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A Novel Inspection of a Time-Delayed Rolling of a Rigid Rod

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Abstract. The work examines the stability configuration of a rolling rigid rod in the presence of a time-delayed (TD) in a square position as well as velocity. Examining time-delayed rolling rigid rod bridges presents real engineering challenges and raises significant theoretical questions, making it a desirable problem in applied and theoretical contexts. It is recommended to use the non-perturbative approach (NPA) to find an equivalent linearized differential equation. Actually, the NPA is based on the He's frequency formula (HFF). The Mathematica Software (MS) is used to compare and assess of this suitability. Using the appropriate numerical methodology (NM), a matching between the strong nonlinear ordinary differential equation (ODE) and the equivalent analytical linear one is obtained. This matching has revealed a very significant agreement for different criteria. Stated differently, the new performance appears powerful, promising, and beneficial, and it may be applied to different classes of nonlinear oscillators. It is well-precision, flexible, and convenient. The new approach has many advantages in contrast to all other perturbed methods. It avoids the usage of the Taylor expansion in expanding the restoring forces; particularly in the topic of the dynamical systems. Therefore, the stability analysis is analyzed and the current work no longer incorporates this shortcoming. Furthermore, the temporal histories of the obtained novel outcomes and their various stable zones are accomplished. It is possible to examine the relevance of the employed parameter and demonstrate the precision of the outcomes through an exploration of the data. It is found that periodic solutions provide reliable and predictable behavior in a system, whereas phase plane diagrams offer a visual and quantitative comprehension of dynamics and stability. This appreciation is vital for ensuring safe operation under different conditions.

2020 Mathematics Subject Classifications: 70K20 , 70Q05, 70K05, 70K40

Key Words and Phrases: Rolling rigid rod, Nonlinear vibrations, Square position-velocity time delay, He's frequency formula, Non-perturbative approach, Stability configuration

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Nomenclature
L: Length of the rod.
θ: Angular displacement.
t: Time.
$\cdot$: Time derivative.
τ: Time-decay.
M: Uniform rigid rod's mass.
g: Gravity's acceleration.
$2r$: Circular surface diameter.
$C.M$: Center of mass of the uniform rod.
a, b : Non-dimensional quantities.

1. Introduction

There are various amounts of nonlinear phenomena present in all practical structures. The nonlinearity's characteristics are usually understated enough to be not ignored. Consequently, the linear theories cannot be used to study the dynamics and behavior of these systems. On the other hand, many structures in the real-world have large nonlinearities that make linear theories out-of-date. Furthermore, the system richer the dynamical behavior, such as period doubling, quasi-periodic behavior, and chaotic response, may be disregarded if linear techniques are applied to highly nonlinear systems. Therefore, a full understanding of nonlinear system dynamics is necessary to comprehend organizational dynamics. Differential equation systems, ordinary or partial, are widely recognized for their ability to represent a wide range of engineering systems. With a few exclusions, their exact solutions seem too challenging and unachievable. In order to make handling numerous nonlinear equations easier, many academics are focused their attention on the approximation solutions for specific weak nonlinear issues, such as the mean approach and the procedure of extremely tiny parameters [9, 10, 35, 42]. Analysis was done on the magnetic spherical pendulum problem [16]. The issue was investigated using the advanced nonlinear frequency, additionally with the homotopy perturbation method (HPM), and the Laplace transforms. It was studied how to move a simple pendulum with a light spring and a rolling wheel attachment [14]. The basic motion equation experienced a transformation into high nonlinear ODE under specific conditions. The complicated behavior of a perturbed Van der Pol-Duffing oscillator was effectively reduced by utilizing TD position and velocity [15]. A time delay was implemented as an additional protection to prevent the system under examination from undergoing the nonlinear vibrations. The issue of this work was especially current since technologies with a TD are the subject of many investigations recently. Over the recent years, the techniques involving TD have increasingly drawn interest. Henceforth, research was conducted on a TD controller for a damped, nonlinear, excited Duffing oscillator (DO) [19]. The modified periodic solution that forms the basis of the current investigation is called the HFF. The examination of the stability analysis concerning the disturbed pendulum movement was conducted using the adapted HPM. The TD for position and velocity are employed in order to reduce the complex oscillation's behavior of the framework under consideration. Additional progress in this field was performed [2, 24]. A precise solution for a nonlinear oscillator with a mass that depends on the position of the coordinate was derived [1]. The findings collected demonstrated that the proposed technique was a promising tool for solving the van der Pol oscillator and providing information on the phase portrait, so proving the stability of the system [3]. The study reported the use of first- and second-order difference techniques to find numerical solutions for initial boundary value issues of both homogeneous and nonhomogeneous Helmholtz equations [5]. The stability of these procedures is thoroughly examined, guaranteeing their dependability and convergence over a broad spectrum of problem situations. An approach that is dependable and utilizes an adapted version of the conventional differential transform method was introduced [4]. The results collected demonstrated that the proposed technique is a highly promising tool for solving the van der Pol oscillator and providing accurate information on the phase portrait, so proving the system's stability.

In real-life, there have been situations where objects are just tangentially attached to their framework, allowing rocks to fall on nearby materials. These include radioactive fuel cells in reactors, air extraction columns, petroleum cracking towers, and liquid petrol tanks. Undeniably, the most lucid illustration of structural responses during seismic activities is their rocking motion. Despite the seemingly straightforward nature of this phenomenon, the swaying and toppling of rigid structures in response to basic stimuli present significant challenges. The primary inspiration for comprehending this rocking issue stems from the potential to utilize it as a preventive measure. This understanding could be instrumental in safeguarding structures, furnishings, and apparatus from collapsing and posing threats to human safety during the tremors of an earthquake. Early and good results were obtained from the shaking of the isolation system [13, 41]. The first-order approximations of the iterative perturbation technique are explored to mimic pertains to the oscillation patterns observed in a stiff bar that sways reciprocally on a cylindrical surface without any slippage, along with the cubic-quintic DO [6]. An investigation of the movement of a rigid rod swaying back and cubic-quintic DO employed He's energy balancing approach [11]. It is found that this approach does not require the linearization process or small perturbations, and it is very practical and efficient. The problem of a stiff rod rocking while slipping on a stiff circular surface was examined [17]. From the principles of the Euler-Lagrange theorem, one can potentially derive the overarching equation governing motion. Analysis was done on the stability analysis of a rocking rigid rod using TD square position as well as velocity [17]. The TD technologies have been the subject of several researches lately; therefore this topic of investigation was especially relevant. The TD served as a supplementary protection against the nonlinearly oscillating system that was being examined. In all the previous efforts, the restorative powers were weakened through the application of the Taylor expansion. On the other hand, the current representation is incomparable in that it maintains these forces. Recently, the HFF were adopted in analyzing several works in the dynamical system as well as the Electrohydrodynamics stability [8, 18, 20–23, 25–34].

