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Some Results of Conformable Fourier Transform

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Abstract. Based on a new definition of α-periodicals functions with $0 < \alpha \leq 1$ introduced by Khalil et al (2014), we introduce a new definition of conformable Fourier transform for such a class of functions. Further, we establish some operational formulas, and we set the relation between the newly defined conformable Fourier transform and the classical Fourier transform. Finally, some classical results of periodical functions are obtained and some illustrative examples are constructed.

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Key Words and Phrases: α-periodic function, Conformable derivative, Conformable Fourier transform, Conformable fractional integral

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1. Introduction

The fractional calculus [11, 14, 17] attracted many researches in the last and present centuries. The impact of this fractional calculus in both pure and applied branches of science and engineering started to increase substantially during the last two decades apparently.

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Traditionally, the arbitrary order of integration and differentiation has been described by nonlocal fractional operators with kernels reflecting their memories. Recently, the conformable derivative operator $T^{(\alpha)}(x)(t) = \lim_{h\to 0} \frac{f(t+ht^{1-\alpha})-f(t)}{h}$ was introduced in the literature by Khalil [9] to allow integrating and differentiating with respect to arbitrary order without having memory in the structure and hence falling in a similar category to local fractional calculus and fractal calculus [16, 21]. Since then, many classical problems have been generalized to the conformable case [12, 13]. Later, several modification of conformable derivatives have been appeared such as: the fractional Beta derivative [15] defined as $D^{\gamma}_{\rho}(f(\rho)) = \lim_{\epsilon\to 0} \frac{f(\rho+\epsilon(\rho+\frac{1}{\Gamma(\gamma)}))-f(\rho)}{\epsilon}$ and the M-truncated derivative [19] defined as $D^{M}_{\alpha,\beta} f(t) = \lim_{\epsilon\to 0} \frac{f(t\mathbb{E}_{\beta,i}(\epsilon t^{-\alpha}))-f(t)}{\epsilon}$ where $\mathbb{E}_{\beta,i}(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(\beta k+1)}$.

Cauchy type problems are very well-known important in many fields of science and engineering. Several results regarding the capture of candidate solutions of the conformable differential equations can be found in [18]. This new definition has been developed by Abdeljawad [1] and by El-Ajou [6]. For more developments on the conformable differentiation, we refer to [3, 5]. The usability of the conformable derivative notion has wide areas of interest in both theoretical and practical aspects (see [10], [20]).

The authors of ([2], [24]) provided some applications through partial differential equations (PDEs) in the conformable sense. Precisely, Maxwell's equations have been considered in the conformable fractional setting to describe electromagnetic fields of media in [23]. The conformable differential equation (CDE) has been used for the description of the subdiffusion process in [24]. Also, some applications in quantum mechanics have been treated in the context of CFD (see for example [2]).

Fourier series is one of the most important tools in applied sciences. For example one can solve partial differential equations using Fourier series. Further one can find the sum of certain numerical series using Fourier series. Fractional partial differential equations appeared to have many applications in physics and engineering. There are many definitions of fractional derivative.

The conformable fractional Fourier series for α-periodical functions is introduced by Khalil et al [8]. They proved that the fractional Fourier series of a piece wise continuous α-periodical function converges pointwise to the average limit of the function at each point of discontinuity, and to the function at each point of continuity.

The rest of this paper is structured as follows : In section 2, we introduce the basic definitions and properties of α-conformable functional derivative $T^{(\alpha)}(f)(t)$ for $0 < \alpha \leq 1$ and $f : [0, +\infty[\to \mathbb{R}$ is α-periodic function, define by khalil et al [9]. In section 3, we prove some results and examples of α-periodic functions which are important for the next section. In Section 4, we give a new definition of conformable Fourier transform for α-periodical functions.

In the first result (Theorem 8), we show that there exists a relationship between the conformable Fourier transform and the classical Fourier transform as follows:

$$\mathcal{F}_\alpha\{f(t)\}(k) = \mathcal{F}\{f((\alpha t)^{\frac{1}{\alpha}})\}(k)$$

for all $k \in \mathbb{Z}$. In the second and third result (Theorem 9 and Theorem 11), we give the results of the conformable Fourier transform for the conformable fractional integral $I_\alpha(f)(t)$ defined by Abdeljawad [1], as follows:

$$\mathcal{F}_\alpha(I_\alpha(f)(t))(k) = (2ik\pi\frac{\alpha}{p^\alpha})\mathcal{F}_\alpha(f(t)(k)$$

for all $k \in \mathbb{Z}^*$ and for the conformable derivative introduced by khalil et al [9] as follows, $\mathcal{F}_\alpha(T^{(\alpha)}(f)(t))(k) = (2ik\pi\frac{\alpha}{p^\alpha})\mathcal{F}_\alpha(f(t)(k)$ and in the general case for $n \in \mathbb{N}$,

$$\mathcal{F}_\alpha(T^{(j\alpha)}(f)(t))(k) = (2ik\pi\frac{\alpha}{p^\alpha})^j\mathcal{F}_\alpha(f(t)(k),\ \forall j \in \{0, 1, ..., n\}.$$

A following classical result is also obtained for α-periodical functions

$$\mathcal{F}_\alpha((a *_\alpha f)_{-\infty}(t))(k) = \mathcal{L}_\alpha(a(t))(2ik\pi\frac{\alpha}{p^\alpha})\mathcal{F}_\alpha(f((t))(k)$$

where $(a *_\alpha f)_{-\infty}(t) = \int_{-\infty}^{\frac{t^\alpha}{\alpha}} a((t^\alpha - \alpha s)^{\frac{1}{\alpha}})f((\alpha s)^{\frac{1}{\alpha}})ds$ and $\mathcal{L}_\alpha(a(t))(\lambda)$ is the conformable Laplace transform of the function $a(t)$, given by Z.Al-Zhouri et al [22]. Many examples are given to support the results presented. Finally, the conclusion is presented in Section 5.

2. Basic definitions and tools

In this section, we introduce the definition of conformable fractional calculus and its important properties.

Definition 1. *[9] Given a function $f : [0, +\infty[\to \mathbb{R}$, the conformable fractional derivative of order α is defined by:*

$$T^{(\alpha)}(f)(t) = \lim_{h \to 0} \frac{f(t + ht^{1-\alpha}) - f(t)}{h}$$

for all $t > 0$ and $0 < \alpha \leq 1$.

Definition 2. *Let $0 < \alpha \leq 1$ and $f : [0, +\infty[\to \mathbb{R}$.*

(i) The function f is called α-differentiable on $[0, +\infty[$, if f is continuous. $T^{(\alpha)}f(t)$ exists for all $t \in]0, +\infty[$ and $T^{(\alpha)}f(0) = \lim_{t \to 0^+} T^{(\alpha)}f(t)$ exists.

(ii) The function f is called continuously α-differentiable on $[0, +\infty)$ if f is α-differentiable on $[0, +\infty)$ and $T^{(\alpha)}f(t)$ is continuous on $[0, +\infty[$.

Definition 3. *Let* $0 < \alpha \leq 1, n \in \mathbb{N}$ *and* $f : [0, +\infty[\to \mathbb{R}$.

(i) The function f *is called* n *times* α*-differentiable on* $[0, +\infty[$ *if* f *is continuous,* $\forall j \in \{0, ...n\}$ $T^{(j\alpha)}f(t) = T^{(\alpha)}(T^{(\alpha)}...(T^{(\alpha)}(f)))(t), j$ *times, exists for all* $t \in]0, +\infty[$ *and* $T^{(j\alpha)}f(0) = \lim_{t\to 0^+} T^{(j\alpha)}f(t)$ *exists.*

(ii) The function f *is called* n *times continuously* α*-differentiable on* $[0, +\infty)$ *if* f *is* n *times* α*-differentiable on* $[0, +\infty)$ *and* $\forall j \in \{0, \ldots, n\}$ $T^{(j\alpha)}f(t)$ *is continuous on* $[0, +\infty[$.

(iii) The function f *is called infinitely continuously* α*-differentiable, if* f *is* n *times continuously* α*-differentiable for all* $n \in \mathbb{N}$.

Note that for $n = 0$, f *is* n *time* α*-differentiable if there is continuous.*

Example 1. *Let* $f(t) = e^t, \ t \in [0, +\infty[$.

(i) For all $t > 0$ *and* $0 < \alpha \leq 1$

$$\begin{aligned} T^{(\alpha)}(f)(t) &= \lim_{h\to 0} \frac{e^{(t+ht^{1-\alpha})} - e^{(t)}}{h} \\ &= t^{1-\alpha}e^t \lim_{h\to 0} \frac{e^{ht^{1-\alpha}} - 1}{ht^{1-\alpha}} \\ &= t^{1-\alpha}e^t \end{aligned}$$

(ii) For all $t > 0$ *and* $0 < \alpha \leq 1$

$$\begin{aligned} T^{(2\alpha)}(f)(t) &= T^{(\alpha)}(T^{(\alpha)}(f)(t)) = T^{(\alpha)}(t^{1-\alpha}e^t) \\ &= \lim_{h\to 0} \frac{(t + ht^{1-\alpha})^{1-\alpha}e^{(t+ht^{1-\alpha})} - t^{1-\alpha}e^t}{h} \\ &= t^{1-\alpha}e^t \lim_{h\to 0} \frac{(1 - ht^{-\alpha})^{1-\alpha}e^{ht^{1-\alpha}} - 1}{h} \\ &= t^{1-\alpha}e^t g'(0), \end{aligned}$$

where $g(t) = (1 - ht^{-\alpha})^{1-\alpha}e^{ht^{1-\alpha}}$ *and* $g'(0) = (1 - \alpha)t^{-\alpha} + t^{1-\alpha}$. *Then, we get*

$$T^{(2\alpha)}(e^t) = t^{1-\alpha}e^t((1 - \alpha)t^{-\alpha} + t^{1-\alpha}).$$

Theorem 1. *[9] Let* $\alpha \in (0, 1]$ *and* f *is* α*-differentiable at a point* $t > 0$. *Then*

(i) $T^{(\alpha)}(f)(t) = t^{1-\alpha}f'(t)$.

(ii) $T^{(\alpha)}(e^{ct}) = c\ t^{1-\alpha}e^{ct}, c \in \mathbb{R}$ *or* $\mathbb{C}$.

(iii) $T^{(\alpha)}(\frac{t^\alpha}{\alpha}) = 1$.

Example 2. *[9]*

(i) $T^{(\alpha)}(t^p) = pt^{\alpha-p}$.

