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Characterizations Regular and Intra-regular Ordered Semigroups by using Generalized Interval Valued Bipolar Fuzzy Quasi-Ideals

Thiti Gaketem[1], Tanaphong Prommai[2,*]

[1,2] *Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand*

Abstract. In this article, we introduce the concept of a generalized interval-valued bipolar fuzzy quasi-ideal and investigate its properties. We explore the relationship between generalized interval-valued bipolar fuzzy quasi-ideals and generalized interval-valued bipolar fuzzy ideals. Furthermore, we characterize regular and intra-regular ordered semigroups by utilizing generalized interval valued bipolar fuzzy quasi-ideals.

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Key Words and Phrases: Ordered semigroup, interval valued bipolar fuzzy quasi-ideal, interval valued bipolar fuzzy ideals

1. Introduction

The tool used phenomena of renowned vagueness and uncertainty of data scientists by L. A. Zadeh in 1965 [15]. The theory of fuzzy semigroups was contained by Kuroki in 1979 [10]. Later the theory of interval valued fuzzy sets was introduced by L. A. Zadeh in 1975 [16] as a generalization of the notion of fuzzy sets. Interval valued fuzzy sets have various applications in several areas like medical science [5], image processing [8], decision making [18], etc. In 2006, Narayanan and Manikantan [13] developed the theory of interval valued fuzzy subsemigroup and studied types interval valued fuzzy ideals in semigroups. In 1994 Zhang [17] introduced the notion of bipolar fuzzy sets with the extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$, and used them for modeling and decision analysis. In 2000, Lee [11] used the term bipolar valued fuzzy sets and applied it to algebraic structures. In 2016, Mumtaz Ali et al. extended the concept of interval valued fuzzy set and bipolar fuzzy set to interval valued bipolar fuzzy set. In 2019, K. Arulmozhi et al. studied interval valued bipolar fuzzy set in

*Corresponding author.
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Email addresses: thiti.ga@up.ac.th (T. Gaketem), , tanaphong.pr@up.ac.th (T. Prommai)

algebra structure. In 2019, A. Salm el at [4] characterize of regular ordered semigroups by $(\varepsilon, \varepsilon, \vee_k, qk)$)-fuzzy quasi-ideals. In 2021, S. Lekkoksung [7] developed interval valued bipolar fuzzy ideal in ordered semigroup and characterized regular ordered semigroup in terms of generalized interval valued bipolar fuzzy ideal and bi-ideal. In 2024 P. Khamrot et. al [14] characterized weakly ordered semigroup in terms of generalized interval valued bipolar fuzzy ideal. There are also research studies related to ordered semigroup like fuzzy (m, n)-substructures [3], fuzzy (m, n)-ideal [1], fuzzy (m, n)-filters [2], fuzzy prime subset [12], etc.

In this paper, we establish the concept of a generalized interval valued bipolar fuzzy quasi ideal. We prove properties of a generalized interval valued bipolar fuzzy quasi ideal in semigroups. Main results, we will characterize regular and intra-regular ordered semigroup in terms of generalized interval valued bipolar fuzzy quasi ideal.

2. Preliminaries

In this section, we give some definitions and theory helpful in later sections.

An ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation.

For a nonempty subset X and Y of ordered semigroup S, we write
$(X] := \{a \in S \mid a \leq b \text{ for some } b \in X\}$ and $XY := \{xy \mid x \in X \text{ and } y \in Y\}$.

A non-empty subset L of an ordered semigroup G is called

(1) a *subsemigroup* of G if $L^2 \subseteq L$,

(2) a *left* (right) ideal of G if $(GL] \subseteq L$ ($(LG] \subseteq L$) and $x \in L$ and $s \in G$ such that $s \leq x$, then $s \in L$, that is $(L] \subseteq L$,

(3) a *bi-ideal* of G if L is a subsemigorup and $LGL \subseteq L$,

(4) an *quasi-ideal* of G if $(LG] \cap (GL] \subseteq L$

An ordered semigroup G is called a *regular* if, for each $u \in G$, there exists $x \in G$ such that $u \leq uxu$. An ordered semigroup G called an *intra-regular* if, for each $u \in G$, there exists $a.b \in G$ such that $u \leq au^2b$.

For any $p_i \in [0, 1]$, where $i \in \mathcal{A}$, define

$$\underset{i\in\mathcal{A}}{\vee} p_i := \sup_{i\in\mathcal{A}}\{p_i\} \quad \text{and} \quad \underset{i\in\mathcal{A}}{\wedge} p_i := \inf_{i\in\mathcal{A}}\{p_i\}.$$

We see that for any $p, q \in [0, 1]$, we have

$$p \vee q = \max\{p, q\} \quad \text{and} \quad p \wedge q = \min\{p, q\}.$$

A *fuzzy set* of a non-empty set T is a function $\omega : L \to [0, 1]$.