Nonlinear oscillations are a part of ordinary life and comparable technical systems. Nonlinear vibrations, as one of the most significant and often used prototypes in dynamic structures, are vital in considering different nonlinear occurrences in electrical manufacturing and industry. The HFF is the most straightforward way to calculate the frequency-amplitude relationship, while analyzing different classes of nonlinear dynamical systems. The calculation of the remaining values is compared with the formulation of frequency-amplitude and its diverse versions [40]. This study suggested introducing a free parameter to offer an accurate estimation of the nonlinear oscillator frequency and to reduce the number of techniques needed for the residual computation. Using residuals from two trial solutions, the HFF is used to find the nonlinear oscillator frequency and amplitude relationship [37]. This approach can still be improved, even in the unlikely event that an extremely precise response is produced. For nonlinear oscillators, the HFF offered the most straightforward and precise frequency formulation [36]. The un-damped DO and its related family are successfully resolved with it [7]. The complexity characteristic in evaluating the DO with advanced nonlinearity, as well as the quadratic damping equation, has marked this subject as a critical area necessitating comprehensive investigation for more accurate solutions. Since linear equations are frequently perfect solutions, the HFF for damped nonlinear oscillation is a hot topic. The extremely accurate solution, also known as the linearized equation solution, illustrated how to approach the nonlinear issue. An exact solution is obtained when equations having constant coefficients are made linear through the application of the HPM. The existing study aims to utilize the HFF in identifying the frequency of a nonlinear oscillation, which may be subject to linear or nonlinear damping forces.

The TD rolling of a rigid rod is the occurrence in which the rolling motion of a rod is affected by a delay in the application of forces or torques. This delay can lead to intricate dynamics, such as oscillations, alterations in stability, or changing rolling trajectories. The TD in mechanical systems frequently occurs as a result of factors such as control system feedback, material deformation, or delayed actuation. Several distinguished uses can be concisely summarized as follows: The TD rolling can have a considerable impact on the movement control of robotic manipulators or mobile robots, especially when precise control is required. Gaining comprehension of this TD can aid in formulating enhanced control algorithms to alleviate unwanted oscillations or instability.

The TD rolling can be employed in engineering structures to serve as vibration damping systems. Engineers can enhance the stability and durability of structures by implementing controlled delays to mitigate or counter undesired vibrations. Understanding the delayed rolling behavior of conveyor systems, particularly when handling elongated objects such as rods, can result in improved design and control strategies to prevent jams or misalignment during material transport.

At the micro and nano scales, the TD can be more pronounced because of the slower response times in actuation and sensing. Precision is crucial when developing instruments, as even slight delays can have a significant impact on the performance and accuracy of the device. Understanding TD rolling can improve the dependability and predictability of deployment mechanisms in military electronics, especially in situations when timing and synchronization are critical. This understanding ensures that systems function appropriately under time-sensitive conditions.

When designing tires or rolling elements for automotive and aerospace applications, including the effects of time-delayed rolling can enhance the precision of simulations and optimize the performance of these components in dynamic settings, especially in high-speed or high-stress environments.

Comprehending and controlling these delays are essential for maximizing performance and guaranteeing the stability and dependability of the system.

Owing to the aforementioned potentials, their exceptional sensitivity to dynamic loading, geometric variations, and dissipation challenges, assessing the rocking and toppling reactions of rigid blocks during motion events presents a substantial challenge. Therefore, examining the dynamic behavior of a rigid rocking rod over a circular surface under the assumption of pure rolling without slippage is the aim of this paper. According to classical mechanics, the motion's governing equation is controlled as a nonlinear ODE with highly nonlinear orders. Regarding the innovative approach or noteworthy outcomes, it is vital to emphasize the following:

The new technique yields highly nonlinear ODE that exactly has the same behavior as the existing nonlinear one.

There must be a harmonious correlation between the mathematical expressions for the new method to work.

All traditional methods utilize the Taylor series expansion to reduce the complexity of the specified issue in the presence of restoring forces. The current paper has no longer encompasses this shortcoming.

We can examine the problem's stability analysis thanks to the current methodology, which is not possible with other conventional techniques.

Lastly, the novel approach seems to be an easy-to-use, worthwhile, and interesting tool. It is useful for the analysis of several types of nonlinear oscillators.

To cartelize the presentation of the paper, the remaining text is organized as follows: The NPA yields an equivalent linear equation in § 2. This Section presents a confirmation using the numerical solution via NM. The graphical plots, including time history and stability, are presented in § 3 together with their interpretations based on the obtained data. Lately, closing thoughts are offered in § 4.

2. Methodology Structure

The improvement of the existing equation is used to analyze the motion of a solid rod that rocks. The suitable response to the periodic motions remains challenging due to the structures of related essential equations. Therefore, the goal of this work is to find periodic

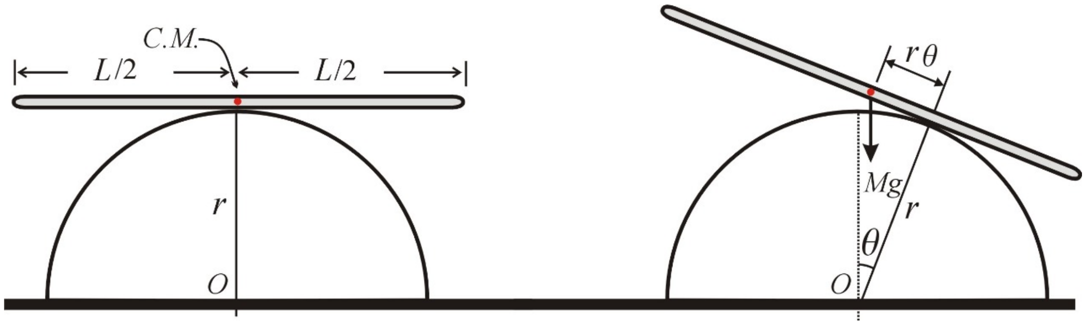

Figure 1: Sketches the model being examined.

analytical solutions using a more straightforward scheme for this type of dynamical system. In this instance, any other conventional perturbation method in addition to the Taylor expansion is not required. The subject under research is established by a consistently inflexible rod actively rolling across a circular surface without sliding. The length and the mass of the rod are both been retained. The rod is first resting horizontally at the tangency point, which is situated at the center of mass (C.M.), as shown in Fig. (1). The arc $r\theta$ with no-slip motion, which is defined as a small movement on the circle, must eventually be used to characterize the distance between the center of mass and the tangency point. Currently, going back to our earlier work [17], we can extract the uniform stiff swinging rod's regulating equation of motion, which took the following form [17]:

$$\frac{ML^2}{12}\ddot{\theta} + Mr^2\left(\dot{\theta}^2\ddot{\theta} + \ddot{\theta}\dot{\theta}^2\right) + Mg\cos\theta = 0. \tag{1}$$

The foundation of this article provides a comprehensive inventory of all the variables employed in Eq. (1). The anticipated model's outline is illustrated in Fig. (1).

Simply, Eq. (1) will be converted to the subsequent simplified form:

$$\ddot{\theta} + a(\theta^2\ddot{\theta} + \dot{\theta}^2) + b\cos\theta = 0, \tag{2}$$

where a,and b are two non-dimensional physical quantities.

