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Dynamical Analysis of a Three Species Food-Web Model With Extended Holling Type II Functional Response

Bakhan Bahman Kamal[1], Arkan Nawzad Mustafa[1,*]

[1] *Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah 46001, Iraq*

Abstract. This paper deals with dynamical analysis of a food-web including three logistically growing interaction species, prey, intermediate predators and apex predators. The intermediate predator species predate the prey species according to the Holling type-II functional response, while the apex predator species predate both prey species and intermediate predator species according to extended Holling type II functional response for two prey species. Firstly, We conduct a thorough analytical examination of the system, demonstrating the positivity and boundedness of the solutions criteria for the persistence of the model are founded, four biologically possible steady states are determined and the local as well as the dynamics around the model' steady states based on the parameters are also investigated. The occurrence of Hopf-bifurcation of thee model near all steady states, are discussed. Finally, with the help of MATLAB program, it is performed numerical simulations to support the evidence of our analytical results.

2020 Mathematics Subject Classifications: 37N25, 34D20, 37G99, 92D25, 92D40

Key Words and Phrases: Food web, functional response, permanence, stability analysis, Hopf-bifurcation

1. Introduction

Mathematical models for the dynamics of interaction between two species of prey and predator, has been derived by many mathematician author [2, 3, 6, 9]. Food web models are important conceptual tool for illustrating the feeding relationships among species within a community, revealing species interactions and community structure, and understanding the dynamics of energy transfer in an ecosystem, Therefore two-species model has been extended to the three-species model by many authors [5, 8, 10, 18].

The important element to represent the dynamics relationship between predator population and prey population is the functional response for the predator, which defined as

*Corresponding author.
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Email addresses: bakhan.kamal@univsul.edu.iq (B. B. Kamal),
arkan.mustafa@univsul.edu.iq (A. N. Mustafa)

the number of consumed prey per predator per unit time [9]. C.S. Holling [12] identified three types of functional response Type I, Type II and Type- III. The most useful functional response is the Holling type II functional response, which is characterized by decelerating intake rate [12]. Many authors used this type of functional response for modeling the dynamics of interactions between predator and prey [4, 14]. Jha and Ghorai [14] proposed a prey-predator model with selective harvesting between the species using a Holling-type functional response. Hastings [11, 16] has studied the chaotic behavior of an ecology model including the three species. Khan et al. [15] investigated bifurcation analysis of a three-species discrete time. Sk et al. [22] explored the impact of the fear of predators in prey and shelter in a three-species food chain model with delays in hunting cooperation. The authors [20, 21] have investigated the dynamic behavior of a three species system with a scavenger. Diana et al. [7] investigated the three species model's dynamic behavior with logistic growth in which disease was included. The behavior of a three-species model with time delay and noise was analyzed stochastically by Danane and Torres [13]. Numerous writers have examined the three species model's dynamics in the presence of both logistic and non-logistic growth as of now. It has been explored the fear effect and stability analysis of the food chain model [1, 17, 19].

In nature, there are many predator species that consume more than one species of prey as well as intermediate predator. For example, lions usually predate a number of large land-based animals, such as antelopes, buffaloes, crocodiles, giraffes, pigs, zebra, wild dogs and wildebeest. In 2022, the Holling type II functional response is extended to more than one prey species [9]. Therefore, this current work is considered and studying the following food web system with extended Holling type II functional response to two preys.

$$\begin{cases} \frac{dX}{dt} = X\left[r_1\left(1-\frac{X}{K}\right)-\frac{\beta Y}{1+\beta T X}-\frac{\alpha_1 Z}{1+\alpha_1 T_1 X+\alpha_2 T_2 Y}\right], \\ \frac{dY}{dt} = Y\left[r_2\left(1-\frac{Y}{X}\right)-\frac{\alpha_2 Z}{1+\alpha_1 T_1 X+\alpha_2 T_2 Y}\right], \\ \frac{dZ}{dt} = r_3 Z\left(1-\frac{Z}{X+Y}\right), \end{cases} \tag{1}$$

where $X(t), Y(t)$ and $Z(t)$ represent the individual numbers of the prey, intermediate predators and apex predators, respectively. The parameters are positive and their descriptions are given in Table.

System (1) is derived on the following assumptions

(i) All the species grow logistically.

(ii) The apex predators predate both prey and intermediate predators according to extended Holling type II functional response [8].

(iii) The prey population predated by intermediate predators according to Holling type II functional response.

The paper organized as follows: in the next section some preliminaries on the model solution property, which are also needed in this work, are given. In the third section, the existence conditions of all feasible and possible steady stat points of system (1) are found and their local as well as globally stability are investigated. In section five, the Hopfbifurcation near to each steady state point is studied. In section six, system (1) numerically solved to observe the impact of parameters and confirm the analytical results in this work. Finally in section eight. a brief conclusion on the total work is given. Where, all other parameter description in Table 1.

Parameters	Description
$r_{\mathbf{1}}, r_2, r_3$	Intrinsic growth rate for prey, intermediate predators and apex predators, respectively
K	Carrying capacity for prey species
β	by intermediate predators' predation rate
T	the predator's average handling time of intermediate predators
$\alpha_{\mathbf{1}}, \alpha_{\mathbf{2}}$	predator's search efficiency of prey, intermediate predators, respectively
$T_{\mathbf{1}}, T_{\mathbf{2}}$	predator's average handling time of prey, intermediate predators, respectively

Table 1: Parameter description of system (1).

2. Preliminaries and permanence

In this section, it is proved some lemma on the model solutions, the lemmas are also needed to obtain the result in this paper. A permanent ecological model, imply that all the species in an ecosystem continues to exist, therefore, the criteria that make the model permanent is important, so in this section, the definition of permanence is reviewed and then it is proved that system (1) is permanent under certain conditions.

Lemma 2.1. *System (1), exhibits a unique solution that is non-negative and satisfy the following inequalities:*

$$\lim_{t\to\infty} \sup(X(t)) \leq K \tag{2}$$

$$\lim_{t\to\infty} \sup(Y(t)) \leq \text{K} \tag{3}$$

$$\lim_{t\to\infty} \sup(Z(t)) \leq 2\ \text{K} \tag{4}$$

Proof. Right side of system (1), are continuous and has partial derivatives on the space R^3, and hence, system (1) satisfies the Lipschitzian condition. Therefore, by uniqueness Theorem, it has unique solution. Further, the time derivative of X, Y and Z are zero in YZ - Plane, XZ - Plane and XY - plane, respectively.

