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Some Inclusion Properties for Hohlov Operator to be in Comprehensive Subfamilies of Analytic Functions

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Abstract. In this paper, the subfamilies $\mathcal{C}_{\kappa_3}(\kappa_1,\kappa_2)$ and $S^*_{\kappa_3}(\kappa_1,\kappa_2)$ investigated through the Hohlov operator. More specifically, a number of sufficient requirements are given in order for the aforementioned functions subfamilies. Moreover, our results will imply several corollaries.

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1. Preliminaries and Definitions

Geometric functions are a basic family of special functions in mathematics that have complex mathematical features and linkages and are widely used in many different domains as solutions to differential equations and recurrence relations. Gaussian hypergeometric function is one of the most important in geometric functions theory, it is foundation is found in the research of 17th-century mathematicians and astrophysicists. In terms of measurement, instrumentation, and statistics, the Gaussian distribution is most likely the most employed. Indeed, celestial sources frequently have a Gaussian nature in radio astronomy [8].

Let $\mathbf{\Pi}$ be the family of all analytic and univalent functions of the form:

$$L(\varsigma) = \varsigma + \sum_{\tau=2}^{\infty} \mathbf{r}_\tau \varsigma^\tau, \quad |\varsigma| < 1, \tag{1}$$

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which are normalized by $L(0) = L'(0) - 1 = 0$. Also, indicate by H the subfamily of $\mathbf{\Pi}$ made up of functions of the form:

$$L(\varsigma) = \varsigma - \sum_{\tau=2}^{\infty} \mathbf{r}_\tau \varsigma^\tau, \quad \mathbf{r}_\tau \geq 0. \tag{2}$$

For two functions $L, \ell \in \mathbf{\Pi}$, L given by (1) and $\ell(\varsigma) = \varsigma + \sum_{\tau=2}^{\infty} \mathbf{s}_\tau \varsigma^\tau$, we define the convolution of L and ℓ by:

$$(L*\ell)(\varsigma) = \varsigma + \sum_{\tau=2}^{\infty} \mathbf{r}_\tau \mathbf{s}_\tau \varsigma^\tau.$$

The subfamily $\xi - \mathcal{ST}$ meeting the requirements

$$\xi - \mathcal{ST} = \left\{ L \in \mathbf{\Pi} : \operatorname{Re}\left(\frac{\varsigma L'(\varsigma)}{L(\varsigma)}\right) > \xi \left|\frac{\varsigma L'(\varsigma)}{L(\varsigma)} - 1\right|, \quad (0 \leq \xi < \infty, \ \ |\varsigma| < 1) \right\}$$

and $L(\varsigma) \in \xi - \mathcal{UCV} \Leftrightarrow \varsigma L'(\varsigma) \in \xi - \mathcal{ST}$, was presented by Kanas and Wiśniowska [13] and [14]. In particular, when $\xi = 1$, we obtain the comprehensive subfamilies of uniformly convex functions $\mathcal{UCV}$ and parabolic starlike functions $\mathcal{SP}$ in $U = \{ \varsigma \in \mathbb{C} : |\varsigma| < 1\}$. Also, when $\xi = 0$, we obtain the renowned subfamilies of convex functions $\mathcal{CV}$ and starlike functions $\mathcal{ST}$ in U (see for details, [10]). Let (see [13] and [14])

$$\Upsilon_1 =: \Upsilon_1(\xi) = \begin{cases} \frac{8\left(\cos^{-1}\xi\right)^2}{\pi^2(1-\xi^2)} & \text{for } 0 \leq \xi < 1 \\ \frac{8}{\pi^2} & \text{for } \xi = 1 \\ \frac{\pi^2}{4\sqrt{v}(v+1)(\xi^2-1)\psi^2(v)} & \text{for } \xi > 1, \end{cases} \tag{3}$$

where $v \in (0,1)$ is determined by $\xi = \cosh\left(\pi F'(v) \big/ 4F(v)\right)$, F is the Legendre's complete Elliptic integral of the first kind $F(v) = \int_0^1 \frac{dt}{(1-t^2)(1-v^2t^2)}$ and $F'(v) = F\left(\sqrt{1-v^2}\right)$ is the complementary integral of $F(v)$.

Let $L \in \mathbf{\Pi}$ be of the form (2) and in the subfamily $\xi - \mathcal{UCV}$, then the following inequalities hold true [13]

$$|\mathbf{r}_\tau| \leq \frac{(\Upsilon_1(\xi))_{\tau-1}}{\tau!}, \quad \tau \in \mathbb{N} - \{1\}. \tag{4}$$

Also, if $L \in \mathbf{\Pi}$ be of the form (2) in the subfamily $\xi - ST$, then [14]

$$|\mathbf{r}_\tau| \leq \frac{(\Upsilon_1(\xi))_{\tau-1}}{(\tau-1)!}, \quad \tau \in \mathbb{N} - \{1\}. \tag{5}$$

The Gaussian hypergeometric function $\mathcal{G}(\beth_1, \beth_2; \beth_3; \varsigma)$ given by:

$$\mathcal{G}(\beth_1, \beth_2; \beth_3; \varsigma) = \sum_{\tau=0}^{\infty} \frac{(\boldsymbol{\beth_1})_\tau (\boldsymbol{\beth_2})_\tau}{(\boldsymbol{\beth_3})_\tau (1)_\tau} \varsigma^\tau, \quad |\varsigma| < 1$$

where $\daleth_1, \daleth_2, \daleth_3 \in \mathbb{C}$ such that $\daleth_3 \neq 0, -1, -2, \cdots$, $(\daleth_1)_0 = 1$ for $\daleth_1 \neq 0$ and for $\tau \in \mathbb{N}$, $(\daleth_1)_\tau = \daleth_1(\daleth_1+1)(\daleth_1+2)\cdots(\daleth_1+\tau-1)$ is the Pochhammer symbol, and it represents the solution of the homogenous differential equation

$$\varsigma(1-\varsigma)y''(\varsigma) + (\daleth_3 - (\boldsymbol{\daleth_1} + \boldsymbol{\daleth_2}+1)\varsigma)\, y'(\varsigma) - \daleth_1\daleth_2 y(\varsigma)=0$$

has several uses in a variety of fields, including continued fractions, quasi-conformal theory, conformal mappings, and more. Using Gauss Summation theorem, it is possible to write

$$\mathcal{G}(\daleth_1, \daleth_2; \daleth_3; 1) = \sum_{\tau=0}^{\infty} \frac{(\boldsymbol{\daleth_1})_\tau (\boldsymbol{\daleth_2})_\tau}{(\boldsymbol{\daleth_3})_\tau (\mathbf{1})_\tau} = \frac{\Gamma(\boldsymbol{\daleth_3} - \boldsymbol{\daleth_2} - \boldsymbol{\daleth_1})\Gamma(\boldsymbol{\daleth_3})}{\Gamma(\boldsymbol{\daleth_3} - \boldsymbol{\daleth_1})\Gamma(\boldsymbol{\daleth_3} - \boldsymbol{\daleth_2})}, \quad \text{for } Re(\boldsymbol{\daleth_3} - \boldsymbol{\daleth_2} - \boldsymbol{\daleth_1}) > 0. \tag{6}$$

