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Characterizations of Faintly (τ_1, τ_2)-Continuous Functions

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Abstract. This paper deals with the concept of faintly (τ_1, τ_2)-continuous functions. Furthermore, some characterizations of faintly (τ_1, τ_2)-continuous functions are investigated. The relationships between faint (τ_1, τ_2)-continuity and other forms of (τ_1, τ_2)-continuity are considered.

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1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Semi-open sets [25], preopen sets [27], α-open sets [29], β-open sets [22] and θ-open sets [38] play an important role in researches of generalizations of continuity. Using these sets several authors introduced and investigated various types of generalizations of continuity in topological spaces. Viriyapong and Boonpok [40] studied some characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [8]. Dungthaisong et al. [21] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [20] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Furthermore, several characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions,

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$(\Lambda, p(\star))$-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, pairwise M-continuous functions, almost quasi (τ_1, τ_2)-continuous functions and weakly quasi (τ_1, τ_2)-continuous functions were presented in [35], [37], [9], [33], [12], [5], [7], [6], [3], [1], [2], [24] and [17], respectively. Long and Herrington [26] introduced the notion of faintly continuous functions. Moreover, some characterizations of faintly continuous functions were investigated in [28] and [30], respectively. Three weak forms of faint continuity were introduced by Noiri and Popa [31]. Nasef and Noiri [28] introduced and studied three strong forms of faint continuity under the names of strongly faint semi-continuity, strongly faint precontinuity and strongly faint β-continuity. Jafari and Noiri [23] introduced and investigated the concept of faintly α-continuous functions. Chananan et al. [15] introduced a new class of functions, called faintly (m, μ)-continuous functions and established the relationships between faint (m, μ)-continuity and other related generalized forms of (m, μ)-continuity. Noiri and Popa [32] introduced the notion of faintly m-continuous functions as functions from a set X satisfying some minimal conditions into a topological space and investigated several characterizations of faintly m-continuous functions. Pue-on et al. [34] introduced the concept of faintly (τ_1, τ_2)-continuous functions. In this paper, we investigate some characterizations of faintly (τ_1, τ_2)-continuous functions. We also discuss the relationships between faintly (τ_1, τ_2)-continuous functions and other forms of (τ_1, τ_2)-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [14] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*clopen* [14] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [14] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [14] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [14] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2$-*Cl*$(X - A) = X - \tau_1\tau_2$-*Int*$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [39] (resp. $(\tau_1, \tau_2)s$-*open* [4], $(\tau_1, \tau_2)p$-*open* [4], $(\tau_1, \tau_2)\beta$-*open* [4]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [41] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [39] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [39] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [39] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [39] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

3. Characterizations of faintly (τ_1, τ_2)-continuous functions

In this section, we investigate several characterizations of faintly (τ_1, τ_2)-continuous functions.

Definition 1. [34] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called faintly (τ_1, τ_2)-continuous at a point $x \in X$ if for each $(\sigma_1, \sigma_2)\theta$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called faintly (τ_1, τ_2)-continuous if f has this property at every point of X.*

Theorem 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is faintly (τ_1, τ_2)-continuous at $x \in X$ if and only if for each $(\sigma_1, \sigma_2)\theta$-open set V of Y containing $f(x)$, $x \in \tau_1\tau_2$-Int$(f^{-1}(V))$.*

Proof. Let $x \in X$ and V be any $(\sigma_1, \sigma_2)\theta$-open set of Y containing $f(x)$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. Thus $x \in U \subseteq f^{-1}(V)$ and hence $x \in \tau_1\tau_2$-Int$(f^{-1}(V))$.

Conversely, let V be any $(\sigma_1, \sigma_2)\theta$-open set of Y containing $f(x)$. By the hypothesis, $x \in \tau_1\tau_2$-Int$(f^{-1}(V))$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq f^{-1}(V)$; hence $f(U) \subseteq V$. This shows that f is faintly (τ_1, τ_2)-continuous at $x \in X$.

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-T_2 [19] if for any pair of distinct points x, y in X, there exist disjoint $\tau_1\tau_2$-open sets U and V of X containing x and y, respectively.

Definition 2. *A bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-T_2 if for each distinct points $x, y \in X$, there there exist $(\tau_1, \tau_2)\theta$-open sets U and V of X containing x and y, respectively, such that $U \cap V = \emptyset$.*

Theorem 2. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a faintly (τ_1, τ_2)-continuous injection and (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)\theta$-$T_2$, then (X, τ_1, τ_2) is (τ_1, τ_2)-T_2.*

Proof. Let x, y be any distinct points of X. Then $f(x) \neq f(y)$. Since (Y, σ_1, σ_2) is $(\sigma_1, \sigma_2)\theta$-$T_2$, there exist $(\sigma_1, \sigma_2)\theta$-open sets U and V of Y containing $f(x)$ and $f(y)$, respectively, such that $U \cap V = \emptyset$. Since f is faintly (τ_1, τ_2)-continuous, there exist $\tau_1\tau_2$-open sets G and W of X containing x and y, respectively, such that $f(G) \subseteq U$ and $f(W) \subseteq V$. This implies that $G \cap W = \emptyset$. Thus, (X, τ_1, τ_2) is (τ_1, τ_2)-T_2.

