathcal{L}-\theta\beta_\lambda} A \Rightarrow \mathsf{y} \overline{\in}^{\mathcal{L}-\eta}_\lambda A \Rightarrow \mathsf{y} \overline{\in}^\eta_\lambda A \Rightarrow \mathsf{y} \overline{\in}_\lambda V$*.*

Proof. We prove (i) and the other similarly. $\mathsf{y} \underline{\in}_\lambda A \Rightarrow \mathsf{y} \underline{\in}^\eta_\lambda A \Rightarrow \mathsf{y} \underline{\in}^{\mathcal{L}-\eta}_\lambda V$ by Proposition 4. Let $\mathsf{y} \underline{\in}^{\mathcal{L}-\eta}_\lambda V$. Then, $\mathsf{y} \in \underline{\mathcal{R}}^{\mathcal{L}-\eta}_\lambda(V) \Rightarrow \mathsf{y} \in \underline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V)$(*by Proposition* 4) $\Rightarrow$ $\mathsf{y} \underline{\in}^{\mathcal{L}-\theta\beta_\lambda} V$.

Remark 8. *The converse of Proposition 9 is not true in general, as it is shown in Example 1*

(i) *if* $V = \{\mathsf{y}_5\}$*, then* $\mathsf{y}_2 \underline{\in}^{\mathcal{L}-\theta\beta_{\mathbf{a}}} V$*, but* $\mathsf{y}_2 \underline{\in}^{\mathcal{L}-\delta\beta_{\mathbf{a}}} V$*.*

(ii) *if* $V = \{\mathsf{y}_1\}$*, then* $\mathsf{y}_2 \underline{\in}^{\mathcal{L}-\theta\beta_{\mathbf{a}}} V$*, but* $\mathsf{y}_2 \underline{\in}^{\mathcal{L}-\bigwedge_{\beta_{\mathbf{a}}}} V$*.*

Definition 21. *Let* $(X, \mathcal{R}, \Xi_\lambda)$ *be a* $\mathbb{G}_\lambda$*-space,* $\mathcal{L}$ *be an ideal on* $X, V \subseteq X$ *and* $\mathsf{y} \in X$*. The* $\mathcal{L}-\theta\beta_\lambda$*-rough membership functions of* V *are defined by* $\mu^{\mathcal{L}-\theta\beta_\lambda}_V \to [0,1]$*, where*

$$\mu^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}) = \{^{1}_{min(\psi^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}))} \quad ^{if\ 1 \in \psi^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}).}_{otherwise.}\}.$$

$$and\ \psi^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}) = \frac{|\mathcal{L}-\theta\beta_\lambda(\mathsf{y}) \cap V|}{|\mathcal{L}-\theta\beta_\lambda(\mathsf{y})|}, \mathsf{y} \in \mathcal{L}-\theta\beta_\lambda(\mathsf{y}),\ \mathcal{L}-\theta\beta_\lambda(\mathsf{y}) \in \mathcal{L}\text{-}\theta\beta_{\lambda O}(X).$$

Remark 9. *The* $\mathcal{L}-\theta\beta_\lambda$*-rough membership functions are used to define the* $\mathcal{L}$*-*$\theta\beta_\lambda$*-lower and* $\mathcal{L}$*-*$\theta\beta_\lambda$*-upper approximations as follows:*

(i) $\underline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V) = \{\mathsf{y} \in X : \mu^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}) = 1\}$.

(ii) $\overline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V) = \{\mathsf{y} \in X : \mu^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}) > 0\}$.

(iii) $\mathcal{BND}^{\mathcal{L}-\theta\beta}_\lambda(V) = \{\mathsf{y} \in X : 0 < \mu^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}) < 1\}$.