Let $\Omega[0, 1]$ be the set of all closed subintervals of $[0, 1]$, i.e.,

$$\Omega[0, 1] = \{\overline{\omega} = [\omega^-, \omega^+] \mid 0 \leq \omega^- \leq \omega^+ \leq 1\}.$$

We note that $[\omega, \omega] = \{\omega\}$ for all $\omega \in [0,1]$. For $\omega = 0$ or 1 we shall denote $[0,0]$ by $\overline{0}$ and $[1,1]$ by $\overline{1}$.

Let $\overline{\omega} = [\omega^-, \omega^+]$ and $\overline{\varpi} = [\varpi^-, \varpi^+] \in \Omega[0,1]$. Define the operations $\preceq$, $=$, $\curlywedge$ and $\curlyvee$ as follows:

(1) $\overline{\omega} \preceq \overline{\varpi}$ if and only if $\omega^- \leq \varpi^-$ and $\omega^+ \leq \varpi^+$

(2) $\overline{\omega} = \overline{\varpi}$ if and only if $\omega^- = \varpi^-$ and $\omega^+ = \varpi^+$

(3) $\overline{\omega} \curlywedge \overline{\varpi} = [(\omega^- \wedge \varpi^-), (\omega^+ \wedge \varpi^+)]$

(4) $\overline{\omega} \curlyvee \overline{\varpi} = [(\omega^- \vee \varpi^-), (\omega^+ \vee \varpi^+)]$.

If $\overline{\omega} \succeq \overline{\varpi}$, we mean $\overline{\varpi} \preceq \overline{\omega}$.

For each interval $\overline{\omega}_i = [\omega_i^-, \omega_i^+] \in \Omega[0,1]$, $i \in \mathcal{A}$ where $\mathcal{A}$ is an index set, we define

$$\curlywedge_{i\in\mathcal{A}} \overline{\omega}_i = [\bigwedge_{i\in\mathcal{A}} \omega_i^-, \bigwedge_{i\in\mathcal{A}} \omega_i^+] \quad \text{and} \quad \curlyvee_{i\in\mathcal{A}} \overline{\omega}_i = [\bigvee_{i\in\mathcal{A}} \omega_i^-, \bigvee_{i\in\mathcal{A}} \omega_i^+].$$

Definition 1. *[13] Let T be a non-empty set. Then the function $\overline{f} : T \to \Omega[0,1]$ is called* interval valued fuzzy set *(shortly, IVF set) of T.*

Definition 2. *[13] Let M be a subset of a non-empty set G. An* interval valued characteristic function *of M is defined to be a function $\overline{\chi}_M : G \to \Omega[0,1]$ by*

$$\overline{\chi}_M(e) = \begin{cases} \overline{1} & if \quad e \in M, \\ \overline{0} & if \quad e \notin M \end{cases}$$

for all $e \in G$.

Now, we review the definition of bipolar valued fuzzy set and the basic properties used in the next section.

Definition 3. *[11] Let T be a non-empty set. A **bipolar fuzzy set** (BF set) ω on T is an object having the form*

$$\omega := \{(k, \omega^p(k), \omega^n(k)) \mid k \in T\},$$

where $\omega^p : T \to [0,1]$ and $\omega^n : T \to [-1,0]$.

Remark 1. *For the sake of simplicity we shall use the symbol $\omega = (T; \omega^p, \omega^n)$ for the BF set $\omega = \{(k, \omega^p(k), \omega^n(k)) \mid k \in T\}$.*

The following example of a BF set.

Example 1. *Let* $T = \{21, 22, 23...\}$*. Define* $\omega^p : T \to [0,1]$ *is a function*

$$\omega^p(u) = \begin{cases} 0 & \textit{if } u \textit{ is old number} \\ 1 & \textit{if } u \textit{ is even number} \end{cases}$$

and $\omega^n : T \to [-1,0]$ *is a function*

$$\omega^n(u) = \begin{cases} -1 & \textit{if } u \textit{ is old number} \\ 0 & \textit{if } u \textit{ is even number.} \end{cases}$$

Then $\omega = (T; \omega^p, \omega^n)$ *is a BF set.*

For $k \in T$, define $F_k = \{(y,z) \in T \times T \mid k = yz\}$.