For $L \in \mathbf{\Pi}$, we recall the operator $HO_{\daleth_1,\daleth_2,\daleth_3}(L) : \mathbf{\Pi} \to \mathbf{\Pi}$ of Hohlov [11] defined as:

$$\begin{aligned} HO_{\daleth_1,\daleth_2,\daleth_3}(L) &\equiv HO_{\daleth_1,\daleth_2,\daleth_3}(L)(\varsigma) = \varsigma\mathcal{G}(\daleth_1, \daleth_2; \daleth_3; \varsigma) * L(\varsigma) \\ &= \varsigma + \sum_{\tau=2}^{\infty} \frac{(\boldsymbol{\daleth_1})_{\tau-\mathbf{1}} (\boldsymbol{\daleth_2})_{\tau-\mathbf{1}}}{(\boldsymbol{\daleth_3})_{\tau-\mathbf{1}} (1)_{\tau-\mathbf{1}}} \mathbf{r}_\tau \varsigma^\tau. \end{aligned}$$

Remark 1. *1) If $\daleth_1 = 1$, $\daleth_2 = \alpha + 1$, $\daleth_3 = \alpha + 2$ with $Re(\alpha) > -1$, then the operator $HO_{\daleth_1,\daleth_2,\daleth_3}(L)$ turns into $HO_{1,\alpha+1,\alpha+2}(L)$ Bernardi operator [7],*
2) $HO_{1,1,2}(L)$ Alexander operator [3],
3) $HO_{1,2,3}(L)$ Libera operator [16].

We examine the following subfamilies of analytic functions investigated by Murugusundaramoorthy et al., [18] and Ali et al., [20], respectively.

Definition 1. *For some $\kappa_1 (0 \le \kappa_1 < 1)$, $\kappa_2 (\kappa_2 \ge 0)$ and $\kappa_3 (0 \le \kappa_3 \le 1)$. Let the subfamily $\mathcal{C}_{\kappa_3}(\kappa_1, \kappa_2)$ consists of functions in $\mathbf{\Pi}$ satisfying the inequality*

$$Re\left(\frac{\varsigma L'(\varsigma) + \varsigma^2 L''(\varsigma)}{(1-\kappa_3)\varsigma + \kappa_3 \varsigma L'(\varsigma)} - \kappa_1\right) > \kappa_2 \left|\frac{\varsigma L'(\varsigma) + \varsigma^2 L''(\varsigma)}{(1-\kappa_3)\varsigma + \kappa_3 \varsigma L'(\varsigma)} - 1\right|, \quad |\varsigma| < 1$$

and the subfamily $\mathrm{S}^*_{\kappa_3}(\kappa_1, \kappa_2)$ consists of functions in $\mathbf{\Pi}$ satisfying the inequality

$$Re\left(\frac{\varsigma L'(\varsigma)}{(1-\kappa_3)\varsigma + \kappa_3 L(\varsigma)} - \kappa_1\right) > \kappa_2 \left|\frac{\varsigma L'(\varsigma)}{(1-\kappa_3)\varsigma + \kappa_3 L(\varsigma)} - 1\right|, \quad |\varsigma| < 1.$$

Example 1. *[12, 19] For some $\kappa_1 (0 \le \kappa_1 < 1)$, $\kappa_2 (\kappa_2 \ge 0)$, $\kappa_3 = 1$ and $L(\varsigma)$ of the form (1), let the subfamily $\mathcal{C}_1(\kappa_1, \kappa_2)$ consists of functions in $\mathbf{\Pi}$ satisfying*

$$Re\left(\frac{\varsigma L''(\varsigma)}{L'(\varsigma)} + 1 - \kappa_1\right) > \kappa_2 \left|\frac{\varsigma L''(\varsigma)}{L'(\varsigma)}\right|, \quad |\varsigma| < 1$$

and the subfamily $\mathrm{S}^*_1(\kappa_1, \kappa_2)$ consists of functions in $\mathbf{\Pi}$ satisfying

$$Re\left(\frac{\varsigma L'(\varsigma)}{L(\varsigma)} - \kappa_1\right) > \kappa_2 \left|\frac{\varsigma L'(\varsigma)}{L(\varsigma)} - 1\right|, \quad |\varsigma| < 1.$$

Example 2. *[18] For some $\kappa_1(0 \leq \kappa_1 < 1)$, $\kappa_2(\kappa_2 \geq 0)$, $\kappa_3 = 0$ and $L(\varsigma)$ of the form (1), let the subfamily $\mathcal{C}_0(\kappa_1, \kappa_2)$ consists of functions in $\mathbf{\Pi}$ satisfying*

$$Re\left(\left(\varsigma L'(\varsigma)\right)' - \kappa_1\right) > \kappa_2 \left|\left(\varsigma L'(\varsigma)\right)' - 1\right|, \ \ |\varsigma| < 1$$

and the subfamily $\mathrm{S}_0^*(\kappa_1, \kappa_2)$ consists of functions in $\mathbf{\Pi}$ satisfying

$$Re\left(L'(\varsigma) - \kappa_1\right) > \kappa_2 \left|L'(\varsigma) - 1\right|, \ \ |\varsigma| < 1.$$

Example 3. *[21] For some $\kappa_1(0 \leq \kappa_1 < 1)$, $\kappa_2 = 0$, $\kappa_3 = 1$ and $L(\varsigma)$ of the form (1), let the subfamily $\mathcal{C}_1^*(\kappa_1, 0) \equiv \mathcal{CV}(\kappa_1)$ consists of functions in $\mathbf{\Pi}$ satisfying*

$$Re\left(\frac{\varsigma L''(\varsigma)}{L'(\varsigma)} + 1\right) > \kappa_1, \ \ |\varsigma| < 1$$

and the subfamily $\mathrm{S}_1^*(\kappa_1, 0) \equiv \mathcal{ST}(\kappa_1)$ consists of functions in $\mathbf{\Pi}$ satisfying

$$Re\left(\frac{\varsigma L'(\varsigma)}{L(\varsigma)}\right) > \kappa_1, \ \ |\varsigma| < 1.$$

Both subfamilies $\mathcal{CV}\ (\kappa_1)$ and $\mathcal{ST}(\kappa_1)$ are well known subfamilies of convex and starlike functions of order κ_1, respectively. Moreover, if $\kappa_1 = \kappa_2 = 0$, $\kappa_3 = 1$, we get the subfamilies of convex functions $\mathcal{CV}$ and starlike functions $\mathcal{ST}$, respectively (see [21]).