As was widely known, TD controllers were proposed to control the nonlinear disturbances. As was previously shown, the loop delay can significantly affect the stability of the system' or instability. However, when the ideal TD was considered, the subsequent analytical and numerical evaluations demonstrated that the nonlinear position and nonlinear velocity are the best at damping the vibration. A simple procedure for determining the ideal loop delay values in a way that enhances the system profile was suggested [38]. Moreover, the effectiveness of six iterations of linear and nonlinear feedback controllers for position, acceleration, and velocity was studied [38]. It was discovered that the best method for reducing vibration and suppressing bifurcations was the TD cubic acceleration control. In view of the previous achievement, we consequently incorporate the square TD in location and velocity into the existing model. Accordingly, the basic equation of motion that controls all future motion is as follows:

$$\ddot{\theta} + a\left[\theta^2\ddot{\theta}(t-\tau) + \theta\dot{\theta}^2(t-\tau)\right] + b\theta\cos\theta = 0. \tag{3}$$

One could ideally imagine the initial conditions (ICs), as in the next form:

$$\theta(0) = A, \quad \dot{\theta}(0) = 0. \tag{4}$$

At this stage, returning again to the fundamental TD as given in Eq. (3), the NPA enables us to transform the nonlinear ODE as shown in Eq. (3) into an equivalent linear one as $\ddot{u} + \Omega^2 u = 0$ under the similar ICs that are given in Eq. (4), where Ω^2 is known as the total frequency that depends on all parameters of the original system.

$$u = A\cos(\Omega t), \quad \dot{u} = -A\Omega\cos(\Omega t), and \quad \ddot{u} = -\Omega^2 u, \tag{5}$$

where A is the initial oscillation amplitude.
Appropriately, the shift of the independent time may be expressed as:

$$\begin{aligned} u(t-\tau) &= A\cos(t-\tau) \\ &= A(\cos\Omega t\cos\Omega\tau + \sin\Omega t\sin\Omega\tau) \\ &= u(t)\cos\Omega t - \frac{1}{\Omega}\dot{u}(t)\sin\Omega\tau. \end{aligned} \tag{6}$$

It follows that

$$\ddot{u}(t-\tau) = \dot{u}(t)\cos\Omega(t) + \Omega u(t)\sin\Omega(\tau). \tag{7}$$

Eq. (3) may be written as follows:

$$\ddot{\theta} + f(\theta, \dot{\theta}, \ddot{\theta}) + f(\theta, \dot{\theta}, \ddot{\theta}) = 0, \tag{8}$$

where

$$\begin{aligned} f(\theta, \dot{\theta}, \ddot{\theta}) =& \alpha(\ddot{\theta}^2 + \dot{\theta}^2)\cos^2\Omega t + \left(\Omega^2\dot{\theta}^2 + \frac{1}{\Omega^2}\ddot{\theta}^2\right)\sin^2\Omega t + b\theta\cos\theta, \\ g(\theta, \dot{\theta}, \ddot{\theta}) =& \alpha\Omega\left(\dot{\theta}\ddot{\theta} - \frac{1}{\Omega^2}\dot{\theta}^3\right)\sin 2\Omega t. \end{aligned} \tag{9}$$

An equivalent frequency ω_{eqv}^2 can by evaluated as shown previously by Moatimid et al. [8, 18, 20–23, 25–34] in the following manner:

$$\omega_{eqv}^2 = \frac{\int_0^{2\pi/\Omega} uf(u, \dot{u}, \ddot{u})dt}{\int_0^{2\pi/\Omega} u^2 dt} = \frac{2b}{A}\left(J_1(A) - AJ_2(A)\right) - \frac{1}{2}aA^2\Omega^2\cos 2\Omega t, \tag{10}$$

where $J_1(A)$, and $J_2(A)$are the first kind of Bessel functions of the argument A of order one and two, respectively. Additionally, following Moatimid et al. [8, 23, 27, 28], the equivalent damping term can be evaluated as follows:

$$\sigma_{eqv} = \frac{\int_0^{2\pi/\Omega} \dot{u}g(u, \dot{u}, \ddot{u})dt}{\int_0^{2\pi/\Omega} \dot{u}^2 dt} = -\frac{1}{2}aA^2\Omega^2\sin 2\Omega t. \tag{11}$$

The equivalent linear ODE can now be constructed as follows:

$$\ddot{u} + \sigma_{eqv}\dot{u} + \omega_{eqv}^2 u = 0. \tag{12}$$

Furthermore, the standard normal form may be achieved concurrently with the transformation $u(t) = h(t)(-\sigma_{eqv}t/2)$. Following the standard calculus, the unknown function satisfies the following damped-simple harmonic differential equation:

$$\ddot{h} + \Omega^2 h = 0, \tag{13}$$

where $\Omega^2 = \omega_{eqv}^2 - \sigma_{eqv}^2/4$.
In other words, the equivalence frequency can be obtained by combining the results in Eqs. (10) and (11) with the previous relation to produce:

$$\Omega^2 = \frac{2b}{A}\left(J_1(A) - AJ_2(A)\right) - \frac{1}{2}aA^2\Omega^2\cos 2\Omega\tau - \frac{1}{4}\left(\frac{1}{2}aA^2\Omega^2\sin 2\Omega\tau\right)^2. \tag{14}$$

Indeed, the governing equation of the total frequency as given in the previous Eq. (14) is a transcendental one. Therefore, the numerical stability discussions are plotted via the Command (PlotRegion) in the MS, for this purpose, see the subsequent Figs. (6), (7),(8), and (9). Additionally, because the time-delay is infinitesimal, employing Taylor expansion, then Eq. (14) will be polynomial in Ω^2. In this case the stability standard requires:

$$\Omega^2 > 0, \quad \text{and} \quad \sigma_{eqv} > 0. \tag{15}$$

To obtain the value of the equivalent frequency, consider the following data sample and the MS:

$$a = 5.0, \quad b = 3.0, \quad \tau = 0.01, \text{and} \quad A = 0.1$$

Utilizing the assistance of the Mathematica Software through the command of FindRoot, the value of the total frequency becomes 1.39425 . Once more, Eq. (14) gives the characteristic equation in the total frequency equation. Actually, this equation is a transcendental one; it has no exact solution. To analyze the stability configuration, we are forced to adopt the MS via the command PlotRegion.