Definition 4. *[6] Let* M *be a non-empty set of a semigroup* T*. A* ***positive characteristic function and a negative characteristic function*** *are respectively defined by*

$$\chi^p_M : T \to [0,1], k \mapsto \lambda^p_M(u) := \begin{cases} 1 & k \in M, \\ 0 & k \notin M, \end{cases}$$

and

$$\chi^n_M : T \to [-1,0], k \mapsto \lambda^n_M(k) := \begin{cases} -1 & k \in M, \\ 0 & k \notin M. \end{cases}$$

Remark 2. *For the sake of simplicity we shall use the symbol* $\chi_M = (T; \chi^p_M, \chi^n_M)$ *for the BF set* $\chi_I := \{(k, \chi^p_I(k), \chi^n_I(k)) \mid k \in I\}$.

Now, we review the definition of an interval valued bipolar fuzzy set and the basic properties used in the next section.

Definition 5. *[7] An* interval valued bipolar fuzzy set *(shortly, IVBF subset)* $\overline{\mathcal{T}}$ *on an ordered semigroup* G *is form*

$$\overline{\mathcal{T}} := \{e, \overline{\omega}^p(e), \overline{\omega}^n(e) \mid e \in G\},$$

where $\overline{\omega}^p : G \to \Omega[0,1]$ *and* $\overline{\omega}^n : G \to \Omega[-1,0]$.

In this page we shall use the symbol $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ instead of the IVBF set $\overline{\mathcal{T}} := \{e, \overline{\omega}^p(e), \overline{\omega}^n(e) \mid e \in G\}$.

For two IVBF sets $\overline{\mathcal{T}}_1 = (\overline{\omega}^p, \overline{\omega}^n)$ and $\overline{\mathcal{T}}_2 = (\overline{\varpi}^p, \overline{\varpi}^n)$ of an ordered semigroup G, define

(1) $\overline{\mathcal{T}}_1 \sqsubseteq \overline{\mathcal{T}}_2$ if and only if $\overline{\omega}^p(e) \le \overline{\varpi}^p(e)$ and $\overline{\omega}^n(e) \le \overline{\varpi}^n(e)$ for all $e \in G$,

(2) $\overline{\mathcal{T}}_1 = \overline{\mathcal{T}}_1$ if and only if $\overline{\mathcal{T}}_1 \sqsubseteq \overline{\mathcal{T}}_2$ and $\overline{\mathcal{T}}_2 \sqsubseteq \overline{\mathcal{T}}_1$,

(3) $\overline{\mathcal{T}}_1 \sqcup \overline{\mathcal{T}}_2$ if and only if $\overline{\omega} \cup \overline{\varpi}$ where $(\overline{\omega}^p \cup \overline{\varpi}^p)(e) = \overline{\omega}^p(e) \vee \overline{\varpi}^p(e)$ and $(\overline{\omega}^n \cup \overline{\varpi}^n)(e) = \overline{\omega}^n(e) \wedge \overline{\varpi}^n(e)$ for all $e \in G$,

(4) $\overline{\mathcal{T}}_1 \sqcap \overline{\mathcal{T}}_2$ if and only if $\overline{\omega} \cap \overline{\varpi}$ where $(\overline{\omega}^p \cap \overline{\varpi}^p)(e) = \overline{\omega}^p(e) \wedge \overline{\varpi}^p(e)$ and $(\overline{\omega}^n \cap \overline{\varpi}^n)(e) = \overline{\omega}^n(e) \vee \overline{\varpi}^n(e)$for all $e \in G$,

(5) $\overline{\mathcal{T}}_1 \bar{\circ} \overline{\mathcal{T}}_2$ if and on if $\overline{\omega} \circ \overline{\varpi}$ where

$$(\overline{\omega}^p \circ \overline{\varpi}^p)(e) = \begin{cases} \bigvee_{(t,h)\in F_e} \{\overline{\omega}^p(t) \wedge \overline{\varpi}^p(h)\} & \text{if } F_e \neq \emptyset, \\ \overline{0} & \text{if } F_e = \emptyset, \end{cases}$$

and

$$(\overline{\omega}^n \circ \overline{\varpi}^n)(e) = \begin{cases} \bigwedge_{(t,h)\in F_e} \{\overline{\omega}^n(t) \vee \overline{\varpi}^n(h)\} & \text{if } F_e \neq \emptyset, \\ \overline{0} & \text{if } F_e = \emptyset, \end{cases}$$

where $F_e := \{(t, h) \in G \times G \mid e \leq th\}$ for all $e \in G$.