(ii) $T^{(\alpha)}(e^{ikt}) = ikt^{1-\alpha}e^{ikt}, k \in \mathbb{Z}$.

(iii) $T^{(\alpha)}(\sin(\frac{1}{\alpha}t^\alpha)) = \cos(\frac{1}{\alpha}t^\alpha)$.

(iv) $T^{(\frac{1}{2})}(2\sqrt{t}) = 1$.

(v) $T^{(2\alpha)}(e^t) = T^{(\alpha)}(t^{1-\alpha}e^t) = t^{1-\alpha}(t^{1-\alpha}e^t)' = t^{1-\alpha}e^t((1-\alpha)t^{-\alpha} + t^{1-\alpha})$.

Definition 4. *[1] The conformable fractional integral of order $0 < \alpha \leq 1$ is defined by*

$$I_\alpha(f)(t) = \int_0^t s^{\alpha-1}f(s)\,ds, t \in [0, +\infty[.$$

Lemma 1. *[1] Assume that $f : [0, +\infty) \to \mathbb{R}$ is continuous and $0 < \alpha \leq 1$. Then, for all $t > 0$, we have*

$$T^{(\alpha)}(I_\alpha(f))(t) = f(t)$$

Lemma 2. *[1] Let $f : [0, +\infty) \to \mathbb{R}$ be α-differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have*

$$I_\alpha(T^{(\alpha)}(f))(t) = f(t) - f(0).$$

Let X be a Banach space, and f is a periodic function with period T on $\mathbb{R}$. For a function $f \in L^1(0, T; X)$, the k^{th} fourier coefficient of f is given by

$$\mathcal{F}(f(t))(k) = \frac{1}{T}\int_0^T e^{-ik\frac{2\pi}{T}t}f(t)dt.$$

Definition 5. *[22] Let $f : [0; +\infty[\to \mathbb{R}$ be a given function and $0 < \alpha \leq 1$. Then the conformable fractional Laplace transform of f is defined as:*

$$\mathcal{L}_\alpha(f(t))(\lambda) = \int_0^{+\infty} e^{-\lambda\frac{t^\alpha}{\alpha}}t^{\alpha-1}f(t)dt$$

provided the integral exists.

Theorem 2. *[22] Let $a : [0; +\infty[\to \mathbb{R}$ be a function and $0 < \alpha \leq 1$. Then*

$$\mathcal{L}_\alpha(a(t))(\lambda) = \mathcal{L}(a((\alpha t)^{\frac{1}{\alpha}}))(\lambda), \lambda \in \mathbb{C}.$$

where $\mathcal{L}(a(t))(\lambda) = \int_0^{+\infty} e^{-\lambda t}a(t)dt$ denotes the Laplace transform of $a(t)$.

Theorem 3. *[7] Given $a \in L^1(\mathbb{R}^+)$ and $g : [0, 2\pi] \to X$ is a periodic function with period 2π (extended by periodicity to $\mathbb{R}$), where X is a Banach space. We find that*

$$\mathcal{F}(F(t))(k) = \mathcal{L}(a(t))(ik)\mathcal{F}(g(t))(k), \ k \in \mathbb{Z} \tag{1}$$

where the function F is defined by $F(t) = \int_{-\infty}^t a(t-s)g(s)ds = \int_0^{+\infty} a(s)g(t-s)ds$ is continuous and bounded on $\mathbb{R}$.

This theorem has been used by many authors to solve some integro-differential equations using the fourier transform ([7], [4]).

In the next section, we present some results of α-periodic functions.

3. Some results of α-periodic functions

Definition 6. *[8] (α-periodic function)*
Let $0 < \alpha \leq 1$. The function $f : [0, +\infty) \to \mathbb{R}$ is called α-periodical with period $p > 0$, if there exists a continuous function $g : [0, +\infty) \to \mathbb{R}$ such that

$$f(t) = g\Big(\frac{t^\alpha}{\alpha}\Big) = g\Big(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha}\Big)$$

for all $t \in [0, +\infty)$.

Remark 1. *:*

(i) Note that the continuity of g implies that of f.

(ii) The function $g(t) = f((\alpha t)^{\frac{1}{\alpha}})$ is periodic with period $\frac{p^\alpha}{\alpha}$.

Example 3. *Let $0 < \alpha \leq 1$. For all $t \in [0, (\frac{1}{\alpha})^{\frac{1}{\alpha}}]$, let us consider the following functions $f_1(t)$ and $f_2(t)$*

$$f_1(t) = \begin{cases} \frac{t^\alpha}{\alpha}, & 0 \leq t \leq (\frac{1}{2\alpha})^{\frac{1}{\alpha}} \\ \frac{1}{\alpha^2} - \frac{t^\alpha}{\alpha}, & (\frac{1}{2\alpha})^{\frac{1}{\alpha}} < t \leq (\frac{1}{\alpha})^{\frac{1}{\alpha}} \end{cases} \tag{2}$$

and

$$f_2(t) = \begin{cases} \frac{t^\alpha}{\alpha}, & 0 \leq t \leq (\frac{1}{4\alpha})^{\frac{1}{\alpha}} \\ \frac{1}{2\alpha^2} - \frac{t^\alpha}{\alpha}, & (\frac{1}{4\alpha})^{\frac{1}{\alpha}} < t \leq (\frac{3}{4\alpha})^{\frac{1}{\alpha}} \\ \frac{t^\alpha}{\alpha} - \frac{1}{\alpha^2}, & (\frac{3}{4\alpha})^{\frac{1}{\alpha}} < t \leq (\frac{1}{\alpha})^{\frac{1}{\alpha}} \end{cases} \tag{3}$$

We have $f_1(t) = g_1(\frac{t^\alpha}{\alpha})$ and $f_2(t) = g_2(\frac{t^\alpha}{\alpha})$, where

$$g_1(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2\alpha^2} \\ \frac{1}{\alpha^2} - t, & \frac{1}{2\alpha^2} < t \leq \frac{1}{\alpha^2} \end{cases} \tag{4}$$

and

$$g_2(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{4\alpha^2} \\ \frac{1}{2\alpha^2} - t, & \frac{1}{4\alpha^2} < t \leq \frac{3}{4\alpha^2} \\ t - \frac{1}{\alpha^2}, & \frac{3}{4\alpha^2} < t \leq \frac{1}{\alpha^2} \end{cases} \tag{5}$$

for all $t \in [0, \frac{1}{\alpha^2}]$. $g_1(t)$ *and* $g_2(t)$ *are countinuous periodic functions with period* $\frac{1}{\alpha^2}$ *for all* $t \in [0, +\infty[$ *(extended by periodicity to* $[0, +\infty[$*). Then* $f_1(t)$ *and* $f_2(t)$ *are* α*-periodic with period* $p = (\frac{1}{\alpha})^{\frac{1}{\alpha}}$ *for all* $t \in [0, +\infty[$.

Theorem 4. *Let* $0 < \alpha \leq 1$ *and assume that* $f : [0, +\infty[\to \mathbb{R}$ *is a* α*-periodic function with period* p *such that*

$$I_\alpha(f)(p) = 0 \tag{6}$$

then $I_\alpha(f)(t)$ *is a* α*-periodic function with period* p*, for all* $t \in [0, +\infty[$.

Proof. Let $0 < \alpha \leq 1$ and assume that $f : [0, +\infty[\to \mathbb{R}$ is a α-periodic function with period p. By Definition 4 and using variable change $u = \frac{p^\alpha}{\alpha}$, we have for all $t \in [0, +\infty[$

$$I_\alpha(f)(t) = \int_0^t s^{\alpha-1} f(s)ds = \int_0^{\frac{t^\alpha}{\alpha}} f((\alpha u)^{\frac{1}{\alpha}})du =: g_1\Big(\frac{t^\alpha}{\alpha}\Big)$$

with $g_1(t)$ is the continuous function defined by $g_1(t) = \int_0^t f((\alpha u)^{\frac{1}{\alpha}})du$. Then, we have

$$g_1\Big(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha}\Big) = I_\alpha(f)(p) + g_1\Big(\frac{t^\alpha}{\alpha}\Big)$$

using the condition given in Equation 6, we obtain:

$$g_1(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha}) = g_1(\frac{t^\alpha}{\alpha}).$$

Then the function g_1 is a continuous periodic function with period $\frac{p^\alpha}{\alpha}$. Thus $I_\alpha(f)(t)$ is α-periodic with period p for all $t \in [0, +\infty[$.

Example 4. *1. The function* f_2 *defined by Example 3 is* α*-periodic with period* $p = (\frac{1}{\alpha})^{\frac{1}{\alpha}}$ *and we have*

$$I_\alpha(f_2)(t) = \begin{cases} \int_0^{\frac{t^\alpha}{\alpha}} sds, & 0 \leq t \leq (\frac{1}{4\alpha})^{\frac{1}{\alpha}} \\ \int_0^{(\frac{1}{4\alpha})^{\frac{1}{\alpha}}} sds + \int_{(\frac{1}{4\alpha})^{\frac{1}{\alpha}}}^{\frac{t^\alpha}{\alpha}} (\frac{1}{2\alpha^2} - s)ds, & (\frac{1}{4\alpha})^{\frac{1}{\alpha}} < t \leq (\frac{3}{4\alpha})^{\frac{1}{\alpha}} \\ \int_0^{(\frac{1}{4\alpha})^{\frac{1}{\alpha}}} sds + \int_{(\frac{1}{4\alpha})^{\frac{1}{\alpha}}}^{(\frac{3}{4\alpha})^{\frac{1}{\alpha}}} (\frac{1}{2\alpha^2} - s)ds + \int_{(\frac{3}{4\alpha})^{\frac{1}{\alpha}}}^{\frac{t^\alpha}{\alpha}} (s - \frac{1}{\alpha^2}), & (\frac{3}{4\alpha})^{\frac{1}{\alpha}} < t \leq (\frac{1}{\alpha})^{\frac{1}{\alpha}} \end{cases}$$

then

$$I_\alpha(f_2)(t) = \begin{cases} \frac{1}{2}(\frac{t^\alpha}{\alpha})^2, & 0 \leq t \leq (\frac{1}{4\alpha})^{\frac{1}{\alpha}} \\ \frac{-1}{16\alpha^4} + \frac{1}{2\alpha^2}(\frac{t^\alpha}{\alpha}) - \frac{1}{2}(\frac{t^\alpha}{\alpha})^2, & (\frac{1}{4\alpha})^{\frac{1}{\alpha}} < t \leq (\frac{3}{4\alpha})^{\frac{1}{\alpha}} \\ \frac{1}{2}(\frac{t^\alpha}{\alpha})^2 - \frac{1}{\alpha^2}(\frac{t^\alpha}{\alpha}) + \frac{1}{2\alpha^4}, & (\frac{3}{4\alpha})^{\frac{1}{\alpha}} < t \leq (\frac{1}{\alpha})^{\frac{1}{\alpha}} \end{cases} \tag{7}$$

and

$$g_2(t) = I_\alpha(f_2)((\alpha t)^{\frac{1}{\alpha}}) = \begin{cases} \frac{1}{2}t^2, & 0 \le t \le \frac{1}{4\alpha^2} \\ \frac{-1}{16\alpha^4} + \frac{1}{2\alpha^2}t - \frac{1}{2}t^2, & \frac{1}{4\alpha^2} < t \le \frac{3}{4\alpha^2} \\ \frac{1}{2}t^2 - \frac{1}{\alpha^2}t + \frac{1}{2\alpha^4}, & \frac{3}{4\alpha^2} < t \le \frac{1}{\alpha^2} \end{cases} \tag{8}$$