3. Graphical Representations and Discussions of the Results

The purpose of the current section is to highlight the graphical representations and discuss the achieved results in light of the following data of the parameters:

$$A = (0.8, 0.9, 1), \quad b = (0.1, 0.5, 1, 1.5), \quad a = (0.1, 0.5, 1, 1.5), \quad \tau = (0.1, 0.5, 1).$$

It is easy to access to match the numerical solution (NS) of the previous Eq. (3) via the knowledge of NDSolve of the MS with the solution of the equivalent linear ODE. Therefore, Fig.(2) is graphed for $A = 1, a = 0.1$, and $b = 1$ when the TD parameter τ has the value 0.1. As seen from this comparison, there is an excellent consistency

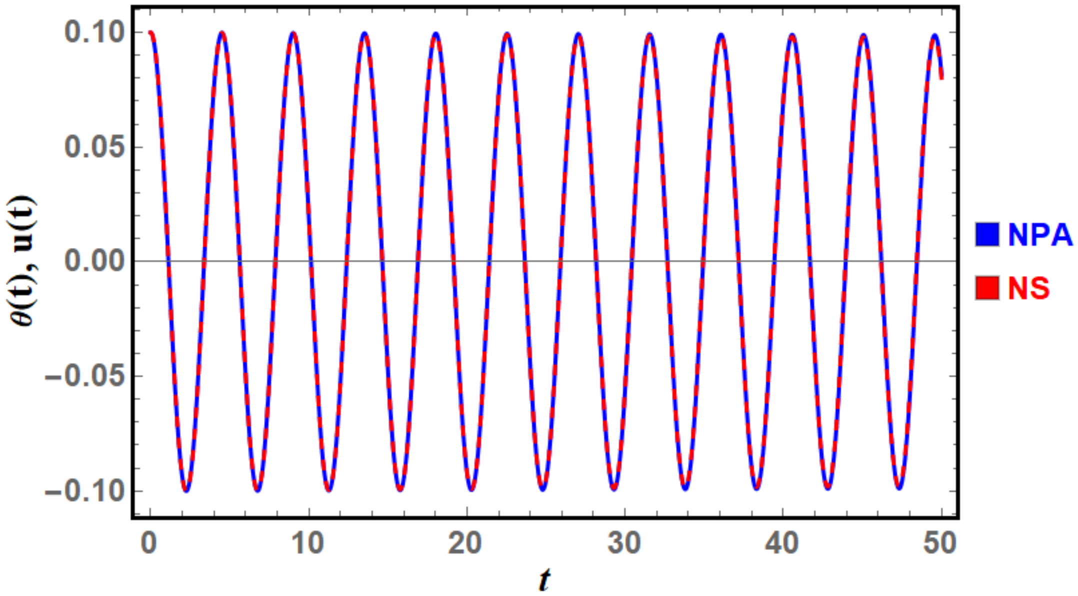

Figure 2: Explores the comparison connecting the AS and NS.

concerning the analytical solution and numerical one, which demonstrates good accuracy of the NPA. Additionally, up to the time of 50 units, the MS showed that the absolute error between the analytical and numerical solutions is 0.002025. To validate the absolute error between them, Table (3) is presented. The inspection of the drawn curves of this figure explores their periodic forms which give an impression of the stability and steady behavior of mutual solutions.

Time	Analytical Solution	Approximate Solution	Absolute Error
0	1	1	0.0000000000000000
5	-0.6367368984628483	-0.6840673392798011	0.04733044081695281
10	-0.1891322442718248	-0.165428295011344	0.02370394927069039
15	0.8775918556967678	0.8671456885746593	0.01044616712210844
20	-0.9284579883534055	-0.9463244488562964	0.017866460211890867
25	0.30437506401763743	0.37993757885446146	0.0755625143862673
30	0.540334930370593	0.521267116029197	0.01906781434157334
35	-0.9928774394082572	-0.9734314088762631	0.01944603040994164
40	0.7240684722745052	0.758824169662992	0.0347556793179396
45	0.0707952127866591	0.0599615698265702	0.01083361589400279
50	-0.8142243207060901	-0.8215978344057133	0.007373513699623246

The solutions of Eq.(3) at various values of A, a, and b are plotted, respectively, in Figs. ((3), (4), and (5)). The curves in Fig. (3) are calculated when $A = 0.8, 0.9$, and 1 and with the similar amounts of other factors as shown in Fig. (2), while curves in Fig. (4) are drawn at $a = (0, 1, 0.5, 1)$, $A = 1, b = 1$, and $\tau = 0.1$. In addition, the variation of the value of b is examined in Fig. (5) at $a = A = b = 1$, and $\tau = 0.1$.

The parts (a) of these figures show the temporal history of θ according to the variation of the above-mentioned parameters. These curves have a decaying form, where the amplitudes of the waves are improved with the rise of A values, and the wave number increases with time, as seen in Fig. (3a). The number of oscillations and their wave's number, are shown in Fig. 4(a), remain stationary with the change of a values, in which their amplitudes decrease with the increase of the values of a. However, the change of the values b of has a positive impact on the behavior of the waves describing θ, as seen in Fig. (5a). Looking at the curves of this figure, in more detail, reveals that the fluctuation numbers increase with the increase of the values of b, i.e., as the wavelength decreases, even though the amplitudes of these curves persist unaffected. When we going to compare the curves of Fig. (3a), Fig. (4a), and Fig. (5a) with each other, one can say that the number of wave fluctuations rises with the change of A than with the difference between both a and b. Moreover, these waves have variable amplitudes, as seen in Figs. (4a) and (5a), but in Fig. (5a), they remain constant. Figs. (3b), (4b), and (5b) reveal the corresponding diagrams of phase planes which have the forms of closed curves to maintain the stability of the solutions.

The drawn curves in Figs. (3b) and (4b) are close to each other and take the direction inward, which indicates the stability and stationary behavior of the solution. The reason is due to the change in the amplitudes of the waves. On the other hand, one finds that these curves take symmetrical closed forms, as shown in Fig. (5b) because the amplitudes of the drawn waves in this figure don not change to some extent. However, this interpretation confirms the validity of what has been discussed in the previous discussion.

The symmetry in the phase plane curves for periodic waves arises due to the nature of the underlying differential equations governing the system. These curves represent the relationship between a system's position and its velocity as it evolves over time. The symmetry of these curves backs to the following reasons: In a periodic system, the motion repeats itself after a certain time period. This implies that for every point in the phase plane, the system will return to the same point after one period. This repetitive motion is a key factor leading to the closed curves in the phase plane. In conservative systems, the total mechanical energy is constant. This constancy is mirrored in the symmetry of the phase plane. As the system oscillates, energy continuously shifts between kinetic and potential forms, resulting in symmetric trajectories in the phase plane. If the system is linear, the phase plane trajectories are elliptical and perfectly symmetric about the origin. This symmetry is due to the linear relationship between the restoring force and displacement in harmonic motion. Even in nonlinear systems, if the nonlinearities are symmetric, the phase plane curves can remain symmetric. Although the type and degree of nonlinearity may change the shape of the curves, symmetry in the potential function frequently leads to symmetric trajectories in the phase plane.