Lemma 1. *[18] A function $L \in \mathcal{C}_{\kappa_3}(\kappa_1, \kappa_2)$ if and only if*

$$\sum_{\tau=2}^{\infty} \tau \left[(\tau(\kappa_2 + 1) - \kappa_3(\kappa_1 + \kappa_2)\right] |\mathbf{r}_\tau| \leq 1 - \kappa_1 \tag{7}$$

and $L \in \mathrm{S}_{\kappa_3}^*(\kappa_1, \kappa_2)$ if and only if

$$\sum_{\tau=2}^{\infty} \left[(\tau(\kappa_2 + 1) - \kappa_3(\kappa_1 + \kappa_2)\right] |\mathbf{r}_\tau| \leq 1 - \kappa_1. \tag{8}$$

The geometric characteristics of numerous types of special functions are covered in a substantial body of literature (see [1], [6], [9], [4], [11], [2]).

The current paper aims to create connections between Hohlov operator and geometric function theory. By findings the relationships between different subfamilies of analytic univalent functions. Inspired by numerous works, for example Ahmad et al. [17] deduced sufficient conditions and some properties for Mittag-Leffler. Frasin et al. [5] deduced necessary and sufficient conditions for Struve functions to be in some classes. Kasthuri et al. [15] introduced a new class by Hohlov Operator and obtain some inclusion relations. Murugusundaramoorthy et al. [18] introduced a class of starlike functions and subordination results for some classes of starlike functions. Swaminathan [22] introduced some conditions for normalized Gaussian hypergeometric function.

2. Main Results

This section will provide sufficient conditions for Hohlov operator function to be in the subfamilies $\mathcal{C}_{\kappa_3}(\kappa_1, \kappa_2)$ and $S^*_{\kappa_3}(\kappa_1, \kappa_2)$.

Theorem 1. *If* $L \in \mathcal{ST}$ *and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$, *then* $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{C}_{\kappa_3}(\kappa_1, \kappa_2)$ *if the following condition is satisfied:*

$$\frac{(\kappa_2+1)\,|\boldsymbol{\beth_1\beth_2}|\,(|\boldsymbol{\beth_1}|+1)\,(|\boldsymbol{\beth_2}|+1)\,(|\boldsymbol{\beth_1}|+2)\,(|\boldsymbol{\beth_2}|+2)}{\boldsymbol{\beth_3(\beth_3+1)(\beth_3+2)}}\mathcal{G}_{2,1}(|\boldsymbol{\beth_1}|+3, |\boldsymbol{\beth_2}|+3; \beth_3+3, \mathbf{1})$$
$$+\frac{(6(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\,|\boldsymbol{\beth_1\beth_2}|\,(|\boldsymbol{\beth_1}|+1)\,(|\boldsymbol{\beth_2}|+1)}{\boldsymbol{\beth_3(\beth_3+1)}}\mathcal{G}_{2,1}(|\boldsymbol{\beth_1}|+2, |\boldsymbol{\beth_2}|+2; \beth_3+2, \mathbf{1})$$
$$+\frac{(7(\kappa_2+1)-3\kappa_3(\kappa_1+\kappa_2))\,|\boldsymbol{\beth_1\beth_2}|}{\boldsymbol{\beth_3}}\mathcal{G}_{2,1}(|\boldsymbol{\beth_1}|+1, |\boldsymbol{\beth_2}|+1; \beth_3+1, \mathbf{1})$$
$$+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\,[\mathcal{G}_{2,1}\,|\boldsymbol{\beth_1}|\,, |\boldsymbol{\beth_2}|\,; \beth_3, \mathbf{1})-1] \le 1-\kappa_1.$$

Proof. By equation (7), to prove $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{C}_{\kappa_3}(\kappa_1, \kappa_2)$, it suffices to show that

$$\sum_{\tau=2}^{\infty} \tau\,[(\tau(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2)]\left|\frac{(\boldsymbol{\beth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\beth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\beth_3})_{\boldsymbol{\tau-1}}(\mathbf{1})_{\boldsymbol{\tau-1}}}\mathbf{r}_\tau\right| \le 1-\kappa_1.$$

Since $L \in ST$ we have $|\mathbf{r}_\tau| \le \tau$, from above equation we get

$$\sum_{\tau=2}^{\infty} \left[\left(\tau^3(\kappa_2+1\right)-\tau^2\kappa_3(\kappa_1+\kappa_2)\right]\left|\frac{(\boldsymbol{\beth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\beth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\beth_3})_{\boldsymbol{\tau-1}}(\mathbf{1})_{\boldsymbol{\tau-1}}}\right| \le 1-\kappa_1.$$

Writing

$$\begin{cases} \tau = (\tau-1)+1, \\ \tau^2 = (\tau-1)(\tau-2)+3(\tau-1)+1, \\ \tau^3 = (\tau-1)(\tau-2)(\tau-3)+6(\tau-1)(\tau-2)+7(\tau-1)+1 \end{cases}$$

and use the relations

$$(\boldsymbol{\beth_1})_\tau = \boldsymbol{\beth_1}(\boldsymbol{\beth_1}+\mathbf{1})_{\boldsymbol{\tau-1}} \text{ and } |(\boldsymbol{\beth_1})_\tau| \le (|\boldsymbol{\beth_1}|)_\tau\,, \tag{9}$$

we have

$$\sum_{\tau=2}^{\infty} \left[\left(\tau^3(\kappa_2+1\right)-\tau^2\kappa_3(\kappa_1+\kappa_2)\right]\left|\frac{(\boldsymbol{\beth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\beth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\beth_3})_{\boldsymbol{\tau-1}}(\mathbf{1})_{\boldsymbol{\tau-1}}}\right|$$
$$= (\kappa_2+1)\sum_{\tau=2}^{\infty}(\tau-1)(\tau-2)(\tau-3)\frac{(|\boldsymbol{\beth_1}|)_{\tau-1}\,(|\boldsymbol{\beth_2}|)_{\tau-1}}{(\boldsymbol{\beth_3})_{\tau-1}\,(\mathbf{1})_{\boldsymbol{\tau-1}}}$$
$$+(6(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\sum_{\tau=2}^{\infty}(\tau-1)(\tau-2)\frac{(|\boldsymbol{\beth_1}|)_{\tau-1}\,(|\boldsymbol{\beth_2}|)_{\tau-1}}{(\boldsymbol{\beth_3})_{\tau-1}\,(\mathbf{1})_{\boldsymbol{\tau-1}}}$$