Therefore, we have

$$I_\alpha(f_2)(p) = \frac{1}{2}(\frac{p^\alpha}{\alpha})^2 - \frac{1}{\alpha^2}(\frac{p^\alpha}{\alpha}) + \frac{1}{2\alpha^4} = \frac{1}{2\alpha^4} - \frac{1}{\alpha^4} + \frac{1}{2\alpha^4} = 0.$$

The condition 6 is satisfied, then the function g_2 is continuous periodic with period $\frac{1}{\alpha^2}$, thus $I_\alpha(f_2)$ is α-periodic function with period $(\frac{1}{\alpha})^{\frac{1}{\alpha}}$.

2. The function f_1 defined by Example 3 is α-periodic with period $p = (\frac{1}{\alpha})^{\frac{1}{\alpha}}$, and we have

$$I_\alpha(f_1)(t) = \begin{cases} \int_0^{\frac{t^\alpha}{\alpha}} s ds, \quad 0 \le t \le (\frac{1}{2\alpha})^{\frac{1}{\alpha}} \\ \int_0^{(\frac{1}{2\alpha})^{\frac{1}{\alpha}}} s ds + \int_{(\frac{1}{2\alpha})^{\frac{1}{\alpha}}}^{\frac{t^\alpha}{\alpha}} (\frac{1}{\alpha^2} - s) ds, \ (\frac{1}{2\alpha})^{\frac{1}{\alpha}} < t \le (\frac{1}{\alpha})^{\frac{1}{\alpha}} \end{cases} \tag{9}$$

then

$$I_\alpha(f_1)(t) = \begin{cases} \frac{1}{2}(\frac{t^\alpha}{\alpha})^2, \quad 0 \le t \le (\frac{1}{2\alpha})^{\frac{1}{\alpha}} \\ -\frac{1}{2}(\frac{t^\alpha}{\alpha})^2 + \frac{1}{\alpha^2}\frac{t^\alpha}{\alpha} - \frac{1}{4\alpha^4}, \quad (\frac{1}{2\alpha})^{\frac{1}{\alpha}} < t \le (\frac{1}{\alpha})^{\frac{1}{\alpha}}. \end{cases} \tag{10}$$

and

$$g_1(t) = I_\alpha(f_1)((\alpha t)^{\frac{1}{\alpha}}) = \begin{cases} \frac{1}{2}t^2, \quad 0 \le t \le \frac{1}{2\alpha^2} \\ -\frac{1}{2}t^2 + \frac{1}{\alpha^2}t - \frac{1}{4\alpha^4}, \quad \frac{1}{2\alpha^2} < t \le \frac{1}{\alpha^2}. \end{cases} \tag{11}$$

We have $I_\alpha(f_1)(p) = \frac{1}{4\alpha^4} \neq 0$, then g_1 is not a continuous periodic function with period $\frac{1}{\alpha^2}$, therfore $I_\alpha(f_1)$ is not a α-periodic function with period $(\frac{1}{\alpha})^{\frac{1}{\alpha}}$.

Theorem 5. *Let $0 < \alpha \le 1$ and assume that the function $f : [0, +\infty[\to \mathbb{R}$ is continuously α-differentiable on $[0, +\infty[$, and α-periodic with period p. Then we have*

(i) $T^{(\alpha)}(f)(t) = g'(\frac{t^\alpha}{\alpha})$ and $g \in \mathcal{C}^1([0, +\infty[)$, where $g(t) = f((\alpha t)^{\frac{1}{\alpha}})$,

(ii) $T^{(\alpha)}(f)(t)$ is α-periodic function with period p for all $t \in [0, +\infty[$.

Proof. Let $0 < \alpha \leq 1$ and $f : [0, +\infty[\to \mathbb{R}$ is α-periodic function with period p and continuously α-differentiable on $[0, +\infty[$. Then $f(t)$ is α-differentiable and $T^{(\alpha)}(f)(t)$ is continuous, for all $t \in [0, +\infty[$.

By Definition 6, there exists a continuous function $g : [0, +\infty[\to \mathbb{R}$ such that

$$f(t) = g(\frac{t^\alpha}{\alpha}) = g(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha})$$

Case 1: $t > 0$

(1) By Theorem 1,

$$T^{(\alpha)}(f)(t) = t^{1-\alpha} f'(t) = g'(\frac{t^\alpha}{\alpha}) := g_1\Big(\frac{t^\alpha}{\alpha}\Big) \tag{12}$$

with $g_1(t) = g'(t)$. If $f(t)$ is α-differentiable, then $T^{(\alpha)}(f)(t)$ exists. Therefore $g(t)$ is differentiable and $g'(t) = T^{(\alpha)}(f)((\alpha t)^{\frac{1}{\alpha}})$. On the other hand, if $T^{(\alpha)}(f)(t)$ is continuous, then $g \in \mathcal{C}^1(]0, +\infty[)$.

Case 2: $t = 0$

If $f(t)$ is α-differentiable for all $t \in [0, +\infty[$ especially for $t = 0$, then $T^{(\alpha)}(f)(0) = lim_{t\to 0^+} T^{(\alpha)}(f)(t)$ exists and by continuity of $T^{(\alpha)}(f)(t)$ and $g'(t)$ we have

$$\lim_{t\to 0^+} g'(t) = \lim_{t\to 0^+} T^{(\alpha)}(f)((\alpha t)^{\frac{1}{\alpha}}) = T^{(\alpha)}(f)(0) = g'(0).$$

Finally

$$T^{(\alpha)}(f)(t) = g'(\frac{t^\alpha}{\alpha}) \text{ for all } t \in [0, +\infty[\text{ and } g \in \mathcal{C}^1([0, +\infty[)$$

(2) If f is α-periodic then $g(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha}) = g(\frac{t^\alpha}{\alpha})$ for all $t \in [0, +\infty[$. If $g \in \mathcal{C}^1([0, +\infty[)$, then $g'(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha}) = g'(\frac{t^\alpha}{\alpha})$. Thus $g_1(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha}) = g_1(\frac{t^\alpha}{\alpha})$, for all $t \in [0, +\infty[$. Finally $T^{(\alpha)}(f)(t)$ is α-periodic with period p for all $t \in [0, +\infty[$.

Example 5. *Let $0 < \alpha \leq 1$ and $t \in [0, (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}]$. Let us consider the function*

$$f(t) = \begin{cases} f_1(t) = \sin(\alpha t^\alpha), & 0 \leq t < (\frac{\pi}{\alpha})^{\frac{1}{\alpha}} \\ \\ f_2(t) = -\frac{1}{2}\sin(2\alpha t^\alpha), & (\frac{\pi}{\alpha})^{\frac{1}{\alpha}} \leq t \leq (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}} \end{cases} \tag{13}$$

with

$$g(t) = f((\alpha t)^{\frac{1}{\alpha}}) = \begin{cases} g_1(t) = \sin(\alpha^2 t), & 0 \leq t < \frac{\pi}{\alpha^2} \\ \\ g_2(t) = -\frac{1}{2}\sin(2\alpha^2 t), & \frac{\pi}{\alpha^2} \leq t \leq \frac{3\pi}{2\alpha^2} \end{cases} \tag{14}$$

The function $g(t)$ is continuous periodic with period $\frac{3\pi}{2\alpha^2}$ for all $t \in [0, +\infty[$ (extended by periodicity to $[0, +\infty[$) and $f(t)$ is α-periodic with period $(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$ for all $t \in [0, +\infty[$. Therefore, we have

$$T^{(\alpha)}(f)(t) = \begin{cases} T^{(\alpha)}(f_1)(t) = \alpha^2\cos(\alpha t^\alpha), & 0 < t < (\frac{\pi}{\alpha})^{\frac{1}{\alpha}} \\ \\ T^{(\alpha)}(f_2)(t) = -\alpha^2\cos(2\alpha t^\alpha), & (\frac{\pi}{\alpha})^{\frac{1}{\alpha}} < t < (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}} \end{cases} \tag{15}$$

and

$$g'(t) = T^{(\alpha)}(f)((\alpha t)^{\frac{1}{\alpha}}) = \begin{cases} g_1'(t) = \alpha^2\cos(\alpha^2 t), & 0 < t < \frac{\pi}{\alpha^2} \\ g_2'(t) = -\alpha^2\cos(2\alpha^2 t), & \frac{\pi}{\alpha^2} < t < \frac{3\pi}{2\alpha^2} \end{cases} \tag{16}$$

The function f_1 is continuously α-differentiable on $[0, (\frac{\pi}{\alpha})^{\frac{1}{\alpha}}]$ and $g_1 \in \mathcal{C}^1([0, \frac{\pi}{\alpha^2}])$. The function f_2 is continuously α-differentiable on $[(\frac{\pi}{\alpha})^{\frac{1}{\alpha}}, (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}]$ and $g_2 \in \mathcal{C}^1([\frac{\pi}{\alpha^2}, \frac{3\pi}{2\alpha^2}])$.
On the other hand, we have

$$T^{(\alpha)}(f_1)((\frac{\pi}{\alpha})^{\frac{1}{\alpha}}) = g_1'(\frac{\pi}{\alpha^2}) = T^{(\alpha)}(f_2)((\frac{\pi}{\alpha})^{\frac{1}{\alpha}}) = g_2'(\frac{\pi}{\alpha^2}) = -\alpha^2$$

and

$$T^{(\alpha)}(f_1)(0) = g_1'(0) = T^{(\alpha)}(f_1)((\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}) = g_2'(\frac{3\pi}{2\alpha^2}) = \alpha^2.$$

Then f is continuously α-differentiable on $[0, (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}]$ and $g \in \mathcal{C}^1([0, \frac{3\pi}{2\alpha^2}])$. Therefore f is continuously α-differentiable on $[0, +\infty[$ and $g \in \mathcal{C}^1([0, +\infty[)$. So, we have $g'(t)$ is periodic with period $\frac{3\pi}{2\alpha^2}$ for all $t \in [0, +\infty[$ (extended by periodicity to $[0, +\infty[$) and $T^{\alpha}(f)(t)$ is α-periodic with period $(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$ for all $t \in [0, +\infty[$.

