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Relations between G-part and Atoms in Q-algebras

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Abstract. In this work the concepts of G-part $G(X)$, atoms and strong atoms in Q-algebras are discussed. We provide some connections among $G(X)$, set of all atoms and set of all strong atoms of X which related to the concept of ideals. We prove that a Q-algebra X does not contain a strong atom whenever it contains a non-zero ideal $G(X)$. In addition, we provide some conditions that make a set of atoms an abelian group.

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Key Words and Phrases: Q-algebra, ideals, G-part, $G(X)$, $A(X)$, atoms, strong atom, abelian group

1. Introduction and Preliminaries

In 1996, two Japanese mathematicians Y. Imai and K. Iseki [6] introduced a class of logical algebra which is called a BCK-algebra. In the same year the notion of BCI-algebra was introduced by K. Iseki [7], which is a super class of BCK-algebra. For more informations of BCK-algebra and BCI-algebra see also [[16], [8]]. It is natural to study a generalization of these algebras. Later on there is a rich literature involved with BCK-algebra and BCI-algebra. A BCH-algebra was emerged in 1983 by Q. P. Hu and X. Li which is a generalization of BCK, BCI-algebras. Later, J. Neggers et al. introduced many algebras which related to BCK, BCI-algebras such as d-algebra, B-algebra and Q-algebra. They examined some relations and some properties of theses algebras. In 2001, J. Neggers et al. [9] introduced a new generalization of BCI-algebra and BCK-algebra. This new algebra was known as Q-algebra which is also a generalization of BCH-algebra. In [9] the authors generalized some properties and theorems discussed in BCI-algebra. The concept of quadratic Q-algebra is also offered in [9]. A Q-algebra consists of a non-empty set X and a constant $0 \in X$ together with a binary operation $*$ on X that yields the following: for all $x, y, z \in X$
(Q_1) $x * x = 0$,

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(Q_2) $x * 0 = x$,
(Q_3) $(x * y) * z = (x * z) * y$.
For convenience, we write xy instead of $x * y$. Since then, there are many authors working on Q-algebras [see [2], [1], [3], [14], [12], [10], [11], [13]]. In [9], the concepts of ideal and G-part were established. A non-empty subset I of a Q-algebra X is an ideal if the following conditions were fulfilled: (I_1) $0 \in I$, (I_2) $xy \in I$ and $y \in I$ imply $x \in I$. A set $\{0\}$ and X are always ideals of X. A subset $G(X) := \{x \in X \mid 0x = x\}$ of a Q-algebra X is called a G-part of X. In 2004, S. S. Ahn et al. [14] introduced the notion of implicative Q-algebra which is a Q-algebra with $(xy)(yz) = (xy)z$ for all $x, y, z \in X$. Many mathematicians from Korea and Egypt studied on mappings of Q-algebras, namely: R-maps, L-maps, right fixed maps and fuzzy set [see [11], [14], [13]]. In 2010, S. S. Ahn and K. So [4] considered homomorphims and congruence in Q-algebras. The authors in [4] provided some decompositions of ideals in Q-algebras. Recently, the concept of ideal is again in a spotlight. Various kinds of ideals were discussed. Q-ideal, prime ideal, fuzzy ideal, intuitionistic fuzzy prime ideal, G-part ideal were studied in [2], [12], [10], [13]. In the year 2001, D. Sun [15] introduced the concept of atom and strong atom in BCK-algebra. He proved that a set of all strong atoms and together with zero element is an ideal of BCK-algebra X. In 2010, S. S. Ahn and S. E. Kang [3] introduced the concepts of atoms in Q-algebra. An atom of X is an element $a \in X$ satisfying: for $x \in X$, $xa = 0$ implies $x = a$. A set of all atoms of X is denoted by $A(X)$. Some properties of atoms are provided in [3]. The authors showed that if every non-zero element of X is an atom, then any subalgebra of X is an ideal. A subalgebra of a Q-algebra X is a non-empty subset I of X with $ab \in I$ for all $a, b \in I$. Moreover, the authors in [3] proved that if every non-zero element of X is an atom, then any subalgebra of X is an ideal of X.