$$
\begin{aligned}
&+\left(7(\kappa_2+1)-3\kappa_3(\kappa_1+\kappa_2)\right)\sum_{\tau=2}^{\infty}(\tau-1)\frac{(|\beth_1|)_{\tau-1}\,(|\beth_2|)_{\tau-1}}{(\beth_3)_{\tau-1}\,(1)_{\tau-1}}\\
&+\left(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1\right)\sum_{\tau=2}^{\infty}\frac{(|\beth_1|)_{\tau-1}\,(|\beth_2|)_{\tau-1}}{(\beth_3)_{\tau-1}\,(1)_{\tau-1}}\\
\leq\ &\frac{(\kappa_2+1)\,|\beth_1\beth_2|\,(|\beth_1|+1)\,(|\beth_2|+1)\,(|\beth_1|+2)\,(|\beth_2|+2)}{\beth_3(\beth_3+1)(\beth_3+2)}\sum_{\tau=4}^{\infty}\frac{(|\beth_1|+3)_{\tau-4}\,(|\beth_2|+3)_{\tau-4}}{(\beth_3+3)_{\tau-4}\,(1)_{\tau-4}}\\
&+\frac{(6(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\,|\beth_1\beth_2|\,(|\beth_1|+1)\,(|\beth_2|+1)}{\beth_3(\beth_3+1)}\sum_{\tau=3}^{\infty}\frac{(|\beth_1|+2)_{\tau-3}\,(|\beth_2|+2)_{\tau-3}}{(\beth_3+2)_{\tau-3}\,(1)_{\tau-3}}\\
&+\frac{(7(\kappa_2+1)-3\kappa_3(\kappa_1+\kappa_2))\,|\beth_1\beth_2|}{\beth_3}\sum_{\tau=2}^{\infty}\frac{(|\beth_1|+1)_{\tau-2}\,(|\beth_2|+1)_{\tau-2}}{(\beth_3+1)_{\tau-2}\,(1)_{\tau-2}}\\
&+\left(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1\right)\sum_{\tau=2}^{\infty}\frac{(|\beth_1|)_{\tau-1}\,(|\beth_2|)_{\tau-1}}{(\beth_3)_{\tau-1}\,(1)_{\tau-1}}.
\end{aligned}
$$

By Gauss Summation Theorem, we can write

$$
\begin{aligned}
&\sum_{\tau=2}^{\infty}\left[\left(\tau^2(\kappa_2+1)-\tau\kappa_3(\kappa_1+\kappa_2)\right)\right]\left|\frac{(\beth_1)_{\tau-1}(\beth_2)_{\tau-1}}{(\beth_3)_{\tau-1}(1)_{\tau-1}}\right|\\
&\leq\frac{(\kappa_2+1)\,|\beth_1\beth_2|\,(|\beth_1|+1)\,(|\beth_2|+1)\,(|\beth_1|+2)\,(|\beth_2|+2)}{\beth_3(\beth_3+1)(\beth_3+2)}\mathcal{G}_{2,1}(|\beth_1|+3,|\beth_2|+3;\beth_3+3,1)\\
&+\frac{(6(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\,|\beth_1\beth_2|\,(|\beth_1|+1)\,(|\beth_2|+1)}{\beth_3(\beth_3+1)}\mathcal{G}_{2,1}(|\beth_1|+2,|\beth_2|+2;\beth_3+2,1)\\
&+\frac{(7(\kappa_2+1)-3\kappa_3(\kappa_1+\kappa_2))\,|\beth_1\beth_2|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,1)\\
&+\left(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1\right)\left[\mathcal{G}_{2,1}(|\beth_1|,|\beth_2|;\beth_3,1)-1\right]. \qquad (10)
\end{aligned}
$$

But the expression *(10)* is bounded above by $1-\kappa_1$, thus the proof is completed.

Theorem 2. *If* $L\in\mathcal{CV}$ *and* $\beth_1,\beth_2\in\mathbb{C}-\{0\}$, $\beth_3\in\mathbb{R}$, *then* $HO_{\beth_1,\beth_2,\beth_3}(L)\in\mathcal{C}_{\kappa_3}(\kappa_1,\kappa_2)$ *if the following condition is satisfied:*

$$
\begin{aligned}
&\frac{(\kappa_2+1)\,|\beth_1\beth_2|\,(|\beth_1|+1)\,(|\beth_2|+1)}{\beth_3(\beth_3+1)}\mathcal{G}_{2,1}(|\beth_1|+2,|\beth_2|+2;\beth_3+2,1)\\
&+\frac{(3(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\,|\beth_1\beth_2|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,1)\\
&+\left(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1\right)\left[\mathcal{G}_{2,1}(|\beth_1|,|\beth_2|;\beth_3,1)-1\right]\leq 1-\kappa_1.
\end{aligned}
$$

Proof. Since $L\in\mathcal{CV}$ we have $|\mathbf{r}_\tau|\leq 1$ and by equation (8), to prove $HO_{\beth_1,\beth_2,\beth_3}(L)\in\mathcal{C}_{\kappa_3}(\kappa_1,\kappa_2)$, it suffices to show that

$$
\sum_{\tau=2}^{\infty}\tau\left[(\tau(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\right]\left|\frac{(\beth_1)_{\tau-1}(\beth_2)_{\tau-1}}{(\beth_3)_{\tau-1}(1)_{\tau-1}}\mathbf{r}_\tau\right|
$$

$$\leq \sum_{\tau=2}^{\infty}\left[\left(\tau^2(\kappa_2+1\right)-\tau\kappa_3(\kappa_1+\kappa_2)\right]\left|\frac{(\beth_1)_{\tau-1}(\beth_2)_{\tau-1}}{(\beth_3)_{\tau-1}(1)_{\tau-1}}\right|\leq 1-\kappa_1.$$