Definition 6. *[7] Let M be a non-empty set of an ordered semigroup G. An* ***interval valued bipolar characteristic function*** *are respectively defined by*

$$\overline{\chi}^p_M : G \to \Omega[0, 1], e \mapsto \overline{\chi}^p_I(e) := \begin{cases} \overline{1} & e \in M, \\ \overline{0} & e \notin M, \end{cases}$$

and

$$\overline{\chi}^n_M : G \to \Omega[-1, 0], e \mapsto \overline{\chi}^n_I(e) := \begin{cases} -\overline{1} & e \in M, \\ \overline{0} & e \notin M. \end{cases}$$

Remark 3. *For the sake of simplicity we shall use the symbol $\overline{\chi}_m = (G; \overline{\chi}^p_M, \overline{\chi}^n_M)$ for the IVBF set $\overline{\chi}_M := \{(k, \overline{\chi}^p_M(k), \overline{\chi}^n_M(k)) \mid k \in M\}$.*

Now, we let $\overline{\lambda}^p$, $\overline{\delta}^p \in \Omega[0, 1]$ be such that $\overline{0} \leq \overline{\lambda}^p < \overline{\delta}^p \leq \overline{1}$ and $\overline{\lambda}^n$, $\overline{\delta}^n \in \Omega[-1, 0]$ be such that $-\overline{1} \leq \overline{\delta}^n < \overline{\lambda}^n \leq \overline{1}$. Both $\overline{\lambda}, \overline{\delta}$ are arbitrary but fixed.

Definition 7. *[7] Let G be an ordered semigroup and $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ be an IVBF set of G is called an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of G if*

(1) $\overline{\omega}^p(e_1e_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(e_1) \wedge \overline{\omega}^p(e_2) \wedge \overline{\delta}^p$.

(2) $\overline{\omega}^n(e_1e_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(e_1) \vee \overline{\omega}^n(e_2) \curlywedge \overline{\delta}^n$.

for all $e_1, e_2 \in G$.

Definition 8. *[7] Let G be an ordered semigroup and $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ be an IVBF set of G is called an $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal of G if*

(1) $\overline{\omega}^p(e_1e_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(e_2) \curlywedge \overline{\delta}^p$.

(2) $\overline{\omega}^n(e_1e_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(e_2) \vee \overline{\delta}^n$.

(3) *If $e_1 \leq e_2$, then $\overline{\omega}^p(e_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(e_2) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(e_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(e_2) \vee \overline{\delta}^n$.*

for all $e_1, e_2 \in G$.

Definition 9. *[7] Let* G *be an ordered semigroup and* $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *be an IVBF set of* G *is called an* $(\overline{\lambda}, \overline{\delta})$*-IVBF right ideal of* G *if*

(1) $\overline{\omega}^p(e_1e_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(e_1) \wedge \overline{\delta}^p$,

(2) $\overline{\omega}^n(e_1e_2) \wedge \overline{\lambda}^n \geq \overline{\omega}^n(e_1) \vee \overline{\delta}^n$.

(3) *If* $e_1 \leq e_2$*, then* $\overline{\omega}^p(e_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(e_2) \wedge \overline{\delta}^p$ *and* $\overline{\omega}^n(e_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(e_2) \vee \overline{\delta}^n$,

for all $e_1, e_2 \in G$.

An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ of an ordered semigroup E is called $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of G if it is both $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal and $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of G.

Definition 10. *[7] Let* G *be an ordered semigroup and* $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *be an IVBF set of* G *is called an* $(\overline{\lambda}, \overline{\delta})$*-IVBF bi-ideal of* G *if*

(1) $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *is an* $(\overline{\lambda}, \overline{\delta})$*-IVBF subsemigroup of* G

(2) $\overline{\omega}^p(e_1e_2e_3) \vee \overline{\lambda}^p \geq \overline{\omega}^p(e_1) \wedge \overline{\omega}^p(e_3) \wedge \overline{\delta}^p$

(3) $\overline{\omega}^n(e_1e_2e_3) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(e_1) \vee \overline{\omega}^n(e_3) \vee \overline{\delta}^n$,

for all $e_1, e_2, e_3 \in G$.

For two IVBF sets $\overline{\mathcal{T}}_1 = (\overline{\omega}^p, \overline{\omega}^n)$ and $\overline{\mathcal{T}}_2 = (\overline{\varpi}^p, \overline{\varpi}^n)$ of an ordered semigroup G, define

(1) $\overline{\mathcal{T}}_{\overline{\delta}}^{\overline{\lambda}}(x) := \Big((\overline{\omega}^p)_{\overline{\delta}}^{\overline{\lambda}}(x), (\overline{\omega}^n)_{\overline{\delta}}^{\overline{\lambda}}(x)\Big) = \Big(((\overline{\omega}^p)(x) \wedge \overline{\lambda}^p) \vee \overline{\delta}^p, ((\overline{\omega}^n)(x) \vee \overline{\lambda}^n) \wedge \overline{\delta}^p\Big)$,