Example 1. *Let $X = \{0, a, b, c, d\}$ and $Y = \{0, a, b\}$. The binary operations $*$ and $\bullet$ be defined on X and Y as the following tables:*

$*$	0	a	b	c	d
0	0	a	c	b	b
a	a	0	b	c	c
b	b	c	0	a	a
c	c	b	a	0	0
d	d	b	a	0	0

$\bullet$	0	a	b
0	0	a	a
a	a	0	0
b	b	0	0

It is a routine to check that $(X; *, 0)$ and $(Y; \bullet, 0)$ are Q-algebras. It is easy to see that $G(X) = \{0, a\}$, $A(X) = \{0, a, b\}$ and $G(Y) = \{0, a\}$, $A(Y) = \{0\}$. Moreover, we get that $G(X)$ is an ideal and a subalgebra of X.

In this paper, we examine the properties of atoms and strong atoms. We also show some relations between a set G-part $G(X)$, a set of all atoms $A(X)$ and a set of all strong atoms of a Q-algebra X that involve with ideal property. Now we will review some properties and theorems that we will use later. In [9] and [3] gave us some calculations and showed a left cancellation law in a Q-algebra X.

Lemma 1. *[9] Let X be a Q-algebra and $a, b, c \in X$. If $ab = ac$, then $0b = 0c$.*

Corollary 1. *[9] A left cancellation law holds in $G(X)$, i. e. for all $a, b, c \in G(X)$, $ab = ac$ implies $b = c$.*

Lemma 2. *[3] Every Q-algebra X satisfies the following property: $0(xy) = (0x)(0y)$ for all $x, y \in X$.*

In [10], some informations and calculations in $G(X)$ are presented.

Proposition 1. *[10] Let X be a Q-algebra and $x \in X$. Then $0x \in G(X)$ if and only if $(0x)x = 0$.*

Proposition 2. *[10] Let X be a Q-algebra. If $a, b \in G(X)$, then $ab = ba$.*

Proposition 3. *[10] Let X be a Q-algebra and $a, b, c \in G(X)$. Then the following three properties hold:*

(i) If $a \neq b$, then $ab \notin \{0, a, b\}$ for $a \neq 0$ and $b \neq 0$.

(ii) If $ab = c$, then $ac = b$ and $bc = a$.

(iii) $xa \neq x$ for all $0 \neq x \in X$ and $a \neq 0$.

2. G-part and Atoms

In this section, we investigate some properties of a set G-part $G(X)$, atoms and strong atoms of a Q-algebra X. We also present some connections among them. First, we will mention some results of atoms in [3].

Theorem 1. *[3] Let X be a Q-algebra. Then for all x, z, u of X, the following conditions are equivalent:*
(i) x is atom;
(ii) $x = z(zx)$;
(iii) $(zu)(zx) = xu$.

Theorem 2. *[3] Let X be a Q-algebra and $x \in X$. If x is an atom of X, then the following properties are satisfied:*
(iv) $0(zx) = xz$ for all $z \in X$.
(v) $0(0x) = x$.

The converse of Theorem 2 is not true. The following example is a counterexample.

Example 2. *Consider a Q-algebra X from Example 1. For all $z \in X$, we get that $0(zc) = cz$ but an element c is not an atom of X. Hence, the converse of Theorem 2(iv) is not ture. Beside that, the converse of (v) is also not ture since $0(0c) = 0b = c$ but c is not an atom.*

From Theorem 2 we get that every atom of Q-algebra X is a product of 0 and some element of X as the following:

Corollary 2. *Let X be a Q-algebra. If a is an atom of X, then $a = 0x$ for some $x \in X$.*

The following proposition shows some more properties of atoms in Q-algebras.

Proposition 4. *Let X be a Q-algebra and let a, b be atoms of X. Then the following properties hold:*

(i) $a(xb) = b(xa)$ *for all* $x \in X$.

(ii) $(ax)(yb) = (bx)(ya)$ *for all* $x, y \in X$.

Proof. Assume that a and b are atoms of X.

(i): Let $x \in X$. Then by Theorem 1(ii), we get that $a = x(xa)$ and $b = x(xb)$. Then by (Q_3) there follows that $a(xb) = (x(xa))(xb) = (x(xb))(xa) = b(xa)$.

(ii): Let $x, y \in X$. Then by $(Q3)$ and (i) we get

$$(ax)(yb) = (a(yb))x = (b(ya))x = (bx)(ya).$$

The converse of Proposition 4 is not true as seen in the following example.

Example 3. *Consider a Q-algebra X from Example 1. Let us focus on elements c and b of X. We get that $b(xc) = c(xb)$ for all $x \in X$ but c is not an atom of X. Hence, the converse of Proposition 4(i) is not true.*

Proposition 5. *Let X be a Q-algebra. Every element of X is an atom if and only if $a(xb) = b(xa)$ for all $a, b, x \in X$.*

Proof. ($\Rightarrow$) Follows from Proposition 4(i).