Therefore, if the solution initiates at a non-negative point, then the component X, Y and Z of the solution points of system (1), cannot cross any coordinates of the solution points. Hence components X, Y and Z of solution points is always non negative. The first equation in system (1) gives that

$$\frac{dX}{dt} \leq r_1 X \left(1 - \frac{X}{K}\right).$$

Solving above differential inequality, it gets

$$\lim_{t\to\infty} \sup(X(t)) \leq K.$$

Apply above inequality at the second equation of system (1), it gets

$$\frac{dY}{dt} \leq r_2 Y \left(1 - \frac{X}{K}\right).$$

Again Solving above differential inequality, it gets

$$\lim_{t\to\infty} \sup(Y(t)) \leq \text{K}.$$

So, inequalities (2) and (3) are guaranteed.

Apply inequalities $(2, 3)$ in the third equation of system (1) , it gets

$$\frac{dZ}{dt} \leq r_1 Z \left(1 - \frac{Z}{2K}\right)$$

So,

$$\lim_{t\to\infty} \sup(Z(t)) \leq 2K.$$

This completes the proof.

Lemma 2.2. *(i)* $\lim_{t\to\infty} \inf X(t) \geq K_1$, *if*

$$r_1 > (\beta + 2\alpha_1) K \tag{5}$$

(ii) $\lim_{t\to\infty} \inf Y(t) \geq K_2$, *if Inq.(5) and the following inequality are satisfied*

$$r_2 > \frac{2\alpha_2}{(1 + \alpha_1 T_1 K_1)} K \tag{6}$$

(iii) $\lim_{t\to\infty} \inf Z(t) \geq K_1 + K_2$, *if both conditions (5) and (6) are provided.*
where, K_1 *and* K_2 *are positive number to be determined in the proof.*

Proof. Suppose t approaches infinity, then

(i) Apply inequalities (2)–(4) in Lemma 2.1, at the first equation of system (1), it gets

$$\frac{dX(t)}{dt} \geq X\left[r_1\left(1-\frac{X}{K}\right)-(\beta+2\alpha_1)\,K\right].$$

Therefore under condition (5), it gets

$$\lim_{t\to\infty}\inf X(t) \geq .\frac{[r_1-(\beta+2\alpha_1)\,K]\,K}{r_1} = K_1 > 0$$

(ii) From condition(4) in Lemma 2.1 and inequality (7), it follows

$$\frac{dY}{dt} \geq Y\left[r_2\left(1-\frac{Y}{K_1}\right)-\frac{2\alpha_2 K}{1+\alpha_1 T_1 K_1}\right],$$

so under condition (6), it is concluded that:

$$\lim_{t\to\infty}\inf Y(t) \geq \left[1-\frac{2\alpha_2 K}{r_2\,(1+\alpha_1 T_1 K_1)}\right]K_1 = K_2 > 0$$

(iii) From (7) and (8), it follows

$$\frac{dZ}{dt} \geq r_3 Z\left(1-\frac{Z}{K_1+K_2}\right),$$

and hence

$$\lim_{t\to\infty}\inf Z(t) \geq K_1+K_2.$$

Definition 2.1. *[5] System (1) is said to be permanent if there exist positive constants a and b such that*

$$b \geq \max\left\{\lim_{t\to\infty}\sup(X(t)),\lim_{t\to\infty}\sup(Y(t)),\lim_{t\to\infty}\sup(Z(t))\right\}$$
$$\geq \min\left\{\lim_{t\to\infty}\inf(X(t)),\lim_{t\to\infty}\inf(Y(t)),\lim_{t\to\infty}\inf(Z(t))\right\} \geq a.$$

Theorem 2.1. *If the following conditions (5) and (6) in Lemma 2.2 are provided, then system (1), is permanent.*

Proof. From Lemma 2.1, it is obtained that

$$\min\left\{\lim_{t\to\infty}\inf(X(t)), \lim_{t\to\infty}\inf(Y(t)), \lim_{t\to\infty}\inf(Z(t))\right\} \geq \min\left\{K_1, K_2\right\}.$$

From Lemma 2.2, it follows

$$2K \geq \max\left\{\lim_{t\to\infty}\sup(X(t)), \lim_{t\to\infty}\sup(Y(t)), \lim_{t\to\infty}\sup(Z(t))\right\}.$$

The proof is completed.

3. Stability analysis

This section including three subsections. In the first subsection, the existence conditions of all feasible and possible steady stat points of system (1) are determined and their local and globally stability are investigated in the second and third subsection, respectively.

3.1. Existence of steady states

System (1) has at most four steady states, they are the only prey existence steady state $S_1 = (K, 0, 0)$, apex predatorfree steady state $S_2 = (\bar{S}, \bar{S}, 0)$, intermediate predators-free steady state $S_3 = (\bar{\bar{S}}, 0, \bar{\bar{S}})$ and coexistence steady state is $S_4 = (X^*, Y^*, Z^*)$ always exist, where

$$\bar{S} = \frac{r_1\beta TK - r_1 - \beta K + \sqrt{(r_1\beta TK - r_1 - \beta K)^2 + 4r_1^2\beta TK}}{2r_1\beta T},$$

$$\bar{\bar{S}} = \frac{r_1\alpha_1 T_1 K - r_1 - \alpha_1 K + \sqrt{(r_1\alpha_1 T_1 K - r_1 - \alpha_1 K)^2 + 4r_1^2\alpha_1 T_1 K}}{2r_1\alpha_1 T_1}, \quad Z^* = X^* + Y^*,$$

$Y^* = a_1X^* + a_2X^{*2} + a_3X^{*3}$ and X^* is root for the following function

$$F(X) = r_1\left(1 - \frac{X}{K}\right) - \frac{\beta Y(X)}{1 + \beta TX} - \frac{\alpha_1(X + Y(X))}{1 + \alpha_1 T_1 X + \alpha_2 T_2 Y(X)}, \tag{10}$$

with $a_1 = \frac{r_1\alpha_2\beta T}{r_2\alpha_1 K}, a_2 = \frac{r_1\alpha_2}{r_2\alpha_1 K} - \frac{r_1\alpha_2\beta T}{r_2\alpha_1}$ and $a_3 = \frac{r_2\beta}{r_2\alpha_1} + 1 - \frac{r_1\alpha_2}{r_2\alpha_1}$ and $Y(X) = a_1X + a_2X^2 + a_3X^3$.