(2) $(\overline{\mathcal{T}}_1 \sqcap \overline{\mathcal{T}}_2)_{\overline{\delta}}^{\overline{\lambda}}(x) := \Big((\overline{\omega}^p \cap \overline{\varpi}^p)_{\overline{\delta}}^{\overline{\lambda}}(x), (\overline{\omega}^n \cap \overline{\varpi}^n)_{\overline{\delta}}^{\overline{\lambda}}(x)\Big)$
$= \Big(((\overline{\omega}^p(x) \wedge \overline{\varpi}^p(x)) \wedge \overline{\lambda}^p) \vee \overline{\delta}^p, ((\overline{\omega}^n(x) \vee \overline{\varpi}^n(x)) \vee \overline{\lambda}^n) \wedge \overline{\delta}^n\Big)$,

(3) $(\overline{\mathcal{T}}_1 \circ \overline{\mathcal{T}}_2)_{\overline{\delta}}^{\overline{\lambda}}(x) := \Big((\overline{\omega}^p \circ \overline{\varpi}^p)_{\overline{\delta}}^{\overline{\lambda}}(x), (\overline{\omega}^n \circ \overline{\varpi}^n)_{\overline{\delta}}^{\overline{\lambda}}(x)\Big)$
$= \Big(((\overline{\omega}^p(x) \circ \overline{\varpi}^p(x)) \wedge \overline{\lambda}^p) \vee \overline{\delta}^p, ((\overline{\omega}^n(x) \circ \overline{\varpi}^n(x)) \vee \overline{\lambda}^n) \wedge \overline{\delta}^n\Big)$

where

$$(\overline{\omega}^p \circ \overline{\varpi}^p)(e) = \begin{cases} \bigvee_{(t,h)\in F_e} \{\overline{\omega}^p(t) \wedge \overline{\varpi}^p(h)\} & \text{if } F_e \neq \emptyset, \\ \overline{0} & \text{if } F_e = \emptyset, \end{cases}$$

and

$$(\overline{\omega}^n \circ \overline{\varpi}^n)(e) = \begin{cases} \bigwedge_{(t,h)\in F_e} \{\overline{\omega}^n(t) \vee \overline{\varpi}^n(h)\} & \text{if } F_e \neq \emptyset, \\ \overline{0} & \text{if } F_e = \emptyset, \end{cases}$$

In the following theorem, we give a relationship between an ideal and the interval valued bipolar characteristic function which is proved easily.

Theorem 1. *Let M be a non-empty subset of an ordered semigroup G. Then M is a left ideal (right ideal, ideal) of G with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$ if and only if $\overline{\chi}_M = (G; \overline{\chi}^p_M, \overline{\chi}^n_M)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal (right ideal, ideal) of G.*

3. Generalized Interval Valued Bipolar Fuzzy Quasi-Ideals

In this section, we give the concept of a generalized interval valued bipolar fuzzy quasi-ideal and investigate properties of generalized interval valued bipolar fuzzy quasi-ideal in ordered semigroups.

Definition 11. *Let G be an ordered semigroup and $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ be an IVBF set of G is called an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G if*

(1) $(\overline{G} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} \sqcap (\overline{\mathcal{T}} \circ \overline{G})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$.

(2) *If $e_1 \leq e_2$, then $\overline{\varpi}^p(e_1) \vee \overline{\lambda}^p \geq \overline{\varpi}^p(e_2) \wedge \overline{\delta}^p$ and $\overline{\varpi}^n(e_1) \wedge \overline{\lambda}^n \leq \overline{\varpi}^n(e_2) \vee \overline{\delta}^n$,*

for all $e_1, e_2 \in G$.

The following example is a $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of a semigroup.

Example 2. *Let G be an ordered semigroup given by the following table.*

$\cdot$	α	κ	ρ
α	α	α	α
κ	α	κ	κ
ρ	α	α	κ

Define IVBF set $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ in G as follows: $\overline{\mu}^p(\alpha) = [0.1, 0.8]$, $\overline{\mu}^p(\kappa) = [0.1, 0.8]$, $\overline{\mu}^p(\rho) = [0.3, 0.6]$ and $\overline{\mu}^n(\alpha) = [-0.1, -0.7]$, $\overline{\mu}^n(\kappa) = [-0.1, -0.7]$, $\overline{\mu}^n(\rho) = [-0.2, -0.5]$. and define a partial order relation $\leq$ on G as follows: $\leq$: $\{(\alpha, \kappa), (\alpha, \rho), (\kappa, \rho)\} \cup \triangle_G$, where $\triangle_G$ is an equality relation on G. By routine calculation, $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ is an $([0.3, 0.3], [0.5, 0.5])$-IVBF quasi-ideal of G.