Writing $\tau=(\tau-1)+1$, $\tau^2=(\tau-1)(\tau-2)+3(\tau-1)+1$ and use of (9), we get

$$\begin{aligned}
&\sum_{\tau=2}^{\infty}\left[\left(\tau^2(\kappa_2+1\right)-\tau\kappa_3(\kappa_1+\kappa_2)\right]\left|\frac{(\beth_1)_{\tau-1}(\beth_2)_{\tau-1}}{(\beth_3)_{\tau-1}(1)_{\tau-1}}\right| \\
= \; & (\kappa_2+1)\sum_{\tau=2}^{\infty}(\tau-1)(\tau-2)\frac{(|\beth_1|)_{\tau-1}\,(|\beth_2|)_{\tau-1}}{(\beth_3)_{\tau-1}\,(1)_{\tau-1}} \\
&+(3(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\sum_{\tau=2}^{\infty}(\tau-1)\frac{(|\beth_1|)_{\tau-1}\,(|\beth_2|)_{\tau-1}}{(\beth_3)_{\tau-1}\,(1)_{\tau-1}} \\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\sum_{\tau=2}^{\infty}\frac{(|\beth_1|)_{\tau-1}\,(|\beth_2|)_{\tau-1}}{(\beth_3)_{\tau-1}\,(1)_{\tau-1}} \\
\leq \; & \frac{(\kappa_2+1)\,|\beth_1\beth_2|\,(|\beth_1|+1)\,(|\beth_2|+1)}{\beth_3(\beth_3+1)}\sum_{\tau=3}^{\infty}\frac{(|\beth_1|+2)_{\tau-3}\,(|\beth_2|+2)_{\tau-3}}{(\beth_3+2)_{\tau-3}\,(1)_{\tau-3}} \\
&+\frac{(3(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\,|\beth_1\beth_2|}{\beth_3}\sum_{\tau=2}^{\infty}\frac{(|\beth_1|+1)_{\tau-2}\,(|\beth_2|+1)_{\tau-2}}{(\beth_3+1)_{\tau-2}\,(1)_{\tau-2}} \\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\sum_{\tau=2}^{\infty}\frac{(|\beth_1|)_{\tau-1}\,(|\beth_2|)_{\tau-1}}{(\beth_3)_{\tau-1}\,(1)_{\tau-1}}.
\end{aligned}$$

By Gauss Summation Theorem, we have

$$\begin{aligned}
&\sum_{\tau=2}^{\infty}\left[\left(\tau^2(\kappa_2+1\right)-\tau\kappa_3(\kappa_1+\kappa_2)\right]\left|\frac{(\beth_1)_{\tau-1}(\beth_2)_{\tau-1}}{(\beth_3)_{\tau-1}(1)_{\tau-1}}\right| \\
&\leq \frac{(\kappa_2+1)\,|\beth_1\beth_2|\,(|\beth_1|+1)\,(|\beth_2|+1)}{\beth_3(\beth_3+1)}\mathcal{G}_{2,1}(|\beth_1|+2,|\beth_2|+2;\beth_3+2,1) \\
&+\frac{(3(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\,|\beth_1\beth_2|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,1) \\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\left[\mathcal{G}_{2,1}(|\beth_1|,|\beth_2|;\beth_3,1)-1\right]. \qquad (11)
\end{aligned}$$

But the expression *(11)* is bounded above by $1-\kappa_1$, thus the proof is completed.

Theorem 3. *Let* Υ_1 *given by (3) and* $\beth_1,\beth_2\in\mathbb{C}-\{0\}$, $\beth_3\in\mathbb{R}$. *If* $L\in\xi-\mathcal{UCV}$ *for some* $\xi(0\leq\xi<\infty)$ *and satisfies the inequality*

$$\begin{aligned}
&\frac{(\kappa_2+1)\,|\beth_1\beth_2|\,\Upsilon_1}{\beth_3(1)_1}\mathcal{G}_{3,2}(|\beth_1|+1,|\beth_2|+1,\Upsilon_1+1;\beth_3+1,2;1) \\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\left[\mathcal{G}_{3,2}(|\beth_1|,|\beth_2|,\Upsilon_1;\beth_3,1;1)-1\right]\leq 1-\kappa_1,
\end{aligned}$$

then $HO_{\beth_1,\beth_2,\beth_3}(L)\in\mathcal{C}_{\kappa_3}(\kappa_1,\kappa_2)$.

Proof. To prove $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{C}_{\kappa_3}(\kappa_1,\kappa_2)$, it suffices to show that

$$\sum_{\tau=2}^{\infty} \tau\left[(\tau(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2)\right]\left|\frac{(\boldsymbol{\beth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\beth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\beth_3})_{\boldsymbol{\tau-1}}(\mathbf{1})_{\boldsymbol{\tau-1}}}\mathbf{r}_\tau\right| \leq 1-\kappa_1.$$

Applying the inequality (4) and use of (9), and then write $\tau = (\tau-1)+1$, we get

$$\begin{aligned}
&\sum_{\tau=2}^{\infty} \tau\left[(\tau(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2)\right]\left|\frac{(\boldsymbol{\beth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\beth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\beth_3})_{\boldsymbol{\tau-1}}(\mathbf{1})_{\boldsymbol{\tau-1}}}\mathbf{r}_\tau\right| \\
\leq\ & \sum_{\tau=2}^{\infty}\left[(\tau(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2)\right]\frac{(|\boldsymbol{\beth_1}|)_{\tau-1}\,(|\boldsymbol{\beth_2}|)_{\tau-1}}{(\boldsymbol{\beth_3})_{\tau-1}\,(\mathbf{1})_{\boldsymbol{\tau-1}}}\frac{(\Upsilon_1)_{\tau-1}}{(\mathbf{1})_{\boldsymbol{\tau-1}}} \\
\leq\ & (\kappa_2+1)\sum_{\tau=2}^{\infty}(\tau-1)\frac{(|\boldsymbol{\beth_1}|)_{\tau-1}\,(|\boldsymbol{\beth_2}|)_{\tau-1}}{(\boldsymbol{\beth_3})_{\tau-1}\,(\mathbf{1})_{\boldsymbol{\tau-1}}} \\
& +(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\sum_{\tau=2}^{\infty}\frac{(|\boldsymbol{\beth_1}|)_{\tau-1}\,(|\boldsymbol{\beth_2}|)_{\tau-1}}{(\boldsymbol{\beth_3})_{\tau-1}\,(\mathbf{1})_{\boldsymbol{\tau-1}}} \\
\leq\ & \frac{(\kappa_2+1)\,|\boldsymbol{\beth_1}\boldsymbol{\beth_2}|\,\Upsilon_1}{\boldsymbol{\beth_3}(\mathbf{1})_1}\sum_{\tau=2}^{\infty}\frac{(|\boldsymbol{\beth_1}|+1)_{\tau-2}\,(|\boldsymbol{\beth_2}|+1)_{\tau-2}\,(\Upsilon_1+1)_{\tau-2}}{(\boldsymbol{\beth_3}+1)_{\tau-2}\,(\mathbf{1})_{\boldsymbol{\tau-2}}(\mathbf{2})_{\boldsymbol{\tau-2}}} \\
& +(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\sum_{\tau=2}^{\infty}\frac{(|\boldsymbol{\beth_1}|)_{\tau-1}\,(|\boldsymbol{\beth_2}|)_{\tau-1}\,(\Upsilon_1)_{\tau-1}}{(\boldsymbol{\beth_3})_{\tau-1}\,(\mathbf{1})_{\boldsymbol{\tau-1}}(\mathbf{1})_{\boldsymbol{\tau-1}}}.
\end{aligned}$$