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*compact* [14] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover. A subset K of X is said to be $\tau_1\tau_2$-*compact relative to* (X, τ_1, τ_2) if every cover of K by $\tau_1\tau_2$-open sets of X has a finite subcover.

Definition 3. *A subset K of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-compact relative to (X, τ_1, τ_2) if every cover of K by $(\tau_1, \tau_2)\theta$-open sets of X has a finite subcover. A bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-compact if the set X is $(\tau_1, \tau_2)\theta$-compact relative to (X, τ_1, τ_2).*

Theorem 3. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a faintly (τ_1, τ_2)-continuous function and K is $\tau_1\tau_2$-compact relative to (X, τ_1, τ_2), then $f(K)$ is $(\sigma_1, \sigma_2)\theta$-compact relative to (Y, σ_1, σ_2).*

Proof. Let $\{V_\gamma : \gamma \in \Gamma\}$ be any cover of $f(K)$ by $(\sigma_1, \sigma_2)\theta$-open sets of Y. For each $x \in K$, there exists $\gamma(x) \in \Gamma$ such that $f(x) \in V_{\gamma(x)}$. Since f is faintly (τ_1, τ_2)-continuous, there exist a $\tau_1\tau_2$-open set $U(x)$ of X containing x such that $f(U(x)) \subseteq V_{\gamma(x)}$. The family $\{U(x) : x \in K\}$ is a cover of K by $\tau_1\tau_2$-open sets of X. Since K is $\tau_1\tau_2$-compact relative to (X, τ_1, τ_2), there exists a finite number of points, say, $x_1, x_2, x_3, ..., x_n$ in K such that $K \subseteq \cup\{U(x_k) : x_k \in K, 1 \leq k \leq n\}$. Thus,

$$\begin{aligned} f(K) &\subseteq \cup\{f(U(x_k)) : x_k \in K, 1 \leq k \leq n\} \\ &\subseteq \cup\{V_{\gamma(x_k)} : x_k \in K, 1 \leq k \leq n\}. \end{aligned}$$

This shows that $f(K)$ is $(\sigma_1, \sigma_2)\theta$-compact relative to (Y, σ_1, σ_2).

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*connected* [14] if X cannot be written as the union of two disjoint nonempty $\tau_1\tau_2$-open sets.

Lemma 2. [34] *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is faintly (τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for each $(\sigma_1, \sigma_2)\theta$-open set V of Y;

(3) $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for each $(\sigma_1, \sigma_2)\theta$-closed set K of Y;

(4) for each $x \in X$ and for each $(\sigma_1, \sigma_2)\theta$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$.

Theorem 4. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a faintly (τ_1, τ_2)-continuous surjection and (X, τ_1, τ_2) is $\tau_1\tau_2$-connected, then (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-connected.*

Proof. Assume that (Y, σ_1, σ_2) is not $\sigma_1\sigma_2$-connected. Then, there exist nonempty $\sigma_1\sigma_2$-open sets V and W such that $V \cap W = \emptyset$ and $V \cup W = Y$. Thus, $f^{-1}(V) \cap f^{-1}(W) = \emptyset$ and $f^{-1}(V) \cup f^{-1}(W) = X$. Since f is surjective, $f^{-1}(V)$ and $f^{-1}(W)$ are nonempty. Since V and W are $\sigma_1\sigma_2$-open and $\sigma_1\sigma_2$-closed, we have V and W are $(\sigma_1, \sigma_2)\theta$-open sets of Y. Since f is faintly (τ_1, τ_2)-continuous, by Lemma 2, $f^{-1}(V)$ and $f^{-1}(W)$ are $\tau_1\tau_2$-open in X. Thus, (X, τ_1, τ_2) is not $\tau_1\tau_2$-connected. This is a contradiction and hence (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-connected.

