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Plithogenic Crisp Hypersoft Topology

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Abstract. In this paper, we deal with the plithogenic crisp hypersoft set. This notion is more adaptable than the hypersoft set and more suited to challenges involving decision-making. Consequently, the topology defined by the collection of this type of set will be of great importance. Through this paper, first we redefine the set operations on this type of set (set theoretic). Then, we introduce plithogenic crisp hypersoft topological spaces, which are defined over an initial universal set with a fixed set of parameters. The plithogenic crisp hypersoft set considers the degree of appurtenance of the elements with respect to the attribute system. Further, the notions of plithogenic crisp hypersoft open sets, plithogenic crisp hypersoft closed sets, plithogenic crisp hypersoft neighborhood, plithogenic crisp hypersoft limit point, and plithogenic crisp hypersoft subspace are introduced, and their basic properties are investigated. Finally, we introduce the concepts of plithogenic crisp hypersoft closure and plithogenic crisp hypersoft interior.

2020 Mathematics Subject Classifications: 54C50

Key Words and Phrases: Hypersoft sets, Plithogenic hypersoft sets, Plithogenic crisp hypersoft sets, Plithogenic crisp hypersoft topology.

1. Introduction

In 1999, Molodtsov [18] introduced the concept of a soft set to deal with the difficult problems in economics, engineering, and the environment, where no mathematical methods could effectively deal with the many types of uncertainty. Biswas et al. in [15] introduced various operators for soft sets (see [3, 5, 8, 25]). Also, Finite soft-open sets introduced by Abd El-latif et al. [4] and Supra finite soft-open sets and applications to operators and continuity introduced by Arar et al. [26] . It is known that topology is a branch of mathematics that has numerous applications in the physical and computer sciences. Topology is the study of the qualitative properties of particular objects, known as topological spaces, which are invariant under specific transformations, known as continuous mappings. Open sets are commonly used to describe these characteristics. By replacing open sets with more general ones, the concept of topological space is frequently

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generalized. A classic example of this form of generalization is fuzzy topology, proposed by Chang [10] and later fuzzy topology introduced by Lowen [14]. Topological structures on soft sets, in a similar manner, are more generalized methods that can be used to measure the similarities and differences between the objects in a universe that are soft sets (for more, see [11, 12, 24]).

In 2018, Smarandache [27] expanded the concept of soft set to a hypersoft set by substituting the function with a multi-argument function described in the Cartesian product with a different set of parameters. This concept is more adaptable than the soft set and more useful when it comes to making decisions. Researchers have been drawn to the hypersoft set structure because it is better suited to decision-making difficulties than the soft set structure. The fundamentals of hypersoft sets are studied by Siddique et al. [23] (also see [22]). The idea of hypersoft sets is combined with topology by Musa and Asaad [20], in which they introduced hypersoft topological spaces, which are defined over an initial universe with a fixed set of parameters. Furthermore, Continuity and compactness via hypersoft open sets introduced by Asaad and Musa [6], Connectedness on hypersoft topological spaces by Musa and Asaad [19] and Hypersoft separation axioms by Asaad and Musa [7] are developed the field of topology. Also, an innovative extension of hypersoft sets and their applications [21] is introduced by Mohammed et al.

In the same paper [27], the author developed the concept of hypersoft into a concept called plithogenic hypersoft set. Furthermore, in the plithogenic crisp hypersoft set, there is a degree (0 or 1) of appurtenance of an element x to the set with respect to each attribute value. The idea of a plithogenic hypersoft set has many applications (see [2, 9, 13, 16, 17]). In [1], Murtaza et al. studied basic operations on hypersoft sets and hypersoft point together some basic properties of plithogenic α hypersoft sets where α can take any value in the (Crisp, Fuzzy, Intuitionistic Fuzzy and Neutrosophic) sets. When these two advanced concepts—plithogenic logic by Smarandache [28] and hypersoft topology by Musa and Asaad [20] are combined, they form the basis for the plithogenic crisp hypersoft topology. This emerging field allows for the modeling of topological spaces where each point can be characterized by a set of attributes, with each attribute having a degree of belongingness that is influenced by multiple, possibly conflicting criteria. In essence, plithogenic crisp Hypersoft topology provides a robust framework for analyzing and interpreting complex systems where uncertainty, indeterminacy, and multi-criteria decision-making play crucial roles. This framework has potential applications in areas such as decision theory, artificial intelligence, and data science, where traditional topological concepts may fall short in capturing the intricacies of real-world phenomena.

Our work is organized as follows: Sections 2 and 3 contain some basic definitions related to the hypersoft set and the plithogenic hypersoft set that are required in our work. In Section 4, we introduce plithogenic crisp hypersoft topological spaces, which are defined over an initial universe set with a fixed set of parameters, and investigate the concepts of plithogenic crisp hypersoft neighborhood and plithogenic crisp hypersoft limit points. In Section 5, the notions of plithogenic crisp hypersoft closure and plithogenic crisp hypersoft interior are introduced. One thing that must be mentioned is that in this paper we have redefined the definitions that relate to plithogenic crisp hypersoft set in both aspects: symbolic and expression, and this is for the sake of the study.

2. Preliminaries

Since the plithogenic crisp hypersoft is the extension of hypersoft set, we put the basic definitions here.

Definition 1. *[18] Let U be a universe of discourse, $P(U)$ the power set of U, and A a set of attributes. Then, the pair (F,U) where $F : A \to P(U)$ is called a soft set over U.*

Definition 2. *[20] Let U be a universal set and $P(U)$ be the power set of U. Let $\psi=\{r_1, r_2, \ldots, r_n\}$ be a set of n-distinct attributes with attribute value sets respectively as $E_1, E_2, \ldots, E_n$, where $E_i \cap E_j = \phi$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. Also, let D_i be the nonempty subset of E_i for each $i \in \{1, 2, \ldots, n\}$ and $V_\psi = D_1 \times D_2 \times \cdots \times D_n$. The pair (Γ, V_ψ) where $\Gamma : V_\psi \to P(U)$ is called a hypersoft (in short, HS) set. That is, $(\Gamma, V_\psi) = \{(\alpha, \Gamma(\alpha)) : \alpha \in V_\psi\}$.*

Definition 3. *[23] Let (Γ_1, F_ψ) and (Γ_2, H_ψ) be two hypersoft sets over U. Then (Γ_1, F_ψ) is a hypersoft subsets of (Γ_2, H_ψ) if:*

(i) $F_\psi \subseteq H_\psi$, *and*

(ii) $\Gamma_1(\alpha) \subseteq \Gamma_2(\alpha)$, $\forall \alpha \in F_\psi$.

We write $(\Gamma_1, F_\psi) \ \tilde{\subseteq} \ (\Gamma_2, H_\psi)$. And (Γ_1, F_ψ) is said to be a hypersoft superset of (Γ_2, H_ψ), if (Γ_2, H_ψ) is a hypersoft subset of (Γ_1, F_ψ). We write it as $(\Gamma_1, F_\psi) \tilde{\supseteq} (\Gamma_2, H_\psi)$.

Definition 4. *[23] Two hypersoft sets (Γ_1, F_ψ) and (Γ_2, H_ψ) over a common universe U are said to be hypersoft equal if (Γ_1, F_ψ) is a hypersoft subset of (Γ_2, H_ψ) and (Γ_2, H_ψ) is a hypersoft subset of (Γ_1, F_ψ).*

Definition 5. *[23] Let $\psi=\{r_1, r_2, \ldots, r_n\}$ be a set of parameters (attributes). The NOT set of ψ denoted by $\neg\psi$ is defined by $\neg\psi = \{\neg r_1, \neg r_2, \ldots, \neg r_n\}$ where $\neg r_i = \mathrm{not} r_i$ for $i \in \{1, 2, \ldots, n\}$.*

Definition 6. *[23] Let U be a universal set. The complement of a hypersoft set (Γ, V_ψ) is denoted by $(\Gamma, V_\psi)^c$ and is defined by $(\Gamma, V_\psi)^c = (\Gamma^c, V_\psi)$ where $\Gamma^c : V_\psi \to P(U)$ is a mapping given by $\Gamma^c(\alpha) = U \setminus \Gamma(\alpha)$, for all $\alpha \in V_\psi$.*

Definition 7. *[20] Let U be a universal set. A hypersoft set (Γ, V_ψ) over U is said to be a null hypersoft set and denoted by (ϕ, V_ψ), if for all $\alpha \in V_\psi, \Gamma(\alpha) = \phi$.*

Definition 8. *[20] Let U be a universal set. A hypersoft set (Γ, V_ψ) over U is said to be a whole hypersoft set and denoted by (U, V_ψ), if for all $\alpha \in V_\psi, \Gamma(\alpha) = U$.*

Definition 9. *[20] Difference of two hypersoft sets (Γ_1, F_ψ) and (Γ_2, H_ψ) over a universe U is a hypersoft set (Γ, V_ψ) where $V_\psi = F_\psi \cap H_\psi$ and for all $\alpha \in V_\psi$, $\Gamma(\alpha) = \Gamma_1(\alpha) \setminus \Gamma_2(\alpha)$. We write $(\Gamma_1, F_\psi) \setminus (\Gamma_2, H_\psi) = (\Gamma, V_\psi)$.*

Definition 10. *[20] Union of two hypersoft sets* (Γ_1, F_ψ) *and* (Γ_2, H_ψ) *over a universe* U *is a hypersoft set* (Γ, V_ψ) *where* $V_\psi = F_\psi \cup H_\psi$ *and for all* $\alpha \in V_\psi, \Gamma(\alpha) = \Gamma_1(\alpha) \cup \Gamma_2(\alpha)$ *and will be written as* $(\Gamma_1, F_\psi) \tilde{\sqcup} (\Gamma_2, H_\psi) = (\Gamma, V_\psi)$.

