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Differential Sandwich Theorems of Analytic Functions Defined by Linear Operators

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Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results involving a linear operator and other linear operators for certain normalized analytic functions. Some of our results generalize previously known results.

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1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} \ldots \ (a \in \mathbb{C}). \tag{1}$$

For simplicity $H[a] = H[a,1]$. Also, let $\mathscr{A}$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

If f, $g \in H(U)$, we say that f is subordinate to g or f is superordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$, such that $f(z) = g(\omega(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, [cf., e.g., 6, 16, 17]:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

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Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\phi\left(p(z), zp^{'}(z); z\right) \prec h(z), \tag{3}$$

then $p(z)$ is a solution of the differential subordination (3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (3) is called the best dominant. If $p(z)$ and $\phi\left(p(z), zp^{'}(z); z\right)$ are univalent in U and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi\left(p(z), zp^{'}(z); z\right), \tag{4}$$

then $p(z)$ is a solution of the differential superordination (4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (4) is called the best subordinant.Using the results of Miller and Mocanu [17], Bulboaca [5] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [6]. Ali et al. [1], have used the results of Bulboaca [5] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Tuneski [25] obtained a sufficient condition for starlikeness of f in terms of the quantity $\dfrac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [24] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They [24] also obtained results for functions defined by using Carlson-Shaffer operator [7], Ruscheweyh derivative [20] and Sălăgean operator [22].

For functions f given by (1) and $g \in \mathscr{A}$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{5}$$

For functions $f, g \in \mathscr{A}$, we define the linear operator $D_\lambda^n : \mathscr{A} \to \mathscr{A}$ $(\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\})$ by:

$$D_\lambda^0 (f * g)(z) = (f * g)(z),$$

$$D_{\lambda}^{1}(f*g)(z)=D_{\lambda}(f*g)(z)=(1-\lambda)(f*g)(z)+\lambda z\left(\left(f*g\right)(z)\right)', \tag{6}$$

and (in general)

$$\begin{aligned} D_{\lambda}^{n}(f*g)(z) &= D_{\lambda}(D_{\lambda}^{n-1}(f*g)(z)) \\ &= z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^{n}a_{k}b_{k}z^{k} \quad (\lambda\geq 0; n\in\mathbb{N}_{0}). \end{aligned} \tag{7}$$

From (7), we can easily deduce that

$$\lambda z\left(D_{\lambda}^{n}(f*g)(z)\right)'=D_{\lambda}^{n+1}(f*g)(z)-(1-\lambda)D_{\lambda}^{n}(f*g)(z)\ (\lambda>0). \tag{8}$$

The linear operator $D_{\lambda}^{n}(f*g)(z)$ was introduced by Aouf and Seoudy [3] and we observe that $D_{\lambda}^{n}(f*g)(z)$ reduces to several interesting many other linear operators considered earlier for different choices of n, λ and the function $g(z)$:

(i) For $b_k=1$ (or $g(z)=\dfrac{z}{1-z}$), we have $D_{\lambda}^{n}(f*g)(z)=D_{\lambda}^{n}f(z)$, where D_{λ}^{n} is the generalized Sălăgean operator (or Al-Oboudi operator [2] which yield Sălăgean operator D^n for $\lambda=1$ introduced and studied by Sălăgean [21];

(ii) For $n=0$ and

$$g(z)=z+\sum_{k=2}^{\infty}\frac{(a_1)_{k-1}\ldots(a_l)_{k-1}}{(b_1)_{k-1}\ldots(b_m)_{k-1}(1)_{k-1}}z^k \tag{9}$$

$\left(a_i\in\mathbb{C}; i=1,\ldots,l; b_j\in\mathbb{C}\backslash\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}; j=1,\ldots,m; l\leq m+1; l,m\in\mathbb{N}_0; z\in U\right)$,

where

$$(x)_k=\begin{cases} 1 & (k=0; x\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}) \\ x(x+1)\ldots(x+k-1) & (k\in N; x\in\mathbb{C}), \end{cases}$$

we have $D_{\lambda}^{0}(f*g)(z)=(f*g)(z)=H_{l,m}\left(a_1;b_1\right)f(z)$, where the operator $H_{l,m}\left(a_1;b_1\right)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [10] ([see also 11, 12]). The operator $H_{l,m}\left(a_1;b_1\right)$, contains in turn many interesting operators such as, Hohlov linear operator (see [13]), the Carlson-Shaffer linear operator (see [7, 21]), the Ruscheweyh derivative operator (see [20]), the Bernardi-Libera-Livingston operator (see [4, 14, 15]) and Owa-Srivastava fractional derivative operator (see [19]);