Proposition 10. *Let* $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$ *be an* $\mathcal{L}-\mathbb{G}_\lambda$*-space and* $V, W \subseteq X$*. Then*

(i) *if* $\mu^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}) = 1 \Leftrightarrow \mathsf{y} \underline{\in}^{\mathcal{L}-\theta\beta_\lambda} V$*.*

(ii) *if* $\mu^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}) = 0 \Leftrightarrow \mathsf{y} \in X - \overline{\mathcal{R}}^{\mathcal{L}-\theta\beta}_\lambda(V)$*.*

(iii) *if* $0 < \mu^{\mathcal{L}-\theta\beta_\lambda}_V(\mathsf{y}) < 1 \Leftrightarrow \mathsf{y} \in \mathcal{BND}^{\mathcal{L}-\theta\beta}_\lambda(V)$*.*

(iv) *if* $\mu_{A'}^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}) = 1 - \mu_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}), \forall\, \mathsf{y} \in X$.

(v) *if* $\mu_{V\cup B}^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}) \geq max(\mu_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}), \mu_B^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y})), \forall\, \mathsf{y} \in X$.

(vi) *if* $\mu_{V\cap B}^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}) \leq min(\mu_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}), \mu_B^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y})), \forall\, \mathsf{y} \in X$.

Proof. We prove (i), and the others similarly. $\mathsf{y} \,\underline{\in}^{\mathcal{L}-\theta\beta_\lambda} V \Leftrightarrow \mathsf{y} \in \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$. Since $\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)$ is $\mathcal{L}-\theta\beta_\lambda$-open set contained in V, thus $\frac{|\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)\cap V|}{|\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)|} = \frac{|\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)|}{|\underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V)|} = 1$. Then, $1 \in \psi_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y})$ and accordingly $\mu_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}) = 1$.

In the next, we prove an important result showing the interrelations between the relations of λ-rough membership [35] 12, λ-nearly rough membership [45] 13, λ-nearly rough membership w.r.t $\mathcal{L}$ [22, 23] 14, and $\mathcal{L}$-$\theta\beta_\lambda$-rough membership functions.

Lemma 3. *Let $(X, \mathcal{R}, \Xi_\lambda, \mathcal{L})$ be an $\mathcal{L} - \mathbb{G}_\lambda$-space and $V \subseteq X$. Then*

(i) $\mu_V^{\lambda}(\mathsf{y}) = 1 \Rightarrow \mu_V^{\eta_\lambda}(\mathsf{y}) = 1 \Rightarrow \mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = 1 \Rightarrow \mu_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}) = 1, \forall\, \mathsf{y} \in X$.

(ii) $\mu_V^{\lambda}(\mathsf{y}) = 0 \Rightarrow \mu_V^{\eta_\lambda}(\mathsf{y}) = 0 \Rightarrow \mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = 0 \Rightarrow \mu_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}) = 0, \forall\, \mathsf{y} \in X$.

Proof.

(i) $\mu_V^{\lambda}(\mathsf{y}) = 1 \Rightarrow \mu_V^{\eta_\lambda}(\mathsf{y}) = 1 \Rightarrow \mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = 1$ directly from Lemma 1. Let $\mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = 1$, then $\mathsf{y} \in \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\eta}(V) \Rightarrow \mathsf{y} \in \underline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \Rightarrow \mu_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}) = 1, \forall\, \mathsf{y} \in X$.

(ii) $\mu_V^{\lambda}(\mathsf{y}) = 0 \Rightarrow \mu_V^{\eta_\lambda}(\mathsf{y}) = 0 \Rightarrow \mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = 0$ directly from Lemma 1. Let $\mu_V^{\mathcal{L}-\eta_\lambda}(\mathsf{y}) = 0$, then $\mathsf{y} \in X - \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\eta}(V) \Rightarrow \mathsf{y} \in X - \overline{\mathcal{R}}_\lambda^{\mathcal{L}-\theta\beta}(V) \Rightarrow \mu_V^{\mathcal{L}-\theta\beta_\lambda}(\mathsf{y}) = 0, \forall\, \mathsf{y} \in X$.