Theorem 6. *Let $0 < \alpha \le 1$. Assume that the function $f : [0, +\infty[\to \mathbb{R}$ is n times continuously α-differentiable on $[0, +\infty[$ for $n \in \mathbb{N}$ and α-periodic with period p. Then for all $j \in \{0, \ldots, n\}$ and for all $t \in [0, +\infty[$, we have*

- *(i) $T^{(j\alpha)}(f)(t) = g^{(j)}(\frac{t^\alpha}{\alpha})$ and $g \in \mathcal{C}^j([0, +\infty[)$ where $g(t) = f((\alpha t)^{\frac{1}{\alpha}})$.*
- *(ii) $T^{(j\alpha)}(f)(t)$ is α-periodic function with period p.*

Note that $T^{(0)}(f)(t) = f(t)$ and $g^{(0)}(t) = g(t)$.

Proof. Let $0 < \alpha \le 1$ and f is n times continuously α-differentiable on $[0, +\infty[$ for $n \in \mathbb{N}$ and α-periodic with period p. Let $j \in \{0, \ldots, n\}$ and by recurrence, we have the following:
For $j = 0$, f is α-periodic, then by Definition 6 there exists a continuous function $g : [0, +\infty[\to \mathbb{R}$ such that $f(t) = g\big(\frac{t^\alpha}{\alpha}\big) = g\big(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha}\big)$. Thus (1) and (2) are satisfied.
For $j = 1$, see Theorem 5. Suppose that for all $j \in \{2, ..., n\}$ and for all $t \in [0, +\infty[$,
(*) $T^{((j-1)\alpha)}(f)(t) = g^{(j-1)}(\frac{t^\alpha}{\alpha})$ and $g \in \mathcal{C}^{j-1}([0, +\infty[)$
(*) $T^{((j-1)\alpha)}(f)(t)$ is α-periodic with period p.

(1) For all $t \in [0, +\infty[$, we have $f(t)$ is j times continuously α-differentiable, then $T^{(j\alpha)}(f)(t)$ exists and continuous.
Case 1: $t > 0$
By hypothesis $T^{((j-1)\alpha)}(f)(t)$ is α-periodic and $g \in \mathcal{C}^{j-1}([0, +\infty[)$, then

$$T^{(j\alpha)}(f)(t) := T^{(\alpha)}(T^{((j-1)\alpha)}(f))(t) = T^{(\alpha)}(g^{(j-1)})(\frac{t^\alpha}{\alpha})$$

$$= t^{1-\alpha}(g^{(j-1)}(\frac{t^\alpha}{\alpha}))' = g^{(j)}(\frac{t^\alpha}{\alpha})$$

and the function $g^{(j)}(t) = T^{(j\alpha)}(f)((\alpha t)^{\frac{1}{\alpha}})$ exists and continuous for all $t \in]0, +\infty[$. Thus

$$g \in \mathcal{C}^j(]0, +\infty[)$$

Case 2: $t = 0$

We have $T^{(j\alpha)}(f)(0) = \lim_{t\to 0^+} T^{(j\alpha)}(f)(t)$ exists. The functions $T^{(j\alpha)}(f)(t)$ and $g^{(j)}(t)$ are continuous, then

$$\lim_{t\to 0^+} g^{(j)}(t) = \lim_{t\to 0^+} T^{(j\alpha)}(f)(t) = T^{(j\alpha)}(f)(0) = g^{(j)}(0).$$

Finally, $T^{(j\alpha)}(f)(t) = g^{(j)}(\frac{t^\alpha}{\alpha})$ for all $t \in [0, +\infty[$ and $g \in \mathcal{C}^j([0, +\infty[)$.

(2) We have $g^{(j-1)}(\frac{t^\alpha}{\alpha}) = g^{(j-1)}(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha})$ for all $t \in [0, +\infty[$ and $g \in \mathcal{C}^j([0, +\infty[)$, then $g^{(j)}(\frac{t^\alpha}{\alpha}) = g^{(j)}(\frac{t^\alpha}{\alpha} + \frac{p^\alpha}{\alpha})$. Thus $T^{(j\alpha)}(f)(t)$ is α-periodic with period p for all $t \in [0, +\infty[$.

Example 6. *Let us consider the Example 5. The function f is α-periodic with period $(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$ and g is continuous periodic with period $\frac{3\pi}{2\alpha^2}$. Then, we have for $n \in \mathbb{N}$ and $t \in [0, (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}]$*

$$T^{(n\alpha)}(f)(t) = \begin{cases} T^{(n\alpha)}(f_1)(t) = \alpha^{2n}\sin(\alpha t^\alpha + n\frac{\pi}{2}), & 0 < t < (\frac{\pi}{\alpha})^{\frac{1}{\alpha}} \\ \\ T^{(n\alpha)}(f_2)(t) = -2^{n-1}\alpha^{2n}\sin(2\alpha t^\alpha + \frac{n\pi}{2}), & (\frac{\pi}{\alpha})^{\frac{1}{\alpha}} < t < (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}} \end{cases}$$

and for $t \in [0, \frac{3\pi}{2\alpha^2}]$

$$g^{(n)}(t) = \begin{cases} g_1^{(n)}(t) = \alpha^{2n}\sin(\alpha^2 t + n\frac{\pi}{2}), & 0 < t < \frac{\pi}{\alpha^2} \\ \\ g_2^{(n)}(t) = -2^{n-1}\alpha^{2n}\sin(2\alpha^2 t + \frac{n\pi}{2}), & \frac{\pi}{\alpha^2} < t < \frac{3\pi}{2\alpha^2} \end{cases}$$

The function f_1 is n times continuously α-differentiable on $[0, (\frac{\pi}{\alpha})^{\frac{1}{\alpha}}]$ and $g_1 \in \mathcal{C}^n([0, \frac{\pi}{\alpha^2}])$. The function f_2 is n times continuously α-differentiable on $[(\frac{\pi}{\alpha})^{\frac{1}{\alpha}}, (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}]$ and $g_2 \in \mathcal{C}^n([\frac{\pi}{\alpha^2}, \frac{3\pi}{2\alpha^2}])$. To study the continuity of $T^{(n\alpha)}(f)$ and of $g^{(n)}$ on $[0, +\infty[$, we put

$$\Delta_n^\alpha = T^{(n\alpha)}(f_1)(0) - T^{(n\alpha)}(f_2)((\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}) = g_1^{(n)}(0) - g_2^{(n)}(\frac{3\pi}{2\alpha^2})$$

and

$$\delta_n^\alpha = T^{(n\alpha)}(f_1)((\frac{\pi}{\alpha})^{\frac{1}{\alpha}}) - T^{(n\alpha)}(f_2)((\frac{\pi}{\alpha})^{\frac{1}{\alpha}}) = g_1^{(n)}(\frac{\pi}{\alpha^2}) - g_2^{(n)}(\frac{\pi}{\alpha^2}).$$

Now, we have

$$\Delta_n^\alpha = \alpha^{2n}(1 - 2^{n-1})\sin(n\frac{\pi}{2})$$

and

$$\delta_n^\alpha = -\alpha^{2n}(1-2^{n-1})\sin(n\frac{\pi}{2}).$$

The continuity conditions of $T^{(n\alpha)}(f)$ *on* $[0,(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}]$ *of* $g^{(n)}$ *on* $[0,\frac{3\pi}{2\alpha^2}]$ *and their extention by periodicity to* $[0,+\infty[$ *are* $\Delta_n^\alpha=\delta_n^\alpha=0$.
Therefore under this condition, f *is* n *times continuously* α*-differentiable on* $[0,+\infty[$ *and* $g\in\mathcal{C}^n([0,+\infty[)$*. On the other hand*

$$\Delta_n^\alpha=\delta_n^\alpha=0 \Leftrightarrow n\in\{0,1,2\}$$

Then the function f *is twice continuously* α*-differentiable on* $[0,+\infty[$ *and we have for all* $j\in\{0,1,2\}$ *and* $t\in[0,+\infty[$, $T^{(j\alpha)}f(t)=g^{(j)}(\frac{t^\alpha}{\alpha})$, $g\in\mathcal{C}^{(j)}([0,+\infty[)$ *and* $T^{(j\alpha)}f(t)$ *is* α*-periodic function with period* $(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$.

We conclude this section with the following theorem.

Theorem 7. *Let* $0<\alpha\le 1$*. Assume that* $f\in L^1(\mathbb{R}^+,\mathbb{R})$ *is* α*-periodic function with period* p *and* $a(t)=a_1(\frac{t^\alpha}{\alpha})$ *with* $a_1\in L^1(\mathbb{R}^+)$*. The function* $(a*_\alpha f)_{-\infty}(t)$ *is defined by*

$$(a*_\alpha f)_{-\infty}(t)=\int_{-\infty}^{\frac{t^\alpha}{\alpha}} a((t^\alpha-\alpha s)^{\frac{1}{\alpha}})f((\alpha s)^{\frac{1}{\alpha}})ds,\ t\in[0,+\infty[$$

is α*-periodic with period* p.