Definition 11. *[20] Intersection of two hypersoft sets* (Γ_1, F_ψ) *and* (Γ_2, H_ψ) *over a universe* U, *is a hypersoft set* (Γ, V_ψ) *where* $V_\psi = F_\psi \cap H_\psi$ *and for all* $\alpha \in V_\psi$, $\Gamma(\alpha) = \Gamma_1(\alpha) \cap \Gamma_2(\alpha)$ *and will be written as* $(\Gamma_1, F_\psi) \tilde{\sqcap} (\Gamma_2, H_\psi) = (\Gamma, V_\psi)$.

3. Set-theoretic Operations on Plithogenic Crisp Hypersoft Sets and Their Properties

The results of this section is appear in [1], but we have redefined.

Definition 12. *Let* U_P *be a universal set and* $\psi = \{r_1, r_2, \ldots, r_n\}$ *be a set of* n*-distinct attributes with attribute value sets respectively as* $E_1, E_2, \ldots, E_n$, *where* $E_i \cap E_j = \phi$ *for* $i \neq j$ *and* $i, j \in \{1, 2, \ldots, n\}$. *Also, let* D_i *be the nonempty subset of* E_i *for each* $i \in \{1, 2, \ldots, n\}$ *and* $V_\psi = D_1 \times D_2 \times \cdots \times D_n$. *The triple* $(\Gamma, C, V_\psi)_{PC}$ *is called a Plithogenic crisp hypersoft (in short, PCHS) set where* $\Gamma : V_\psi \to P(U_P)$ *and* $C : P(U_P) \times D_i \to \{0, 1\}$, *for all* $x \in P(U_P)$, *for each* $i \in \{1, 2, \ldots, n\}$. *That is,* $(\Gamma, C, V_\psi)_{PC} = \{< (\beta),\ \{x\ (C(x, d_i))\};\ \beta \in V_\psi$ *and* $x \in \Gamma(\beta)\} >\}$. *Note that for* $x \notin \Gamma(\beta)$, $C(x, d_i) = 0$ *for each* $i \in \{1, 2, ..., n\}$. *The set of all the PCHS sets over* U_P *will be denoted as* $P_{PC}(U_P)$.

Definition 13. *A PCHS set* $(\Gamma, C, V_\psi)_{PC}$ *over* U_P *is called a null plithogenic crisp hypersoft (in short, null PCHS) set if* $\forall \beta \in V_\psi$, $C(x, d_i) = 0_{PC}$ *for each* $i \in \{1, 2, \ldots, n\}$ *and for all* $x \in U_P$. *The null PCHS set will be denoted by* $(\Phi, C, V_\psi)_{PC}$.

Definition 14. *A PCHS set* $(\Gamma, C, V_\psi)_{PC}$ *is called a whole plithogenic crisp hypersoft (in short, whole PCHS) set if* $\forall \beta \in V_\psi$, $C(x, d_i) = 1_{PC}$ *for each* $i \in \{1, 2, \ldots, n\}$ *and for all* $x \in U_P$. *The whole PCHS set will be denoted by* $(\Psi, C, V_\psi)_{PC}$.

Definition 15. *Let* $(\Gamma_1, C_1, F_\psi)_{PC}$ *and* $(\Gamma_2, C_2, H_\psi)_{PC}$ *be two PCHS sets over* U_P. *Then* $(\Gamma_1, C_1, F_\psi)_{PC}$ *is a PCHS subset of* $(\Gamma_2, C_2, H_\psi)_{PC}$ *if* $F_\psi \subseteq H_\psi$ *and* $\Gamma_1(\beta) \subseteq \Gamma_2(\beta)$ *for all* $\beta \in F_\psi$ *and* $C_1(x, d_i) \leq C_2(x, d_i)$, *for each* $i \in \{1, 2, \ldots, n\}$ *and for all* $x \in \Gamma_1(\beta)$.
And it will be denoted by $(\Gamma_1, C_1, F_\psi)_{PC} \ \breve{\breve{\sqsubseteq}}\ (\Gamma_2, C_2, H_\psi)_{PC}$. *Thus* $(\Gamma_1, C_1, F_\psi)_{PC}$ *and* $(\Gamma_2, C_2, H_\psi)_{PC}$ *are equal, if* $(\Gamma_1, C_1, F_\psi)_{PC} \ \breve{\breve{\sqsubseteq}}\ (\Gamma_2, C_2, H_\psi)_{PC}$ *and* $(\Gamma_2, C_2, H_\psi)_{PC} \ \breve{\breve{\sqsubseteq}}\ (\Gamma_1, C_1, F_\psi)_{PC}$ *and denoted by* $(\Gamma_1, C_1, F_\psi)_{PC} \ \breve{\breve{=}}\ (\Gamma_2, C_2, H_\psi)_{PC}$.

Definition 16. *Let* $(\Gamma_1, C_1, F_\psi)_{PC}$ *and* $(\Gamma_2, C_2, H_\psi)_{PC}$ *be two PCHS sets over* U_P. *The intersection of* $(\Gamma_1, C_1, F_\psi)_{PC}$ *and* $(\Gamma_2, C_2, H_\psi)_{PC}$ *is a PCHS set as defined the follows: Let* $\beta \in F_\psi \cap H_\psi$, *then* :

$$(\Gamma_1, C_1, V_\psi)_{PC} \ \breve{\breve{\sqcap}}\ (\Gamma_2, C_2, H_\psi)_{PC} \ \breve{\breve{=}}\ \{< \beta, \{x(Min\{C_1(x, d_i), C_2(x, d_i)\})\} >\}.$$

Definition 17. *Let* $(\Gamma_1, C_1, F_\psi)_{PC}$ *and* $(\Gamma_2, C_2, H_\psi)_{PC}$ *be two PCHS sets over* U_P. *The union of* $(\Gamma_1, C_1, F_\psi)_{PC}$ *and* $(\Gamma_2, C_2, H_\psi)_{PC}$ *is a PCHS set as defined the follows: Let*

$\beta \in F_\psi \cup H_\psi, then:$

$$(\Gamma_1, C_1, F_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, H_\psi)_{PC} \breve{\tilde{=}} \begin{cases} \{< (\beta),\ \{x\ \ (C_1(x,d_i))\} >\} & if\ \beta \in F_\psi \setminus H_\psi \\ \{< (\beta),\ \{x\ \ (C_2(x,d_i))\} >\} & if\ \beta \in H_\psi \setminus F_\psi \\ \{< (\beta), \{x\ (Max\{C_1(x,d_i), C_2(x,d_i)\} >\} & if\ \beta \in H_\psi \cap F_\psi \end{cases}$$

Definition 18. *Let $(\Gamma_1, C_1, F_\psi)_{PC}$ and $(\Gamma_2, C_2, H_\psi)_{PC}$ be two PCHS sets over U_P. The PCHS difference of $(\Gamma_1, C_1, F_\psi)_{PC}$ and $(\Gamma_2, C_2, H_\psi)_{PC}$ is denoted by $(\Gamma, C, V\psi)_{PC}$ where $V\psi = F_\psi \cap H_\psi$ and for all $\beta \in V_\psi$, $V_\psi(\beta) = F_\psi(\beta) \setminus H_\psi(\beta)$. We write $(\Gamma, C, V\psi)_{PC} \breve{\tilde{=}} (\Gamma_1, C_1, F_\psi)_{PC} \breve{\setminus} (\Gamma_2, C_2, H_\psi)_{PC}$.*

Definition 19. *The complement of a PCHS set $(\Gamma,C,V_\psi)_{PC} = \{< (\beta),\ \{x\ \ (C(x,d_i))\};\ \beta \in V_\psi$ and $x \in \Gamma(\beta)\} >\}$ is denoted by $\left(\Gamma,C,V_\psi\right)^c_{PC}$ and defined by $(\Gamma, C, V_\psi)^c_{PC}\ \breve{\tilde{=}}\ \{<\ (\beta),\ \ \{x\ (C(x,d_i))^c\}; \beta\ \in\ V_\psi$ and for all $x\ \in\ U_P\}$. That is, if $C(x,d_i) = 0$, then $(C(x,d_i))^c = 1$ or the reverse.*

Proposition 1. *Let $\left(\Gamma,C,V_\psi\right)_{PC}$ be a PCHS set over U_P. Then the following are true:*

(i) $(\Gamma,C,V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Phi,C,V_\psi)_{PC} \breve{\tilde{=}} \left(\Gamma,C,V_\psi\right)_{PC}$

(ii) $\left(\Gamma,C,V_\psi\right)_{PC} \breve{\tilde{\sqcap}} (\Phi,C,V_\psi)_{PC} \breve{\tilde{=}} (\Phi,C,V_\psi)_{PC}$

(iii) $\left(\Gamma,C,V_\psi\right)_{PC} \breve{\tilde{\sqcup}} (\Psi,C,V_\psi)_{PC} \breve{\tilde{=}} (\Psi,C,V_\psi)_{PC}$

(iv) $\left(\Gamma,C,V_\psi\right)_{PC} \breve{\tilde{\sqcap}} (\Psi,C,V_\psi)_{PC} \breve{\tilde{=}} \left(\Gamma,C,V_\psi\right)_{PC}$

(v) $(\Psi,C,V_\psi)_{PC} \breve{\setminus} (\Gamma,C,V_\psi)_{PC} \breve{\tilde{=}} \left(\Gamma,C,V_\psi\right)^c_{PC}$

(vi) $(\Gamma,C,V_\psi)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma,C,V_\psi\right)^c_{PC} \breve{\tilde{=}} (\Psi,C,V_\psi)_{PC}$

(vii) $(\Gamma,C,V_\psi)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma,C,V_\psi\right)^c_{PC} \breve{\tilde{=}} (\Phi,C,V_\psi)_{PC}$.