Using Gauss Summation Theorem, we can write

$$\begin{aligned}
&\sum_{\tau=2}^{\infty} \tau\left[(\tau(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2)\right]\left|\frac{(\boldsymbol{\beth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\beth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\beth_3})_{\boldsymbol{\tau-1}}(\mathbf{1})_{\boldsymbol{\tau-1}}}\right| \\
&\leq \frac{(\kappa_2+1)\,|\boldsymbol{\beth_1}\boldsymbol{\beth_2}|\,\Upsilon_1}{\boldsymbol{\beth_3}(\mathbf{1})_1}\mathcal{G}_{3,2}(|\boldsymbol{\beth_1}|+1,|\boldsymbol{\beth_2}|+1,\Upsilon_1+1;\beth_3+1,2;1) \\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\left[\mathcal{G}_{3,2}(|\boldsymbol{\beth_1}|,|\boldsymbol{\beth_2}|,\Upsilon_1;\beth_3,1;1)-1\right]. \qquad (12)
\end{aligned}$$

But the expression *(12)* is bounded above by $1-\kappa_1$, thus the proof is completed.

Theorem 4. *Let* Υ_1 *given by (3) and* $\beth_1,\beth_2 \in \mathbb{C}-\{0\}$, $\beth_3 \in \mathbb{R}$. *If* $L \in \xi-\mathcal{ST}$ *for some* $\xi(0 \leq \xi < \infty)$ *and satisfies the inequality*

$$\begin{aligned}
&\frac{(\kappa_2+1)\,|\boldsymbol{\beth_1}\boldsymbol{\beth_2}|\,(|\boldsymbol{\beth_1}|+1)\,(|\boldsymbol{\beth_2}|+1)\,(\Upsilon_1+1)\Upsilon_1}{\boldsymbol{\beth_3}(\boldsymbol{\beth_3}+1)(2)_1}\mathcal{G}_{3,2}(|\boldsymbol{\beth_1}|+2,|\boldsymbol{\beth_2}|+2,\Upsilon_1+2;\beth_3+2,3;1) \\
&+\frac{(3(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\,|\boldsymbol{\beth_1}\boldsymbol{\beth_2}|\,\Upsilon_1}{\boldsymbol{\beth_3}(\mathbf{1})_1}\mathcal{G}_{3,2}(|\boldsymbol{\beth_1}|+1,|\boldsymbol{\beth_2}|+1,\Upsilon_1+1;\beth_3+1,2;1) \\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\left[\mathcal{G}_{3,2}(|\boldsymbol{\beth_1}|,|\boldsymbol{\beth_2}|,\Upsilon_1;\beth_3,1;1)-1\right] \leq 1-\kappa_1,
\end{aligned}$$

then $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{C}_{\kappa_3}(\kappa_1,\kappa_2)$.

Proof. To prove $HO_{\daleth_1,\daleth_2,\daleth_3}(L) \in \mathcal{C}_{\kappa_3}(\kappa_1, \kappa_2)$, it suffices to show that

$$\sum_{\tau=2}^{\infty} \tau \left[(\tau(\kappa_2+1) - \kappa_3(\kappa_1+\kappa_2)\right] \left| \frac{(\boldsymbol{\daleth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\daleth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\daleth_3})_{\boldsymbol{\tau-1}}(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \mathbf{r}_\tau \right| \leq 1-\kappa_1.$$

Applying the inequality (5) and use of (9), and then write $\tau = (\tau-1)+1$, $\tau^2 = (\tau-1)(\tau-2)+3(\tau-1)+1$, we get

$$\begin{aligned}
&\sum_{\tau=2}^{\infty} \tau \left[(\tau(\kappa_2+1) - \kappa_3(\kappa_1+\kappa_2)\right] \left| \frac{(\boldsymbol{\daleth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\daleth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\daleth_3})_{\boldsymbol{\tau-1}}(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \mathbf{r}_\tau \right| \\
\leq\ & \sum_{\tau=2}^{\infty} \left[\left(\tau^2(\kappa_2+1\right) - \tau\kappa_3(\kappa_1+\kappa_2)\right] \frac{(|\boldsymbol{\daleth_1}|)_{\tau-1}\,(|\boldsymbol{\daleth_2}|)_{\tau-1}}{(\boldsymbol{\daleth_3})_{\tau-1}\,(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \frac{(\Upsilon_1)_{\tau-1}}{(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \\
\leq\ & (\kappa_2+1)\sum_{\tau=2}^{\infty}(\tau-1)(\tau-2) \frac{(|\boldsymbol{\daleth_1}|)_{\tau-1}\,(|\boldsymbol{\daleth_2}|)_{\tau-1}}{(\boldsymbol{\daleth_3})_{\tau-1}\,(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \frac{(\Upsilon_1)_{\tau-1}}{(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \\
&+ (3(\kappa_2+1) - \kappa_3(\kappa_1+\kappa_2)) \sum_{\tau=2}^{\infty}(\tau-1) \frac{(|\boldsymbol{\daleth_1}|)_{\tau-1}\,(|\boldsymbol{\daleth_2}|)_{\tau-1}}{(\boldsymbol{\daleth_3})_{\tau-1}\,(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \frac{(\Upsilon_1)_{\tau-1}}{(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \\
&+ (\kappa_2 - \kappa_3(\kappa_1+\kappa_2)+1) \sum_{\tau=2}^{\infty} \frac{(|\boldsymbol{\daleth_1}|)_{\tau-1}\,(|\boldsymbol{\daleth_2}|)_{\tau-1}}{(\boldsymbol{\daleth_3})_{\tau-1}\,(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \frac{(\Upsilon_1)_{\tau-1}}{(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \\
\leq\ & \frac{(\kappa_2+1)\,|\boldsymbol{\daleth_1\daleth_2}|\,(|\boldsymbol{\daleth_1}|+1)\,(|\boldsymbol{\daleth_2}|+1)\,(\Upsilon_1+1)\Upsilon_1}{\boldsymbol{\daleth_3}(\boldsymbol{\daleth_3}{+}1)(2)_1} \sum_{\tau=3}^{\infty} \frac{(|\boldsymbol{\daleth_1}|+2)_{\tau-3}\,(|\boldsymbol{\daleth_2}|+2)_{\tau-3}\,(\Upsilon_1+2)_{\tau-3}}{(\boldsymbol{\daleth_3}+2)_{\tau-3}\,(1)_{\tau-\boldsymbol{3}}(3)_{\boldsymbol{\tau-3}}} \\
&\frac{(3(\kappa_2+1) - \kappa_3(\kappa_1+\kappa_2))\,|\boldsymbol{\daleth_1\daleth_2}|\,\Upsilon_1}{\boldsymbol{\daleth_3}(\boldsymbol{1})_1} \sum_{\tau=2}^{\infty} \frac{(|\boldsymbol{\daleth_1}|+1)_{\tau-2}\,(|\boldsymbol{\daleth_2}|+1)_{\tau-2}\,(\Upsilon_1+1)_{\tau-2}}{(\boldsymbol{\daleth_3}+1)_{\tau-2}\,(1)_{\tau-\boldsymbol{2}}(2)_{\boldsymbol{\tau-2}}} \\
&+ (\kappa_2 - \kappa_3(\kappa_1+\kappa_2)+1) \sum_{\tau=2}^{\infty} \frac{(|\boldsymbol{\daleth_1}|)_{\tau-1}\,(|\boldsymbol{\daleth_2}|)_{\tau-1}\,(\Upsilon_1)_{\tau-1}}{(\boldsymbol{\daleth_3})_{\tau-1}\,(\boldsymbol{1})_{\boldsymbol{\tau-1}}(\boldsymbol{1})_{\boldsymbol{\tau-1}}}.
\end{aligned}$$