Proof. Let $0<\alpha\le 1$ and $f\in L^1(\mathbb{R}^+,\mathbb{R})$ is α-periodic function with period p and $a(t)=a_1(\frac{t^\alpha}{\alpha})$ with $a_1\in L^1(\mathbb{R}^+)$. For all $t\in\mathbb{R}^+$, we have

$$\begin{aligned}(a*_\alpha f)_{-\infty}(t)&=\int_{-\infty}^{\frac{t^\alpha}{\alpha}} a((t^\alpha-\alpha s)^{\frac{1}{\alpha}})f((\alpha s)^{\frac{1}{\alpha}})ds\\ &=\int_{-\infty}^{\frac{t^\alpha}{\alpha}} a_1\Big(\frac{t^\alpha}{\alpha}-s\Big)g(s)ds=F\Big(\frac{t^\alpha}{\alpha}\Big)\end{aligned}$$

where F is the continuous function given by Theorem 3

$$F(t)=\int_{-\infty}^{t} a_1(t-s)g(s)ds.$$

On the other hand

$$F\Big(\frac{t^\alpha}{\alpha}+\frac{p^\alpha}{\alpha}\Big)=\int_{-\infty}^{\frac{t^\alpha}{\alpha}+\frac{p^\alpha}{\alpha}} a_1(\frac{p^\alpha}{\alpha}+\frac{t^\alpha}{\alpha}-s)g(s)ds.$$

By making a change of variable $u=s-\frac{p^\alpha}{\alpha}$, we obtain

$$F\Big(\frac{t^\alpha}{\alpha}+\frac{p^\alpha}{\alpha}\Big)=\int_{-\infty}^{\frac{t^\alpha}{\alpha}} a_1\Big(\frac{t^\alpha}{\alpha}-u\Big)g\Big(u+\frac{p^\alpha}{\alpha}\Big)ds$$

the function g is continuous periodic with period $\frac{p^\alpha}{\alpha}$, then

$$F\Big(\frac{t^\alpha}{\alpha}+\frac{p^\alpha}{\alpha}\Big)=\int_{-\infty}^{\frac{t^\alpha}{\alpha}}a_1\Big(\frac{t^\alpha}{\alpha}-u\Big)g(u)ds=F\Big(\frac{t^\alpha}{\alpha}\Big)$$

and $(a *_\alpha f)_{-\infty}(t)$ is α-periodic function with period p for all $t\in[0,+\infty[$.

Example 7. *Let f_1 defined in Example 3 and $a(t)=a_1(\frac{t^\alpha}{\alpha})$ such that $a_1(t)=e^{-t}\in L^1(\mathbb{R}^+)$. The function f_1 is α-periodic function with period $(\frac{1}{\alpha})^{\frac{1}{\alpha}}$ and we have for all $t\in[0,+\infty[$*

$$\begin{aligned}(a *_\alpha f)_{-\infty}(t)&=\int_{-\infty}^{\frac{t^\alpha}{\alpha}}a_1\Big(\frac{t^\alpha}{\alpha}-s\Big)f((\alpha s)^{\frac{1}{\alpha}})ds\\&=\int_{-\infty}^{\frac{t^\alpha}{\alpha}}e^{-\frac{t^\alpha}{\alpha}+s}g(s)ds=F\Big(\frac{t^\alpha}{\alpha}\Big).\end{aligned}$$

where $F(t)=e^{-t}\int_{-\infty}^{t}e^s g(s))ds$ is a continuous function. We have

$$\begin{aligned}F\Big(\frac{t^\alpha}{\alpha}+\frac{1}{\alpha^2}\Big)&=\int_{-\infty}^{\frac{t^\alpha}{\alpha}+\frac{1}{\alpha^2}}e^{-\frac{t^\alpha}{\alpha}+s-\frac{1}{\alpha^2}}g(s)ds\\&=\int_{-\infty}^{\frac{t^\alpha}{\alpha}}e^{-\frac{t^\alpha}{\alpha}+s}g\Big(s+\frac{1}{\alpha^2}\Big)ds=F\Big(\frac{t^\alpha}{\alpha}\Big)\end{aligned}$$

Then $(a *_\alpha f)_{-\infty}(t)$ is α-periodic function with period $(\frac{1}{\alpha})^{\frac{1}{\alpha}}$.

In the next section, we present some results of conformable fourier transforms.

4. Result of conformable fourier transform

For investigating the property of the classical fourier transform, the following new definition of the conformable fourier transform for α-periodic function is introduced.

Definition 7. *(Conformable fourier Transform)*
Assume that $f:[0,+\infty[\to\mathbb{R}$ is α-periodic function with period p and $0<\alpha\leq 1$. The k-th conformable Fourier coefficient of f denoted by $\mathcal{F}_\alpha(f(t))(k)$ is defined by

$$\mathcal{F}_\alpha(f(t))(k)=\frac{\alpha}{p^\alpha}\int_0^p e^{-ik\frac{2\pi}{p^\alpha}t^\alpha}f(t)t^{\alpha-1}dt,\ \forall k\in\mathbb{Z}$$

Remark 2. *: For $k=0$, $\mathcal{F}_\alpha(f(t))(0)=\frac{\alpha}{p^\alpha}I_\alpha(f)(p)$*

The next theorem gives a relationship between fourier conformable transform and classical fourier transform applied to α-periodic functions.

Theorem 8. *Assume that* $f : [0, +\infty[\to \mathbb{R}$ *is* α*-periodic function with period* p *and* $0 < \alpha \leq 1$*. Then for all* $k \in \mathbb{Z}$*,*

$$\mathcal{F}_\alpha\{f(t)\}(k) = \mathcal{F}\{f((\alpha t)^{\frac{1}{\alpha}})\}(k)$$

Proof. Let $0 < \alpha \leq 1$ and $f : [0, +\infty[\to \mathbb{R}$ is α-periodic function with period p, then by Remark 1, $f((\alpha t)^{\frac{1}{\alpha}})$ is periodic with period $\frac{p^p}{\alpha}$ and for all $k \in \mathbb{Z}$,

$$\mathcal{F}_\alpha\{f(t)\}(k) = \frac{\alpha}{p^\alpha}\int_0^p e^{-ik\frac{2\pi}{p^\alpha}t^\alpha} f(t)t^{\alpha-1}dt.$$

By variable change $\frac{t^\alpha}{\alpha}$, we obtain

$$\mathcal{F}_\alpha\{f(t)\}(k) = \frac{\alpha}{p^\alpha}\int_0^{\frac{p^\alpha}{\alpha}} e^{-ik\frac{2\pi\alpha}{p^\alpha}t} f((\alpha t)^{\frac{1}{\alpha}})dt,$$

the function $t \in [0, +\infty[\to e^{-ik\frac{2\pi\alpha}{p^\alpha}t} f((\alpha t)^{\frac{1}{\alpha}})$ is periodic with period $\frac{p^\alpha}{\alpha}$, then

$$\mathcal{F}_\alpha\{f(t)\}(k) = \mathcal{F}\{f((\alpha t)^{\frac{1}{\alpha}})\}(k).$$

Example 8. *The functions* f_1 *and* f_2 *defined in Example 3 are* α*-periodic with period* $p = (\frac{1}{\alpha})^{\frac{1}{\alpha}}$*. For* $k \neq 0$*,*

$$\begin{aligned}
\mathcal{F}_\alpha(f_1(t))(k) = \mathcal{F}(f_1((\alpha t)^{\frac{1}{\alpha}}))(k) &= \alpha^2\int_0^{\frac{1}{\alpha^2}} e^{-2ik\pi\alpha^2 t} f_1((\alpha t)^{\frac{1}{\alpha}})dt \\
&= \alpha^2[\int_0^{\frac{1}{2\alpha^2}} te^{-2ik\pi\alpha^2 t}dt + \int_{\frac{1}{2\alpha^2}}^{\frac{1}{\alpha^2}} (\frac{1}{\alpha^2} - t)e^{-2ik\pi\alpha^2 t}dt] \\
&= \frac{(-1)^k - 1}{2\pi^2 k^2\alpha^2}.
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_\alpha(f_2(t))(k) = \mathcal{F}(f_2((\alpha t)^{\frac{1}{\alpha}}))(k) &= \alpha^2\int_0^{\frac{1}{\alpha^2}} e^{-2ik\pi\alpha^2 t} f_2((\alpha t)^{\frac{1}{\alpha}})dt \\
&= \alpha^2[\int_0^{\frac{1}{4\alpha^2}} te^{-2ik\pi\alpha^2 t}dt + \int_{\frac{1}{4\alpha^2}}^{\frac{3}{4\alpha^2}} (\frac{1}{\alpha^2} - t)e^{-2ik\pi\alpha^2 t}dt \\
&\quad + \int_{\frac{3}{4\alpha^2}}^{\frac{1}{\alpha^2}} (t - \frac{1}{\alpha^2})e^{-2ik\pi\alpha^2 t}dt] \\
&= \frac{(-1)^{-\frac{k}{2}}(1 - (-1)^{-k})}{2\alpha^2 k^2\pi^2}.
\end{aligned}$$

For $k=0$, we have

$$\mathcal{F}_\alpha(f_1(t))(0)=\alpha^2\int_0^{\frac{1}{\alpha^2}} f_1((\alpha t)^{\frac{1}{\alpha}}))dt=\frac{1}{4\alpha^2}$$

and

$$\mathcal{F}_\alpha(f_2(t))(0)=\alpha^2\int_0^{\frac{1}{\alpha^2}} f_2((\alpha t)^{\frac{1}{\alpha}}))dt=0.$$

Then

$$\mathcal{F}_\alpha(f_1(t))(k)=\begin{cases}\frac{(-1)^k-1}{2\pi^2k^2\alpha^2},\ \forall k\in\mathbb{Z}^*\\ \\ \frac{1}{4\alpha^2},\ k=0.\end{cases}\tag{17}$$

and

$$\mathcal{F}_\alpha(f_2(t))(k)=\begin{cases}\frac{(-1)^{-\frac{k}{2}}(1-(-1)^{-k})}{2\alpha^2k^2\pi^2},\ \forall k\in\mathbb{Z}^*\\ \\ 0, \qquad\qquad k=0.\end{cases}\tag{18}$$

As a classical fourier transform, we apply the conformable fourier transform to the conformable fractional integral given by Definition 4. The following theorem is obtained.

Theorem 9. *Assume that $f:[0,+\infty[\to\mathbb{R}$ is α-periodic function with period p such that $I_\alpha(f)(p)=0$ and $0<\alpha\leq 1$. Then for all $t\in[0,+\infty[, I_\alpha(f)(t)$ is α-periodic with period p and*

$$\mathcal{F}_\alpha(I_\alpha(f)(t))(k)=\begin{cases}\frac{p^\alpha}{2ik\pi\alpha}\mathcal{F}_\alpha(f(t))(k),\ \forall k\in\mathbb{Z}^*\\ \\ \mathcal{F}(f_\alpha((\alpha t)^{\frac{1}{\alpha}}))(0),\ k=0.\end{cases}$$

where $f_\alpha(t)=-\frac{t^\alpha}{\alpha}f(t)$.