Note the steady states S_1, S_2 and S_3 are always uniquely exists. But the existence criteria for the coexistence steady state S_4 is determined in the following Theorem

Theorem 3.1. *The coexistence steady state $S_4 = (X^*, Y^*, Z^*)$ exist uniquely, if*

$$a_i \geq 0, i = 2, 3 \tag{11}$$

Proof. From (10), it gets $F(0) = r_1 > 0$ and

$$F(K) = -\frac{\beta Y(K)}{1+\beta TK} - \frac{\alpha_1(K+Y(K))}{1+\alpha_1 T_1 K + \alpha_2 T_2 Y(K)} < 0.$$

Intermediate value Theorem, guarantees that $F(X)$ has a root, namely X^* in $(0, K)$. Further $\frac{dF(X)}{dX} < 0$, for all $x \geq 0$, due to condition (11), that is $F(X)$ is decreasing on the positive real line, and hence X^* is unique root of $F(X)$, and this completes the proof.

3.2. Local stability

Here, local asymptotical stability (LAS) of all the steady states of system (1) is studied. Suppose $\lambda_{iX}, \lambda_{iY}$ and λ_{iZ} represent the eigenvalues of the Variational matrix at S_i, in the X-, Y-, and Z-directions, respectively; $i = 1, 2, 3, 4$. Then,

(i) $\lambda_{1X} = -r_1, \lambda_{0Y} = r_2$ and $\lambda_{1Z} = r_3$

(ii) $\lambda_{2Z} = r_3$ and λ_{2X} and λ_{2X} are roots of the equation $\lambda^2 + A_1\lambda + B_1 = 0$, where,

$$A_1 = \frac{r_1}{K}\bar{S} + r_2 - \left(\beta^2 T\bar{S}^2\right)/(1+\beta T\bar{S})^2 \text{ and } B_1 = \frac{r_2 r_1}{K}\bar{S} + \left(r_2\beta\bar{S}\right)/(1+\beta T\bar{S})^2$$

(iii) $\lambda_{3X}, \lambda_{3Y}$ and λ_{3Z} are roots of the equation $\lambda^3 + A_2\lambda^2 + B_2\lambda + r_3 C_2 = 0$, where,

$$A_2 = -\left(R_1 + R_5\right), B_2 = R_1R_5 - R_2R_4 - r_3\left(R_1 + R_3 + R_5\right) \text{ and } C_2 = R_1R_5 - R_2R_4 - R_3R_4 + R_3R_5$$

with

$$\begin{aligned}
R_1 &= -\frac{r_1}{K}\overline{\overline{S}} + \left(\alpha_1^2 T_1 \overline{\overline{S}}^2\right)/\left(1+\alpha_1 T_1 \overline{\overline{S}}\right)^2, \\
R_2 &= -\frac{\beta\overline{\overline{S}}}{(1+\beta T\bar{s})} - \alpha_1\overline{\overline{S}}/\left(1+\alpha_1 T_1\overline{\overline{S}}\right), \\
R_3 &= -\alpha_2\overline{\overline{S}}/\left(1+\alpha_1 T_1\overline{\overline{S}}\right), \\
R_4 &= -\alpha_2\overline{\overline{S}}/\left(1+\alpha_1 T_1\overline{\overline{S}}\right)^2 \text{ and} \\
R_5 &= r_2 - \left(\alpha_2\overline{\overline{S}}\right)/\left(1+\alpha_1 T_1\overline{\overline{S}}\right)
\end{aligned}$$

(iv) $\lambda_{4X}, \lambda_{4Y}$ and λ_{4Z} are roots of the equation $\lambda^3 + A_3\lambda^2 + B_3\lambda + r_3 C_3 = 0$, where

$$\begin{aligned}
&A_3 = -\left(E_1 + E_5\right), B_2 = E_1E_5 - E_2E_4 - r_3\left(E_1 + E_3 + E_5 + E_6\right) \text{ and} \\
&C_3 = E_1E_5 + E_1E_6 - E_2E_4 - E_2E_6 - E_3E_4 + E_3E_5
\end{aligned}$$

with

$$E_1 = r_1 - 2\frac{r_1}{K}X^* - \frac{\beta Y^*}{(1+\beta T X^*)^2} - \frac{\alpha_1(1+\alpha_2 T_2 Y^*)Z^*}{(1+\alpha_1 T_1 X^* + \alpha_2 T_2 Y^*)^2},$$
$$E_2 = -\frac{\beta x^*}{1+\beta T x^*} - \frac{\alpha_1(1+\alpha_1 T_1 X^*)Z^*}{(1+\alpha_1 T_1 X^* + \alpha_2 T_2 Y^*)^2},$$
$$E_3 = -\frac{\alpha_1 x^*}{1+\alpha_1 T_1 X^* + \alpha_2 T_2 Y^*}$$
$$E_4 = r_2\left(\frac{Y^*}{X^*}\right)^2 - \frac{\alpha_2(1+\alpha_2 T_2 Y^*)Z^*}{(1+\alpha_1 T_1 X^* + \alpha_2 T_2 Y^*)^2},$$
$$E_5 = r_2 - 2r_2\frac{Y^*}{X^*} - \frac{\alpha_2(1+a_1 T_1 X^*)Z^*}{(1+\alpha_1 T_1 X^* + \alpha_2 T_2 Y^*)^2} \text{ and}$$
$$E_6 = -\frac{\alpha_2 Y^*}{1+\alpha_1 T_1 X^* + \alpha_2 T_2 Y^*}$$

Therefore, the following Theorem can be derived based on the above argument.

Theorem 3.2. *(i) The only prey existence steady state and the apex predator-free steady state are unstable.*

(ii) The intermediate predators-free steady state is LAS if and only if, all the following Routh-Hurwize criteria hold.

$$A_2 > 0, C_2 > 0 \text{ and } A_2 B_2 > r_3 C_2 \tag{12}$$

(iii) The coexistence steady state it is LAS if and only if, all the following Routh-Hurwize criteria hold.

$$A_3 > 0, C_3 > 0 \text{ and } A_3 B_3 > r_3 C_3 \tag{13}$$

3.3. Global stability

Global stability (or globally asymptotically stable GAS) means that any trajectories finally tend to the attractor of the system, regardless of initial conditions. Therefore, Most of biological systems, especially prey predator system, are needed to be globally stable. Since S_1 and S_2 are not LAS, so they cannot be GAS. However GAS for both S_3 and S_4 of the system (1) is established in Theorem 3.3 and Theorem 3.4, respectively.