By Gauss Summation Theorem, we can write

$$\begin{aligned}
&\sum_{\tau=2}^{\infty} \tau \left[(\tau(\kappa_2+1) - \kappa_3(\kappa_1+\kappa_2)\right] \left| \frac{(\boldsymbol{\daleth_1})_{\boldsymbol{\tau-1}}(\boldsymbol{\daleth_2})_{\boldsymbol{\tau-1}}}{(\boldsymbol{\daleth_3})_{\boldsymbol{\tau-1}}(\boldsymbol{1})_{\boldsymbol{\tau-1}}} \right| \\
&\leq \frac{(\kappa_2+1)\,|\boldsymbol{\daleth_1\daleth_2}|\,(|\boldsymbol{\daleth_1}|+1)\,(|\boldsymbol{\daleth_2}|+1)\,(\Upsilon_1+1)\Upsilon_1}{\boldsymbol{\daleth_3}(\boldsymbol{\daleth_3}{+}1)(2)_1} \mathcal{G}_{3,2}(|\boldsymbol{\daleth_1}|+2, |\boldsymbol{\daleth_2}|+2, \Upsilon_1+2; \daleth_3+2, 3; 1) \\
&+ \frac{(3(\kappa_2+1) - \kappa_3(\kappa_1+\kappa_2))\,|\boldsymbol{\daleth_1\daleth_2}|\,\Upsilon_1}{\boldsymbol{\daleth_3}(\boldsymbol{1})_1} \mathcal{G}_{3,2}(|\boldsymbol{\daleth_1}|+1, |\boldsymbol{\daleth_2}|+1, \Upsilon_1+1; \daleth_3+1, 2; 1) \\
&+ (\kappa_2 - \kappa_3(\kappa_1+\kappa_2)+1) \left[\mathcal{G}_{3,2}(|\boldsymbol{\daleth_1}|, |\boldsymbol{\daleth_2}|, \Upsilon_1; \daleth_3, 1; 1) - 1\right]. \qquad (13)
\end{aligned}$$

But the expression *(13)* is bounded above by $1-\kappa_1$, thus the proof is completed.

Using the same proceeding used to prove Theorem 1, Theorem 2 and Theorem 4, we get the following Theorems for subfamily $S^*_{\kappa_3}(\kappa_1, \kappa_2)$, respictivlly.

Theorem 5. *If* $L \in \mathcal{ST}$ *and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$, *then* $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathrm{S}^*_{\kappa_3}(\kappa_1, \kappa_2)$ *if the following condition is satisfied:*

$$\begin{aligned}
&\frac{(\kappa_2+1)\left|\beth_1\beth_2\right|(\left|\beth_1\right|+1)(\left|\beth_2\right|+1)}{\beth_3(\beth_3+1)}\mathcal{G}_{2,1}(|\beth_1|+2,|\beth_2|+2;\beth_3+2,1)\\
&+\frac{(3(\kappa_2+1)-\kappa_3(\kappa_1+\kappa_2))\left|\beth_1\beth_2\right|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,1)\\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\left[\mathcal{G}_{2,1}(|\beth_1|,|\beth_2|;\beth_3,1)-1\right]\le 1-\kappa_1.
\end{aligned}$$

Theorem 6. *If* $L \in \mathcal{CV}$ *and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$, *then* $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathrm{S}^*_{\kappa_3}(\kappa_1, \kappa_2)$ *if the following condition is satisfied:*

$$\begin{aligned}
&\frac{(\kappa_2+1)\left|\beth_1\beth_2\right|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,1)\\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\left[\mathcal{G}_{2,1}(|\beth_1|,|\beth_2|;\beth_3,1)-1\right]\le 1-\kappa_1.
\end{aligned}$$

Theorem 7. *Let* Υ_1 *given by (3) and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$. *If* $L \in \xi - \mathcal{ST}$ *for some* $\xi(0 \le \xi < \infty)$ *and satisfies the inequality*

$$\begin{aligned}
&\frac{(\kappa_2+1)\left|\beth_1\beth_2\right|\Upsilon_1}{\beth_3(1)_1}\mathcal{G}_{3,2}(|\beth_1|+1,|\beth_2|+1,\Upsilon_1+1;\beth_3+1,2;1)\\
&+(\kappa_2-\kappa_3(\kappa_1+\kappa_2)+1)\left[\mathcal{G}_{3,2}(|\beth_1|,|\beth_2|,\Upsilon_1;\beth_3,1;1)-1\right]\le 1-\kappa_1,
\end{aligned}$$

then $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathrm{S}^*_{\kappa_3}(\kappa_1, \kappa_2)$.

3. Some Corollaries

In this section, by suitable choices for parameters $\beth_1$, $\beth_2$ and $\beth_3$, we can conclude many subresults from our main results related to Bernardi operator $HO_{1,\alpha+1,\alpha+2}(L)$, Alexander operator $HO_{1,1,2}(L)$ and Libera operator $HO_{1,2,3}(L)$. Also, by suitable choices for parameters κ_1, κ_2 and κ_3, we have many subresults. For example, if we set $\kappa_2 = 0$ and $\kappa_3 = 1$ in our main results, we get the following corollaries for subfamilies $\mathcal{CV}(\kappa_1)$ and $\mathcal{ST}(\kappa_1)$.