For more convenience, since the TD must be of a positive real value, the inequality Eq. (15) has been drawn to reflect the influences of the physical parameters $A.a$, and b, as explored in Figs (6,7,8), and (9). These figures are graphed to show the drawn

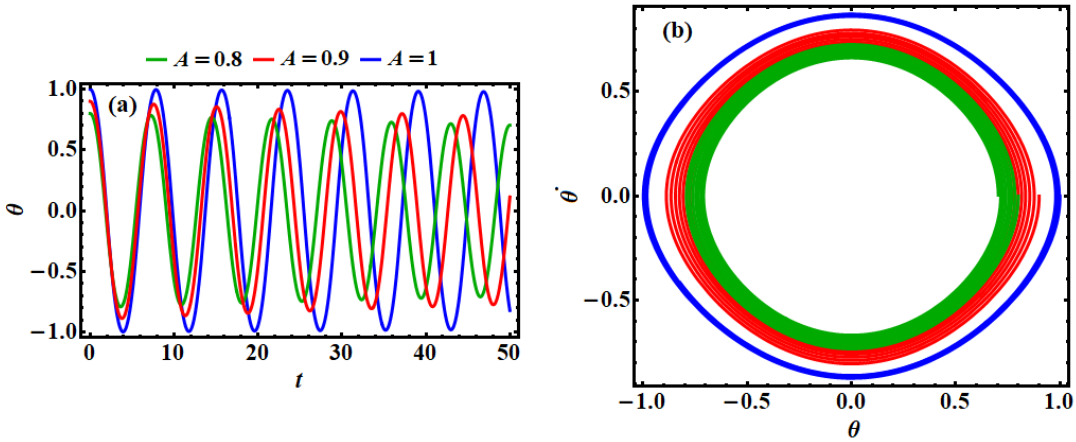

Figure 3: (a) Explores curves of at different amounts of $\theta(t)$, and (b) Reveals the phase plane trajectories in (a).

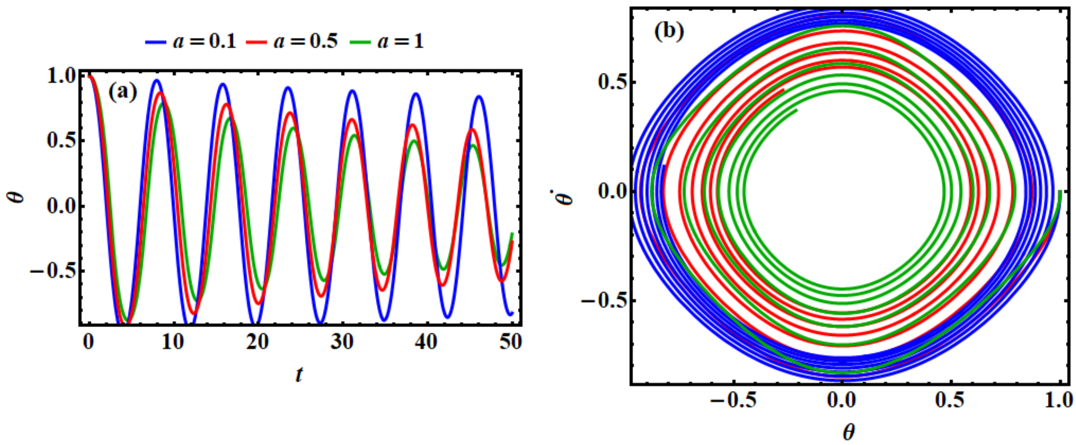

Figure 4: (a) Demonstrates curves of $\theta(t)$ at different amounts of a, and (b) Shows the phase plane diagrams in (a).

stable/unstable zones, in which the green areas denote the stabile zones, while the white ones express the unstable areas. These zones are plotted in view the intersection of the instability and stability regions of the inequalities Eq. (15).

Fig. (6) show the variation of these zones according to the various values of $A = (0.4, 0.7, 1)$ at $a = b = 0.01$, while parts of Fig. (7) are drawn when $A = (0.4, 0.7)$ and $a = b = 1$. Looking at the parts (a) and (b) of these figures, besides the part (c) of Fig. (7), we conclude that the stability regions decrease with the increase of the values of A. This means that the stable areas shrink to account for the unstable areas. The comparison of Fig. (6) with Fig. (7), shows that more deformation is observed with the increase of a and b values.

The graphed areas in the potions of Fig. (8) are calculated at $A = b = 1$ when $a = (0.1, 0.5)$. The increase of the values produces a decrease in the stability areas, which is

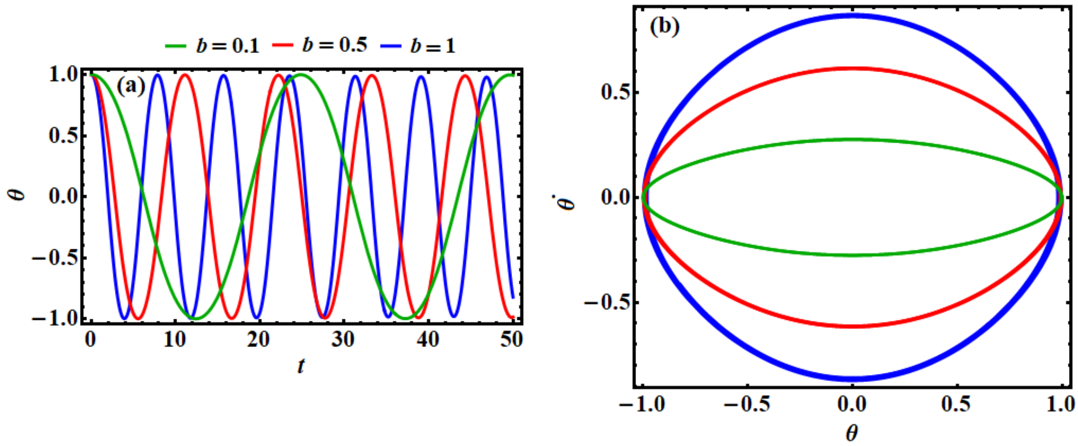

Figure 5: (a) Shows curves of $\theta(t)$ at different amounts of b, and (b) Presents the phase plane paths in (a).

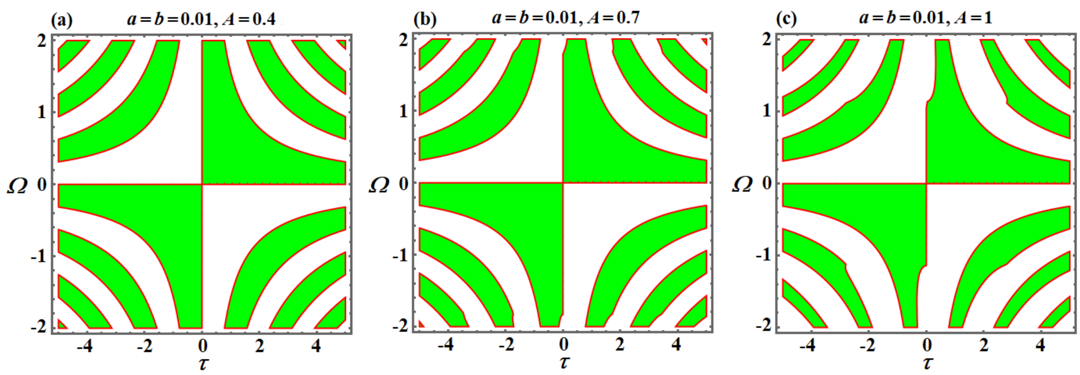

Figure 6: Show the stability/instability areas in the plane $(\tau - \Omega)$ at $a = b = 0.01$ when (a) $A = 0.4$, (b) $A = 0.7$, and (c) $A = 1$.

the reason backs to the mathematical form of the first inequality in Eq. (15), where the terms conclude the parameter a in Eq. (15) have negative signs. The inspection in the drawn areas in Fig. (9) shows that they are graphed at $A = a = 1$ when $b = (0.01, 0.1)$. A closer look at the Fig. (9a) and Fig. (9b), shows the positive influence of the b values, i.e. when the values of b grows, the stability areas increases. We find that the reason behind this increase is that the term containing the parameter b in the first inequalities Eq. (15) has a positive sign.