The $\tau_1\tau_2$-*frontier* [13] of a subset A of a bitopological space (X, τ_1, τ_2), denoted by $\tau_1\tau_2$-fr(A), is defined by

$$\tau_1\tau_2\text{-fr}(A) = \tau_1\tau_2\text{-Cl}(A) \cap \tau_1\tau_2\text{-Cl}(X - A) = \tau_1\tau_2\text{-Cl}(A) - \tau_1\tau_2\text{-Int}(A).$$

Theorem 5. *The set of all points $x \in X$ at which a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not faintly (τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the inverse images of $(\sigma_1, \sigma_2)\theta$-open sets of Y containing $f(x)$.*

Proof. Suppose that f is not faintly (τ_1, τ_2)-continuous at $x \in X$. Then, there exists a $(\sigma_1, \sigma_2)\theta$-open set V of Y containing $f(x)$ such that $f(U)$ is not contained in V for every $\tau_1\tau_2$-open set U of X containing x. Then, $U \cap (X - f^{-1}(V)) \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. Thus, $x \in \tau_1\tau_2\text{-Cl}(X - f^{-1}(V))$. On the other hand, we have $x \in f^{-1}(V) \subseteq \tau_1\tau_2\text{-Cl}(f^{-1}(V))$ and hence $x \in \tau_1\tau_2\text{-fr}(A)$.

Conversely, suppose that f is faintly (τ_1, τ_2)-continuous at $x \in X$. Let V be any $(\sigma_1, \sigma_2)\theta$-open set of Y containing $f(x)$. Then by Theorem 1, $x \in \tau_1\tau_2\text{-Int}(f^{-1}(V))$. Thus, $x \notin \tau_1\tau_2\text{-fr}(f^{-1}(V))$ for each $(\sigma_1, \sigma_2)\theta$-open set V of Y containing $f(x)$. This completes the proof.

Definition 4. [36] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be slightly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-clopen set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$.*

Theorem 6. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is faintly (τ_1, τ_2)-continuous, then f is slightly (τ_1, τ_2)-continuous.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-clopen set of Y containing $f(x)$. Then, V is $(\sigma_1, \sigma_2)\theta$-open in Y. Since f is faintly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. This shows that f is slightly (τ_1, τ_2)-continuous.

4. On faint (τ_1, τ_2)-continuity and other forms of (τ_1, τ_2)-continuity

In this paper, we investigate the relationships between faintly (τ_1, τ_2)-continuous functions and other forms of (τ_1, τ_2)-continuous functions.

Definition 5. [10] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\tau_1\tau_2$-open set V of Y containing $f(x)$, there* *exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous if f has this property at each point of X.*

Theorem 7. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly (τ_1, τ_2)-continuous, then f is faintly (τ_1, τ_2)-continuous.*

Proof. Let $x \in X$ and V be any $(\sigma_1, \sigma_2)\theta$-open set of Y containing $f(x)$. There exists a $\sigma_1\sigma_2$-open set W of Y such that $f(x) \in W \subseteq \sigma_1\sigma_2$-Cl$(W) \subseteq V$. Since f is weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(W) \subseteq V$. Thus, f is faintly (τ_1, τ_2)-continuous.

Definition 6. [13] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous if f has this property at each point of X.*

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-*regular* [16] if for each $\tau_1\tau_2$-closed set F and each point $x \in X - F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 3. [13] *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $f(\tau_1\tau_2$-Cl$(A)) \subseteq \sigma_1\sigma_2$-Cl$(f(A))$ for every subset A of X;

(4) $\tau_1\tau_2$-Cl$(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2$-Int$(B)) \subseteq \tau_1\tau_2$-Int$(f^{-1}(B))$ for every subset B of Y;

(6) $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y.

Theorem 8. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is faintly (τ_1, τ_2)-continuous and (Y, σ_1, σ_2) is a (σ_1, σ_2)-regular space, then f is (τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is a (σ_1, σ_2)-regular space, V is $(\sigma_1, \sigma_2)\theta$-open in Y. Since f is faintly (τ_1, τ_2)-continuous, by Lemma 2 we have $f^{-1}(V)$ is $\tau_1\tau_2$-open in X and hence by Lemma 3, f is (τ_1, τ_2)-continuous.

Definition 7. [11] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost (τ_1, τ_2)-continuous if f has this property at each point of X.*

Lemma 4. [11] *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost (τ_1, τ_2)-continuous at $x \in X$;

(2) $x \in \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$;

(3) $x \in \tau_1\tau_2\text{-}Int(f^{-1}(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$;

(4) for each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$.

Recall that a bitopological space (X, τ_1, τ_2) is said to be *almost (τ_1, τ_2)-regular* [18] if for each $(\tau_1, \tau_2)r$-closed set F and each $x \notin F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 5. *Let (X, τ_1, τ_2) be an almost (τ_1, τ_2)-regular space. Then, every $(\tau_1, \tau_2)r$-open set is $(\tau_1, \tau_2)\theta$-open.*

Theorem 9. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is faintly (τ_1, τ_2)-continuous and (Y, σ_1, σ_2) is almost (σ_1, σ_2)-regular, then f is almost (τ_1, τ_2)-continuous.*

Proof. Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Then by Lemma 5, V is $(\sigma_1, \sigma_2)\theta$-open in Y. Since f is faintly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. It follows from Lemma 4 that f is almost (τ_1, τ_2)-continuous.

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