($\Leftarrow$) Let $z \in X$. Then by assumption we get that $z(xx) = x(xz)$ for all $x \in X$. There follows that $z = z0 = z(xx) = x(xz)$ for all $x \in X$. Then by Theorem 1(ii), z is an atom of X.

In 2001, D. Sun [15] introduced the concept of strong atoms in BCK-algebra. We will apply a concept of strong atom to Q-algebras in a similar way. Let a be an atom of a Q-algebra X. An element a is called a strong atom if $a \neq 0$ and $ax = a$ for all $x \in X$ and $x \neq a$. We denote a set $SA(X)$ as follows:

$$SA(X) = \{a \in A(X) \mid a \text{ is a strong atom of } X\} \cup \{0\}.$$

There is a connection between strong atoms and G-part of X. The following properties show that X does not contain any strong storm whenever X contains G-part which is an ideal with the cardinality greater or equal to 2. First, we need the following proposition:

Proposition 6. *[5] Let X be a Q-algebra with $|X| = n$ and $G(X) \neq X$. If $G(X)$ is an ideal of X, then $|G(X)| \leq \frac{n}{2}$.*

Proposition 7. *Let X be a Q-algebra. If $G(X)$ is an ideal and $|G(X)| = 2$, then $SA(X) = \{0\}$.*

Proof. Assume that $G(X)$ is an ideal of X and $|G(X)| = 2$. We assume that $G(X) = \{0, a\}$. Suppose that a is a strong atom of X. Since $G(X)$ is an ideal, then by Proposition 6, we get $|G(X)| \leq \frac{|X|}{2}$. Therefore, $|X| \geq 4$. Let $b \in X$ such that $b \notin G(X)$. Since a is a strong atom, then $ab = a$. It follows that $0b = (aa)b = (ab)a = aa = 0$. Since $b \notin G(X)$, $a \in G(X)$ and $G(X)$ is an ideal, then $ba \notin G(X)$. Moreover, by Proposition 3(iii), we get that $ba \neq b$. Therefore, $ba \notin \{0, a, b\}$. We may assume that $ba = c$ for some $c \in X \setminus \{0, a, b\}$. Similarly, we get $ac = a$ and $ca \notin \{0, a, c\}$. Then we get $cb = (ba)b = (bb)a = 0a = a$. It follows that $a = ac = (cb)c = (cc)b = 0b = 0$, a contradiction. Hence, $a \notin SA(X)$. Let $x \in X \setminus G(X)$. Since $a \in G(X)$ and $x \neq 0$, then $xa \neq x$ by by Proposition 3(iii). It follows that $x \notin SA(X)$. Altogether, $SA(X) = \{0\}$.

Proposition 8. *Let X be a Q-algebra. If $|G(X)| \geq 3$, then $SA(X) = \{0\}$.*

Proof. Assume that $|G(X)| \geq 3$. Then there are $a, b \in G(X)$ such that $a, b \notin \{0\}$ and $a \neq b$. Let $x \in X$ and $x \neq 0$. If $x = a$, then $xb = ab \notin \{0, a, b\}$ by Proposition 3(i). Therefore, $xb \neq x$ there follows that $x \notin SA(X)$. If $x \neq a$, then by Proposition 3(i), $xa \neq x$. Thus, $x \notin SA(X)$. Altogether, we get $SA(X) = \{0\}$.

Proposition 7 and Proposition 8 give the following theorem:

Theorem 3. *Let X be a Q-algebra and $G(X) \neq \{0\}$. If $G(X)$ is an ideal, then $SA(X) = \{0\}$.*

It is clear that a set of all atoms of a Q-algebra X is not closed. But if we focus on a set of strong atoms, we get that the product of strong atoms is again a strong atom. It follows that $SA(X)$ is a subalgebra of X.

Proposition 9. *Let X be a Q-algebra. Then $SA(X)$ is a subalgebra of X.*

Proof. If $|SA(X)| \leq 2$, then it is clear that $SA(X)$ is a subalgebra. Assume now that $|SA(X)| \geq 3$. Let $a, b \in SA(X)$. If $b = 0$, then $ab = a0 = a \in SA(X)$. If $a \neq 0$ and $b \neq 0$, then $ab = a$ since a is a strong atom. Therefore, $ab = a \in SA(X)$. If $a = 0$ and $b \neq 0$, then $ab = 0b$. Since $|SA(X)| \geq 3$, then there is a strong atom c such that $c \notin \{0, b\}$. Then $cb = c$. By Proposition 4(i) we get that $ab = 0b = 0(bc) = c(b0) = cb = c$. Therefore, $ab \in SA(X)$. Altogether, we get $SA(X)$ is a subalgebra of X.