Referring to the previous analysis, one can say that periodic solutions provide insights into the regular and predictable behavior of systems. The diagrams of phase plane Offer a visual and qualitative understanding of system dynamics and stability, while stability areas are crucial for ensuring systems remain stable and operate safely under different conditions.

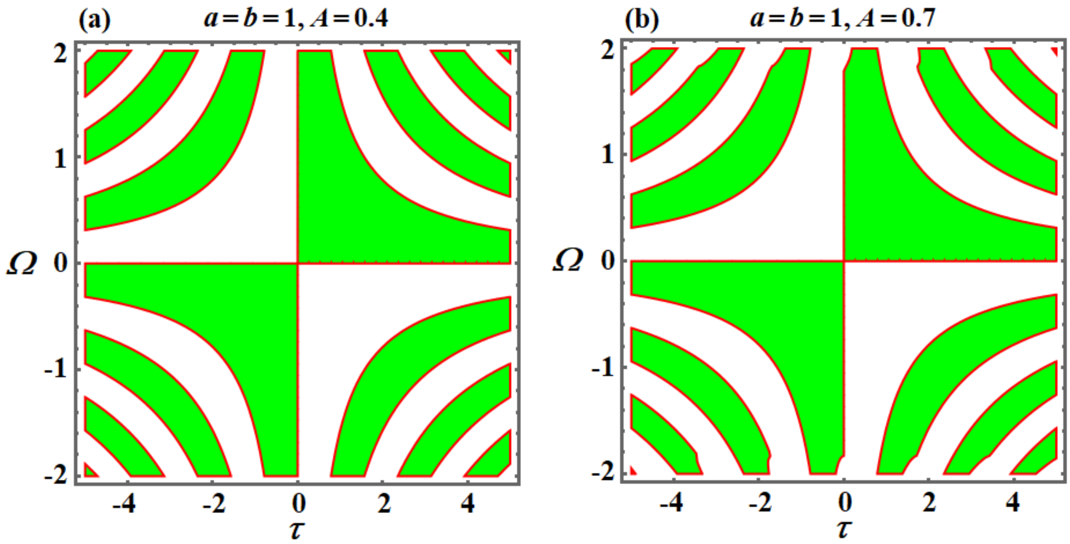

Figure 7: Presents the stability/instability zones in the plane $(\tau - \Omega)$ at $a = b = 1$ when (a) $A = 0.4$, and (b) $A = 0.7$.

4. Evaluation of the Impact of Outcomes

Assessing the impact of solving dynamical systems with periodic solutions requires a nuanced understanding of both the theoretical aspects and practical implications. The periodic nature of the solution implies that the system exhibits repeating behavior over time, which can have distinct short-term and long-term effects [12, 39]. Below is a comprehensive analysis of the impact assessment.

(i) Short-Term Effects:

Solving a dynamical system with periodic solutions allows for an immediate assessment of system stability. In the short term, it becomes possible to determine whether the system will remain in a stable periodic orbit or if there is a risk of divergence. This is crucial for predicting short-term behavior, particularly in systems where stability is paramount. In systems where periodic motion is a factor, short-term solutions can lead to immediate gains in energy efficiency by optimizing the periodic cycles. For instance, in resonance-based systems, ensuring that the system operates at or near resonance can minimize energy losses.

(ii) Long-Term Effects: Periodic solutions allow for better prediction of long-term behavior, including the identification of potential failure points due to fatigue in mechanical systems or wear and tear in repetitive processes. Once a periodic solution is established, long-term optimization strategies can be employed to fine-tune system performance. This could involve adjusting system parameters to improve overall system performance. Over time, systems may need to adapt to changing conditions, such as varying external forces. The periodic nature of the solution can guide the adaptation process, ensuring that the system remains in a desirable state despite these changes.

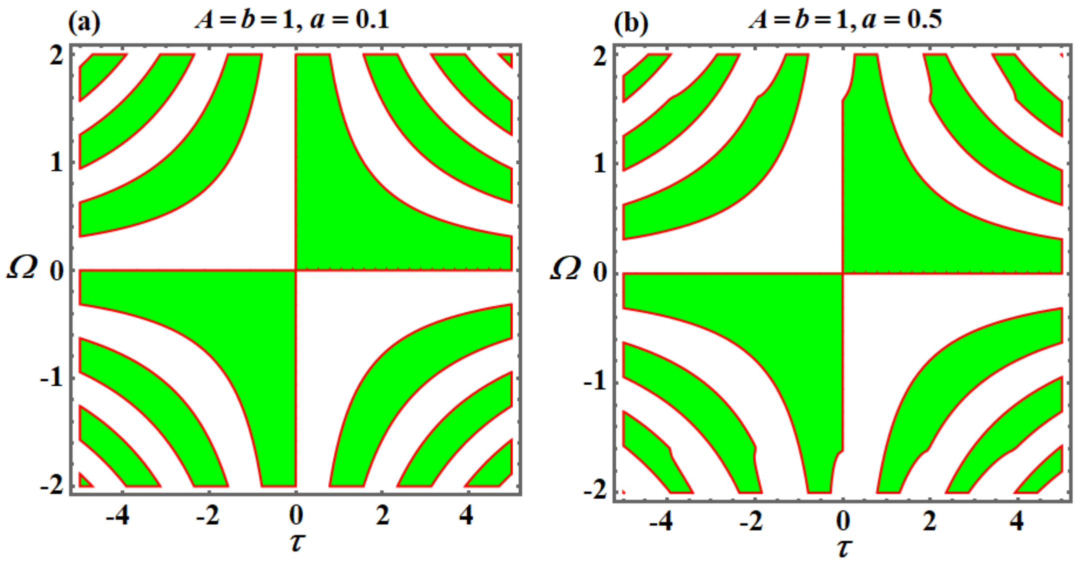

Figure 8: Presents the time variation ofΩ at $A = b = 1$ when: (a)$a = 0.1$, and (b) $a = 0.5$.