Proof. Straightforward.

Definition 20. *A PCHS set $(\Gamma,C,V_\psi)_{PC}$ is said to be a PCHS point, if* range(Γ) $=\{x\}$ *and $\exists i \in \{1, 2, \ldots, n\}$ such that $C(x,d_i) = 1$ and will be denote by $P_C P^{(\beta,x)}$ where $\beta \in V_\psi$.*

Proposition 2. *Let $(\Gamma,C,V_\psi)_{PC}, (\Gamma_1, C_1, F_\psi)_{PC}$ and $(\Gamma_2, C_2, H_\psi)_{PC}$ be PCHS sets over U_P. Then the following hold:*

(i) If $(\Gamma,C,V_\psi)_{PC}$ is not a null PCHS point, then $(\Gamma,C,V_\psi)_{PC}$ contains at least one non-null PCHS point.

(ii) $(\Gamma_1, C_1, F_\psi)_{PC} \breve{\tilde{\sqsubseteq}} (\Gamma_2, C_2, H_\psi)_{PC} \iff P_cP^{(\boldsymbol{\beta},x)} \in (\Gamma_1, C_1, F_\psi)_{PC}$ *implies that* $P_cP^{(\boldsymbol{\beta},\boldsymbol{x})} \in (\Gamma_2, C_2, H_\psi)_{PC}$.

(iii) $P_cP^{(\boldsymbol{\beta},\boldsymbol{x})} \in (\Gamma_1, C_1, F_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, H_\psi)_{PC} \iff P_cP^{(\boldsymbol{\beta},x)} \in (\Gamma_1, C_1, F_\psi)_{PC}$ *or* $P_cP^{(\boldsymbol{\beta},x)} \in (\Gamma_2, C_2, H_\psi)_{PC}$.

(iv) $P_cP^{(\boldsymbol{\beta},x)} \in (\Gamma_1, C_1, F_\psi)_{PC} \breve{\tilde{\sqcap}} (\Gamma_2, C_2, H_\psi)_{PC} \iff P_cP^{(\boldsymbol{\beta},\boldsymbol{x})} \in (\Gamma_1, C_1, F_\psi)_{PC}$ *and* $P_cP^{(\boldsymbol{\beta},x)} \in (\Gamma_2, C_2, H_\psi)_{PC}$.

(v) $P_cP^{(\boldsymbol{\beta},x)} \in (\Gamma_1, C_1, F_\psi)_{PC} \breve{\tilde{\setminus}} (\Gamma_2, C_2, H_\psi)_{PC} \iff P_cP^{(\boldsymbol{\beta},x)} \in (\Gamma_1, C_1, F_\psi)_{PC}$ *and* $P_cP^{(\boldsymbol{\beta},x)} \notin (\Gamma_2, C_2, H_\psi)_{PC}$.

Proof. Straightforward.

Remark 1.

(i) If $C(x, d_i) = 0_{PC}$ *for all* $i \in \{1, 2, \ldots, n\}$, *then* $P_C P^{(\beta,x)}$ *is called null PCHS point and denoted by* $P_C P^{(\beta,x,0_{PC})}$. *Also, if* $\left(\Gamma, C, V_\psi\right)_{PC}$ *is null PCHS set, then it can be considered as a null PCHS point.*

(ii) $\left(\Gamma, C, V_\psi\right)_{PC} \breve{\tilde{=}} \breve{\tilde{\sqcup}} \left\{P_C P^{(\beta,x)};\ P_C P^{(\beta,x)} \in (\Gamma, C, V_\psi)_{PC}\right\}$

4. Plithogenic Crisp Hypersoft Topology

Definition 21. *Let* $\left(\Gamma, C, V_\psi\right)_{PC}$ *be a PCHS set over* U_P *and* $x \in U_P$. *Then* $x \in \left(\Gamma, C, V_\psi\right)_{PC}$ *if* $x \in \Gamma(\beta)$ *for all* $\beta \in V_\psi$ *and* $C(x, d_i) \neq 0$ *for some* $i \in \{1, 2, ..., n\}$. *Otherwise,* $x \notin \left(\Gamma, C, V_\psi\right)_{PC}$.

Definition 22. *Let* Y *be a non-empty subset of* U_P. *Then* $(Y,\ C,\ V_\psi)_{PC}$ *denoted a PCHS set over* U_P *and defined by* $Y(\beta) = Y$, *for all* $\beta \in V_\psi$.

Definition 23. *Let* $(\Gamma, C, V_\psi)_{PC}$ *be a PCHS set over* U_P, *and let* Y *be a non-empty subset of* U_P. *The sub-PCHS set of* $(\Gamma, C, V_\psi)_{PC}$ *over* Y *is denoted by* $(\Gamma_Y, C, V_\psi)_{PC}$ *and is defined as* $\Gamma_Y(\beta) = Y \cap \Gamma(\beta)$ *for each* $\beta \in V_\psi$. *That is,*

$$(\Gamma_Y, C, V_\psi)_{PC} \breve{\tilde{=}} (Y, C, V_\psi)_{PC} \breve{\tilde{\sqcap}} (\Gamma, C, V_\psi)_{PC}.$$

Definition 24. *Let* τ_{PC} *be the collection of PCHS sets over* U_P. *Then* τ_{PC} *is said to be plithogenic crisp hypersoft (in short, PCHS) topology over* U_P, *if the following holds:*

(i) $(\Phi, C, V_\psi)_{PC}$, $(\Psi, C, V_\psi)_{PC}$ *belong to* τ_{PC},

(ii) The intersection of any two PCHS sets in τ_{PC} *belongs to* τ_{PC},

(iii) The union of any number of PCHS sets in τ_{PC} *belongs to* τ_{PC}.

Then (U_P, τ_{PC}, V_ψ) is called a plithogenic crisp hypersoft topological (in short, PCHST) space over U_P. Also, the members of (U_p, τ_{PC}, V_ψ) are said to be plithogenic crisp hypersoft (in short, PCHS) open sets over U_P.

Definition 25. *In a PCHST space (U_p, τ_{PC}, V_ψ) a PCHS set $\left(\Gamma, C, V_\psi\right)_{PC}$ over U_P is said to be plithogenic crisp hypersoft (in short, PCHS) closed set if its complement belongs to τ_{PC}.*

Example 1. *Let $U_P = \{x_1,\ x_2,\ x_3,\ x_4\}$, $E_1 = \{e_1,\ e_2\}$, $E_2 = \{e_3\}$, $E_3 = \{e_4\}$. Let $V_\psi = E_1 \times E_2 \times E_3$ and $(\alpha) = (e_1, e_3, e_4)$ and $(\beta) = (e_2, e_3, e_4)$. Define the following PCHS sets:*

$(\Gamma_1, C_1, V_\psi)_{PC} = \{< (\alpha), \{x_1\,(1,0,1)\} >, < (\beta)\,, \{x_2\,(1,0,1)\} >\}$
$(\Gamma_2, C_2, V_\psi)_{PC} = \{< (\alpha), \{x_1\,(1,0,1)\} >, < (\beta)\,, 1_{PC} >\}$
$(\Gamma_3, C_3, V_\psi)_{PC} = \{< (\alpha), 1_{PC} >, < (\beta)\,, \{x_2\,(1,0,1)\} >\}$.

Then the collection:
$\tau_{PC} = \{(\Phi, C, V_\psi)_{PC}, (\Gamma_1, C_1, V_\psi)_{PC}, (\Gamma_2, C_2, V_\psi)_{PC}, (\Gamma_3, C_3, V_\psi)_{PC}, (\Psi, C, V_\psi)_{PC}\}$ forms a PCHS topology over U_P.

Remark 2. *Let (U_p, τ_{PC}, V_ψ) be a PCHST space over U_P. Then the following holds:*

(i) $(\Phi, C, V_\psi)_{PC}$, $(\Psi, C, V_\psi)_{PC}$ are PCHS closed sets over U_P.

(ii) The intersection of any number of PCHS closed sets is PCHS closed set over U_P.

(iii) The union of any two number of PCHS closed sets is PCHS closed set over U_P.

Definition 26. *Let U_P be the plithogenic crisp universal set. Then*

(i) $\tau^I_{PC} = \{(\Phi, C, V_\psi)_{PC}, (\Psi, C, V_\psi)_{PC}\}$ is called Plithogenic crisp hypersoft indiscrete (in short, PCHSI) topology over U_P and $\left(\Gamma, \tau^I_{PC}, V_\psi\right)$ is called plithogenic crisp hypersoft indiscrete topological (in short, PCHSIT) space over U_P.