Theorem 2. *Every $(\overline{\lambda}, \overline{\delta})$-IVBF left (right) ideal of an ordered semigroup G is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi ideal of G.*

Proof. Suppose that $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal of $\mathfrak{F}$ and let $e_1, e_2 \in G$ with $e_1 \geq e_2$. Then $\overline{\varpi}^p(e_1) \vee \overline{\lambda}^p \geq \overline{\varpi}^p(e_2) \wedge \overline{\delta}^p$ and $\overline{\varpi}^n(e_1) \wedge \overline{\lambda}^n \leq \overline{\varpi}^n(e_2) \vee \overline{\delta}^n$. Let $e \in \mathfrak{F}$.

If $A_e = \emptyset$, then it is easy to verify that $(\overline{G}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \wedge (\overline{\varpi}^p \circ \overline{G}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \leq (\overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e)$ and $(\overline{G}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \vee (\overline{\varpi}^n \circ \overline{G}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \geq (\overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e)$.

If $A_e \neq \emptyset$, then

$$\begin{aligned}
(\overline{G}^p \circ \overline{\omega}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) &= (\bigvee_{(k,o)\in A_e} \{\overline{G}^p(k) \wedge \overline{\omega}^p(o)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\bigvee_{(k,o)\in A_e} \{\overline{1} \wedge \overline{\omega}^p(o)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\bigvee_{(k,o)\in A_e} \{\overline{\omega}^p(o)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\bigvee_{(k,o)\in A_e} \{\overline{\omega}^p(o) \wedge \overline{\delta}^p\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&\leq (\bigvee_{(k,o)\in A_e} \{\overline{\omega}^p(ko) \vee \overline{\lambda}^p\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= ((\overline{\omega}^p(e) \vee \overline{\lambda}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p = ((\overline{\omega}^p(e) \vee \overline{\lambda}^p \vee \overline{\lambda}^p) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \\
&= ((\overline{\omega}^p(e) \vee \overline{\lambda}^p) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) = (\overline{\omega}^p(e) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p = (\overline{\omega}^{\,p})^{\overline{\delta}}_{\overline{\lambda}}(e)
\end{aligned}$$

and

$$\begin{aligned}
(\overline{G}^n \circ \overline{\omega}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) &= (\bigwedge_{(k,o)\in A_e} \{\overline{G}^n(k) \vee \overline{\omega}^n(o)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\bigwedge_{(k,o)\in A_e} \{-\overline{1} \vee \overline{\omega}^n(o)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\bigwedge_{(k,o)\in A_e} \{\overline{\omega}^n(o)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\bigwedge_{(k,o)\in A_e} \{\overline{\omega}^n(o) \vee \overline{\delta}^n\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&\geq (\bigwedge_{(k,o)\in A_e} \{\overline{\omega}^n(ko) \wedge \overline{\lambda}^n\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= ((\overline{\omega}^p(e) \wedge \overline{\lambda}^n) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n = (\overline{\omega}^p(r) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n = (\overline{\omega}^{\,n})^{\overline{\lambda}}_{\overline{\delta}}(e)
\end{aligned}$$

Thus, $(\overline{G}^p \circ \overline{\omega}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \leq (\overline{\omega}^{\,p})^{\overline{\delta}}_{\overline{\lambda}}(e)$ and $(\overline{G}^n \circ \overline{\omega}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \geq (\overline{\omega}^{\,n})^{\overline{\lambda}}_{\overline{\delta}}(e)$ implies that, $(\overline{G}^p \circ \overline{\omega}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \wedge (\overline{\omega}^p \circ \overline{G}^p)^{\overline{\delta}}_{\overline{\lambda}}(r) \leq (\overline{\omega}^{\,p})^{\overline{\delta}}_{\overline{\lambda}}(e)$ and $(\overline{G}^n \circ \overline{\omega}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \vee (\overline{\omega}^n \circ \overline{G}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \geq (\overline{\omega}^{\,n})^{\overline{\lambda}}_{\overline{\delta}}(e)$. Hence $\overline{\mathcal{T}} = (\overline{\omega}^{\,p}, \overline{\omega}^{\,n})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G.

The following theorem show that the $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal and $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroups.

Theorem 3. *Every $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of an ordered semigroup G is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of G.*

Proof. Assume that $\overline{\mathcal{T}} = (\overline{\omega}^{\,p}, \overline{\omega}^{\,n})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G and let $e_1, e_2 \in G$ with $e_1 \geq e_2$. Then $\overline{\omega}^{\,p}(e_1) \vee \overline{\lambda}^{\,p} \geq \overline{\omega}^{\,p}(e_2) \wedge \overline{\delta}^{\,p}$ and $\overline{\omega}^{\,n}(e_1) \wedge \overline{\lambda}^{\,n} \leq \overline{\omega}^{\,n}(e_2) \vee \overline{\delta}^{\,n}$.