Proof. Let $0<\alpha\leq 1$. f is α-periodic function with period p such that $I_\alpha(f(p))=0$, then by Theorem 4, for all $t\in[0,+\infty[, I_\alpha(f)(t)$ is α-periodic with period p.
For $k\neq 0$,

$$\begin{aligned}\mathcal{F}_\alpha(I_\alpha(f)(t))(k)&=\mathcal{F}(I_\alpha(f)((\alpha t)^{\frac{1}{\alpha}}))(k)\\ &=\frac{\alpha}{p^\alpha}\int_0^{\frac{p^\alpha}{\alpha}} e^{-2ik\pi\frac{\alpha}{p^\alpha}t}I_\alpha(f)((\alpha t)^{\frac{1}{\alpha}})dt\end{aligned}$$

By Definition 4, we have

$$I_\alpha(f)(t)=\int_0^t s^{\alpha-1}f(s)ds.$$

Using variable change $\frac{t^\alpha}{\alpha}$, we obtain

$$I_\alpha(f)(t) = \int_0^{\frac{t^\alpha}{\alpha}} f((\alpha s)^{\frac{1}{\alpha}})ds$$

then

$$I_\alpha(f)((\alpha t)^{\frac{1}{\alpha}}) = \int_0^t f((\alpha s)^{\frac{1}{\alpha}})ds.$$

By integrating by parts, we find that

$$\mathcal{F}_\alpha(I_\alpha(f)(t))(k) = -\frac{1}{2ik\pi}[I_\alpha(f)(p) - \int_0^{\frac{p^\alpha}{\alpha}} e^{-2ik\pi\frac{\alpha}{p^\alpha}t} f((\alpha t)^{\frac{1}{\alpha}})dt]$$

and using the condition $I_\alpha(f)(p) = 0$, the result is obtained.
For $k = 0$, using Remark 2, we have

$$\mathcal{F}_\alpha(I_\alpha(f(t)))(0) = \frac{\alpha}{p^\alpha} I_\alpha(I_\alpha(f))(p) = \frac{\alpha}{p^\alpha}\int_0^{\frac{p^\alpha}{\alpha}} I_\alpha(f)((\alpha t)^{\frac{1}{\alpha}})dt.$$

Using integration by parts and the condition $I_\alpha(f)(p) = 0$, we find that

$$\mathcal{F}_\alpha(I_\alpha(f)(t))(0) = \frac{\alpha}{p^\alpha}\int_0^{\frac{p^\alpha}{\alpha}} -tf((\alpha t)^{\frac{1}{\alpha}})dt = \mathcal{F}(f_\alpha((\alpha t)^{\frac{1}{\alpha}}))(0)$$

where $f_\alpha(t) = -\frac{t^\alpha}{\alpha}f(t)$.

Example 9. *Consider the function f_2 defined by Example 3. We have showed in Example 4 that $I_\alpha(f_2)(\frac{1}{\alpha^2}) = 0$ and $I_\alpha(f_2)$ is α-periodic with period $\frac{1}{\alpha^2}$. For $k \in \mathbb{Z}^*$,*

$$\mathcal{F}_\alpha(I_\alpha(f_2)(t))(k) = \mathcal{F}(I_\alpha(f_2)((\alpha t)^{\frac{1}{\alpha}})(k) = \alpha^2\int_0^{\frac{1}{\alpha^2}} e^{-2ik\pi\alpha^2 t} I_\alpha(f_2)((\alpha t)^{\frac{1}{\alpha}})dt.$$

Using integration by parts and the condition $I_\alpha(f_2)(\frac{1}{\alpha^2}) = 0$, we have

$$\begin{aligned}
\mathcal{F}_\alpha(I_\alpha(f_2)(t))(k) &= \frac{1}{2ik\pi}\int_0^{\frac{1}{\alpha^2}} e^{-2ik\pi\alpha^2 t} f_2((\alpha t)^{\frac{1}{\alpha}})dt \\
&= \frac{1}{2ik\pi}[\int_0^{\frac{1}{4\alpha^2}} te^{-2ik\pi\alpha^2 t}dt + \int_{\frac{1}{4\alpha^2}}^{\frac{3}{4\alpha^2}} (\frac{1}{2\alpha^2} - t)e^{-2ik\pi\alpha^2 t}dt] \\
&+ \frac{1}{2ik\pi}\int_{\frac{3}{4\alpha^2}}^{\frac{1}{\alpha^2}} (t - \frac{1}{\alpha^2})e^{-2ik\pi\alpha^2 t}dt \\
&= \frac{(-1)^{-\frac{k}{2}}(1 - (-1)^{-k})}{4i\alpha^4 k^3\pi^3}
\end{aligned}$$

On the other hand, by Example 8, we have

$$\mathcal{F}(f_2((\alpha t)^{\frac{1}{\alpha}}))(k) = \frac{(-1)^{-\frac{k}{2}}(1-(-1)^{-k})}{2\alpha^2 k^2 \pi^2},$$

Then

$$\mathcal{F}_\alpha(I_\alpha(f_2)(t))(k) = \frac{1}{2i\alpha^2 k\pi}\mathcal{F}(f_2((\alpha t)^{\frac{1}{\alpha}}))(k).$$

For $k=0$, we have

$$\begin{aligned}\mathcal{F}((f_2)_\alpha((\alpha t)^{\frac{1}{\alpha}}))(0) &= -\alpha^2\int_0^{\frac{1}{\alpha^2}} tf_2((\alpha t)^{\frac{1}{\alpha}})dt\\ &= -\alpha^2[\int_0^{\frac{1}{4\alpha^2}} t^2 dt + \int_{\frac{1}{4\alpha^2}}^{\frac{3}{4\alpha^2}} t(\frac{1}{2\alpha^2}-t)dt + \int_{\frac{3}{4\alpha^2}}^{\frac{1}{\alpha^2}} t(t-\frac{1}{\alpha^2})dt]\\ &= \frac{1}{32\alpha^4}.\end{aligned}$$

On the other hand, by Example 4, we have

$$\begin{aligned}\mathcal{F}_\alpha(I_\alpha(f_2)(t))(0) &= \alpha^2\int_0^{\frac{1}{\alpha^2}} I_\alpha(f_2)((\alpha t)^{\frac{1}{\alpha}})dt\\ &= \alpha^2\left\{\int_0^{\frac{1}{4\alpha^2}}\frac{t^2}{2} + \int_{\frac{1}{4\alpha^2}}^{\frac{3}{4\alpha^2}}(-\frac{1}{16\alpha^4}+\frac{1}{2\alpha^2}-\frac{t^2}{2})dt\right.\\ &\quad\left. + \int_{\frac{3}{4\alpha^2}}^{\frac{1}{\alpha^2}}(\frac{t^2}{2}-\frac{t}{\alpha^2}+\frac{1}{2\alpha^4})dt\right\}\\ &= \frac{1}{32\alpha^4}.\end{aligned}$$

Then $\mathcal{F}_\alpha(I_\alpha(f_2)(t))(0) = \mathcal{F}(f_\alpha((\alpha t)^{\frac{1}{\alpha}}))(0)$ where $f_\alpha(t) = -\frac{t^\alpha}{\alpha}f(t)$.

In order to establish a similar relationship between conformable fourier transform and conformable fractional derivative as a classical fourier transform of order α, the following two theorems are obtained.

Theorem 10. *Let $0<\alpha\leq 1$, and assume that $f:[0,+\infty[\to\mathbb{R}$ is α-periodic function with period p and continuously α-differentiable on $[0,+\infty[$. Then $T^{(\alpha)}(f)$ is α-periodic function with period p and for all $k\in\mathbb{Z}$:*

$$\mathcal{F}_\alpha(T^{(\alpha)}(f)(t))(k) = (2ik\pi\frac{\alpha}{p^\alpha})\mathcal{F}_\alpha(f(t))(k)$$

Proof. Let $0<\alpha\leq 1$, f is α-periodic function with period p and continuously α-differentiable on $[0,+\infty[$. By Theorem 5, $T^{(\alpha)}(f)(t) = g'(\frac{t^\alpha}{\alpha})$, $g\in\mathcal{C}^1([0,+\infty[)$ where $g(t) = f((\alpha t)^{\frac{1}{\alpha}})$ and $T^{(\alpha)}(f)$ is α-periodic function with period p. For $k\in\mathbb{Z}$, we have

$$\mathcal{F}_\alpha(T^{(\alpha)}(f)(t))(k) = \mathcal{F}(T^{(\alpha)}(f)((\alpha t)^{\frac{1}{\alpha}}))(k)$$

$$= \frac{\alpha}{p^\alpha}\int_0^{\frac{p^\alpha}{\alpha}} e^{-2ik\pi\frac{\alpha}{p^\alpha}t} T^{(\alpha)}(f)((\alpha t)^{\frac{1}{\alpha}})dt$$
$$= \frac{\alpha}{p^\alpha}\int_0^{\frac{p^\alpha}{\alpha}} e^{-2ik\pi\frac{\alpha}{p^\alpha}t} g'(t)dt$$

Using integration by parts the periodicity of g, we have

$$\mathcal{F}_\alpha(T^{(\alpha)}(f)(t)(k) = (2ik\pi\frac{\alpha}{p^\alpha})\mathcal{F}_\alpha(f(t))(k).$$