(ii) $\tau^D_{PC} = P_{PC}(U_P)$ is called Plithogenic crisp hypersoft discrete (in short, PCHSD) topology over U_P and $\left(\Gamma, \tau^D_{PC}, V_\psi\right)$ is called plithogenic crisp hypersoft discrete (in short, PCHSDT) topological space over U_P.

Definition 27. *Let $(\Gamma, \tau_{PC_1}, V_\psi)$ and $(\Gamma, \tau_{PC_2}, V_\psi)$ be two PCHST spaces over U_P. If $\tau_{PC_1} \,\breve{\breve{\sqsubseteq}}\, \tau_{PC_2}$, then τ_{PC_2} is said to be finer than τ_{PC_1}. If $\tau_{PC_2} \,\breve{\breve{\sqsubseteq}}\, \tau_{PC_1}$, then τ_{PC_1} is said to be finer than τ_{PC_2}. If $\tau_{PC_1} \,\breve{\breve{\sqsubseteq}}\, \tau_{PC_2}$ or $\tau_{PC_2} \,\breve{\breve{\sqsubseteq}}\, \tau_{PC_1}$, then τ_{PC_1} and τ_{PC_2} are said to be comparable PCHS topologies over U_P.*

Proposition 3. *Let $(\Gamma, \tau_{PC}, V_\psi)$ and $(\Gamma, \tau^*_{PC}, V_\psi)$ be two PCHST spaces over U_P, then $\left(\Gamma, \tau_{PC} \,\breve{\breve{\sqcap}}\, \tau^*_{PC}, V_\psi\right)$ is a PCHS topological space over U_P.*

Proof.

(i) Clearly $(\Phi,C,V_{\psi})_{PC}$, $(\Psi,C,V_{\psi})_{PC} \in \tau_{PC} \,\breve{\breve{\sqcap}}\, \tau^*_{PC}$.

(ii) Let $(\Gamma_1,C_1,V_\psi)_{PC}$, $(\Gamma_2,C_2,V_\psi)_{PC} \in \tau_{PC} \,\breve{\breve{\sqcap}}\, \tau^*_{PC}$, then $(\Gamma_1,C_1,V_\psi)_{PC}$, $(\Gamma_2,C_2,V_\psi)_{PC} \in \tau_{PC}$ and $(\Gamma_1,C_1,V_\psi)_{PC}$, $(\Gamma_2,C_2,V_\psi)_{PC} \in \tau^*_{PC}$. Since $(\Gamma_1,C_1,V_\psi)_{PC} \,\breve{\breve{\sqcap}}\, (\Gamma_2,C_2,V_\psi)_{PC} \in \tau_{PC}$ and $(\Gamma_1,C_1,V_\psi)_{PC} \,\breve{\breve{\sqcap}}\, (\Gamma_2,C_2,V_\psi)_{PC} \in \tau^*_{Pc}$, so $(\Gamma_1,C_1,V_\psi)_{PC} \,\breve{\breve{\sqcap}}\, (\Gamma_2,C_2,V_\psi)_{PC} \in \tau_{PC} \,\breve{\breve{\sqcap}}\, \tau^*_{PC}$.

(iii) Let $\{(\Gamma_i,C_i,V_\psi)_{PC}; i \in I\}$ be a family of PCHS sets in $\tau_{PC} \,\breve{\breve{\sqcap}}\, \tau^*_{PC}$.
Then $(\Gamma_i,C_i,V_\psi)_{PC} \in \tau_{PC}$ and $(\Gamma_i,C_i,V_\psi)_{PC} \in \tau^*_{PC}$ for each $i \in I$, so $\breve{\breve{\bigsqcup}}_{i\in I}(\Gamma_i,C_i,V_\psi)_{PC} \in \tau_{PC}$ and $\breve{\breve{\bigsqcup}}_{i\in I}(\Gamma_i,C_i,V_\psi)_{PC} \in \tau^*_{PC}$. Therefore, $\breve{\breve{\bigsqcup}}_{i\in I}(\Gamma_i,C_i,V_\psi)_{PC} \in \tau_{PC} \,\breve{\breve{\sqcap}}\, \tau^*_{Pc}$.

Thus, $\tau_{PC} \,\breve{\breve{\sqcap}}\, \tau^*_{PC}$ forms a PCHS topology over U_P and $\left(\Gamma, \tau_{PC} \,\breve{\breve{\sqcap}}\, \tau^*_{PC}, V_\psi\right)$ is a PCHST space over U_P.

Remark 3. *The union of two PCHS topologies on U_P may not be a PCHS topology on U_P. See the next example.*

Example 2. *Let $U_P = \{x_1,\ x_2,\ x_3,\ x_4\}$, $E_1 = \{e_1,\ e_2\}$, $E_2 = \{e_3\}$, $E_3 = \{e_4\}$. Let $V_\psi = E_1 \times E_2 \times E_3$ and $(\alpha) = (e_1, e_3, e_4)$ and $(\beta) = (e_2, e_3, e_4)$. Define the following PCHS sets:*

$(\Gamma_1,C_1,V_{\psi})_{PC} = \{< (\alpha), \{x_3\,(1,1,0),\ x_4\,(1,1,1)\} >,\ < (\beta), \{x_2\,(1,1,1),\ x_3\,(1,1,1)\} >\}$

$(\Gamma_2,C_2,V_{\psi})_{PC} = \{< (\alpha), \{x_1\,(1,1,1),\ x_2\,(1,1,1), x_3\,(1,1,1)\} >,\ < (\beta), \{x_1\,(1,1,1),\ x_4\,(1,1,1)\} >\}$

$(\Gamma_3,C_3,V_{\psi})_{PC} = \{< (\alpha), \{x_3\,(1,1,0)\} >,\ < (\beta), 0_{PC} >\}$

$(\Gamma^*_1,C^*_1,V_\psi)_{PC} = \{< (\alpha), \{x_1\,(1,0,1)\} >, < (\beta), \{x_2\,(1,1,0)\} >\}$

$(\Gamma^*_2,C^*_2,V_\psi)_{PC} = \{< (\alpha), \{x_1\,(1,0,0)\} >, < (\beta), \{x_2\,(1,0,0)\} >\}$

$(\Gamma^*_3,C^*_3,V_\psi)_{PC} = \{< (\alpha), \{x_1\,(1,0,0)\} >, < (\beta), \{x_2\,(1,1,0)\} >\}$.

Then the collections:

$\tau_{PC} = \left\{(\Phi,C,V_{\psi})_{PC},\ (\Gamma_1,C_1,V_\psi)_{PC},\ (\Gamma_2,C_2,V_\psi)_{PC},\ (\Gamma_3,C_3,V_\psi)_{PC},\ (\Psi,C,V_{\psi})_{PC}\right\}$

and $\tau^*_{PC} = \left\{(\Phi,C,V_{\psi})_{PC},\ (\Gamma^*_1,C^*_1,V_\psi)_{PC},\ (\Gamma^*_2,C^*_2,V_\psi)_{PC},\ (\Gamma^*_3,C^*_2,V_\psi)_{PC},\ (\Psi,C,V_{\psi})_{PC}\right\}$

forms a PCHS topological spaces on U_P.

Now, we see that:

$(\Gamma^*_3,C^*_3,V_\psi)_{PC} \,\breve{\breve{\sqcup}}\, (\Gamma_3,C_3,V_\psi)_{PC} = \{< (\alpha), \{x_1\,(1,1,0), x_3\,(1,1,0)\} >, < (\beta), \{x_2\,(1,1,0)\} >\}$

not in $\tau_{PC} \,\breve{\breve{\sqcup}}\, \tau^*{}_{PC}$. Hence, $\tau_{PC} \,\breve{\breve{\sqcup}}\, \tau^*_{PC}$ does not form a PCHS topology over U_P.

Definition 28. *Let (U_p, τ_{PC}, V_ψ) be a PCHST space over U_P, $(\Gamma,C,V_{\psi})_{PC}$ be a PCHS set over U_P and $x \in U_P$. Then $(\Gamma,C,V_{\psi})_{PC}$ is said to be a PCHS neighborhood of x if there exists a PCHS open* set $(\Gamma^*,C^*,V_{\psi})_{PC}$ *such that $x \in (\Gamma^*,C^*,V_{\psi})_{PC} \,\breve{\breve{\sqsubseteq}}\, (\Gamma,C,V_{\psi})_{PC}$.*

Remark 4. *Let (U_p, τ_{PC}, V_ψ) be a PCHST space over U_P, then:*

(i) If $\left(\Gamma,C,V_{\psi}\right)_{PC}$ *is a PCHS neighborhood of* $x \in U_P$*, then* $x \in \left(\Gamma,C,V_{\psi}\right)_{PC}$.

(ii) Each $x \in U_P$ *has a PCHS neighborhood.*

(iii) If $\left(\Gamma,C,V_{\psi}\right)_{PC}$ *and* $\left(\Gamma^*,C^*,V_{\psi}\right)_{PC}$ *are PCHS neighborhoods of some* $x \in U_P$*, then* $\left(\Gamma,C,V_{\psi}\right)_{PC} \,\breve{\tilde{\sqcap}}\, \left(\Gamma,C,V_{\psi}\right)_{PC}$ *is also a PCHS neighborhood of* x.