(iii) For $n=0$ and

$$g(z)=z+\sum_{k=2}^{\infty}\left[\frac{1+l+\lambda(k-1)}{1+l}\right]^{s}z^{k}\ (\lambda\geq 0; l,s\in N_0), \tag{10}$$

we see that $D_{\lambda}^{0}(f*g)(z)=(f*g)(z)=I(s,\lambda,l)f(z)$, where $I(s,\lambda,l)$ is the generalized multiplier transformations which was introduced and studied by Cătaş et al. [8]. The operator $I(s,\lambda,l)$, contains as special cases, the multiplier transformation $I(s,l)$ (see [9]) for $\lambda=1$, the generalized Sălăgean operator D_{λ}^{n} introduced and studied by Al-Oboudi [2] which in turn contains as special case the Sălăgean operator D^n (see [21]);

(iv) For $g(z)$ of the form (9), the operator $D_{\lambda}^{n}(f*g)(z)=D_{\lambda}^{n}(a_1,b_1)f(z)$, introduced and studied by Selvaraj and Karthikeyan [23].

In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator $D_{\lambda}^{n}(f*g)(z)$ and some of its special operators by some choices of n,λ and the function $g(z)$.

2. Preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

Definition 1. *[17] Denote by Q, the set of all functions f that are analytic and injective on $U\backslash E(f)$, where*

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f\left(z\right)=\infty\right\},$$

and are such that $f'\left(\zeta\right)\neq 0$ for $\zeta\in\partial U\backslash E\left(f\right)$.

Lemma 1. *[17] Let $q\left(z\right)$ be univalent in the unit disk U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi\left(w\right)\neq 0$ when $w\in q(U)$. Set*

$$\psi\left(z\right)=zq'\left(z\right)\varphi\left(q\left(z\right)\right)\text{ and }h\left(z\right)=\theta\left(q\left(z\right)\right)+\psi\left(z\right).\tag{11}$$

Suppose that

(i) $\psi\left(z\right)$ is starlike univalent in U,

(ii) $\Re\left\{\dfrac{zh'\left(z\right)}{\psi\left(z\right)}\right\}>0$ for $z\in U$.

If $p\left(z\right)$ is analytic with $p(0)=q(0)$, $p(U)\subset D$ and

$$\theta\left(p\left(z\right)\right)+zp'\left(z\right)\varphi\left(p\left(z\right)\right)\prec\theta\left(q\left(z\right)\right)+zq'\left(z\right)\varphi\left(q\left(z\right)\right),\tag{12}$$

then $p(z)\prec q(z)$ and $q\left(z\right)$ is the best dominant.

Taking $\theta\left(w\right)=\alpha w$ and $\varphi\left(w\right)=\gamma$ in Lemma 1, Shanmugam et al. [24] obtained the following lemma.

Lemma 2. *[24] Let $q\left(z\right)$ be univalent in U with $q(0)=1$. Let $\alpha\in\mathbb{C}$; $\gamma\in\mathbb{C}^{*}$, further assume that*

$$\Re\left\{1+\frac{zq''\left(z\right)}{q'\left(z\right)}\right\}>\max\left\{0,-\Re\left(\frac{\alpha}{\gamma}\right)\right\}.\tag{13}$$

If $p\left(z\right)$ is analytic in U, and

$$\alpha p\left(z\right)+\gamma zp'\left(z\right)\prec\alpha q\left(z\right)+\gamma zq'\left(z\right),$$

then $p\left(z\right)\prec q\left(z\right)$ and $q\left(z\right)$ is the best dominant.

Lemma 3. *[5] Let $q(z)$ be convex univalent in U and ϑ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\Re\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$,

(ii) $\Psi(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.

If $p(z) \in H[q(0),1] \cap Q$, with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{14}$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Taking $\vartheta(w) = \alpha w$ and $\phi(w) = \gamma$ in Lemma 3, Shanmugam et al. [24] obtained the following lemma.