Corollary 1. *If* $L \in \mathcal{ST}$ *and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$, *then* $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{CV}(\kappa_1)$ *if*

$$\begin{aligned}
&\frac{\left|\beth_1\beth_2\right|(\left|\beth_1\right|+1)(\left|\beth_2\right|+1)(\left|\beth_1\right|+2)(\left|\beth_2\right|+2)}{\beth_3(\beth_3+1)(\beth_3+2)}\mathcal{G}_{2,1}(|\beth_1|+3,|\beth_2|+3;\beth_3+3,\mathbf{1})\\
&+\frac{(6-\kappa_1)\left|\beth_1\beth_2\right|(\left|\beth_1\right|+1)(\left|\beth_2\right|+1)}{\beth_3(\beth_3+1)}\mathcal{G}_{2,1}(|\beth_1|+2,|\beth_2|+2;\beth_3+2,\mathbf{1})\\
&+\frac{(7-3\kappa_1)\left|\beth_1\beth_2\right|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,\mathbf{1})\\
&+(1-\kappa_1)\left[\mathcal{G}_{2,1}\left|\beth_1\right|,\left|\beth_2\right|;\beth_3,\mathbf{1})-1\right]\le 1-\kappa_1.
\end{aligned}$$

Corollary 2. *If* $L \in \mathcal{CV}$ *and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$, *then* $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{CV}(\kappa_1)$ *if*

$$\begin{aligned}&\frac{|\beth_1\beth_2|\left(|\beth_1|+1\right)\left(|\beth_2|+1\right)}{\beth_3(\beth_3+\mathbf{1})}\mathcal{G}_{2,1}(|\beth_1|+2,|\beth_2|+2;\beth_3+2,\mathbf{1})\\&+\frac{(3-\kappa_1)\,|\beth_1\beth_2|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,\mathbf{1})\\&+(1-\kappa_1)\left[\mathcal{G}_{2,1}(|\beth_1|,|\beth_2|;\beth_3,\mathbf{1})-1\right]\leq 1-\kappa_1.\end{aligned}$$

Corollary 3. *Let* Υ_1 *given by (3) and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$. *If* $L \in \xi - \mathcal{UCV}$ *for some* $\xi(0 \leq \xi < \infty)$ *and satisfies*

$$\begin{aligned}&\frac{|\beth_1\beth_2|\,\Upsilon_1}{\beth_3(\mathbf{1})_1}\mathcal{G}_{3,2}(|\beth_1|+1,|\beth_2|+1,\Upsilon_1+1;\beth_3+1,2;1)\\&+(1-\kappa_1)\left[\mathcal{G}_{3,2}(|\beth_1|,|\beth_2|,\Upsilon_1;\beth_3,1;1)-1\right]\leq 1-\kappa_1,\end{aligned}$$

then $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{K}(\kappa_1)$.

Corollary 4. *Let* Υ_1 *given by (3) and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$. *If* $L \in \xi - \mathcal{ST}$ *for some* $\xi(0 \leq \xi < \infty)$ *and satisfies*

$$\begin{aligned}&\frac{|\beth_1\beth_2|\left(|\beth_1|+1\right)\left(|\beth_2|+1\right)(\Upsilon_1+1)\Upsilon_1}{\beth_3(\beth_3+1)(2)_1}\mathcal{G}_{3,2}(|\beth_1|+2,|\beth_2|+2,\Upsilon_1+2;\beth_3+2,3;1)\\&+\frac{(3-\kappa_1)\,|\beth_1\beth_2|\,\Upsilon_1}{\beth_3(\mathbf{1})_1}\mathcal{G}_{3,2}(|\beth_1|+1,|\beth_2|+1,\Upsilon_1+1;\beth_3+1,2;1)\\&+(\mathbf{1}-\kappa_1)\left[\mathcal{G}_{3,2}(|\beth_1|,|\beth_2|,\Upsilon_1;\beth_3,1;1)-1\right]\leq 1-\kappa_1,\end{aligned}$$

then $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{CV}(\kappa_1)$.

Corollary 5. *If* $L \in \mathcal{ST}$ *and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$, *then* $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{ST}(\kappa_1)$ *if*

$$\begin{aligned}&\frac{|\beth_1\beth_2|\left(|\beth_1|+1\right)\left(|\beth_2|+1\right)}{\beth_3(\beth_3+1)}\mathcal{G}_{2,1}(|\beth_1|+2,|\beth_2|+2;\beth_3+2,1)\\&+\frac{(3-\kappa_1)\,|\beth_1\beth_2|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,1)\\&+(1-\kappa_1)\left[\mathcal{G}_{2,1}(|\beth_1|,|\beth_2|;\beth_3,1)-1\right]\leq 1-\kappa_1.\end{aligned}$$

Corollary 6. *IIf* $L \in \mathcal{CV}$ *and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$, *then* $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{ST}(\kappa_1)$ *if*

$$\begin{aligned}&\frac{|\beth_1\beth_2|}{\beth_3}\mathcal{G}_{2,1}(|\beth_1|+1,|\beth_2|+1;\beth_3+1,1)\\&+(1-\kappa_1)\left[\mathcal{G}_{2,1}(|\beth_1|,|\beth_2|;\beth_3,1)-1\right]\leq 1-\kappa_1.\end{aligned}$$

Corollary 7. *Let* Υ_1 *given by (3) and* $\beth_1, \beth_2 \in \mathbb{C} - \{0\}$, $\beth_3 \in \mathbb{R}$. *If* $L \in \xi - \mathcal{ST}$ *for some* $\xi(0 \leq \xi < \infty)$ *and satisfies*

$$\begin{aligned}&\frac{|\beth_1\beth_2|\,\Upsilon_1}{\beth_3(\mathbf{1})_1}\mathcal{G}_{3,2}(|\beth_1|+1,|\beth_2|+1,\Upsilon_1+1;\beth_3+1,2;1)\\&+(1-\kappa_1)\left[\mathcal{G}_{3,2}(|\beth_1|,|\beth_2|,\Upsilon_1;\beth_3,1;1)-1\right]\leq 1-\kappa_1,\end{aligned}$$

then $HO_{\beth_1,\beth_2,\beth_3}(L) \in \mathcal{ST}(\kappa_1)$.

4. Conclusions

Using of the Hohlov operator $HO_{\beth_1,\beth_2,\beth_3}(L)$, we find necessary condition for this operator to be in the subfamilies $\mathcal{C}_{\kappa_3}(\kappa_1, \kappa_2)$ and $S^*_{\kappa_3}(\kappa_1, \kappa_2)$ of analytic functions with negative coefficients. Furthermore, we investigate several inclusion properties for these subfamilies. Also, our results will imply a number of corollaries. Hohlov operator $HO_{\beth_1,\beth_2,\beth_3}(L)$ can be used to derive new necessary and sufficient condition for analytic functions in various subfamilies in the open unit disk.

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