(iv) If $\left(\Gamma,C,V_{\psi}\right)_{PC}$ *is a PCHS neighborhood of* $x \in U_P$ *and* $\left(\Gamma,C,V_{\psi}\right)_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma^*,C^*,V_{\psi}\right)_{PC}$*, then* $\left(\Gamma^*,C^*,V_{\psi}\right)_{PC}$ *is also a PCHS neighborhood of* $x \in U_P$.

Remark 5. *Let Let* (U_p, τ_{PC}, V_ψ) *be a PCHST space over* U_P *and* $\left(\Gamma,C,V_{\psi}\right)_{PC} \in \tau_{PC}$*. Then for any* x *in image of* $\Gamma(\beta)$ *for* $\beta \in V_\psi$*, we have* $x \in \left(\Gamma,C,V_{\psi}\right)_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma,C,V_{\psi}\right)_{PC}$ *and so* $\left(\Gamma,C,V_{\psi}\right)_{PC}$ *is a PCHS neighborhood of each its points. But the converse of it may not be true. See the next example.*

Example 3. *Consider the PCHST space* (U_P, τ_{PC}, V_ψ) *in Example 2.*
Now, cosider the following PCHS set:
$(\Gamma_*, C_*, V_\psi)_{PC} = \{< (\alpha), \{x_1\,(1,1,0)\,,\ x_3\,(1,1,0)\,, x_4\,(1,1,0)\} >, < (\beta)\,, \{x_2\,(1,1,0)\,, x_3\,(1,1,0) >\}$
is a PCHS neighborhood of each of its points, but it is not a PCHS open set.

Definition 29. *Let* (U_p, τ_{PC}, V_ψ) *be a PCHST space over* U_P *and Let* $\left(\Gamma,C,V_{\psi}\right)_{PC}$ *be a PCHS set over* U_P*. A point* $x \in U_P$ *is called a PCHS limit point of* $\left(\Gamma,C,V_{\psi}\right)_{PC}$ *if* $\left(\Gamma,C,V_{\psi}\right)_{PC} \breve{\tilde{\sqcap}} \left[\left(\Gamma_*,C_*,V_{\psi}\right)_{PC} \breve{\tilde{\setminus}} \{x\}\right] \breve{\tilde{\neq}} \left(\Phi,C,V_{\psi}\right)_{PC}$ *for every PCHS open set* $\left(\Gamma_*,C_*,V_{\psi}\right)_{PC}$ *containing* x*. The set of all PCHS limit points of* $\left(\Gamma,C,V_{\psi}\right)_{PC}$ *is denoted by* $\left(\Gamma,C,V_{\psi}\right)^d_{PC}$.

Proposition 4. *Let* (U_p, τ_{PC}, V_ψ) *be a PCHS space over* U_P *and let* $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}$ *,* $\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}$ *be two PCHS sets over* U_P*. Then:*

(i) $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma_2,C_2,V_{\psi}\right)_{PC}$ *implies* $\left(\Gamma_1,C_1,V_{\psi}\right)^d_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma_2,C_2,V_{\psi}\right)^d_{PC}$.

(ii) $\left[\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2,C_2,V_{\psi}\right)_{PC}\right]^d \breve{\tilde{\sqsubseteq}} \left(\Gamma_1,C_1,V_{\psi}\right)^d_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2,C_2,V_{\psi}\right)^d_{PC}$.

(iii) $\left[\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2,C_2,V_{\psi}\right)_{PC}\right]^d \breve{\tilde{\cong}} \left(\Gamma_1,C_1,V_{\psi}\right)^d_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2,C_2,V_{\psi}\right)^d_{PC}$.

Proof.

(i) Let $x \in \left(\Gamma_1,C_1,V_{\psi}\right)^d_{PC}$, so that x is a PCHS limit point of $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}$, then, it follows that $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\tilde{\sqcap}} \left[\left(\Gamma_*,C_*,V_{\psi}\right)_{PC} \breve{\tilde{\setminus}} \{x\}\right] \breve{\tilde{\cong}} \left(\Phi,C,V_{\psi}\right)_{PC}$ for every PCHS open set $\left(\Gamma_*,C_*,V_{\psi}\right)_{PC}$ containing x. But since $\left(\Gamma_1,C_1,V_{\psi}\right) \breve{\tilde{\sqsubseteq}} \left(\Gamma_2,C_2,V_{\psi}\right)_{PC}$, it follows that $\left(\Gamma_2,C_2,V_{\psi}\right)_{PC} \breve{\tilde{\sqcap}} \left[\left(\Gamma_*,C_*,V_{\psi}\right)_{PC} \breve{\tilde{\setminus}} \{x\}\right] \breve{\tilde{\cong}} \left(\Phi,C,V_{\psi}\right)_{PC}$. Thus, $x \in \left(\Gamma_2,C,V_{\psi}\right)^d_{PC}$. Therefore, $\left(\Gamma_1,C_1,V_{\psi}\right)^d_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma_2,C_2,V_{\psi}\right)^d_{PC}$.

(ii) Since $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2, C_2, V_\psi\right)_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma_1, C_1, V_\psi\right)_{PC}$ and $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2, C_2, V_\psi\right)_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$,
then by part (i) follows that $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d \breve{\tilde{\sqsubseteq}} \left(\Gamma_1, C_1, V_\psi\right)^d_{PC}$ and $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d \breve{\tilde{\sqsubseteq}} \left(\Gamma_2, C_1, V_\psi\right)^d_{PC}$.
Hence $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d \breve{\tilde{\sqsubseteq}} \left(\Gamma_1, C_1, V_\psi\right)^d_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2, C_2, V_\psi\right)^d_{PC}$.

(iii) Since $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$ and $\left(\Gamma_2, C_2, V_\psi\right)_{PC} \breve{\tilde{\sqsubseteq}} \left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$, by part (i)
we have $\left(\Gamma_1, C_1, V_\psi\right)^d_{PC} \breve{\tilde{\sqsubseteq}} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d$ and $\left(\Gamma_2, C_2, V_\psi\right)^d_{PC} \breve{\tilde{\sqsubseteq}} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d$.
So, $\left(\Gamma_1, C_1, V_\psi\right)^d_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)^d_{PC} \breve{\tilde{\sqsubseteq}} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d$. Now,
Let $x \in \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d$,
Then $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right] \breve{\tilde{\sqcap}} \left[\left(\Gamma_*, C_*, V_\psi\right)_{PC} \breve{\setminus} \{x\}\right] \breve{\tilde{\neq}} (\Phi, C, V_\psi)_{PC}$ for every PCHS set $\left(\Gamma_*, C_*, V_\psi\right)_{PC}$ containing x.
Therefore, $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left[\left(\Gamma_*, C_*, V_\psi\right)_{PC} \breve{\tilde{\setminus}} \{x\}\right] \breve{\tilde{=}} (\Phi, C, V_\psi)_{PC}$ or $\left(\Gamma_2, C_2, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left[\left(\Gamma_*, C_*, V_\psi\right)_{PC} \breve{\tilde{\setminus}} \{x\}\right] \breve{\tilde{=}} (\Phi, C, V_\psi)_{PC}$. Thus, $x \in \left(\Gamma_1, C_1, V_\psi\right)^d_{PC}$ or $x \in \left(\Gamma_2, C_2, V_\psi\right)^d_{PC}$ and then $x \in \left(\Gamma_1, C_1, V_\psi\right)^d_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)^d_{PC}$. Therefore, $\left(\Gamma_1, C_1, V_\psi\right)^d_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)^d_{PC} \breve{\tilde{\sqsupseteq}} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d$. Hence, $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^d \breve{\tilde{=}} \left(\Gamma_1, C_1, V_\psi\right)^d_{PC} \breve{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)^d_{PC}$.

Remark 6. *The converse of above the proposition part (ii), may not be true. See the next example.*

Example 4. *Consider the PCHST space (U_P, τ_{PC}, V_ψ) in Example 2. Let $\left(\Gamma_4, C_4, V_\psi\right)_{PC}$ and $\left(\Gamma_5, C_5, V_\psi\right)_{PC}$ be two PCHS sets defined as follows:*
$\left(\Gamma_4, C_4, V_\psi\right)_{PC} = \{<(\alpha), 0_{PC}>, <(\beta), \{x_4\,(1,1,0)\}>\}$
$\left(\Gamma_5, C_5, V_\psi\right)_{PC} = \{<(\alpha), \{x_2\,(1,1,0)\}>, <(\beta), \{x_3\,(1,1,0)\}>\}$
Now, $\left(\Gamma_4, C_4, V_\psi\right)^d_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_5, C_5, V_\psi\right)^d_{PC} \breve{\tilde{=}} \{x_1, x_2, x_4\}$ *but* $\left[\left(\Gamma_4, C_4, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_5, C_5, V_\psi\right)_{PC}\right]^d \breve{\tilde{=}} (\Phi, C, V_\psi)_{PC}$. *Hence,* $\left[\left(\Gamma_1, C, V_\psi\right)_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2, C, V_\psi\right)_{PC}\right]^d \breve{\tilde{\not\supseteq}} \left(\Gamma_1, C, V_\psi\right)^d_{PC} \breve{\tilde{\sqcap}} \left(\Gamma_2, C, V_\psi\right)^d_{PC}$.