Remark 10. *By Example 1, one can see that the converse of Lemma 3 fails.*

Remark 11. *According to Lemma 3, the current Definition 21 is also generalization of the approaches in [35] and 11 [42].*

6. Practical application

We allocated this part to examine the proposed models to cope with a real situation in the field of Chemistry. We explain how our models improve the outcomes of generalized approximation spaces over the previous models displayed in [15, 22, 23, 45]. The authors of [19] presented information systems of amino acids (AAs) with some characterizations. To facilitate the mathematical computations, we shall select a sample of that information system as given in Table 1; that is, we choose data of five AAs, say, $\mathcal{C} = \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4, \mathsf{y}_5\}$ described by five attributes as follows ν_1 is PIE, ν_2 is surface area (SAC), ν_3 is molecular refractivity (MR), ν_4 is side chain polarity (LAM), and ν_5 is molecular volume (Vol).

Table 1: Quantitative attributes of five amino acids.

	ν_1	ν_2	ν_3	ν_4	ν_5
y_1	0.23	254.2	2.126	-0.02	82.2
y_2	-0.48	303.6	2.994	-1.24	112.3
y_3	-0.61	287.9	2.994	-1.08	103.7
y_4	0.45	282.9	2.933	-0.11	99.1
y_5	-0.11	335.0	3.458	-0.19	127.5

Table 2: $\mathbb{G}_{\mathbf{a}}$ of each element of $\mathcal{C}$ inspired by each relation $\mathcal{R}_k$.

	$\mathbb{G}_{1\mathbf{a}}(y_i)$	$\mathbb{G}_{2\mathbf{a}}(y_i)$	$\mathbb{G}_{3\mathbf{a}}(y_i)$	$\mathbb{G}_{4\mathbf{a}}(y_i)$	$\mathbb{G}_{5\mathbf{a}}(y_i)$
y_1	$\{y_1, y_4\}$	$\mathcal{C}$	$\mathcal{C}$	$\{y_1, y_4, y_5\}$	$\mathcal{C}$
y_2	$\mathcal{C}$	$\{y_2, y_5\}$	$\{y_2, y_3, y_4, y_5\}$	$\mathcal{C}$	$\{y_2, y_5\}$
y_3	$\mathcal{C}$	$\{y_2, y_3, y_4, y_5\}$	$\{y_2, y_3, y_4, y_5\}$	$\mathcal{C}$	$\{y_2, y_3, y_4, y_5\}$
y_4	$\{y_4\}$	$\{y_2, y_3, y_4, y_5\}$	$\{y_2, y_3, y_4, y_5\}$	$\{y_1, y_4, y_5\}$	$\{y_2, y_3, y_4, y_5\}$
y_5	$\{y_1, y_4, y_5\}$	$\{y_5\}$	$\{y_5\}$	$\{y_1, y_4, y_5\}$	$\{y_3, y_5\}$

Let us take relations on $\mathcal{C}$ as: $\mathcal{R}_k = \{(y_i, y_j) : y_i(\nu_k) - y_j(\nu_k) < \frac{\sigma_{y_k}}{2}\}$ for $i, j, k = 1, 2, 3, 4, 5$ s.t. σ_{y_k} is the standard deviation of the quantitative attributes.

The right neighbourhood $\mathbb{G}_{k\mathbf{a}}$ of each element of $\mathcal{C}$ generated by each one of these relations $\mathcal{R}_k$ is presented in Table 2.

Now, we associate each element of $\mathcal{C}$ with all its $\mathbb{G}_{\mathbf{a}}$ by the following relation

$$\mathbb{H}_{\mathbf{a}}(y_i) = \bigcap_{k=1}^{5} \mathbb{G}_{k\mathbf{a}}(y_i).$$

For the sake of brevity, we conduct the computation for four AAs, say, $\mathcal{Y} = \mathcal{C} \setminus \{y_5\} = \{y_1, y_2, y_3, y_4\}$. Therefore, we first reduce Table 2 to Table 3.

Table 3: $\mathbb{G}_{\mathbf{a}}$ of each element of $\mathcal{Y}$ inspired by each relation $\mathcal{R}_k$.