Lemma 4. *[24] Let $q(z)$ be convex univalent in U, $q(0) = 1$. Let $\alpha \in \mathbb{C}$; $\gamma \in \mathbb{C}^*$ and $\Re\left(\frac{\alpha}{\gamma}\right) > 0$. If $p(z) \in H[q(0),1] \cap Q$, $\alpha p(z) + \gamma zp'(z)$ is univalent in U and*

$$\alpha q(z) + \gamma zq'(z) \prec \alpha p(z) + \gamma zp'(z),$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Sandwich Results

Unless otherwise mentioned, we assume throughout this paper that $\lambda > 0$ and $n \in \mathbb{N}_0$.

Theorem 1. *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further, assume that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{1}{\gamma}\right)\right\}. \tag{15}$$

If $f, g \in \mathscr{A}$ satisfy the following subordination condition:

$$\left(1 + \frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2} + \frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2} - 2\frac{z\left[D_\lambda^{n+1}(f*g)(z)\right]^2}{\left[D_\lambda^n(f*g)(z)\right]^3}\right\}$$

$$\prec q(z) + \gamma zq'(z), \tag{16}$$

then

$$\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by

$$p(z)=\frac{zD_{\lambda}^{n+1}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{2}} \quad (z\in U). \tag{17}$$

Then the function $p(z)$ is analytic in U and $p(0)=1$. Therefore, differentiating (17) logarithmically with respect to z and using the identity (8) in the resulting equation, we have

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_{\lambda}^{n+1}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{2}}+\frac{\gamma}{\lambda}\left\{\frac{zD_{\lambda}^{n+2}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{2}}-2\frac{z\left[D_{\lambda}^{n+1}(f*g)(z)\right]^{2}}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{3}}\right\}=p(z)+\gamma zp^{'}(z),$$

that is,

$$p(z)+\gamma zp^{'}(z)\prec q(z)+\gamma zq^{'}(z).$$

Therefore, Theorem 1 now follows by applying Lemma 2.

Putting $q(z)=\dfrac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let* $\gamma\in\mathbb{C}^{*}$ *and*

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\}>\max\left\{0,-\Re\left(\frac{1}{\gamma}\right)\right\}.$$

If $f,g\in\mathscr{A}$ *satisfy the following subordination condition:*

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_{\lambda}^{n+1}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{2}}+\frac{\gamma}{\lambda}\left\{\frac{zD_{\lambda}^{n+2}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{2}}-2\frac{z\left[D_{\lambda}^{n+1}(f*g)(z)\right]^{2}}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{3}}\right\}$$

$$\prec\frac{1+\mathscr{A}z}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^{2}},$$

then

$$\frac{zD_{\lambda}^{n+1}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{2}}\prec\frac{1+Az}{1+Bz}$$

and the function $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Remark 1. *Taking* $g(z)=\dfrac{z}{1-z}$ *in Theorem 1, we obtain the subordination result of Nechita [18, Theorem 14].*

Remark 2. *Taking* $\lambda=1$ *and* $g(z)=\dfrac{z}{1-z}$ *in Theorem 1, we obtain the subordination result for Sălăgean operator which was obtained by Shanmugam et al. [24, Theorem 5.4] and also obtained by Nechita [18, Corollary 16].*

Taking $n = 0, \lambda = 1$ and $g(z)$ of the form (9) in Theorem 1, we obtain the following subordination result for Dziok-Srivastava operator.

Corollary 2. *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that (15) holds. If $f \in \mathscr{A}$ satisfies the following subordination condition:*

$$\frac{z^2\left(H_{l,m}(a_1;b_1)f(z)\right)'}{\left[H_{l,m}(a_1;b_1)f(z)\right]^2} - \gamma z^2\left(\frac{z}{\left(H_{l,m}(a_1;b_1)f(z)\right)}\right)'' \prec q(z) + \gamma z q'(z),$$

then

$$\frac{z^2\left(H_{l,m}(a_1;b_1)f(z)\right)'}{\left[H_{l,m}(a_1;b_1)f(z)\right]^2} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $g(z)$ of the form (9) in Theorem 1, we obtain the following subordination result for the operator $D_\lambda^n(a_1;b_1)$.