5. Concluding Remarks

The work investigates the stability of a rolling rigid rod with square TD position as well as velocity. The TD and other influential aspects are used to evaluate the approximate analytical outcome technique. The investigation's motivation stems from the fact that TD technology has recently been the subject of numerous technological applications. When attempting to solve an equivalent linearized ODE, it is advised to utilize the NPA. This satisfactoriness is compared and evaluated using the MS. The stability criterion is illustrated physically and explored conceptually. A matching between the strong nonlinear ODE and the corresponding analytical linear ODE is established by using the suitable NM. For several criteria, this matching has shown a highly significant agreement. Put another way, the new performance is robust, promising, and practical, and it could be used with other types of oscillators with high nonlinearity. It is accurate, adaptable, and practical. As is well-known, all traditional methods use the Taylor expansion to reduce their complexity when the restoring forces are present. This flaw is no longer included in the current development. Moreover, different stable and unstable zones are graphed according to the various values of the used parameters. These charts confirm the stability performance of the stability criteria as given in Eq(15). By analyzing the data, it is feasible to assess the applicability of the used parameter and show how accurate the results are. This inquiry scrutinizes the problem of an inflexible rotating rod without slipping on a rigid circular surface. The Euler-Lagrange theorem worked as the foundation for the development of the governing equation of motion in our earlier work [17]. The obtained Gaylord's oscillator has a semi-precise periodic analytic solution because the expected non-linear frequency was anticipated based on the NPA. The below mentioned specifics ought to be highlighted pertaining to the innovative method or results that are important:

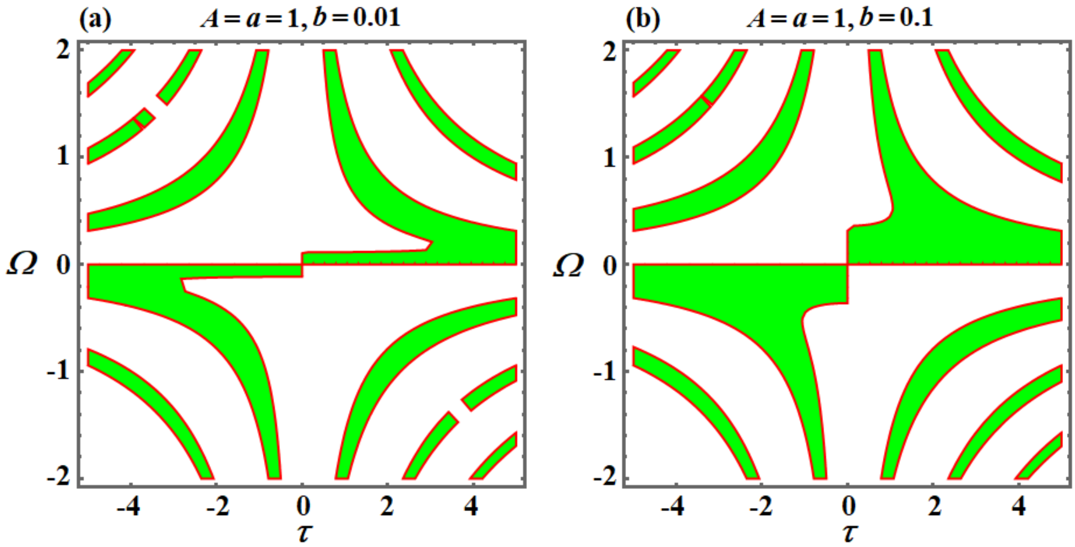

Figure 9: Describes the time variation of Ω at $A = a = 1$ when: (a) $b = 0.01$, and (b) $b = 0.1$.

- In essence, the novel approach creates a parallel linear ODE that corresponds to the existing nonlinear one.
- A good match between these two equations is achieved to ensure the effectiveness of the novel method.
- Every traditional approach employed the Taylor expansion to simplify the given problem in the case of existence of the restoring forces. The current strategy has eliminated this shortcoming.
- Away from the traditional perturbations, the current methodology enables us to investigate the stability analysis of the problem.
- Finally, NPA appears to be a straightforward, effective, and promising tool. This can be utilized to examine different nonlinear oscillator classes.

The implications of the original nonlinear Gaylord oscillator led to the establishing of the equivalent frequency. Consequently, this strategy enables us to tackle a wide range of issues related to the use of various oscillators in mechanical systems. To analyze the stability analysis in the current work, a TD square position and velocity are used. To achieve an analogous frequency and subsequently an equivalent linear ODE, the new NPA has been used. It becomes clear that the total frequency includes every physical variable in the fundamental governing equation of motion. The equivalent frequency of the alternative linear ODE that includes trigonometric and Bessel functions is expressed as a transcendental relationship. A comparison between the numerical and the theoretical outcomes reveals a precise accuracy. The study purposes a careful analysis of the conclusions reached utilizing the analytical strategy of practical approximation. A secondary argument together with the system potential for nonlinear vibration is provided by the

TD. Additionally, the time history of the achieved results and their stabilities are plotted to show the positive effect of the used parameter.
Coupled dynamical systems constitute interrelated systems that mutually affect one other's behavior, offering valuable insights into intricate phenomena like as biological rhythms, disease transmission, brain activity, and social dynamics. These systems have the potential to exhibit emergent behavior, which can offer valuable insights into their functioning, enable predictions of future behavior, and provide suggestions for controlling or optimizing them in practical applications. Therefore, in subsequent papers, the NPA will be developed to analyze such systems.

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Conflict of Interest
There are no conflicts of interest declared by the authors.

Data Availability
All data generated or analysed during this study are included in this published article.

References

[1] Ata Abu-As' ad and Jihad Asad. Exact solution for nonlinear oscillators with coordinate-dependent mass. *European Journal of Pure and Applied Mathematics*, 15(2):496–510, 2022.

[2] TS Amer, Galal M Moatimid, and WS Amer. Dynamical stability of a 3-dof auto-parametric vibrating system. *Journal of Vibration Engineering & Technologies*, 11(8):4151–4186, 2023.

[3] VS Erturk, B Rath, Taqwa M Al-Khader, Noorhan Alshaikh, P Mallick, and Jihad Asad. Two-dimensional coupled asymmetric van der pol oscillator. *European Journal of Pure and Applied Mathematics*, 17(2):1254–1264, 2024.

[4] Bawar Mohammed Faraj, Shnyar Karim Rahman, Deni Adnan Mohammed, Hozan Dlshad Hilmi, and Ali Akgul. Efficient finite difference approaches for solving

initial boundary value problems in helmholtz partial differential equations. *Contemporary Mathematics*, pages 569–580, 2023.

[5] Bawar Mohammed Faraj, Dana Taha Mohammed Salih, Bnar Hama Zaman Hama Ali, Bahadin Muhammad Hussien, Sarkhel Akbar Mahmood, and Shewa Abid Hama. On the numerical solution for two dimensional laplace equation with initial boundary conditions by using finite difference methods. *Journal of Studies in Science and Engineering*, 2(4):50–59, 2022.

[6] Seyed S Ganji, Amin Barari, S Karimpour, and G Domairry. Motion of a rigid rod rocking back and forth and cubic-quintic duffing oscillators. *Journal of theoretical and applied mechanics*, 50(1):215–229, 2012.

[7] Ji-Huan He. Amplitude-frequency relationship for conservative nonlinear oscillators with odd nonlinearities. *International Journal of Applied and Computational Mathematics*, 3:1557–1560, 2017.

[8] Gamal Ismail, Galal Moatimid, and Mohammed Yamani. Periodic solutions of strongly nonlinear oscillators using he's frequency formulation. *European Journal of Pure and Applied Mathematics*, 17(3):2155–2172, 2024.

[9] Goran Janevski, Predrag Kozić, Ratko Pavlović, and Strain Posavljak. Moment lyapunov exponents and stochastic stability of a thin-walled beam subjected to axial loads and end moments. *Facta Universitatis, Series: Mechanical Engineering*, 19(2):209–228, 2021.