	$\mathbb{G}_{1\mathbf{a}}(y_i)$	$\mathbb{G}_{2\mathbf{a}}(y_i)$	$\mathbb{G}_{3\mathbf{a}}(y_i)$	$\mathbb{G}_{4\mathbf{a}}(y_i)$	$\mathbb{G}_{5\mathbf{a}}(y_i)$
y_1	$\{y_1, y_4\}$	$\mathcal{Y}$	$\mathcal{Y}$	$\{y_1, y_4\}$	$\mathcal{Y}$
y_2	$\mathcal{Y}$	$\{y_2\}$	$\{y_2, y_3, y_4\}$	$\mathcal{Y}$	$\{y_2\}$
y_3	$\mathcal{Y}$	$\{y_2, y_3, y_4\}$	$\{y_2, y_3, y_4\}$	$\mathcal{Y}$	$\{y_2, y_3, y_4\}$
y_4	$\{y_4\}$	$\{y_2, y_3, y_4\}$	$\{y_2, y_3, y_4\}$	$\{y_1, y_4\}$	$\{y_2, y_3, y_4\}$

Now, we associate each element of $\mathcal{Y}$ with all its $\mathbb{G}_{\mathbf{a}}$ by the following relation

$$\mathbb{H}_{\mathbf{a}}(y_i) = \bigcap_{k=1}^{4} \mathbb{G}_{k\mathbf{a}}(y_i).$$

Accordingly, we obtain the following neighbourhoods:

- $\mathbb{H}_{\mathbf{a}}(y_1) = \{y_1, y_4\}$,

- $\mathbb{H}_{\mathbf{a}}(\mathsf{y}_2) = \{\mathsf{y}_2\}$,
- $\mathbb{H}_{\mathbf{a}}(\mathsf{y}_3) = \{\mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4\}$, and
- $\mathbb{H}_{\mathbf{a}}(\mathsf{y}_4) = \{\mathsf{y}_4\}$.

Thus, the topology initiated by these neighbourhoods (using the formula $\vartheta_{\mathbf{a}} = \{V \subseteq \mathcal{Y} : \forall \mathsf{y} \in V, \mathbb{H}(\mathsf{y}) \subseteq V\}$) is:

$$\vartheta_{\mathbf{a}} = \{\emptyset, \mathcal{Y}, \{\mathsf{y}_2\}, \{\mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4\}\}.$$

The family of all β-open, δ-open and θ-open subsets of this topology respectively are:

$$\beta_{\mathbf{a}O}(\mathcal{Y}) = \{\emptyset, \mathcal{Y}, \{\mathsf{y}_2\}, \{\mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3\}, \{\mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4\}\},$$

$$\delta_{\mathbf{a}O}(\mathcal{Y}) = \{\emptyset, \mathcal{Y}, \{\mathsf{y}_2\}, \{\mathsf{y}_1, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_4\}\}, \text{ and}$$
$$\theta_{\mathbf{a}O}(\mathcal{Y}) = \{\emptyset, \mathcal{Y}\}.$$

If we take $\mathcal{L} = \{\emptyset, \{\mathsf{y}_1\}\}$ as an ideal structure on $\mathcal{Y}$, then we find the following:

- $\mathcal{L}$-$\beta_{\mathbf{a}O}(\mathcal{Y}) = \{\emptyset, \mathcal{Y}, \{\mathsf{y}_2\}, \{\mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2\}, \{\mathsf{y}_2, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3\}, \{\mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_3\}\} = \beta_{\mathbf{a}O}(\mathcal{Y}) \cup \{\{\mathsf{y}_1, \mathsf{y}_2\}, \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_3\}\}$,
- $\mathcal{L}$-$\delta\beta_{\mathbf{a}O}(\mathcal{Y}) = \{\emptyset, \mathcal{Y}, \{\mathsf{y}_1\}, \{\mathsf{y}_2\}, \{\mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2\}, \{\mathsf{y}_2, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3\}, \{\mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_2, \mathsf{y}_3, \mathsf{y}_4\}, \{\mathsf{y}_1, \mathsf{y}_2, \mathsf{y}_3\}\} = \mathcal{L}$-$\beta_{\mathbf{a}O}(\mathcal{Y}) \cup \{\{\mathsf{y}_1\}, \{\mathsf{y}_1, \mathsf{y}_3\}\}$, and
- $\mathcal{L}$-$\theta\beta_{\mathbf{a}O}(\mathcal{Y}) = P(\mathcal{Y})$.