Corollary 3. *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that [15] holds. If $f \in \mathscr{A}$ satisfies the following subordination condition:*

$$\left(1 + \frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(a_1;b_1)f(z))}{\left[D_\lambda^n(a_1;b_1)f(z)\right]^2} + \frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(a_1;b_1)f(z)}{\left[D_\lambda^n(a_1;b_1)f(z)\right]^2} - 2\frac{z\left[D_\lambda^{n+1}(a_1;b_1)f(z)\right]^2}{\left[D_\lambda^n(a_1;b_1)f(z)\right]^3}\right\}$$

$$\prec q(z) + \gamma z q'(z),$$

then

$$\frac{zD_\lambda^{n+1}(a_1;b_1)f(z))}{\left[D_\lambda^n(a_1;b_1)f(z)\right]^2} \prec q(z)$$

and $q(z)$ is the best dominant.

Taking $n = 0, \lambda = 1$ and

$$g(z) = z + \sum_{k=2}^{\infty}\left(\frac{l+k}{1+l}\right)^s z^k \; (l, s \in N_0), \tag{18}$$

in Theorem 1, we obtain the following subordination result for the multiplier transformations $I(s,l)$.

Corollary 4. *Let $q(z)$ be univalent in U with $q(0) = 1$, and $\gamma \in \mathbb{C}^*$. Further assume that* (15) *holds. If $f \in \mathscr{A}$ satisfies the following subordination condition:*

$$\frac{z^2\left(I(s,l)f(z)\right)'}{\left[I(s,l)f(z)\right]^2} - \gamma z^2\left(\frac{z}{I(s,l)f(z)}\right)'' \prec q(z) + \gamma z q'(z),$$

then

$$\frac{z^2\left(I(s,l)f(z)\right)'}{\left[I(s,l)f(z)\right]^2}\prec q(z)$$

and $q(z)$ *is the best dominant.*

Remark 3. *Taking* $n=0,\lambda=1$ *and* $g(z)=\dfrac{z}{1-z}$ *in Theorem 1, we obtain the subordination result of Shanmugam et al. [24, Theorem 3.4] and also obtained by Nechita [18, Corollary 17].*

Now, by appealing to Lemma 4 it can be easily prove the following theorem.

Theorem 2. *Let* $q(z)$ *be convex univalent in* U *with* $q(0)=1$. *Let* $\gamma\in\mathbb{C}$ *with* $\Re(\gamma)>0$. *If* $f,g\in\mathscr{A}$, $\dfrac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}\in H[1,1]\cap Q$,

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}-2\frac{z\left[D_\lambda^{n+1}(f*g)(z)\right]^2}{\left[D_\lambda^n(f*g)(z)\right]^3}\right\}$$

is univalent in U*, and the following superordination condition*

$$q(z)+\gamma zq'(z)\prec\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}-2\frac{z\left[D_\lambda^{n+1}(f*g)(z)\right]^2}{\left[D_\lambda^n(f*g)(z)\right]^3}\right\}$$

holds, then

$$q(z)\prec\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}$$

and $q(z)$ *is the best subordinant.*

Taking $q(z)=\dfrac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 2, we have the following corollary.

Corollary 5. *Let* $\gamma\in\mathbb{C}$ *with* $\Re(\gamma)>0$. *If* $f,g\in\mathscr{A}$, $\dfrac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}\in H[1,1]\cap Q$,

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}-2\frac{z\left[D_\lambda^{n+1}(f*g)(z)\right]^2}{\left[D_\lambda^n(f*g)(z)\right]^3}\right\}$$

is univalent in U*, and the following superordination condition*

$$\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2}\prec\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^n(f*g)(z)\right]^2}-2\frac{z\left[D_\lambda^{n+1}(f*g)(z)\right]^2}{\left[D_\lambda^n(f*g)(z)\right]^3}\right\}$$

holds, then

$$\frac{1+Az}{1+Bz} \prec \frac{zD_{\lambda}^{n+1}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^{2}}$$

and $q(z)$ is the best subordinant.

Remark 4. *Taking $g(z)=\dfrac{z}{1-z}$ in Theorem 2, we obtain the superordination result of Nechita [18, Theorem 19].*

Remark 5. *Taking $\lambda=1$ and $g(z)=\dfrac{z}{1-z}$ in Theorem 2, we obtain the following superordination result for Sălăgean operator which is obtained Shanmugam et al. [24, Theorem 5.5].*

Taking $n=0, \lambda=1$ and $g(z)$ of the form (9) in Theorem 2, we obtain the following superordination result for Dziok-Srivastava operator.