[10] Weiming Ji, Hui Wang, and Meng Liu. Dynamics analysis of an impulsive stochastic model for spruce budworm growth. *Applied and Computational Mathematics*, 19(3):336–359, 2020.

[11] H Ebrahimi Khah and DD Ganji. A study on the motion of a rigid rod rocking back and cubic-quintic duffing oscillators by using he's energy balance method. *International Journal of Nonlinear Science*, 10(4):447–451, 2010.

[12] Craig A Kluever. *Dynamic systems: modeling, simulation, and control.* John Wiley & Sons, 2020.

[13] JW Meek. Dynamic response of tipping core buildings. *Earthquake Engineering & Structural Dynamics*, 6(5):437–454, 1978.

[14] Galal M Moatimid and TS Amer. Analytical solution for the motion of a pendulum with rolling wheel: stability analysis. *Scientific Reports*, 12(1):12628, 2022.

[15] Galal M Moatimid and TS Amer. Nonlinear suppression using time-delayed controller to excited van der pol–duffing oscillator: analytical solution techniques. *Archive of Applied Mechanics*, 92(12):3515–3531, 2022.

[16] Galal M Moatimid and TS Amer. Analytical approximate solutions of a magnetic spherical pendulum: Stability analysis. *Journal of Vibration Engineering & Technologies*, 11(5):2155–2165, 2023.

[17] Galal M Moatimid and TS Amer. Dynamical system of a time-delayed of rigid rocking rod: analytical approximate solution. *Scientific Reports*, 13(1):5570, 2023.

[18] Galal M Moatimid and TS Amer. Dynamical system of a time-delayed ϕ 6-van der pol oscillator: a non-perturbative approach. *Scientific Reports*, 13(1):11942, 2023.

[19] Galal M Moatimid, TS Amer, and WS Amer. Dynamical analysis of a damped harmonic forced duffing oscillator with time delay. *Scientific Reports*, 13(1):6507, 2023.

[20] Galal M Moatimid, TS Amer, and YY Ellabban. A novel methodology for a time-delayed controller to prevent nonlinear system oscillations. *Journal of Low Frequency Noise, Vibration and Active Control*, 43(1):525–542, 2024.

[21] Galal M Moatimid, TS Amer, and AA Galal. Studying highly nonlinear oscillators using the non-perturbative methodology. *Scientific Reports*, 13(1):20288, 2023.

[22] Galal M Moatimid, TS Amer, and AA Galal. Studying highly nonlinear oscillators using the non-perturbative methodology. *Scientific Reports*, 13(1):20288, 2023.

[23] Galal M Moatimid, TS Amer, and AA Galal. Inspection of some extremely non-linear oscillators using an inventive approach. *Journal of Vibration Engineering & Technologies*, pages 1–11, 2024.

[24] Galal M Moatimid, TS Amer, and Marwa H Zekry. Analytical and numerical study of a vibrating magnetic inverted pendulum. *Archive of Applied Mechanics*, 93(6):2533–2547, 2023.

[25] Galal M Moatimid, AT El-Sayed, and Hala F Salman. Different controllers for suppressing oscillations of a hybrid oscillator via non-perturbative analysis. *Scientific Reports*, 14(1):307, 2024.

[26] Galal M Moatimid, Mona AA Mohamed, and Khaled Elagamy. Nonlinear kelvin-helmholtz instability of a horizontal interface separating two electrified walters' b liquids: A new approach. *Chinese Journal of Physics*, 85:629–648, 2023.

[27] Galal M Moatimid, Mona AA Mohamed, and Khaled Elagamy. An innovative approach in inspecting a damped mathieu cubic–quintic duffing oscillator. *Journal of Vibration Engineering & Technologies*, pages 1–18, 2024.

[28] Galal M Moatimid, Mona AA Mohamed, and Khaled Elagamy. Insightful inspection of the nonlinear instability of an azimuthal disturbance separating two rotating magnetic liquid columns. *The European Physical Journal Plus*, 139(7):1–23, 2024.

[29] Galal M Moatimid, Mona AA Mohamed, and Khaled Elagamy. Inspection of the nonlinear instability of electrified casson fluids: a novel approach. *Waves in Random and Complex Media*, pages 1–31, 2024.

[30] Galal M Moatimid and Yasmeen M Mohamed. Nonlinear electro-rheological instability of two moving cylindrical fluids: An innovative approach. *Physics of Fluids*, 36(2), 2024.

[31] Galal M Moatimid and Yasmeen M Mohamed. A novel methodology in analyzing nonlinear stability of two electrified viscoelastic liquids. *Chinese Journal of Physics*, 89:679–706, 2024.

[32] Galal M Moatimid and DM Mostafa. Nonlinear stability of two superimposed electrified dusty fluids of type rivlin-ericksen: Non-perturbative approach. *Partial Differential Equations in Applied Mathematics*, page 100745, 2024.

[33] Galal M Moatimid, DM Mostafa, and Marwa H Zekry. A new methodology in evaluating nonlinear electrohydrodynamic azimuthal stability between two dusty viscous fluids. *Chinese Journal of Physics*, 90:134–154, 2024.

[34] Galal M Moatimid and Aya Sayed. Nonlinear ehd stability of a cylindrical interface separating two rivlin-ericksen fluids: A novel analysis. *Chinese Journal of Physics*, 87:379–397, 2024.

[35] Ivan R Pavlović, Ratko Pavlović, Goran Janevski, Nikola Despenić, and Vladimir Pajković. Dynamic behavior of two elastically connected nanobeams under a white noise process. *Facta Universitatis, Series: Mechanical Engineering*, 18(2):219–227, 2020.

[36] Na Qie, Wei-Fan Houa, and Ji-Huan He. The fastest insight into the large amplitude vibration of a string. *Reports in Mechanical Engineering*, 2(1):1–5, 2021.

[37] Zhong-Fu Ren and Gui-Fang Hu. He's frequency–amplitude formulation with average residuals for nonlinear oscillators. *Journal of Low Frequency Noise, Vibration and Active Control*, 38(3-4):1050–1059, 2019.

[38] Nasser A Saeed, Galal M Moatimid, Fawzy M Elsabaa, Yomna Y Ellabban, Mohammed A El-Meligy, and Mohamed Sharaf. Time-delayed nonlinear feedback controllers to suppress the principal parameter excitation. *IEEE Access*, 8:226151–226166, 2020.

[39] Steven H Strogatz. *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering.* CRC press, 2018.

[40] Yue Wu and Yan-Ping Liu. Residual calculation in he's frequency–amplitude formulation. *Journal of Low Frequency Noise, Vibration and Active Control*, 40(2):1040–1047, 2021.

[41] Chik-Sing Yim and Anil K Chopra. Earthquake response of structures with partial uplift on winkler foundation. *Earthquake engineering & structural dynamics*, 12(2):263–281, 1984.

[42] Yuting Zuo. A gecko-like fractal receptor of a three-dimensional printing technology: a fractal oscillator. *Journal of Mathematical Chemistry*, 59(3):735–744, 2021.