Theorem 3.3. *Suppose that condition 5 holds, and then $S_3 = (\overline{\overline{S}}, 0, \overline{\overline{S}})$ is GAS, if the following inequality holds:*

$$r_2 < \frac{\alpha_2 K_1}{1+\alpha_1 T_1 K + \alpha_2 T_2 K} \tag{14}$$
$$\frac{r_1 + r_3}{K^2} > \frac{\alpha_1}{(1+\alpha_1 T_1 K_1)^2} \tag{15}$$

Proof. From Lemma 2.1, $\lim_{t\to\infty}\sup X(t) \le K$ and $\lim_{t\to\infty}\sup Y(t) \le K$. From Lemma 2.2 $\lim_{t\to\infty}\inf X(t) \ge K_1$, if condition 5 , holds. Therefore, the third equation f system (1) gives that

$$\lim_{t\to\infty}\inf X(t) \ge K_1 \text{ and } \lim_{t\to\infty}\inf Y(t) \ge K_1.$$

And hence, from the second equation of system (1), it is obtained that

$$\frac{dY}{dt} \le Y\left[r_2 - \frac{\alpha_2 K_1}{1+\alpha_1 T_1 K + \alpha_2 T_2 K} - \frac{r_2 Y}{K}\right].$$

So, $\frac{dY}{dt}$ is negative due to condition 14 , consequently $\lim_{t\to\infty} Y(t) = 0$.
Therefore as time approaches infinity, system (1) reduced to the following subsystem

$$\begin{cases} \frac{dX}{dt} = X\left[r_1\left(1-\frac{X}{K}\right) - \frac{\alpha_1 Z}{1+\alpha_1 T_1 X}\right] = F(X,Z) \\ \frac{dZ}{dt} = r_3 Z\left(1-\frac{Z}{X}\right) = G(X,Z) \end{cases} \tag{16}$$

Consider now the function $H(X,Z) = 1/XZ$, clearly H is a continuously differentiable function. Further,

$$\frac{\partial(HF)}{\partial X} + \frac{\partial(HG)}{\partial Z} = -\frac{r_1}{KZ} + \frac{\alpha_1}{(1+\alpha_1 T_1 X)^2} - \frac{r_3}{X^2} < \frac{\alpha_1}{(1+\alpha_1 T_1 K_1)^2} - \frac{r_1 + r_3}{K^2}.$$

Under the condition (15), it is clear that $\frac{\partial(HF)}{\partial X} + \frac{\partial(HG)}{\partial Z}$ does not change sign and is not identically zero. So, by BendixsonDulac criterion, there is no periodic curve in of the XZ-plane. Since $(\overline{\overline{S}}, \overline{\overline{S}})$ represent unique positive equilibrium point of the subsystem (16), so

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} Z(t) = \overline{\overline{S}}.$$

This completes the proof.

Theorem 3.4. *Suppose that $S_4 = (X^*, Y^*, Z^*)$ is exist, then it is GAS, if in addition of conditions (5), (6), the following inequalities hold:*

$$\left[\frac{2\alpha_1\alpha_2 Z^*(T_2+T_1)}{G_2(X,Y)} + \frac{2r_2 Y^*}{XX^*} - \frac{2\left(\beta + \beta^2 T X^*\right)}{F_2(X)}\right]^2 < 4\left[\frac{r_1}{K} - \frac{\beta^2 T Y^*}{F_2(K_1)} - \frac{\alpha_1^2 T_1 Z^*}{G_2(K_1,K_2)}\right]\left[\frac{r_2}{K} - \frac{\alpha_2^2 T_2 Z^*}{G_2(K_1,K_2)}\right] \tag{17}$$

$$\left[\frac{2\left(\alpha_1 + \alpha_1^2 T_1 X^* + \alpha_1\alpha_2 T_2 Y^*\right)}{G_2(X,Y)} - \frac{2r_3}{(X+Y)}\right]^2 < \frac{2r_3}{K}\left[\frac{r_1}{K} - \frac{\beta^2 T Y^*}{F_2(K_1)} - \frac{\alpha_1^2 T_1 Z^*}{G_2(K_1,K_2)}\right]$$

$$\left[\frac{2\left(\alpha_2+\alpha_1\alpha_2T_1X^*+\alpha_2^2T_2Y^*\right)}{G_2(X,Y)}-\frac{2r_3}{(X+Y)}\right]^2<\frac{2r_3}{K}\left[\frac{r_2}{K}-\frac{\alpha_2^2T_2Z^*}{G_2\left(K_1,K_2\right)}\right],$$

where, $F_2(X)=(1+\beta TX)\left(1+\beta TX^*\right)$ *and*

$$G_2(X,Y)=\left(1+\alpha_1T_1X+\alpha_2T_2Y\right)\left(1+\alpha_1T_1X^*+\alpha_2T_2Y^*\right) \tag{19}$$

Proof. Consider the function

$$L(X,Y,Z)=2\left[X-X^*-X^*\ln\left(\frac{X}{X^*}\right)\right]+2\left[Y-Y^*-Y^*\ln\left(\frac{Y}{Y^*}\right)\right]+2\left[Z-Z^*-Z^*\ln\left(\frac{Y}{Z^*}\right)\right].$$

Clearly, $L(X,Y,Z)\in C^1\left(R_+^3,R\right)$ With $L\left(X^*,Y_6,Z_6\right)=0$ and $L(X,Y,Z)>0$,for all $(X,Y,Z)\in R_+^3$ with $(X,Y,Z)\neq\left(X^*,Y_6,Z_6\right)$. Further,

$$\begin{aligned}
\frac{dL}{dt}=&-\left[\frac{r_1}{K}-\frac{\beta^2TY^*}{F_2(X)}-\frac{\alpha_1^2T_1Z^*}{G_2(X,Y)}\right]\left(X-X^*\right)^2-\left[\frac{r_2}{X}-\frac{\alpha_2^2T_2Z^*}{G_2(X,Y)}\right]\left(Y-Y^*\right)^2\\
&+\left[\frac{22_1\alpha_2Z^*\left(T_2+T_1\right)}{G_2(X,Y)}+\frac{2r_2Y^*}{XX^*}-\frac{2\left(\beta+\beta^2TX^*\right)}{F_2(X)}\right]\left(Y-Y^*\right)\left(X-X^*\right)\\
&-\left[\frac{r_1}{K}-\frac{\beta^2TY^*}{F_2(X)}-\frac{\alpha_1^2T_1Z^*}{G_2(X,Y)}\right]\left(X-X^*\right)^2-\frac{r_3}{(X+Y)}\left(Z-Z^*\right)^2\\
&-\left[\frac{2\left(\alpha_1+\alpha_1^2T_1X^*+\alpha_1\alpha_2T_2Y^*\right)}{G_2(X,Y)}-\frac{2r_3}{(X+Y)}\right]\left(X-X^*\right)\left(Z-Z^*\right)\\
&-\left[\frac{r_2}{X}-\frac{\alpha_2^2T_2Z^*}{G_2(X,Y)}\right]\left(Y-Y^*\right)^2-\frac{r_3}{(X+Y)}\left(Z-Z^*\right)^2\\
&-\left[\frac{2\left(\alpha_2+\alpha_1\alpha_2T_1X^*+\alpha_2^2T_2Y^*\right)}{G_2(X,Y)}-\frac{2r_3}{(X+Y)}\right]\left(Y-Y^*\right)\left(Z-Z^*\right).
\end{aligned}$$