Corollary 6. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re(\gamma)>0$. If $f\in\mathscr{A}$, $\dfrac{z^2\left(H_{l,m}(a_1;b_1)f(z)\right)'}{\left[H_{l,m}(a_1;b_1)f(z)\right]^2}\in H[1,1]\cap Q$,*

$$\frac{z^2\left(H_{l,m}(a_1;b_1)f(z)\right)'}{\left[H_{l,m}(a_1;b_1)f(z)\right]^2}-\gamma z^2\left(\frac{z}{\left(H_{l,m}(a_1;b_1)f(z)\right)}\right)''$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma z q'(z) \prec \frac{z^2\left(H_{l,m}(a_1;b_1)f(z)\right)'}{\left[H_{l,m}(a_1;b_1)f(z)\right]^2}-\gamma z^2\left(\frac{z}{\left(H_{l,m}(a_1;b_1)f(z)\right)}\right)''$$

holds, then

$$q(z) \prec \frac{z^2\left(H_{l,m}(a_1;b_1)f(z)\right)'}{\left[H_{l,m}(a_1;b_1)f(z)\right]^2}$$

and $q(z)$ is the best subordinant.

Taking $g(z)$ of the form (9) in Theorem 2, we obtain the following superordination result for the operator $D_{\lambda}^{n}(a_1;b_1)$.

Corollary 7. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma\in\mathbb{C}$ with $\Re(\gamma)>0$. If $f,g\in\mathscr{A}$, $\dfrac{zD_{\lambda}^{n+1}(a_1;b_1)f(z)}{\left[D_{\lambda}^{n}(a_1;b_1)f(z)\right]^2}\in H[1,1]\cap Q$,*

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_{\lambda}^{n+1}(a_1;b_1)f(z)}{\left[D_{\lambda}^{n}(a_1;b_1)f(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_{\lambda}^{n+2}(a_1;b_1)f(z)}{\left[D_{\lambda}^{n}(a_1;b_1)f(z)\right]^2}-2\frac{z\left[D_{\lambda}^{n+1}(a_1;b_1)f(z)\right]^2}{\left[D_{\lambda}^{n}(a_1;b_1)f(z)\right]^3}\right\}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq^{'}(z) \prec \left(1+\frac{\gamma}{\lambda}\right)\frac{zD_{\lambda}^{n+1}(a_1;b_1)f(z)}{\left[D_{\lambda}^{n}(a_1;b_1)f(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_{\lambda}^{n+2}(a_1;b_1)f(z)}{\left[D_{\lambda}^{n}(a_1;b_1)f(z)\right]^2}-2\frac{z\left[D_{\lambda}^{n+1}(a_1;b_1)f(z)\right]^2}{\left[D_{\lambda}^{n}(a_1;b_1)f(z)\right]^3}\right\}$$

holds, then

$$q(z) \prec \frac{zD_{\lambda}^{n+1}(a_1;b_1)f(z)}{\left[D_{\lambda}^{n}(a_1;b_1)f(z)\right]^2}$$

and $q(z)$ is the best subordinant.

Taking $n=0, \lambda=1$ and $g(z)$ of the form (18) in Theorem 2, we obtain the following supordination result for the multiplier transformations $I(s,l)$.

Corollary 8. *Let $q(z)$ be convex univalent in U with $q(0)=1$. Let $\gamma \in \mathbb{C}$ with $\Re(\gamma)>0$. If $f \in \mathscr{A}$, $\dfrac{z^2\left(I(s,l)f(z)\right)^{'}}{\left[I(s,l)f(z)\right]^2} \in H[1,1] \cap Q$,*

$$\frac{z^2\left(I(s,l)f(z)\right)^{'}}{\left[I(s,l)f(z)\right]^2}-\gamma z^2\left(\frac{z}{I(s,l)f(z)}\right)^{''}$$

is univalent in U, and the following superordination condition

$$q(z)+\gamma zq^{'}(z) \prec \frac{z^2\left(I(s,l)f(z)\right)^{'}}{\left[I(s,l)f(z)\right]^2}-\gamma z^2\left(\frac{z}{I(s,l)f(z)}\right)^{''}$$

holds, then

$$q(z) \prec \frac{z^2\left(I(s,l)f(z)\right)^{'}}{\left[I(s,l)f(z)\right]^2}$$

and $q(z)$ is the best subordinant.