Definition 30. *Let* (U_p, τ_{PC}, V_ψ) *be a PCHST space over* U_P *and* Y *be a non-empty subset of* U_P*. Then* $\tau_{PC_Y} = \left\{ \left(\Gamma_{\mathrm{Y}}, C, V_\psi\right)_{PC} \,\middle|\, \left(\Gamma, C, V_\psi\right)_{PC} \in \tau_{PC} \right\}$ *is said to be the relative PCHS topology on* Y *and* (Y, τ_{PC_Y}, V_ψ) *is called a PCHS subspace of* (U_p, τ_{PC}, V_ψ)*. One can verify that* τ_{PC_Y} *is a PCHS topology on* Y*.*

Example 5. *Consider the PCHST space* (U_P, τ_{PC}, V_ψ) *in Example 2. and Definition* 23*. Let* $Y = \{x_2, x_3\}$*, then:*
$\left(\Gamma_{\mathrm{Y}_1}, C_1, V_\psi\right)_{PC} = \{< (\alpha), \{x_3\,(1,1,0)\} >,\ < (\beta), \{x_2\,(1,1,1),\ x_3\,(1,1,1)\} >\}$
$\left(\Gamma_{\mathrm{Y}_2}, C_2, V_\psi\right)_{PC} = \{< (\alpha), \{x_2\,(1,1,1), x_3\,(1,1,1)\} >,\ < (\beta), 0_{PC} >\}$.
$\left(\Gamma_{\mathrm{Y}_3}, C_3, V_\psi\right)_{PC} = \{< (\alpha), \{x_3\,(1,1,0)\} >,\ < (\beta), 0_{PC} >\}$.
Then, $\tau_{PC_Y} = \left\{ (\Phi, C, V_\psi)_{PC},\ (\Gamma_{\mathrm{Y}_1}, C_1, V_\psi)_{PC},\ (\Gamma_{\mathrm{Y}_2}, C_2, V_\psi)_{PC},\ (\Gamma_{\mathrm{Y}_3}, C_3, V_\psi)_{PC},\ (\Psi_{\mathrm{Y}}, C, V_\psi)_{PC} \right\}$

Proposition 5. *Let* (Y, τ_{PC_Y}, V_ψ) *be a PCHS subspace of PCHST space* (U_p, τ_{PC}, V_ψ) *and* $\left(\Gamma_{\mathrm{Y}}, C, V_\psi\right)_{PC}$ *be a PCHS open set in* Y*. If* $\left(Y, C, V_\psi\right)_{PC} \in \tau_{PC}$*, then* $\left(\Gamma_{\mathrm{Y}}, C, V_\psi\right)_{PC} \in \tau_{PC}$*.*

Proof. Let $\left(\Gamma_{\mathrm{Y}}, C, V_\psi\right)_{PC}$ be a PCHS open set in Y, then there exist a PCHS open set $\left(\Gamma, C, V_\psi\right)_{PC}$ in U_P such that $\left(\Gamma_{\mathrm{Y}}, C, V_\psi\right)_{PC} \,\breve{\breve{=}}\, \left(Y, C, V_\psi\right)_{PC} \,\breve{\breve{\sqcap}}\, \left(\Gamma, C, V_\psi\right)_{PC}$. Now, if $\left(Y, C, V_\psi\right)_{PC} \in \tau_{PC}$, then $\left(Y, C, V_\psi\right)_{PC} \,\breve{\breve{\sqcap}}\, \left(\Gamma, C, V_\psi\right)_{PC} \in \tau_{PC}$. Hence, $\left(\Gamma_{\mathrm{Y}}, C, V_\psi\right)_{PC} \in \tau_{PC}$.

Proposition 6. *Let* (Y, τ_{PC_Y}, V_ψ) *and* (Z, τ_{PC_Z}, V_ψ) *be two PCHS subspace of* (U_P, τ_{PC}, V_ψ) *and let* $Y \subseteq Z$*. Then* (Y, τ_{PC_Y}, V_ψ) *is a PCHS subspace of* (Z, τ_{PC_Z}, V_ψ)*.*

Proof. Let $\left(\Gamma_{\mathrm{Y}}, C, V_\psi\right)_{PC}$ be a PCHS open set in Y, then there exists a PCHS open set $\left(\Gamma, C, V_\psi\right)_{PC}$ in U_P such that $\left(\Gamma_{\mathrm{Y}}, C, V_\psi\right)_{PC} \,\breve{\breve{=}}\, (Y, \tau_{PC_Y}, V_\psi) \,\breve{\breve{\sqcap}}\, \left(\Gamma, C, V_\psi\right)_{PC}$, or equivalently, for each $\beta \in V_\psi$, $\Gamma_{\mathrm{Y}}(\beta) = Y \sqcap \Gamma(\beta)$. Since $Y \sqsubseteq Z$, then $Y = Y \sqcap Z$. Now, $\Gamma_{\mathrm{Y}}(\beta) = Y \sqcap \Gamma_{\mathrm{Z}}(\beta) = (Y \sqcap Z) \sqcap \Gamma(\beta) = Y \sqcap \Gamma_{\mathrm{Z}}(\beta)$. Hence, (Y, τ_{PC_Y}, V_ψ) is a PCHS subspace of (Z, τ_{PC_Z}, V_ψ).

5. PCHS Closure and PCHS Interior

Definition 31. *Let* (U_P, τ_{PC}, V_ψ) *be a PCHST space and* $\left(\Gamma, C, V_\psi\right)_{PC}$ *be a PCHS set over* U_P*. The intersection of all PCHS closed supersets of* $\left(\Gamma, C, V_\psi\right)_{PC}$ *is called the PCHS closure of* $\left(\Gamma, C, V_\psi\right)_{PC}$ *and is denoted by* $\overline{\left(\Gamma, C, V_\psi\right)_{PC}}$*. In other words:*
$$\overline{\left(\Gamma, C, V_\psi\right)_{PC}} \,\breve{\breve{=}}\, \breve{\breve{\sqcap}} \left\{ \left(\Gamma^*, C^*, V_\psi\right)_{PC} \,\middle|\, \left(\Gamma^*, C^*, V_\psi\right)^c_{PC} \in \tau_{PC},\ \left(\Gamma, C, V_\psi\right)_{PC} \,\breve{\breve{\sqsubseteq}}\, \left(\Gamma^*, C^*, V_\psi\right)_{PC} \right\}.$$

Proposition 7. *Let* (U_P, τ_{PC}, V_ψ) *be a PCHST space and* $\left(\Gamma, C, V_\psi\right)_{PC}$ *be a PCHS set over* U_P*, then:*

(i) $\overline{\left(\Gamma, C, V_\psi\right)_{PC}}$ *is the smallest PCHS closed set containing* $\left(\Gamma, C, V_\psi\right)_{PC}$*.*

(ii) $\left(\Gamma,C,V_{\psi}\right)_{PC}$ *is a PCHS closed sets if and only if* $\left(\Gamma,C,V_{\psi}\right)_{PC} \breve{\cong} \overline{\left(\Gamma,C,V_{\psi}\right)_{PC}}$.

Proof.

(i) Obvious.

(ii) Let $\left(\Gamma,C,V_{\psi}\right)_{PC}$ be a PCHS closed set. So, $\left(\Gamma,C,V_{\psi}\right)_{PC}$ itself is the the smallest PCHS closed set over U_P containing $\left(\Gamma,C,V_{\psi}\right)_{PC}$, hence, $\left(\Gamma,C,V_{\psi}\right)_{PC} \breve{\cong} \overline{\left(\Gamma,C,V_{\psi}\right)_{PC}}$. Conversely, let $\left(\Gamma,C,V_{\psi}\right)_{PC} \breve{\cong} \overline{\left(\Gamma,C,V_{\psi}\right)_{PC}}$, by part (i), $\overline{\left(\Gamma,C,V_{\psi}\right)_{PC}}$ is a PCHS closed set, so $\left(\Gamma,C,V_{\psi}\right)_{PC}$ is a PCHS closed set over U_P.

Proposition 8. *Let* $(U_P, \tau_{PC}, V_{\psi})$ *be a PCHS topological space and let* $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}$, $\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}$*be two PCHS set over* U_P*, then:*

(i) $(\Phi,C,V_{\psi})_{\ PC} \breve{\cong} \overline{(\Phi,C,V_{\psi})_{\ PC}}$ *and*$(\Psi,C,V_{\psi})_{\ PC} \breve{\cong} \overline{(\Psi,C,V_{\psi})_{\ PC}}$.

(ii) $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\sqsubseteq} \overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}}$.

(iii) $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\sqsubseteq} \left(\Gamma_2,C_2,V_{\psi}\right)_{PC}$ *implies* $\overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}} \breve{\sqsubseteq} \overline{\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}}$.

(iv) $\overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\sqcup} \left(\Gamma_2,C_2,V_{\psi}\right)_{PC}} \breve{\cong} \overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}} \breve{\sqcup} \overline{\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}}$.

(v) $\overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\sqcap} \left(\Gamma_2,C_2,V_{\psi}\right)_{PC}} \breve{\sqsubseteq} \overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}} \breve{\sqcap} \overline{\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}}$.

(vi) $\overline{\overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}}} \breve{\cong} \overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}}$.

Proof.

(i) Obvious.