Next we will examine some properties of a set of all atoms $A(X)$ of any Q-algebra X. In general, a set $A(X)$ need not to be closed and also need not to be an ideal of X as the following example.

Example 4. *Let $X = \{0, a, b, c, d, f\}$ and let a binary operation $*$ be defined on X as the following:*

$*$	0	a	b	c	d	f
0	0	a	c	b	b	b
a	a	0	b	c	c	c
b	b	c	0	a	a	a
c	c	b	a	0	0	0
d	d	b	a	0	0	0
f	f	b	a	0	0	0

It is a routine to check that $(X; *, 0)$ *is a* Q*-algebra. It is easy to see that* $A(X) = \{0, a, b\}$. *We get that* $A(X)$ *is not a subalgebra since* $0, b \in A(X)$ *but* $0b = c \notin A(X)$. *Moreover,* $A(X)$ *is not an ideal of* X. *Indeed,* $db = a \in A(X)$ *and* $b \in A(X)$ *but* $d \notin A(X)$.

From Example 4, let we mention some errors in [3], namely [Corollary 3.6]: " Let X be a Q-algebra. If a is an atom of X, then for all x in X, ax is an atom. Hence, $A(X)$ is a subalgebra of X. For every x of X, there is an atom a such that $ax = 0$, i.e. every Q-algebra is generated by atoms." is invalid. The mistakes show in Example 4.

Next, we investigate some relations between G-part $G(X)$ and set of all atoms $A(X)$. We know that $G(X) \cap A(X) \neq \emptyset$ since an element 0 is an atom and $00 = 0 \in G(X)$. The set $G(X)$ need not to be a subset of $A(X)$ and vice versa. From Example 1 we have that $G(X) \subseteq A(X)$ and $A(Y) \subseteq G(Y)$. Our aim is to find some conditions that yield previous inclusions. Next proprosition shows a sufficient condition of an element of $G(X)$ to be an atom of X.

Proposition 10. *Let* X *be a* Q*-algebra. If* $G(X)$ *is an ideal of* X*, then* $G(X) \subseteq A(X)$.

Proof. Assume that $G(X)$ is an ideal of X. Let $a \in G(X)$. If $a = 0$, then $a \in A(X)$. Now we assume that $a \neq 0$. Suppose that there is an element $w \in X, w \neq a$ such that $wa = 0$. Since $a \in G(X)$, then $0a = a$ there follows that $w \neq 0$. Since $wa = 0 \in G(X)$, $a \in G(X)$ and $G(X)$ is ideal, then $w \in G(X)$. Now, there are $0, a$ and w belong to $G(X)$ and $wa = 0$, then by Proposition 3(ii) we get that $a0 = w$. Thus, by (Q_2) we get that $w = a0 = a$, a contradiction. Hence, $wa = 0$ implies $w = a$. This gives a is an atom of X. Altogether, we get $G(X) \subseteq A(X)$.

The converse of Proposition 10 is not true, i.e. if all members of $G(X)$ are atoms of X, then $G(X)$ need not to be an ideal of X. The following example is the counterexample of the converse.

Example 5. *[13] Let consider a* Q*-algebra* X*, defined as the following table:*

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	0	3	0	4
4	4	4	4	4	0

It is not difficult to verify that $G(X) = \{0, 4\}$ *and* $A(X) = \{0, 4\}$. *Then* $G(X) \subseteq A(X)$. *But* $G(X)$ *is not an ideal of* X. *Indeed,* $2(4) = 2 \in G(X)$ *and* $4 \in G(X)$ *but* $2 \notin G(X)$.

Let A and B be non-empty subset of a Q-algebra X. We define AB as following: $AB = \{ab \mid a \in A, b \in B\}$. Then we get some important informations of $G(X)$ and $A(X)$:

Remark 1. *Let* X *be a* Q*-algebra. Then we get:*

(i) $G(X) \subseteq G(X)G(X)$, $A(X) \subseteq A(X)A(X)$ *and* $G(X) \subseteq G(X)A(X)$.