Consider

$$\begin{aligned}
\overline{\varpi}^p(e_1e_2) \vee \overline{\lambda}^p &\geq \overline{\varpi}^p(e_1e_2) \wedge \overline{\delta}^p \vee \overline{\lambda}^p \\
&\geq (\overline{\varpi}^p \circ \overline{G}^p)^{\overline{\delta}}_{\overline{\lambda}}(e_1e_2) \wedge (\overline{G}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e_1e_2) \\
&= (\bigvee_{(i,j)\in A_{e_1e_2}} \{\overline{\varpi}^p(i) \wedge \overline{G}^p(j)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \wedge \\
&\quad (\bigvee_{(k,o)\in A_{e_1e_2}} \{\overline{G}^p(k) \wedge \overline{\varpi}^p(o)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&\geq (\overline{\varpi}^p(e_1) \wedge \overline{G}^p(e_2) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \wedge (\overline{G}^p(e_1) \wedge \overline{\varpi}^p(e_2) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \\
&= (\overline{\varpi}^p(e_1) \wedge \overline{1} \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \wedge (\overline{1} \wedge \overline{\varpi}^p(e_2) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \\
&= (\overline{\varpi}^p(e_1) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \wedge (\overline{\mu}^p(e_2) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \\
&= (\overline{\varpi}^p(e_1) \wedge \overline{\varpi}^p(e_2)) \wedge \overline{\delta}^p \vee \overline{\lambda}^p \\
&= \overline{\varpi}^p(e_1) \wedge \overline{\varpi}^p(e_2) \wedge \overline{\delta}^p \vee \overline{\lambda}^p \\
&\geq \overline{\varpi}^p(e_1) \wedge \overline{\varpi}^p(e_2) \wedge \overline{\delta}^p
\end{aligned}$$

and

$$\begin{aligned}
\overline{\varpi}^n(e_1e_2) \wedge \overline{\lambda}^n &\leq \overline{\varpi}^p(e_1e_2) \wedge \overline{\lambda}^n \vee \overline{\delta}^n \\
&\leq (\overline{\varpi}^n \circ \overline{G}^n)^{\overline{\lambda}}_{\overline{\delta}}(e_1e_2) \vee (\overline{G}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e_1e_2) \\
&= (\bigwedge_{(i,j)\in A_{e_1e_2}} \{\overline{\varpi}^n(i) \vee \overline{G}^n(j)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \vee \\
&\quad (\bigwedge_{(k,o)\in A_{e_1e_2}} \{\overline{G}^n(k) \vee \overline{\varpi}^n(o)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&\leq (\overline{\varpi}^n(e_1) \vee \overline{G}^n(e_2) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \vee (\overline{G}^n(e_1) \vee \overline{\varpi}^n(e_2) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \\
&= (\overline{\varpi}^n(e_1) \vee -\overline{1} \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \vee (-\overline{1} \vee \overline{\varpi}^n(e_2) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \\
&= (\overline{\varpi}^n(e_1) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \vee (\overline{\varpi}^n(e_2) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \\
&= (\overline{\varpi}^n(e_1) \vee \overline{\varpi}^n(e_2)) \wedge \overline{\lambda}^n \vee \overline{\delta}^n \\
&= \overline{\varpi}^n(e_1) \vee \overline{\varpi}^n p(e_2) \wedge \overline{\lambda}^n \vee \overline{\delta}^n \\
&\leq \overline{\varpi}^n(e_1) \vee \overline{\varpi}^n(e_2) \vee \overline{\delta}^n.
\end{aligned}$$

Thus, $\overline{\varpi}^p(e_1e_2) \vee \overline{\lambda}^p \geq \overline{\varpi}^p(e_1) \wedge \overline{\varpi}^p(e_2) \wedge \overline{\delta}^p$ and $\overline{\varpi}^n(e_1e_2) \wedge \overline{\lambda}^n \leq \overline{\varpi}^n(e_1) \vee \overline{\varpi}^n(e_2) \vee \overline{\delta}^n$. Hence $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of G.

The following theorem show that the $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal and $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal in semigroup.

Theorem 4. *Every $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of an ordered semigroup G is a $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of G.*

Proof. Assume that $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G and let $e_1, e_2, \in G$. Then by Theorem 3, $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of G.