Example 10. *Consider the same function from Example 5, then f is α-periodic function with period $(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$ and continuously α-differentiable on $[0,+\infty[$. By Theorem 5, $T^{(\alpha)}(f)(t) = g'(\frac{t^\alpha}{\alpha})$, $g \in \mathcal{C}^1([0,+\infty[)$ where $g(t) = f((\alpha t)^{\frac{1}{\alpha}})$ and $T^{(\alpha)}(f)$ is α-periodic function with period $(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$. For all $k \in \mathbb{Z}$, we have*

$$\begin{aligned}
\mathcal{F}_\alpha(f(t))(k) &= \mathcal{F}(f((\alpha t)^{\frac{1}{\alpha}}))(k)\\
&= \frac{2\alpha^2}{3\pi}\int_0^{\frac{3\pi}{2\alpha^2}} e^{-\frac{4}{3}ik\alpha^2 t} f((\alpha t)^{\frac{1}{\alpha}})dt\\
&= \frac{2\alpha^2}{3\pi}[\int_0^{\frac{\pi}{\alpha^2}} \sin(\alpha^2 t)e^{-\frac{4}{3}ik\alpha^2 t}dt - \frac{1}{2}\int_{\frac{\pi}{\alpha^2}}^{\frac{3\pi}{2\alpha^2}} \sin(2\alpha^2 t)e^{-\frac{4}{3}ik\alpha^2 t}dt]\\
&= \frac{81((-1)^{-\frac{4k}{3}}+1)}{2\pi(64k^4 - 180k^2 + 81)}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathcal{F}_\alpha(T^{(\alpha)}(f)(t))(k) &= \mathcal{F}(T^{(\alpha)}(f)((\alpha t)^{\frac{1}{\alpha}})(k)\\
&= \frac{2\alpha^2}{3\pi}\int_0^{\frac{3\pi}{2\alpha^2}} e^{-\frac{4}{3}ik\alpha^2 t} T^{(\alpha)}(f)((\alpha t)^{\frac{1}{\alpha}})dt\\
&= \frac{2\alpha^2}{3\pi}[\int_0^{\frac{\pi}{\alpha^2}} \alpha^2\cos(\alpha^2 t)e^{-\frac{4}{3}ik\alpha^2 t}dt - \int_{\frac{\pi}{\alpha^2}}^{\frac{3\pi}{2\alpha^2}} \alpha^2\cos(2\alpha^2 t)e^{-\frac{4}{3}ik\alpha^2 t}dt]\\
&= \frac{54ik\alpha^2((-1)^{-\frac{4k}{3}}+1)}{\pi(64k^4 - 180k^2 + 81)}
\end{aligned}$$

then

$$\mathcal{F}_\alpha(T^{(\alpha)}(f)(t))(k) = (\frac{4}{3}ik\alpha^2)\mathcal{F}_\alpha(f(t))(k).$$

Theorem 11. *Let $0 < \alpha \leq 1, n \in \mathbb{N}$ and assume that the function $f : [0,+\infty[\to \mathbb{R}$ is α-periodic with period p and n times continuously α-differentiable on $[0,+\infty[$. Then for all $j \in \{0,\dots,n\}$, $T^{(j\alpha)}(f)$ is α-periodic with period p and for $k \in \mathbb{Z}$*

$$\mathcal{F}_\alpha(T^{(j\alpha)}(f)(t))(k) = (2ik\pi\frac{\alpha}{p^\alpha})^j\mathcal{F}_\alpha(f(t))(k).$$

Note that $T^{(0)}f(t) = f(t)$.

Proof. Let $0 < \alpha \leq 1, n \in \mathbb{N}$ and assume that the function f is α-periodic with period p and n times continuously α-differentiable on $[0, +\infty[$. Then by theorem 6, we have for all $j \in \{0, \dots, n\}$, $T^{(j\alpha)}(f)(t) = g^{(j)}(\frac{t^\alpha}{\alpha})$, $g \in \mathcal{C}^j([0, +\infty[)$ where $g(t) = f((\alpha t)^{\frac{1}{\alpha}})$, and $T^{(j\alpha)}(f)$ is α-periodic function with period p.
Let $j \in \{0, \dots, n\}$, by recurence.
For $j = 0$, the property is true.
For $j = 1$, the property is true (see Theorem 10).
Suppose that

$$\mathcal{F}_\alpha(T^{((j-1)\alpha)}(f)(t))(k) = (2ik\pi\frac{\alpha}{p^\alpha})^{j-1}\mathcal{F}(f((\alpha t)^{\frac{1}{\alpha}}))(k)$$

and we show that

$$\mathcal{F}_\alpha(T^{(j\alpha)}(f)(t))(k) = (2ik\pi\frac{\alpha}{p^\alpha})^{j}\mathcal{F}(f((\alpha t)^{\frac{1}{\alpha}}))(k).$$

The function f is n times continuously α-differentiable on $[0, +\infty[$ implies that $T^{((j-1)\alpha)}(f)$ is continuously α-differentiable. Moreover, by Theorem 10 and the recurrence hypothesis

$$\begin{aligned}\mathcal{F}_\alpha(T^{(j\alpha)}(f)(t))(k) &= \mathcal{F}_\alpha(T^{\alpha}(T^{((j-1)\alpha)}(f)(t)))(k)\\ &= (2ik\pi\frac{\alpha}{p^\alpha})\mathcal{F}_\alpha(T^{((j-1)\alpha)}(f)(t))(k)\\ &= (2ik\pi\frac{\alpha}{p^\alpha})^{j}\mathcal{F}_\alpha(f(t))(k).\end{aligned}$$

Then the property is true for all $j \in \{0, \dots, n\}$.

Example 11. *Consider the same function from Example 6, we have f is α-periodic function with period $(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$. We showed that f is twice continuously α-differentiable on $[0, +\infty[$, $g \in \mathcal{C}^2([0, +\infty[)$ where $g(t) = f((\alpha t)^{\frac{1}{\alpha}})$ and $T^{(j\alpha)}(f)$ is α-periodic function with period $(\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$ for $j \in \{0, 1, 2\}$.*
For k in $\mathbb{Z}$, we have
1. For $j = 0$, the property is true.
2. For $j = 1$, the property is true by Example 10.
3. For $j = 2$, the function $T^{(\alpha)}(f)$ is α-differentiable and α-periodic with period $p = (\frac{3\pi}{2\alpha})^{\frac{1}{\alpha}}$.
Then, we have

$$\mathcal{F}_\alpha(T^{(2\alpha)}(f)(t))(k) = \frac{-72\alpha^4k^2[(-1)^{-\frac{4k}{3}} + 1]}{\pi(64k^4 - 180k^2 + 81)}$$

and by Example 10, we have

$$\mathcal{F}_\alpha(f(t))(k) = \frac{81((-1)^{-\frac{4k}{3}} + 1)}{2\pi(64k^4 - 180k^2 + 81)}.$$

Then

$$\mathcal{F}_\alpha(T^{(2\alpha)}(f)(t))(k) = (\frac{4}{3}ik\alpha^2)^2\mathcal{F}_\alpha(f(t))(k).$$

Corollary 1. *If f is α-periodic function with period $(2\pi\alpha)^{\frac{1}{\alpha}}$, then we obtain the following classical fourier property*

$$\mathcal{F}(f^{(n)}((\alpha t)^{\frac{1}{\alpha}}))(k) = (ik)^n \mathcal{F}(f((\alpha t)^{\frac{1}{\alpha}}))(k).$$

We conclude this section with a result which has been used by several authors to solve certain integro-differential equations.

Lemma 3. *Assume that $g \in \L^1(\mathbb{R}^+, \mathbb{R})$ is a continuous periodic function with period T and $a_1 \in L^1(\mathbb{R}^+)$. Then for $t \in [0, +\infty[$*

$$\int_{-\infty}^{0} a_1(t-s)g(s)ds = \sum_{N=1}^{+\infty} \int_{0}^{T} a_1(t-u+NT)g(u)du. \tag{19}$$

Proof. Let $g \in \L^1(\mathbb{R}^+, \mathbb{R})$ is a continuous periodic function with period T. We have

$$\begin{aligned}
\int_{-\infty}^{0} a_1(t-s)g(s)ds &= \sum_{N=1}^{+\infty} \int_{-NT}^{-(N-1)T} a_1(t-s)g(s)ds \\
&= \sum_{N=1}^{+\infty} \int_{-NT}^{-(N-1)T} a_1(t-s)g(s+NT)ds \\
&= \sum_{N=1}^{+\infty} \int_{0}^{T} a_1(t-u+NT)g(u)du.
\end{aligned}$$

Example 12. *Let f_1 defined by Example 3 and $a_1(t) = e^{-t} \in L^1(\mathbb{R}^+)$. The function f_1 is α-periodic with period $(\frac{1}{\alpha})^{\frac{1}{\alpha}}$ for all $t \in [0, +\infty[$, and the associated function g satisfies*

$$g_1(t) = f_1((\alpha t)^{\frac{1}{\alpha}}) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2\alpha^2} \\ \frac{1}{\alpha^2} - t, & \frac{1}{2\alpha^2} < t \leq \frac{1}{\alpha^2} \end{cases}$$

is periodic with period $\frac{1}{\alpha^2}$ and continuous for all $t \in [0, +\infty[$. Then, we have

$$\begin{aligned}
\int_{-\infty}^{0} a_1(t-s)g_1(s)ds &= \sum_{n=1}^{+\infty} \int_{0}^{\frac{1}{\alpha^2}} e^{-(t-u-\frac{n}{\alpha^2})} g_1(u)du \\
&= e^{-t} (\sum_{n=1}^{+\infty} e^{-\frac{n}{\alpha^2}}) \int_{0}^{\frac{1}{\alpha^2}} e^u g_1(u)du \\
&= (\frac{e^{-t}}{e^{\frac{1}{\alpha^2}} - 1}) \int_{0}^{\frac{1}{\alpha^2}} e^u g_1(u)du \\
&= (\frac{e^{-t}}{e^{\frac{1}{\alpha^2}} - 1}) [\int_{0}^{\frac{1}{2\alpha^2}} ue^u du + \int_{\frac{1}{2\alpha^2}}^{\frac{1}{\alpha^2}} (\frac{1}{\alpha^2} - u)e^u du]
\end{aligned}$$

$$= (\frac{e^{\frac{1}{2\alpha^2}} - 1}{e^{\frac{1}{2\alpha^2}} + 1})e^{-t}.$$

Theorem 12. *Let $0 < \alpha \leq 1$, assume that $f \in Ł^1(\mathbb{R}^+, \mathbb{R})$ is α-periodic function with period p, and $a(t) = a_1(\frac{t^\alpha}{\alpha})$ such that $a_1 \in L^1(\mathbb{R}^+)$. Then $(a *_\alpha f)_{-\infty}$ is α-periodic with period p and for $k \in \mathbb{Z}$*

$$\mathcal{F}_\alpha((a *_\alpha f)_{-\infty}(t))(k) = \mathcal{L}_\alpha(a(t))(2ik\pi\frac{\alpha}{p^\alpha})\mathcal{F}_\alpha(f((t))(k)$$

where $\mathcal{L}_\alpha(a(t))(\lambda)$ is the conformable Laplace transform of $a(t)$ given by the Definition 5.