Table 4: Boundary regions and accuracy measures calculated with respect to Amer et al. [15] method, Hosny [23] method, and the present method.

Methods	Amer et al. method $\beta_{\mathbf{a}O}(\mathcal{Y})$		Hosny methods $\mathcal{L}$-$\beta_{\mathbf{a}O}(\mathcal{Y})$		Hosny methods $\mathcal{L}$-$\delta\beta_{\mathbf{a}O}(\mathcal{Y})$		The present method $\mathcal{L}$-$\theta\beta_{\mathbf{a}O}(\mathcal{Y})$	
$V \subseteq \mathcal{Y}$	$\mathcal{BND}^{\beta}_{\mathbf{a}}(V)$	$ACC^{\beta}_{\mathbf{a}}(V)$	$\mathcal{BND}^{\mathcal{L}-\beta}_{\mathbf{a}}(V)$	$ACC^{\mathcal{L}-\beta}_{\mathbf{a}}(V)$	$\mathcal{BND}^{\mathcal{L}-\delta\beta}_{\mathbf{a}}(V)$	$ACC^{\mathcal{L}-\delta\beta}_{\mathbf{a}}(V)$	$\mathcal{BND}^{\mathcal{L}-\theta\beta}_{\mathbf{a}}(V)$	$ACC^{\mathcal{L}-\theta\beta}_{\mathbf{a}}(V)$
$\{y_1\}$	$\{y_1\}$	0	$\{y_1\}$	0	$\emptyset$	1	$\emptyset$	1
$\{y_2\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{y_3\}$	$\{y_3\}$	0	$\{y_3\}$	0	$\{y_3\}$	0	$\emptyset$	1
$\{y_4\}$	$\{y_1\}$	1/2	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{y_1, y_2\}$	$\{y_1\}$	1/2	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{y_1, y_3\}$	$\{y_1, y_3\}$	0	$\{y_1, y_3\}$	0	$\{y_2\}$	2/3	$\emptyset$	1
$\{y_1, y_4\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{y_2, y_3\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{y_2, y_4\}$	$\{y_1, y_3\}$	1/4	$\{y_1, y_3\}$	1/2	$\emptyset$	1	$\emptyset$	1
$\{y_3, y_4\}$	$\{y_1\}$	1/3	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{y_1, y_2, y_3\}$	$\{y_1\}$	2/3	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{y_1, y_2, y_4\}$	$\{y_3\}$	3/4	$\{y_3\}$	3/4	$\{y_3\}$	3/4	$\emptyset$	1
$\{y_1, y_3, y_4\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{y_2, y_3, y_4\}$	$\{y_1\}$	3/4	$\{y_1\}$	3/4	$\emptyset$	1	$\emptyset$	1

According to the computations of boundary regions and accuracy measures of subsets displayed in Table 4, we remark the following points: there are different techniques introduced in the literature to approximate subsets using some forms of subsets of topological spaces. Our rough approximation space minimizes the upper approximation and maximizes the lower approximation, which leads to downsizing (or removing) the boundary regions. As a result, it outperforms other rough models given in the published literature like [15, 22, 23, 45], which makes it the most refined technique. For instance, the above table shows that a subset $\{y_3\}$ is considered a rough set according to the models of [15, 22, 23, 45], whereas this subset and all other subsets are exact according to the model investigated herein. This observation confirms that the current model is more beneficial for coping with real-life scenarios since it extracts a greater amount of information and reduces data ambiguity.