Condition $(5,6)$ guaranteeing that: $\lim_{t\to\infty}\inf X(t)\geq K_1$ and $\lim_{t\to\infty}\inf Y(t)\geq K_2$. Therfore,

$$
\begin{aligned}
\frac{dL}{dt} < &-\left[\frac{r_1}{K}-\frac{\beta^2 T Y^*}{F_2\left(K_1\right)}-\frac{\alpha_1^2 T_1 Z^*}{G_2\left(K_1, K_2\right)}\right]\left(X-X^*\right)^2-\left[\frac{r_2}{K}-\frac{\alpha_2^2 T_2 Z^*}{G_2\left(K_1, K_2\right)}\right]\left(Y-Y^*\right)^2 \\
&+\left[\frac{2 \alpha_1 \alpha_2 Z^*\left(T_2+T_1\right)}{G_2(X, Y)}+\frac{2 r_2 Y^*}{X X^*}-\frac{2\left(\beta+\beta^2 T X^*\right)}{F_2(X)}\right]\left(Y-Y^*\right)\left(X-X^*\right) \\
&-\left[\frac{r_1}{K}-\frac{\beta^2 T Y^*}{F_2\left(K_1\right)}-\frac{\alpha_1^2 T_1 Z^*}{G_2\left(K_1, K_2\right)}\right]\left(X-X^*\right)^2-\frac{r_3}{2 K}\left(Z-Z^*\right)^2 \\
&-\left[\frac{2\left(\alpha_1+\alpha_1^2 T_1 X^*+\alpha_1 \alpha_2 T_2 Y^*\right)}{G_2(X Y)}-\frac{2 r_3}{(X+Y)}\right]\left(X-X^*\right)\left(Z-Z^*\right) \\
&-\left[\frac{r_2}{K}-\frac{\alpha_2^2 T_2 Z^*}{G_2\left(K_1, K_2\right)}\right]\left(Y-Y^*\right)^2-\frac{r_3}{2 K}\left(Z-Z^*\right)^2 \\
&-\left[\frac{2\left(\alpha_2+\alpha_1 \alpha_2 T_1 X^*+\alpha_2^2 T_2 Y^*\right)}{G_2(X, Y)}-\frac{2 r_3}{(X+Y)}\right]\left(Y-Y^*\right)\left(Z-Z^*\right).
\end{aligned}
$$

Provide both inequalities (17-19), it gets

$$
\begin{aligned}
\frac{dL}{dt} < &-\left[\sqrt{\frac{r_1}{K}-\frac{\beta^2 T Y^*}{F_2\left(K_1\right)}-\frac{\alpha_1^2 T_1 Z^*}{G_2\left(K_1, K_2\right)}}+\sqrt{\frac{r_2}{K}-\frac{\alpha_2^2 T_2 Z^*}{G_2\left(K_1, K_2\right)}}\right]^2-\left[\sqrt{\frac{r_3}{2K}}-\sqrt{\frac{r_2}{K}-\frac{\alpha_2^2 T_2 Z^*}{G_2\left(K_1, K_2\right)}}\right]^2 \\
&-\left[\sqrt{\frac{r_1}{K}-\frac{\beta^2 T Y^*}{F_2\left(K_1\right)}-\frac{\alpha_1^2 T_1 Z^*}{G_2\left(K_1, K_2\right)}}-\sqrt{\frac{r_3}{2K}}\right]^2 .
\end{aligned}
$$

So, $\frac{dL}{dt}$ is negative, and hence L_2 is Lyapunov function with respect to $S_4=(X^*, Y^*, Z^*)$, this proves the Theorem.

4. Hopf-bifurcation

Here, the occurrence of Hopf bifurcation in system1 near all steady states, are discussed as follows:

From Theorem (3) it is observed that the only prey existence steady state $S_1 = (K, 0, 0)$, apex predator-free steady state $S_2 = (\bar{S}, \bar{S}, 0)$, are not LAS, Therefore there is no possibility to have a Hopf bifurcation near S_1 and S_2. The conditions that guarantee the occurring of Hopf bifurcation near, intermediate predators-free steady state $S_3 = \left(\bar{\bar{S}}, 0, \bar{\bar{S}}\ \right)$ and coexistence steady state $S_4 = (X^*, Y^*, Z^*)$ are established in Theorem 4.1 and Theorem 4.2, respectively.

Theorem 4.1. *System (1) exhibits a Hopf bifurcation near intermediate predators-free steady state as the parameter r_3 passes through the value.*

$$\overline{r_3} = \frac{A_2 \left(R_1 R_5 - R_2 R_4\right)}{C_2 + A_2 \left(R_1 + R_3 + R_5\right)}. \tag{20}$$

If one of the following conditions holds
$R_1 + R_5 < 0, \quad R_1 R_5 > R_2 R_4$ *and* $R_1 + R_3 + R_5 > -\frac{C_2}{A_2}, \quad R_1 + R_5 < 0, \quad R_1 R_5 < R_2 R_4$ *and* $R_1 + R_3 + R_5 < -\frac{C_2}{A_2},$ *where* A_2, C_2 *and* $R_i; i = 1, 2, 3, 4, 5$ *are given in previous section.*

Proof. Recall that the eigenvalues of Variational matrix at $S_3 = (\overline{\overline{S}}, 0, \overline{\overline{S}})$ satisfy

$$\lambda^3 + A_2 \lambda^2 + B_2 \lambda + r_3 C_2 = 0. \tag{22}$$

The Hopf bifurcation near intermediate predators-free steady state of system (1) occurs if and only if two roots of Eq. (22) have two complex conjugate eigenvalues with the third eigenvalue real and negative such that there exists a constant parameter value, say $\overline{r_3}$ satisfying:

$$\mathrm{Ree}\left(\lambda\left(r_3\right)\right)_{r_3 = \overline{r_3}} = 0 \text{ and } \left[\frac{d\,\mathrm{Re}\,e\left(\lambda\left(r_3\right)\right)}{dr_3}\right]_{r_3 = \overline{r_3}} \neq 0$$

where λ is a complex root of Eq. (22) if $r_3 = \bar{r}_3$, then $A_2 B_2 = r_3 C_2$ and hence equation () can be written

$$\left(\lambda^2 + B_2\right)\left(\lambda + A_2\right) = 0 \tag{23}$$

Clearly, equation (23) has the following three roots:

$$\lambda_1 = i\sqrt{B_2},\ \lambda_2 = -i\sqrt{B_2} \text{ and } d\lambda_3 = -A_2.$$

However, for all values of r_3 in the neighborhood of $\overline{r_3}$, these roots can be written in general as

$$\lambda_1 = a\left(r_3\right) + ib\left(r_3\right),\ \lambda_2 = a\left(r_3\right) - ib\left(r_3\right) \text{ and } \lambda_3 = -A_2\left(r_3\right).$$

Clearly, $a\left(\overline{r_3}\right) = 0$ which means that the first condition of Hopf bifurcation holds. Now, the proof will follows, if we can verify the above second condition (known as transversality condition) of Hopf bifurcation when $\mathrm{Ree}\left(\lambda\left(r_3\right)\right) = a\left(r_3\right)$.