Remark 6. *Taking $n=0, \lambda=1$ and $g(z)=\dfrac{z}{1-z}$ in Theorem 2, we obtain the superordination result of Shanmugam et al. [24, Theorem 3.5].*

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem for the linear operator $D_{\lambda}^{n}(f*g)$.

Theorem 3. *Let $q_1(z)$ be convex univalent in U with $q_1(0)=1$, $\gamma \in \mathbb{C}$ with $\Re(\gamma)>0, q_2(z)$ be univalent in U with $q_2(0)=1$, and satisfies (15). If $f,g \in \mathscr{A}$, $\dfrac{zD_{\lambda}^{n+1}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^2} \in H[1,1] \cap Q$,*

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_{\lambda}^{n+1}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_{\lambda}^{n+2}(f*g)(z)}{\left[D_{\lambda}^{n}(f*g)(z)\right]^2}-2\frac{z\left[D_{\lambda}^{n+1}(f*g)(z)\right]^2}{\left[D_{\lambda}^{n}(f*g)(z)\right]^3}\right\}$$

is univalent in U, and

$$\begin{aligned} & q_1(z)+\gamma z q_1^{'}(z) \\ \prec \quad & \left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}-2\frac{z\left[D_\lambda^{n+1}(f*g)(z)\right]^2}{\left[D_\lambda^{n}(f*g)(z)\right]^3}\right\} \\ \prec \quad & q_2(z)+\gamma z q_2^{'}(z) \end{aligned}$$

holds, then

$$q_1(z)\prec\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}\prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Taking $q_i(z)=\dfrac{1+A_iz}{1+B_iz}$ $\left(i=1,2;-1\leq B_2\leq B_1<A_1\leq A_2\leq 1\right)$ in Theorem 3, we have the following corollary.

Corollary 9. *Let $\gamma\in\mathbb{C}$ with $\Re\left(\gamma\right)>0$. If $f,g\in\mathscr{A}$, $\dfrac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}\in H\left[1,1\right]\cap Q$,*

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}-2\frac{z\left[D_\lambda^{n+1}(f*g)(z)\right]^2}{\left[D_\lambda^{n}(f*g)(z)\right]^3}\right\}$$

is univalent in U, and

$$\begin{aligned} & \frac{1+A_1z}{1+B_1z}+\gamma\frac{\left(A_1-B_1\right)z}{\left(1+B_1z\right)^2} \\ \prec \quad & \left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}+\frac{\gamma}{\lambda}\left\{\frac{zD_\lambda^{n+2}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}-2\frac{z\left[D_\lambda^{n+1}(f*g)(z)\right]^2}{\left[D_\lambda^{n}(f*g)(z)\right]^3}\right\} \\ \prec \quad & \frac{1+A_2z}{1+B_2z}+\gamma\frac{\left(A_2-B_2\right)z}{\left(1+B_2z\right)^2} \end{aligned}$$

holds, then

$$\frac{1+A_1z}{1+B_1z}\prec\frac{zD_\lambda^{n+1}(f*g)(z)}{\left[D_\lambda^{n}(f*g)(z)\right]^2}\prec\frac{1+A_2z}{1+B_2z}$$

and $\dfrac{1+A_1z}{1+B_1z}$ and $\dfrac{1+A_2z}{1+B_2z}$ are, respectively, the best subordinant and the best dominant.

Remark 7. *Taking* $g(z) = \dfrac{z}{1-z}$ *in Theorem 3, we obtain sandwich result of Nechita [18, Theorem 19].*

Remark 8. *Taking* $\lambda = 1$ *and* $g(z) = \dfrac{z}{1-z}$ *in Theorem 3, we obtain sandwich result of Shanmugam et al. [24, Theorem 5.6].*

Remark 9. *Combining (i) Corollary 2 and Corollary 6; (ii) Corollary 3 and Corollary 7; (iii) Corollary 4 and Corollary 8, we obtain similar sandwich theorems for the corresponding linear operators.*

Remark 10. *Taking* $n = 0, \lambda = 1$ *and* $g(z) = \dfrac{z}{1-z}$ *in Theorem 3, we obtain the sandwich result of Shanmugam et al. [24, Corollary 3.6].*

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