Furthermore, the proposed paradigm adheres to most properties of Pawlak's model without any restrictions, as demonstrated in Proposition 8. In this regard, We emphasize that the methodology of using nearly open sets in topology can achieve some or all properties of the Pawlak model, depending on the frameworks these families of subsets form, whether they are topology, supra topology, infra topology, or minimal structures. To elucidate this point, we note that the family of α-open sets constitutes a topology. Thus, rough set models inspired by this family will fulfill all the properties of the Pawlak model. In contrast, the family of semi-open sets constitutes a supra topology. Therefore, rough set models inspired by this family lose some properties of the Pawlak model related to the distribution of the union and intersection operators to the upper and lower approximations, respectively. On the other hand, we find that families that do not achieve all the properties of the Pawlak model expand the confirmed knowledge and produce a greater accuracy measure than those families that achieve all the properties of the Pawlak model.

7. Conclusions

The notion of rough neighborhoods was introduced in the literature with the aim of removing the strict term of an equivalence relation that limited the application of the classical rough set models. Such rough neighborhoods have shown to be useful in several applications. Some formulas have been proposed to institute a topology from these neighborhoods making topological spaces a vital instrument to represent rough approximation operators and analyze information systems. One of the important topological tools to reduce the vagueness of knowledge is nearly open sets. Despite this tool being applied by many researchers, there remain other types that should be investigated.

This work goes along with this line of research. We have studied generalized approximation spaces using the ideas of $\mathcal{L}$-$\theta\beta_\lambda$-open sets and ideal structures. We have explored their structural properties and pointed out the importance of the present models in maximizing the domain of confirmed information and minimizing the boundary region of uncertainty. Therefore, this work is a foundation for handling complicated paradigms in decision-making. We also showed the superiority of the proposed rough paradigms over different kinds of preceding paradigms induced by some nearly open sets. To facilitate the way of specifying the family of $\mathcal{L}$-$\theta\beta_\lambda$-open sets and exploring whether a subset is $\mathcal{L}$-$\theta\beta_\lambda$-definable or $\mathcal{L}$-$\theta\beta_\lambda$-rough, we have initiated two algorithms. Furthermore, we have defined the relations and functions of rough membership and established their key aspects. In the end, we have applied the current technique in a practical situation concerning classifying some chemical elements.

A promising avenue for upcoming research incorporates extending the present rough set paradigms to involve fuzzy and soft settings to enhance its ability to handle uncertainty. Additionally, considering other approaches like generating topological spaces by ideals first and then applying nearly open subsets of these spaces, could further refine the present paradigms and offer another technique to address imperfect knowledge. Moreover, discussing the current approaches in generalizations of topology [37] opens avenues for a deeper comprehension of their mathematical foundations.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] H. M. Abo-Doniaa and A. S. Salama. β-approximation spaces. *Journal of Hybrid Computing Research*, 1:1–15, 2008.

[2] H. M. Abo-Doniaa and A. S. Salama. Generalization of pawlak's rough approximation spaces by using $\delta\beta$-open sets. *International Journal of Approximate Reasoning*, 53:1094–1105, 2012.

[3] E. A. Abo-Tabl. A comparison of two kinds of definitions of rough approximations based on a similarity relation. *Information Sciences*, 181:2587–2596, 2011.

[4] E. A. Abo-Tabl. Rough sets and topological spaces based on similarity. *International Journal of Machine Learning and Cybernetics*, 4:451–458, 2013.

[5] T. M. Al-shami. An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application. *Information Sciences*, 569:110–124, 2021.

[6] T. M. Al-shami. Improvement of the approximations and accuracy measure of a rough set using somewhere dense sets. *Soft Computing*, 25(23):14449–14460, 2021.

[7] T. M. Al-shami. Topological approach to generate new rough set models. *Complex & Intelligent Systems*, 8:4101–4113, 2022.

[8] T. M. Al-shami. Maximal rough neighborhoods with a medical application. *Journal of Ambient Intelligence and Humanized Computing*, 14(12):16373–16384, 2023.