Thus, by substituting $\lambda_1 = a\left(r_3\right) + ib\left(r_3\right)$ in (22) and calculating the derivative with respect to the r_3, it is obtained that

$$\begin{cases} D_1\left(r_3\right)\frac{da(r_3)}{dr_3} - D_2\left(r_3\right)\frac{db(r_3)}{dr_3} + D_3\left(r_3\right) = 0 \\ D_1\left(r_3\right)\frac{db(r_3)}{dr_3} + D_2\left(r_3\right)\frac{da(r_3)}{dr_3} + D_4\left(r_3\right) = 0 \end{cases} \tag{24}$$

where

$$D_1\left(r_3\right) = 3\left[a\left(r_3\right)\right]^2 + 2A_2\left(r_3\right)a\left(r_3\right) + B_2\left(r_3\right) - 3\left[b\left(r_3\right)\right]^2$$

$$D_2(r_3) = 6a(r_3)\,b(r_3) + 2A_2(r_3)\,b(r_3)$$
$$D_3(r_3) = a(r_3)\,\frac{dB_2(r_3)}{dr_3} + C_2$$
$$D_4(r_3) = b(r_3)\,\frac{dB_2(r_3)}{dr_3}$$

Thus, by solving the linear system (24) for the unknown $\frac{da(r_3)}{dr_3}$,it gets

$$\frac{da(r_3)}{dr_3} = \frac{d\,\text{Ree}(\lambda(r_3))}{dr_3} = -\frac{D_2(r_3)\,D_4(r_3) + D_1(r_3)\,D_3(r_3)}{[D_1(r_3)]^2 + [D_2(r_3)]^2}$$

So for $r_3 = \overline{r_3}$, it is easy to verify that

$$\left[\frac{d\,\text{Ree}(\lambda(r_3))}{dr_3}\right]_{r_3=\overline{r_3}} \neq 0$$

Also, condition (20) or (21) guarantee that A_2 is positive for all values of r_3, and hence $\lambda_3 = -A_2(r_3) < 0$. The proof is completed.

In similar ways of proving above theorem we can get the following theorem.

Theorem 4.2. *System (1) exhibits a Hopf bifurcation coexistence steady state as the parameter r_3 passes through the value*

$$\overline{r_3} = \frac{A_3(E_1E_5 - E_2E_4)}{C_3 + A_3(E_1 + E_3 + E_5)} \tag{25}$$

If one of the following condition holds

$$E_1 + E_5 < 0, \quad E_1E_5 > E_2E_4 \text{ and } E_1 + E_3 + E_5 + E_6 > -\frac{C_3}{A_3}E_1 + E_5 < 0,$$
$$E_1E_5 < E_2E_4 \text{ and } E_1 + E_3 + E_5 + E_6 < -\frac{C_3}{A_3}$$

where A_3, C_3 and $E_i; i = 1,2,3,4,5,6$ are given in previous section

5. Numerical simulation

In order to support the analytical finding in this paper via the phase portrait and time series[22] , some numerical simulations are performed; all the simulations are carried out through runga kutta method of order six method, using MATLAB. First, we assume two set of parameter values as given in (27) and (28).

$$\begin{aligned} r_1 = 0.5,\ r_2 = 0.3,\ r_3 = 0.15,\ K = 1000,\ \beta = 0.001 \\ \text{T} = 1,\ \alpha_1 = 0.005,\ \alpha_2 = 0.004,\ \ \text{T}_1 = 1 \text{ and } \ \text{T}_2 = 1 \end{aligned} \tag{27}$$

$$\begin{aligned} r_1 = 1.2,\ r_2 = 0.9,\ r_3 = 0.15,\ K = 1000,\ \beta = 0.001 \\ \mathrm{T} = 1,\ \alpha_1 = 0.005,\ \alpha_2 = 0.004,\ \ \mathrm{T}_1 = 1 \text{ and } \ \mathrm{T}_2 = 1 \end{aligned} \tag{28}$$

The parameter values in (27) and (28), satisfy the condition for global stability of the intermediate predator-free steady state and coexistence steady state, respectively.

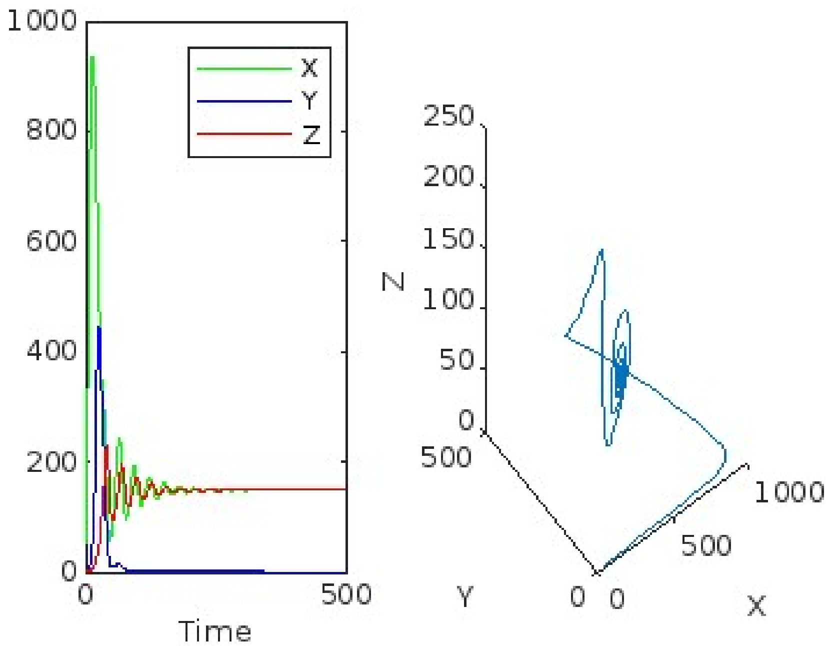

Figure 1: Both the time series and the phase portrait shows that model 1 approaches intermediate predator free Steady state, when the parameter values are as given in (27).