(ii) By proposition 7(i), $\overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}}$ is the smallest PCHS closed set containing $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}$, so it follows $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\sqsubseteq} \overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}}$.

(iii) By part (ii), $\left(\Gamma_2,C_2,V_{\psi}\right)_{PC} \breve{\sqsubseteq} \overline{\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}}$. Since $\left(\Gamma_1,C_2,V_{\psi}\right)_{PC} \breve{\sqsubseteq} \left(\Gamma_2,C_2,V_{\psi}\right)_{PC}$, we have $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC} \breve{\sqsubseteq} \overline{\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}}$, but $\overline{\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}}$ is a PCHS closed set containing $\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}$ and since $\overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}}$ is the smallest PCHS closed set over U_P containing $\left(\Gamma_1,C,V_{\psi}\right)_{PC}$, so it follows that $\overline{\left(\Gamma_1,C_1,V_{\psi}\right)_{PC}} \breve{\sqcup} \overline{\left(\Gamma_2,C_2,V_{\psi}\right)_{PC}}$.

(iv) Since $(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqsubseteq}} (\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC}$ and $(\Gamma_2, C_2, V_\psi)_{PC} \breve{\tilde{\sqsubseteq}} (\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC}$, by part (iii), we have
$\overline{(\Gamma_1, C_1, V_\psi)_{PC}} \breve{\tilde{\sqsubseteq}} \overline{(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_1, V_\psi)_{PC}}$ and
$\overline{(\Gamma_2, C_2, V_\psi)_{PC}} \breve{\tilde{\sqsubseteq}} \overline{(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC}}$.
Hence, $\overline{(\Gamma_1, C_1, V_\psi)_{PC}} \breve{\tilde{\sqcup}} \overline{(\Gamma_2, C_2, V_\psi)_{PC}} \breve{\tilde{\sqsubseteq}} \overline{(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC}}$. Now, since $\overline{(\Gamma_1, C_1, V_\psi)_{PC}}$ and $\overline{(\Gamma_2, C_2, V_\psi)_{PC}}$ are PCHS closed sets over U_P, then $\overline{(\Gamma_1, C_1, V_\psi)_{PC}} \breve{\tilde{\sqcup}} \overline{(\Gamma_2, C_2, V_\psi)_{PC}}$ is also PCHS closed set. Also, $(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqsubseteq}} \overline{(\Gamma_1, C_1, V_\psi)_{PC}}$ and $(\Gamma_2, C_2, V_\psi)_{PC} \breve{\tilde{\sqsubseteq}} \overline{(\Gamma_2, C_2, V_\psi)_{PC}}$ it implies that $(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC} \breve{\tilde{\sqsubseteq}} \overline{(\Gamma_1, C_2, V_\psi)_{PC}} \breve{\tilde{\sqcup}} \overline{(\Gamma_2, C_2, V_\psi)_{PC}}$. Thus, $\overline{(\Gamma_1, C, V_\psi)_{PC}} \breve{\tilde{\sqcup}} \overline{(\Gamma_2, C, V_\psi)_{PC}}$ is a PCHS closed set containing $(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_1, C_1, V_\psi)_{PC}$.
Since $\overline{(\Gamma_1, C_2, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC}}$ is the smallest PCHS closed set containing $(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC}$, we have $\overline{(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC}} \breve{\tilde{\sqsubseteq}} \overline{(\Gamma_1, C_1, V_\psi)_{PC}} \breve{\tilde{\sqcup}} \overline{(\Gamma_2, C_2, V_\psi)_{PC}}$. Hence, $\overline{(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcup}} (\Gamma_2, C_2, V_\psi)_{PC}} \breve{\cong} \overline{(\Gamma_1, C_1, V_\psi)_{PC}} \breve{\tilde{\sqcup}} \overline{(\Gamma_2, C_2, V_\psi)_{PC}}$.

(v) Since $(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcap}} (\Gamma_2, C_2, V_\psi)_{PC} \breve{\tilde{\sqsubseteq}} (\Gamma_1, C_1, V_\psi)_{PC}$ and $(\Gamma_2, C_2, V_\psi)_{PC} \breve{\tilde{\sqcap}} (\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqsubseteq}} (\Gamma_2, C_2, V_\psi)_{PC}$. Therefore, $\overline{(\Gamma_1, C_1, V_\psi)_{PC} \breve{\tilde{\sqcap}} (\Gamma_2, C_2, V_\psi)_{PC}} \breve{\tilde{\sqsubseteq}} \overline{(\Gamma_1, C_1, V_\psi)_{PC}} \breve{\tilde{\sqcap}} \overline{(\Gamma_2, C_2, V_\psi)_{PC}}$.

(vi) Since $\overline{(\Gamma_1, C_1, V_\psi)_{PC}}$ is a PCHS closed set, by proposition 7(ii), it follows that $\overline{\overline{(\Gamma_1, C_1, V_\psi)_{PC}}} \breve{\cong} \overline{(\Gamma_1, C_1, V_\psi)_{PC}}$.

Remark 7. *The equality of above proposition part* (v) *does not hold in general. See the next example.*

Example 6. *Consider the PCHST space* (U_P, τ_{PC}, V_ψ) *in Example 2.*
Define $(\mathit{\Gamma}_4, C_4, V_\psi)_{PC}$ *and* $(\mathit{\Gamma}_5, C_5, V_\psi)_{PC}$ *as the follow:*
$(\mathit{\Gamma}_4, C_4, V_\psi)_{PC} = \{< (\alpha), \{x_1 (0,1,0)\} >, < (\beta), \{x_4 (1,1,1)\} >\}$ *and*
$(\mathit{\Gamma}_5, C_5, V_\psi)_{PC} = \{< (\alpha), \{x_2 (1,1,1)\} >, < (\beta), \{x_1 (1,0,1)\} >\}$.
Then: $\overline{(\mathit{\Gamma}_4, C_4, V_\psi)_{PC}} \breve{\cong} (\mathit{\Gamma}_1, C_1, V_\psi)^c_{PC}$ *and* $\overline{(\mathit{\Gamma}_5, C_5, V_\psi)_{PC}} \breve{\cong} (\mathit{\Gamma}_1, C_1, V_\psi)^c_{PC}$.
Now, $\overline{(\mathit{\Gamma}_4, C_4, V_\psi)_{PC}} \breve{\tilde{\sqcap}} \overline{(\mathit{\Gamma}_5, C_5, V_\psi)_{PC}} \breve{\cong} (\mathit{\Gamma}_1, C_1, V_\psi)^c_{PC}$ *but* $\overline{(\mathit{\Gamma}_4, C_4, V_\psi)_{PC} \breve{\tilde{\sqcap}} (\mathit{\Gamma}_5, C_5, V_\psi)_{PC}} \breve{\cong} (\Phi, C, V_\psi)_{PC}$ *and* $(\mathit{\Gamma}_1, C_1, V_\psi)^c_{PC} \breve{\not\cong} (\Phi, C, V_\psi)_{PC}$.

Definition 32. *Let (U_P, τ_{PC}, V_ψ) be a PCHST space over U_P, $\left(\Gamma, C, V_\psi\right)_{PC}$ be a PCHS set and $x \in U_P$. Then x is said to be a PCHS interior point of $\left(\Gamma, C, V_\psi\right)_{PC}$ if there exist a PCHS open set $\left(\Gamma^*, C^*, V_\psi\right)_{PC}$ such that $x \in \left(\Gamma^*, C^*, V_\psi\right)_{PC} \check{\tilde{\sqsubseteq}} \left(\Gamma, C, V_\psi\right)_{PC}$.*

Definition 33. *Let (U_P, τ_{PC}, V_ψ) be a PCHST space over U_P. Then the PCHS interior of PCHS set $\left(\Gamma, C, V_\psi\right)_{PC}$ over U_P is denoted by $\left(\Gamma, C, V_\psi\right)^o_{PC}$ and is defined as the union of all PCHS open sets contained in $\left(\Gamma, C, V_\psi\right)_{PC}$.*
In other words:
$$\left(\Gamma, C, V_\psi\right)^o_{PC} \check{\tilde{=}} \check{\tilde{\sqcup}} \left\{ \left(\Gamma^*, C^*, V_\psi\right)_{PC} \,\middle|\, \left(\Gamma^*, C^*, V_\psi\right)_{PC} \in \tau_{PC},\ \left(\Gamma^*, C^*, V_\psi\right)_{PC} \check{\tilde{\sqsubseteq}} \left(\Gamma, C, V_\psi\right)_{PC} \right\}$$

Proposition 9. *Let (U_P, τ_{PC}, V_ψ) be a PCHST space and let $\left(\Gamma, C, V_\psi\right)_{PC}$ be a PCHS set over U_P. Then:*

(i) $\left(\Gamma, C, V_\psi\right)^o_{PC}$ *is the largest PCHS open set contained in* $\left(\Gamma, C, V_\psi\right)_{PC}$.

(ii) $\left(\Gamma, C, V_\psi\right)_{PC}$ *is a PCHS open set if and only if* $\left(\Gamma, C, V_\psi\right)_{PC} \check{\tilde{=}} \left(\Gamma, C, V_\psi\right)^o_{PC}$.

Proof.

(i) Follows from the definition.