[9] T. M. Al-shami and I. Alshammari. Rough sets models inspired by supra-topology structures. *Artificial Intelligence Review*, 56(7):6855–6883, 2023.

[10] T. M. Al-shami and D. Ciucci. Subset neighborhood rough sets. *Knowledge-Based Systems*, 237(5):107868, 2022.

[11] T. M. Al-shami and M. Hosny. Generalized approximation spaces generation from $\mathbb{I}_j$-neighborhoods and ideals with application to chikungunya disease. *AIMS Mathematics*, 9(4):10050–10077, 2024.

[12] T. M. Al-shami, H. Işık, A. S. Nawar, and R. A. Hosny. Some topological approaches for generalized rough sets via ideals. *Mathematical Problems in Engineering*, 2021:5642982, 2021.

[13] T. M. Al-shami and A. Mhemdi. Approximation operators and accuracy measures of rough sets from an infra-topology view. *Soft Computing*, 27:1317–1330, 2023.

[14] T. M. Al-shami and A. Mhemdi. Overlapping containment rough neighborhoods and their generalized approximation spaces with applications. *Journal of Applied Mathematics and Computing*, 2024.

[15] W. S. Amer, M. I. Abbas, and M. K. El-Bably. On j-nearly concepts in rough sets with some applications. *International Journal of Fuzzy Intelligence Systems*, 32:1089–1099, 2017.

[16] J. Dai, S. Gao, and G. Zheng. Generalized rough set models determined by multiple neighborhoods generated from a similarity relation. *Soft Computing*, 13:2081–2094, 2018.

[17] S. Demiralp, T. M. Al-shami, A. M. Abd El-latif, and F. A. Abu Shaheen. Topologically indistinguishable relations and separation axioms. *AIMS Mathematics*, 9(6):15701–15723, 2024.

[18] M. M. El-Sharkasy. Minimal structure approximation space and some of its applications. *Journal of Intelligent & Fuzzy Systems*, 40(1):973–982, 2021.

[19] N. E. El-Tayar, R. S. Tsai, P. A. Carrupt, and B. Testa. Octan-1-ol-water partition coefficients of zwitterionic α-amino acids. determination by centrifugal partition chromatography and factorization into steric/hydrophobic and polar components. *Journal of the Chemical Society*, 2:79–84, 1992.

[20] M. Hosny. On generalization of rough sets by using two different methods. *Journal of Intelligent & Fuzzy Systems*, 35:979–993, 2018.

[21] M. Hosny. Idealization of j-approximation spaces. *Filomat*, 34:287–301, 2020.

[22] M. Hosny. Topological approach for rough sets by using j-nearly concepts via ideals. *Filomat*, 34:273–286, 2020.

[23] M. Hosny. Rough sets theory via new topological notions based on ideals and applications. *AIMS Mathematics*, 7:869–902, 2021.

[24] M. Hosny, T. M. Al-shami, and A. Mhemdi. Rough approximation spaces via maximal union neighborhoods and ideals with a medical application. *Journal of Mathematics*, 2022:17 pages, 2022.

[25] R. A. Hosny. Pre-open sets with ideal. *European Journal of Scientific Research*, 104(1):99–101, 2013.

[26] R. A. Hosny and D. Al-Kadi. Types of generalized open sets with ideal. *International Journal of Computer Applications*, 80(4):11–14, 2013.

[27] R. A. Hosny, B. A. Asaad, A. A. Azzam, and T. M. Al-shami. Various topologies generated from e_j-neighbourhoods via ideals. *Complexity*, 2021:4149368, 2021.

[28] D. Jankovic and T. R. Hamlet. New topologies from old via ideals. *American Mathematical Monthly*, 97:295–310, 1990.

[29] A. Kandil, M. M. Yakout, and A. Zakaria. Generalized rough sets via ideals. *Annals of Fuzzy Mathematics and Informatics*, 5(3):525–532, 2013.