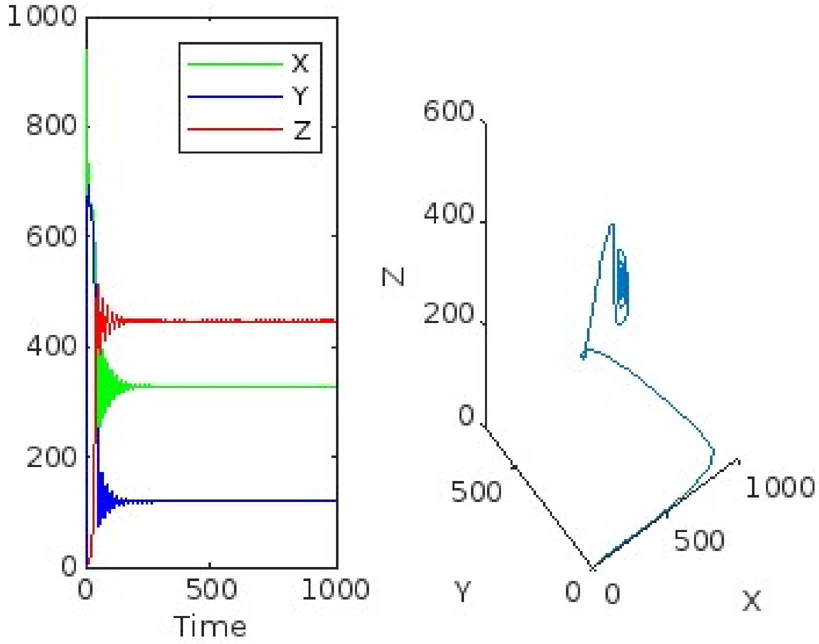

Figure 2: Both the time series and the phase portrait shows that model 1 approaches coexistence Steady state, when the parameter values are as given in (28)

Therefore, Fig. 1 and Fig. 2 confirm the analytical results regarding to global stability of intermediate predator-free steady state and coexistence steady state, respectively.

The critical values $\overline{r_3} \approx 0.11$ and $\overline{r_3} \approx 0.95$ for the parameters given in (27) and (28), respectively.

First let decrease the value of r_3 to 0.1 and fixed another parameter as given in (27), then it is observed that, the dynamics of the system (1) induced a transition from the a stability situation to the state where the prey species and apex predators oscillate periodically oscillate periodically as illustrated in Fig.3. Further, for a range of values $r_3 \in$ [004,0.12], in Fig.4, bifurcation diagram with respect to r_3 is drawn. From Fig. 4, the prey and apex predator populations show oscillatory for the further smaller values of r_3; This indicates that decreasing r_3 may induce a transition from the a stability situation to the state where the populations oscillate periodically.

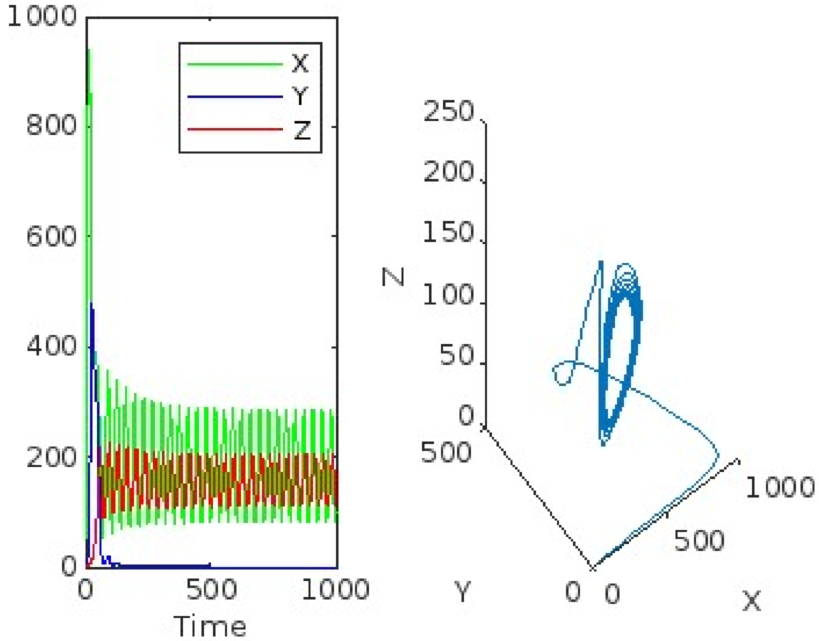

Figure 3: Both the time series and the phase portrait show periodic oscillations around intermediate predators-free steady state, when $r_3 = \mathbf{0.1}$ and other parameter values are as given in (27).

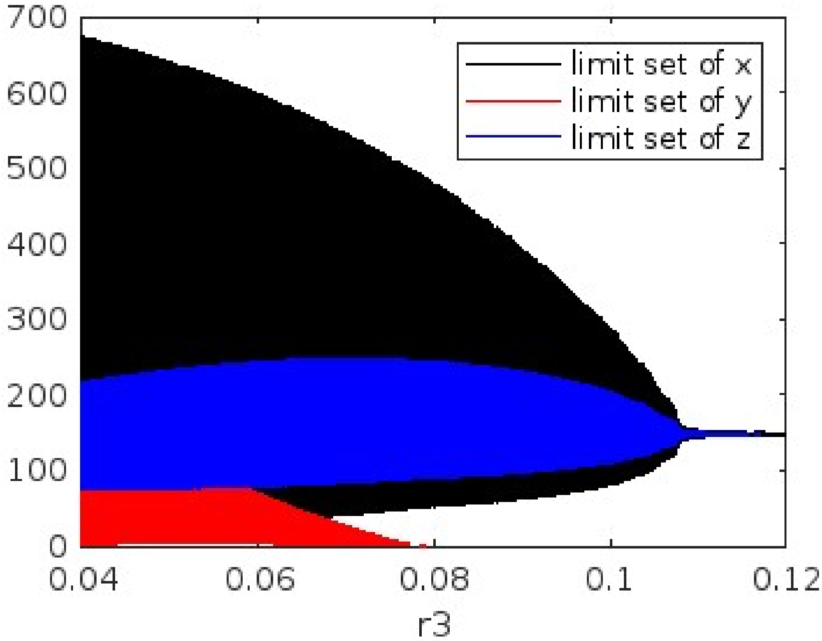

Figure 4: The figure shows the bifurcation diagrams of system (1), which indicates the emergence of Hopf bifurcations as parameter r_3 decreases, and other parameters are fixed as in(27).