Proof. Let $0 < \alpha \leq 1$ and assume that $f \in Ł^1(\mathbb{R}^+, \mathbb{R})$ is α-periodic function with period p. For $t \in [0, +\infty[, g(t) = f((\alpha t)^{\frac{1}{\alpha}})$ is periodic with period $T = \frac{p^\alpha}{\alpha}$.
By Theorem 7, $(a *_\alpha f)_{-\infty}(t)$ is α-periodic function with period p, and we showed that

$$(a *_\alpha f)_{-\infty}(t) = F(\frac{t^\alpha}{\alpha})$$

where the continuous function F is defined by $F(t) = \int_{-\infty}^t a_1(t-s)g(s)ds$. Thus, we have

$$F(t) = \int_{-\infty}^0 a_1(t-s)g(s)ds + \int_0^t a_1(t-s)g(s)ds.$$

By Lemma 3,

$$\begin{aligned}
\int_{-\infty}^0 a_1(t-s)g(s)ds &= \sum_{N=1}^{+\infty} \int_0^T a_1(t-u+NT)g(u)du \\
&= \sum_{N=1}^{+\infty} \int_{t+(n-1)T}^{t+nT} a_1(w)g(t-w)dw \\
&= \lim_{n\to+\infty} \sum_{N=1}^{n} \int_{t+(N-1)T}^{t+NT} a_1(w)g(t-w)dw \\
&= \lim_{n\to+\infty} \int_t^{t+nT} a_1(w)g(t-w)dw \\
&= \int_t^{+\infty} a_1(w)g(t-w)dw
\end{aligned}$$

and

$$F(t) = \int_0^{+\infty} a_1(v)g(t-v)dv$$

Then for all $k \in \mathbb{Z}$

$$\mathcal{F}_\alpha((a *_\alpha f)_{-\infty}(t))(k) = \frac{\alpha}{p^\alpha}\int_0^p e^{-ik\frac{2\pi}{p^\alpha}t^\alpha} F(\frac{t^\alpha}{\alpha})t^{\alpha-1}dt.$$

By making variable change $\frac{t^\alpha}{\alpha} = u$ and $u - s = t$, we have

$$\begin{aligned}
\mathcal{F}_\alpha((a *_\alpha f)_{-\infty}(t))(k) =& \frac{\alpha}{p^\alpha} \int_0^{\frac{p^\alpha}{\alpha}} e^{-ik\frac{2\pi\alpha}{p^\alpha}u} [\int_{-\infty}^{u} a((\alpha(u-s))^{\frac{1}{\alpha}}) f((\alpha s)^{\frac{1}{\alpha}}) ds] du \\
&= \frac{\alpha}{p^\alpha} \int_0^{\frac{p^\alpha}{\alpha}} e^{-ik\frac{2\pi\alpha}{p^\alpha}u} [\int_{0}^{+\infty} a((\alpha s)^{\frac{1}{\alpha}}) f((\alpha(u-s))^{\frac{1}{\alpha}}) ds] du \\
&= [\int_0^{+\infty} a((\alpha s)^{\frac{1}{\alpha}}) e^{2ik\pi\frac{\alpha}{p^\alpha}s} ds][\frac{\alpha}{p^\alpha} \int_0^{\frac{p^\alpha}{\alpha}} e^{-ik\frac{2\pi\alpha}{p^\alpha}t} f((\alpha(t))^{\frac{1}{\alpha}}) dt \\
&= \mathcal{L}_\alpha(a(t))(2ik\pi\frac{\alpha}{p^\alpha}) \mathcal{F}_\alpha(f(t))(k).
\end{aligned}$$

Example 13. *Let f_1 defined by Example 3 and $a(t) = a_1(\frac{t^\alpha}{\alpha})$ such that $a_1(t) = e^{-t} \in L^1(\mathbb{R}^+)$. The function f_1 is α-periodic with period $(\frac{1}{\alpha})^{\frac{1}{\alpha}}$. Let $k \in \mathbb{Z}$ and $t \in [0, +\infty[$, we have*

$$\begin{aligned}
\mathcal{F}_\alpha((a *_\alpha f_1)_{-\infty}(t))(k) &= \mathcal{F}((a *_\alpha f_1)_{-\infty}((\alpha t)^{\frac{1}{\alpha}}))(k) \\
&= \alpha^2 \int_0^{\frac{1}{\alpha^2}} e^{-2ik\pi\alpha^2 t} F(t) dt \\
&= \alpha^2 \int_0^{\frac{1}{\alpha^2}} e^{-2ik\pi\alpha^2 t} [\int_{-\infty}^{0} a_1(t-s) g_1(s) ds + \int_0^t a_1(t-s) g_1(s) ds] dt \\
&= I_1 + I_2
\end{aligned}$$

such that

$$I_1 = \alpha^2 \int_0^{\frac{1}{\alpha^2}} e^{-2ik\pi\alpha^2 t} (\int_{-\infty}^{0} a_1(t-s) g_1(s) ds) dt$$

and

$$I_2 = \alpha^2 \int_0^{\frac{1}{\alpha^2}} e^{-2ik\pi\alpha^2 t} (\int_0^{t} a_1(t-s) g_1(s) ds) dt.$$

By Lemma 3

$$\int_{-\infty}^{0} a_1(t-s) g(s) ds = (\frac{e^{\frac{1}{2\alpha^2}} - 1}{e^{\frac{1}{2\alpha^2}} + 1}) e^{-t}$$

then

$$I_1 = \frac{\alpha^2(-2e^{\frac{1}{2\alpha^2}} + e^{\frac{1}{\alpha^2}} + 1) e^{-\frac{1}{\alpha^2}}}{2ik\pi\alpha^2 + 1}$$

and

$$I_2 = \alpha^2 \int_0^{\frac{1}{\alpha^2}} e^{-(2ik\pi\alpha^2+1)t} (\int_0^t e^s g_1(s) ds) dt$$

$$
\begin{aligned}
&= -\frac{\alpha^2}{2ik\pi\alpha^2+1}\{e^{-\frac{1}{\alpha^2}}\int_0^{\frac{1}{\alpha^2}} e^t g_1(t)dt - \int_0^{\frac{1}{\alpha^2}} e^{-ik2\pi\alpha^2 t} g_1(t)dt\}\\
&= -\frac{\alpha^2}{2ik\pi\alpha^2+1}\{e^{-\frac{1}{\alpha^2}}[\int_0^{\frac{1}{2\alpha^2}} te^t dt + \int_{\frac{1}{2\alpha^2}}^{\frac{1}{\alpha^2}}(\frac{1}{\alpha^2}-t)e^t dt]\\
&\quad - \int_0^{\frac{1}{2\alpha^2}} te^{-2ik\pi\alpha^2 t}dt - \int_{\frac{1}{2\alpha^2}}^{\frac{1}{\alpha^2}}(\frac{1}{\alpha^2}-t)e^{-2ik\pi\alpha^2 t}dt\}\\
&= \frac{-2\pi^2 e^{-\frac{1}{\alpha^2}}\alpha^4k^2 + 4\pi^4 e^{-\frac{1}{2\alpha^2}}\alpha^4k^2 - 2k^2\pi^2\alpha^4 + (-1)^k - 1}{2\alpha^2\pi^2k^2(2ik\pi\alpha^2+1)}.
\end{aligned}
$$

Thus

$$\mathcal{F}_\alpha((a *_\alpha f_1)_{-\infty}(t))(k) = \frac{(-1)^k-1}{2\alpha^2\pi^2k^2(2ik\pi\alpha^2+1)}.$$

On the other hand, we have

$$\mathcal{L}_\alpha(a(t))(2ik\pi\alpha^2) = \frac{1}{2ik\pi\alpha^2+1}$$

and by Example 8

$$\mathcal{F}_\alpha(f_1(t))(k) = \frac{(-1)^k-1}{2\alpha^2\pi^2k^2}$$

then

$$\mathcal{F}_\alpha((a *_\alpha f_1)_{-\infty}(t))(k) = \mathcal{L}_\alpha(a(t))(2ik\pi\alpha^2)\mathcal{F}_\alpha(f_1((t))(k),\ \forall k \in \mathbb{Z}^*.$$

For $k = 0$, we have $\mathcal{F}_\alpha((a *_\alpha f_1)_{-\infty}(t))(0) = I_1 + I_2$ such that

$$I_1 = \alpha^2\int_0^{\frac{1}{\alpha^2}} e^{-t}(\int_{-\infty}^0 e^s g_1(s)ds)dt = -\frac{\alpha^2(e^{-\frac{1}{\alpha^2}}-1)(e^{\frac{1}{2\alpha^2}}-1)}{e^{\frac{1}{2\alpha^2}}+1}$$

and

$$
\begin{aligned}
I_2 &= \alpha^2\int_0^{\frac{1}{\alpha^2}} e^{-t}(\int_0^t e^s g_1(s)ds)dt\\
&= \alpha^2(-e^{-\frac{1}{\alpha^2}}\int_0^{\frac{1}{\alpha^2}} e^t g_1(t)dt + \int_0^{\frac{1}{\alpha^2}} g_1(t)dt)\\
&= -\alpha^2 e^{-\frac{1}{\alpha^2}}(-2e^{\frac{1}{2\alpha^2}} + e^{\frac{1}{\alpha^2}} + 1) + \frac{1}{4\alpha^2}.
\end{aligned}
$$

then by Example 8, we have

$$\mathcal{F}_\alpha((a *_\alpha f_1)_{-\infty}(t))(0) = \frac{1}{4\alpha^2} = \mathcal{L}_\alpha(a(t))(2ik\pi\alpha^2)\mathcal{F}_\alpha(f_1(t))(0).$$

Finally

$$\mathcal{F}_\alpha((a *_\alpha f_1)_{-\infty}(t))(k) = \mathcal{L}_\alpha(a(t))(2ik\pi\alpha^2)\mathcal{F}_\alpha(f_1(t))(k), \forall k \in \mathbb{Z}.$$

5. Conclusion

The definition of α-periodic function introduced by Khalil et al [8] has been investigated. Many results and examples related to this definition have been given and proved. A new definition of conformable Fourier transform for α-periodic function has been given. A relationship between the conformable Fourier transform and the classical Fourier transform have been established. Many results relating to the classical Fourier case have been obtained and demonstrated in the conformable Fourier case. Many examples have been constructed to illustrate these results. Our interest for future work is to apply this results to solve some conformable partial differential equations, conformable ordinary differential equations, conformable integro-differential equations and conformable Cauchy problems. Also, it may be of interest to investigate several modifications of the introduced conformable Fourier transform in this article to serve other modifications of conformable derivatives, such as $M-$truncated fractional derivatives.

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