[30] A. M. Kozae, S. A. El-Sheikh, E. H. Aly, and M. Hosny. Rough sets and its applications in a computer network. *Annals of Fuzzy Mathematics and Informatics*, 6(3):605–624, 2013.

[31] A. M. Kozae, S. A. El-Sheikh, and M. Hosny. On generalized rough sets and closure spaces. *International Journal of Applied Mathematics*, 23(6):997–1023, 2010.

[32] K. Kuratowski. *Topology*, volume I. Academic Press, New York, 1966.

[33] E. F. Lashin, A. M. Kozae, A. A. Abo Khadra, and T. Medhat. Rough set theory for topological spaces. *International Journal of Approximate Reasoning*, 40:35–43, 2005.

[34] Z. Li, T. Xie, and Q. Li. Topological structure of generalized rough sets. *Computers & Mathematics with Applications*, 63:1066–1071, 2012.

[35] T. Y. Lin. Granular computing on binary relation i: Data mining and neighborhood systems. In L. Polkowski and A. Skowron, editors, *Rough Sets in Knowledge Discovery 1*, pages 107–121. Physica-Verlag, Heidelberg, 1998.

[36] R. Mareay. Generalized rough sets based on neighborhood systems and topological spaces. *Journal of the Egyptian Mathematical Society*, 24:603–608, 2016.

[37] A. Mhemdi and T. M. Al-shami. Introduction to temporal topology. *Journal of Mathematics and Computer Science*, 34(3):205–216, 2024.

[38] F. I. Michael. On semi-open sets with respect to an ideal. *European Journal of Pure and Applied Mathematics*, 6(1):53–58, 2013.

[39] A. S. Nawar. Approximations of some near open sets in ideal topological spaces. *Journal of the Egyptian Mathematical Society*, pages 1–11, 2020.

[40] Z. Pawlak. Rough sets. *International Journal of Information and Computer Science*, 11:341–356, 1982.

[41] Z. Pawlak. Rough concept analysis. *Bulletin of the Polish Academy of Sciences Mathematics*, 33:495–498, 1985.

[42] Z. Pawlak and A. Skowron. Rough membership function. In R. E. Yeager, M. Fedrizzi, and J. Kacprzyk, editors, *Advances in the Dempster-Schafer Theory of Evidence*, pages 251–271. Wiley, New York, 1994.

[43] Z. Pei, D. Pei, and Li Zheng. Topology vs generalized rough sets. *International Journal of Approximate Reasoning*, 52:231–239, 2011.

[44] A. S. Salama. Bitopological approximation space with application to data reduction in multi-valued information systems. *Filomat*, 34(1):99–110, 2020.

[45] A. S. Salama and M. M. E. Abd El-Monsef. New topological approach of rough set generalizations. *International Journal of Computer Mathematics*, 88(7):1347–1357, 2011.

[46] P. K. Singh and S. Tiwari. Topological structures in rough set theory: A survey. *Hacettepe Journal of Mathematics and Statistics*, 49(4):1270–1294, 2020.

[47] R. Vaidynathaswamy. The localization theory in set topology. *Proceedings of the Indian Academy of Sciences*, 20:51–61, 1944.

[48] A. Wiweger. On topological rough sets. *Bulletin of the Polish Academy of Sciences Mathematics*, 37:89–93, 1989.

[49] H. Wu and G. Liu. The relationships between topologies and generalized rough sets. *International Journal of Approximate Reasoning*, 119:313–324, 2020.

[50] Y. Y. Yao. Two views of the theory of rough sets in finite universes. *International Journal of Approximate Reasoning*, 15:291–317, 1996.

[51] Y. Y. Yao. Relational interpretations of neighborhood operators and rough set approximation operators. *Information Sciences*, 119:239–259, 1998.

[52] E. D. Yildirim. New topological approaches to rough sets via subset neighborhoods. *Journal of Mathematics*, page 10 pages, 2022.

[53] Y. L. Zhang, J. Li, and C. Li. Topological structure of relational-based generalized rough sets. *Fundamenta Informaticae*, 147(4):477–491, 2016.