Again let further decrease the value of r_3 to 0.07 and fixed other parameters given in (28), then it is observed that, the dynamics of the system (1) induced a transition from the a stability situation to the state where the population oscillate periodically oscillate periodically as illustrated in Fig.5. Further, for a range of values $r_3 \in [0, 0.2]$, in Fig.6, bifurcation diagram with respect to r_3 is drawn. From Fig. 6, all the species show oscillatory for the further smaller values of r_3; This indicates that decreasing r_3 may induce a transition from the a stability situation to the state where the populations oscillate periodically.

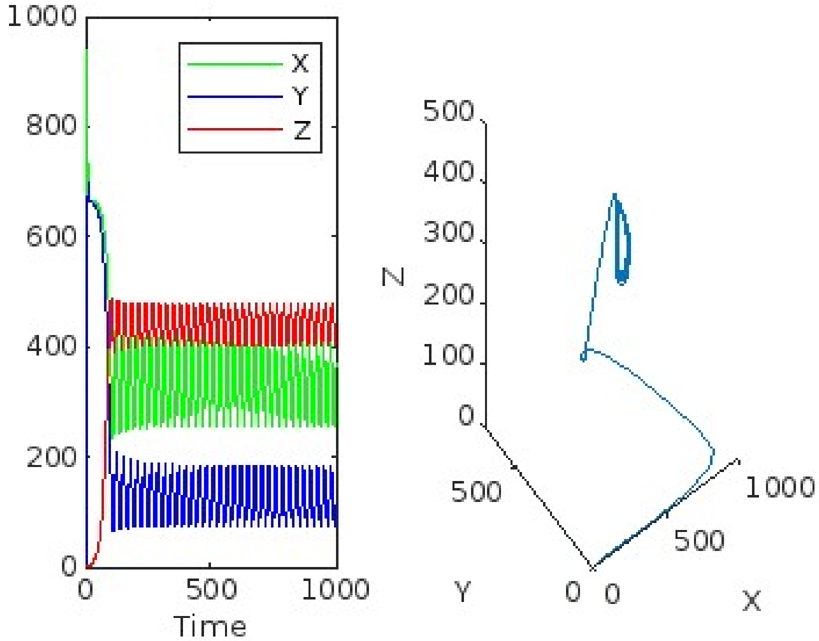

Figure 5: Both the time series and the phase portrait shows periodic oscillations around coexistence Steady state, when $r_3 = \mathbf{0.07}$ and other parameter values are as given in (28).

Note that Fig. 3 and Fig. 5 confirms the analytical finding in Theorem 6 and Theorem 7, respectively.

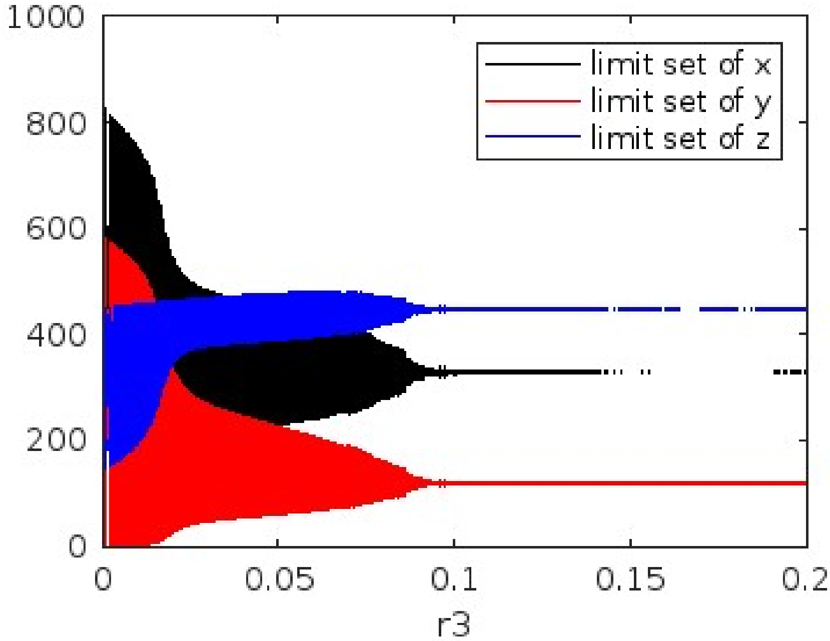

Figure 6: The figure shows the bifurcation diagrams of system (1), which indicates the emergence of Hopf bifurcations as parameter r_3 decreases, and other parameters are fixed as in(28).

6. Conclusion

A three species food-web is considered in this paper, the species are, prey, intermediate predators and apex predators, the intermediate predators predating the prey with the Holling type-II functional response, while the apex predators predating both prey and intermediate predators according to extended Holling type II functional response for two prey species. it is proved that under condition (5)and(6), the model is permanent, it is explored that the model has biologically possible steady states, they are $S_1 = (K, 0, 0), S_2 = (\bar{S}, \bar{S}, 0), S_3 = (\bar{\bar{S}}, 0, \bar{\bar{S}})$ and $S_4 = (X^*, Y^*, Z^*)$, and also it is discovered that both $S_1 = (K, 0, 0), S_2 = (\bar{S}, \bar{S}, 0)$ are unstable, but stability as well as global stability for both $S_3 = (\bar{\bar{S}}, 0, \bar{\bar{S}})$ and $S_4 = (X^*, Y^*, Z^*)$ based on the sample parameters are given. And also it is proved that critical values $\overline{r_3}$ and $\overline{r_3}$, make the occurrence of Hopf-bifurcation of thee model near $S_3 = (\bar{\bar{S}}, 0, \bar{\bar{S}})$ and $S_4 = (X^*, Y^*, Z^*)$, respectively. The model solved numerically by choosing suitable value of the model parameters as given in (27) and (28), the numerical solutions insulated in Fig. 1 and Fig.2, it is explained that both figures confirms analytical results regarding to the stability of both $S_3 = (\bar{\bar{S}}, 0, \bar{\bar{S}})$ and $S_4 = (X^*, Y^*, Z^*)$, respectively. Also we have shown that Fig 3, and Fig.4, confirm analytical results regard to Hopf-bifurcation near $S_3 = (\bar{\bar{S}}, 0, \bar{\bar{S}})$ and $S_4 = (X^*, Y^*, Z^*)$, respectively. Because they show periodic solution of the model.

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