(ii) Let $\left(\Gamma, C, V_\psi\right)_{PC}$ be a PCHS open set. Then $\left(\Gamma, C, V_\psi\right)_{PC}$ is surely identical with the largest PCHS open subset of $\left(\Gamma, C, V_\psi\right)_{PC}$, but by part (i), $\left(\Gamma, C, V_\psi\right)^o_{PC}$ is the largest PCHS open subset of $\left(\Gamma, C, V_\psi\right)_{PC}$. Hence, $\left(\Gamma, C, V_\psi\right)_{PC} \check{\tilde{=}} \left(\Gamma, C, V_\psi\right)^o_{PC}$. Conversely, let $\left(\Gamma, C, V_\psi\right)_{PC} \check{\tilde{=}} \left(\Gamma, C, V_\psi\right)^o_{PC}$, by part (i), $\left(\Gamma, C, V_\psi\right)^o_{PC}$ is a PCHS open set. Therefore, $\left(\Gamma, C, V_\psi\right)_{PC}$ is also a PCHS open set.

Proposition 10. *Let (U_P, τ_{PC}, V_ψ) be a PCHST space over U_P and let $\left(\Gamma_1, C, V_\psi\right)_{PC}$, $\left(\Gamma_2, C, V_\psi\right)_{PC}$ be two PCHS sets over U_P, then:*

(i) $\left(\Phi, C, V_\psi\right)^o_{PC} \check{\tilde{=}} \left(\Phi, C, V_\psi\right)_{PC}$ *and* $\left(\Psi, C, V_\psi\right)^o_{PC} \check{\tilde{=}} \left(\Psi, C, V_\psi\right)_{PC}$.

(ii) $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \check{\tilde{\sqsubseteq}} \left(\Gamma_1, C_1, V_\psi\right)_{PC}$.

(iii) $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \check{\tilde{\sqsubseteq}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$ *implies* $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \check{\tilde{\sqsubseteq}} \left(\Gamma_2, C_2, V_\psi\right)^o_{PC}$.

(iv) $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \check{\tilde{\sqcap}} \left(\Gamma_2, C_2, V_\psi\right)^o_{PC} \check{\tilde{=}} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \check{\tilde{\sqcap}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o$.

(v) $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \check{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)^o_{PC} \check{\tilde{\sqsubseteq}} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \check{\tilde{\sqcup}} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o$.

(vi) $\left[\left(\Gamma_1, C_1, V_\psi\right)^o_{PC}\right]^o \check{\tilde{=}} \left(\Gamma_1, C_1, V_\psi\right)^o_{PC}$.

Proof.

(i) Obvious.

(ii) Let $x \in \left(\Gamma_1, C_1, V_\psi\right)^o_{PC}$, then x is a PCHS interior point of $\left(\Gamma_1, C_1, V_\psi\right)_{PC}$ and this implies that $\left(\Gamma_1, C_1, V_\psi\right)_{PC}$ is PCHS neighborhood of x. Then, $x \in \left(\Gamma_1, C_1, V_\psi\right)_{PC}$. Hence, $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \breve{\sqsubseteq} \left(\Gamma_1, C_1, V_\psi\right)_{PC}$.

(iii) Let $x \in \left(\Gamma_1, C_1, V_\psi\right)^o_{PC}$. Then x is a PCHS interior point of $\left(\Gamma_1, C_1, V_\psi\right)_{PC}$ and so $\left(\Gamma_1, C_1, V_\psi\right)_{PC}$ is PCHS neighborhood of x. Since $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqsubseteq} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$, so $\left(\Gamma_2, C_2, V_\psi\right)_{PC}$ is also a PCHS neighborhood of x.
This implies that $x \in \left(\Gamma_2, C_2, V_\psi\right)^o_{PC}$. Thus, $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \breve{\sqsubseteq} \left(\Gamma_2, C_2, V_\psi\right)^o_{PC}$.

(iv) Since $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right] \breve{\sqsubseteq} \left(\Gamma_1, C_2, V_\psi\right)_{PC}$ and $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right] \breve{\sqsubseteq} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$, by part (ii), we have $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o \breve{\sqsubseteq} \left(\Gamma_1, C_2, V_\psi\right)^o_{PC}$ and $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o \breve{\sqsubseteq} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$.
This implies that $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o \breve{\sqsubseteq} \left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)^o_{PC}$.
Again, let $x \in \left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)^o_{PC}$, then $x \in \left(\Gamma_1, C_1, V_\psi\right)^o_{PC}$ and $x \in \left(\Gamma_2, C_2, V_\psi\right)^o_{PC}$. Hence, x is a PCHS interior point of each of the PCHS sets $\left(\Gamma_1, C_1, V_\psi\right)_{PC}$ and $\left(\Gamma_2, C_2, V_\psi\right)_{PC}$. It follows that $\left(\Gamma_1, C_1, V_\psi\right)_{PC}$ and $\left(\Gamma_2, C_2, V_\psi\right)_{PC}$ are PCHS neighborhood of x, so that, their intersection $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$ is also a PCHS neighborhood of x. Hence, $x \in \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o$.
This, $\left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o \breve{=} \left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \breve{\sqcap} \left(\Gamma_2, C_2, V_\psi\right)^o_{PC}$.

(v) By part (iii), $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqsubseteq} \left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcup} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$ implies that $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \breve{\sqsubseteq} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcup} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o$ and $\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqsubseteq} \left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcup} \left(\Gamma_2, C_2, V_\psi\right)_{PC}$ implies that $\left(\Gamma_2, C_2, V_\psi\right)_{PC} \breve{\sqsubseteq} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcup} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o$. Hence, $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC} \breve{\sqcup} \left(\Gamma_2, C_2, V_\psi\right)^o_{PC} \breve{\sqsubseteq} \left[\left(\Gamma_1, C_1, V_\psi\right)_{PC} \breve{\sqcup} \left(\Gamma_2, C_2, V_\psi\right)_{PC}\right]^o$.

(vi) By Proposition 9(i) $\left(\Gamma_1, C_1, V_\psi\right)^o_{PC}$ is the PCHS open set. Hence by proposition 9(ii) , $\left[\left(\Gamma_1, C_1, V_\psi\right)^o_{PC}\right]^o \breve{=} \left(\Gamma_1, C_1, V_\psi\right)^o_{PC}$.

Remark 8. *The equality of above proposition part* (v) *does not hold in general. See the next example.*

Example 7. *Consider the PCHST space* (U_P, τ_{PC}, V_ψ) *in Example 2.*
Define $(\mathit{\Gamma}_4, C_4, V_\psi)_{PC}$ *and* $(\mathit{\Gamma}_5, C_5, V_\psi)_{PC}$ *as the follow:*

$$(\mathit{\Gamma}_4, C_4, V_\psi)_{PC} = \{< (\alpha), \{x_1\,(1,0,1)\,, x_3\,(1,1,1)\,, x_4\,(1,1,1)\} >, < (\beta), \{x_2\,(1,1,1)\,, x_3\,(1,1,1)\} >\}$$

$(\mathit{\Gamma}_5, C_5, V_\psi)_{PC} = \{< (\alpha), 1_{PC}\} >, < (\beta), \{x_1\,(1,1,1)\,, x_4\,(1,1,1)\} >\}$. *Now,* $(\mathit{\Gamma}_4, C_4, V_\psi)^o_{PC} \breve{\breve{=}} (\mathit{\Gamma}_1, C_1, V_\psi)_{PC}$ *and* $(\mathit{\Gamma}_5, C_5, V_\psi)^o_{PC} \breve{\breve{=}} (\mathit{\Gamma}_2, C_2, V_\psi)_{PC}$ *and* $(\mathit{\Gamma}_4, C_4, V_\psi)^o_{PC} \breve{\breve{\sqcup}} (\mathit{\Gamma}_5, C_5, V_\psi)^o_{PC} \breve{\breve{=}} (\Psi, C, V_\psi)_{PC}$ *but* $\left[(\mathit{\Gamma}_4, C_4, V_\psi)_{PC} \breve{\breve{\sqcup}} (\mathit{\Gamma}_5, C_5, V_\psi)_{PC}\right]^o \breve{\breve{\neq}} (\Psi, C, V_\psi)_{PC}$.

Proposition 11. *Let* (U_P, τ_{PC}, V_ψ) *be a PCHST space over* U_P *and let* $\left(\Gamma, C, V_\psi\right)_{PC}$ *be a PCHS set over* U_P. *Then* $\left(\Gamma, C, V_\psi\right)^o_{PC} \breve{\breve{\sqsubseteq}} \left(\Gamma, C, V_\psi\right)_{PC} \breve{\breve{\sqsubseteq}} \overline{\left(\Gamma, C, V_\psi\right)_{PC}}$.

Proof. Obvious.

6. Conclusion

In this paper, we have introduced the concept of plithogenic crisp hypersoft sets and plithogenic crisp hypersoft topological spaces as an extension of the idea of hypersoft sets which are defined over an initial universal set with a fixed set of parameters. Some concepts such as plithogenic crisp hypersoft closure and plithogenic crisp hypersoft interior which are based on our definition were introduced. For future study, we can study plithogenic crisp hypersoft continuity and the most important fundamental topological properties such as plithogenic crisp hypersoft connectedness. Also, this study will be an entrance to more study such as; plithogenic fuzzy hypersoft sets, plithogenic intuitionistic Fuzzy hypersoft sets and plithogenic neutrosophic hypersoft sets.

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