(ii) $G(X) \subseteq A(X)G(X)$, $A(X) \subseteq A(X)G(X)$ *and* $G(X) \cup A(X) \subseteq A(X)G(X)$.
(iii) $G(X) \subseteq G(X)A(X) \cap A(X)G(X)$.
(iv) $A(X) \subseteq A(X)A(X) \cap A(X)G(X) = A(X)\big(A(X) \cap G(X)\big)$.

A set of all atoms $A(X)$ need not to be closed, i.e. in general $A(X) \neq A(X)A(X)$. From Remark 1(iv), we have that $A(X) \subseteq A(X)\big(A(X) \cap G(X)\big)$. It follows that every atom of X can be written in the form of products of atoms. But the product za of atom z and atom a need not to be atom as seen from Example 4. Next lemma shows the condition that gives equality of Remark 1(iv).

Proposition 11. *Let X be a Q-algebra. If $A(X)$ is an ideal of X, then $A(X) = A(X)\big(A(X) \cap G(X)\big)$.*

Proof. Assume that $A(X)$ is an ideal of X. Let $z \in A(X)$ and $a \in A(X) \cap G(X)$. Then we get $a = 0a = (zz)a = (za)z$. It follows that $(za)z = a \in G(X) \subseteq A(X)$. Since $(za)z \in A(X)$, $a \in A(X)$ and $A(X)$ is an ideal, then $za \in A(X)$. Therefore, $A(X)\big(A(X) \cap G(X)\big) \subseteq A(X)$. The inclusion $A(X) \subseteq A(X)\big(A(X) \cap G(X)\big)$ follows from Remark 1(iv). Hence, $A(X) = A(X)\big(A(X) \cap G(X)\big)$.

As a consequence of Proposition 11, the product ab of atom a and atom b with $0b = b$ is again an atom of X.

Proposition 12. *Let X be a Q-algebra. If $A(X)$ is an ideal of X, then $A(X) \cap G(X)$ is an abelian group.*

Proof. Let $x, y, z \in A(X) \cap G(X)$. Then by Lemma 2 we get that $0(xy) = (0x)(0y) = xy$. Therefore, $xy \in G(X)$. Since $x \in A(X)$ and $y \in A(X) \cap G(X)$, then by Proposition 11 we get $xy \in A(X)$. Thus, $xy \in A(X) \cap G(X)$. The commutative property follows from Proposition 2. Since the commutative property is hold, then we get $(xy)z = (yx)z = (yz)x = x(yz)$. Hence, an associative law is hold. Moreover, $0 \in A(X) \cap G(X)$, by (Q_1) and $x \in G(X)$ we get $x0 = x = 0x$. Therefore, 0 is an identity of $A(X) \cap G(X)$. An inverse property follows from (Q_2). Altogether, we get that $A(X) \cap G(X)$ is an abelian group.

Proposition 13. *Let X be a Q-algebra. If $A(X) \subseteq G(X)$ and $A(X)$ is an ideal of X, then*

(i) *$A(X)$ is a subalgebra of X*

(ii) *$A(X)$ is an abelian group*

Proof. (i) Since $A(X) \subseteq G(X)$, then $A(X) \cap G(X) = A(X)$. Then by Proposition 11 we get that $A(X) = A(X)\big(A(X) \cap G(X)\big) = A(X)A(X)$. Hence, $A(X)$ is a subalgebra of X.

(ii) Since $A(X) \cap G(X) = A(X)$, then by Proposition 12 we get that $A(X)$ is an abelian group.

Proposition 14. *Let X be a Q-algebra. If $G(X)$ and $A(X)$ are ideals of X, then $A(X)G(X)$ is an ideal of X.*

Proof. Assume that $G(X)$ and $A(X)$ are ideals of X. Then by Proposition 10 we get that $G(X) \subseteq A(X)$. Since $A(X)$ is an ideal, then $A(X) = A(X)\big(A(X) \cap G(X)\big)$. There follows that $A(X) = A(X)G(X)$. Hence, $A(X)G(X)$ is an ideal of X.

3. Conclusion

The concept of ideal plays an important role in studying Q-algebra structures. Many mathematicians examine various subsets of a Q-algebra which are ideals. In this work, we obtain information that all elements of $G(X)$ are atoms whenever $G(X)$ is an ideal. Moreover, we get that a Q-algebra X such that $G(X)$ is an ideal and $G(X) \neq \{0\}$, does not contain a strong atom. For future study one can investigate when a set of all atoms $A(X)$ is an ideal of X and which conditions that make X contains both non-zero atoms and strong atoms. Also, for any Q-algebra X one can find the sufficient condition of $A(X)$ to be an ideal of X.

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