Let $e_1, e_2, e_3 \in G$. Then

$$\begin{aligned}
\overline{\omega}^p(e_1e_2e_3) \vee \overline{\lambda}^p &\geq \overline{\omega}^p(e_1e_2e_3) \wedge \overline{\delta}^p \vee \overline{\lambda}^p \\
&\geq (\overline{\omega}^p \circ \overline{G}^p)_{\overline{\lambda}}^{\overline{\delta}}(e_1e_2e_3) \wedge (\overline{G}^p \circ \overline{\omega}^p)_{\overline{\lambda}}^{\overline{\delta}}(e_1e_2e_3) \\
&= \Big(\bigvee_{(i,j)\in A_{e_1e_2e_3}} \{\overline{\omega}^p(i) \wedge \overline{G}^p(j)\} \wedge \overline{\delta}^p\Big) \vee \overline{\lambda}^p \wedge \\
&\quad \Big(\bigvee_{(k,o)\in A_{e_1e_2e_3}} \{\overline{G}^p(k) \wedge \overline{\omega}^p(o)\} \wedge \overline{\delta}^p\Big) \vee \overline{\lambda}^p \\
&\geq (\overline{\omega}^p(e_1) \wedge \overline{G}^p(e_2e_3) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \wedge (\overline{G}^p(e_1e_2) \wedge \overline{\omega}^p(e_3) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \\
&= (\overline{\omega}^p(e_1) \wedge \overline{1} \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \wedge (\overline{1} \wedge \overline{\omega}^p(e_3) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \\
&= (\overline{\omega}^p(e_1) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \wedge (\overline{\mu}^p(e_3) \wedge \overline{\delta}^p \vee \overline{\lambda}^p) \\
&= (\overline{\omega}^p(e_1) \wedge \overline{\omega}^p(e_3)) \wedge \overline{\delta}^p \vee \overline{\lambda}^p = \overline{\omega}^p(e_1) \wedge \overline{\omega}^p(e_3) \wedge \overline{\delta}^p \vee \overline{\lambda}^p \\
&\geq \overline{\omega}^p(e_1) \wedge \overline{\omega}^p(e_3) \wedge \overline{\delta}^p
\end{aligned}$$

and

$$\begin{aligned}
\overline{\omega}^n(e_1e_2e_3) \wedge \overline{\lambda}^n &\leq \overline{\omega}^p(e_1e_2e_3) \wedge \overline{\lambda}^n \vee \overline{\delta}^n \\
&\leq (\overline{\omega}^n \circ \overline{G}^n)_{\overline{\delta}}^{\overline{\lambda}}(e_1e_2e_3) \vee (\overline{G}^n \circ \overline{\omega}^n)_{\overline{\delta}}^{\overline{\lambda}}(e_1e_2e_3) \\
&= \Big(\bigwedge_{(i,j)\in A_{e_1e_2e_3}} \{\overline{\omega}^n(i) \vee \overline{G}^n(j)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \vee \\
&\quad \Big(\bigwedge_{(\mathfrak{k},o)\in A_{e_1e_2e_3}} \{\overline{G}^n(k) \vee \overline{\omega}^n(o)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \\
&\leq (\overline{\omega}^n(e_1) \vee \overline{G}^n(e_2e_3) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \vee (\overline{G}^n(e_1e_2) \vee \overline{\omega}^n(e_3) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \\
&= (\overline{\omega}^n(e_1) \vee -\overline{1} \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \vee (-\overline{1} \vee \overline{\omega}^n(e_3) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \\
&= (\overline{\omega}^n(e_1) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \vee (\overline{\omega}^n(e_3) \wedge \overline{\lambda}^n \vee \overline{\delta}^n) \\
&= (\overline{\omega}^n(e_1) \vee \overline{\omega}^n(e_3)) \wedge \overline{\lambda}^n \vee \overline{\delta}^n = \overline{\omega}^n(e_1) \vee \overline{\omega}^n p(e_3) \wedge \overline{\lambda}^n \vee \overline{\delta}^n \\
&\leq \overline{\omega}^n(e_1) \vee \overline{\omega}^n(e_3) \vee \overline{\delta}^n.
\end{aligned}$$

Thus, $\overline{\omega}^p(e_1e_2e_3) \vee \overline{\lambda}^p \geq \overline{\omega}^p(e_1) \wedge \overline{\omega}^p(e_3) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(e_1e_2e_3) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(e_1) \vee \overline{\omega}^n(e_3) \vee \overline{\delta}^n$. Hence $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of G.

The following theorems are basic properties.

Theorem 5. *Let G be an ordered semigroup. Then the intersection of two $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G.*

Proof. Assume that $\overline{\mathcal{T}}_1 = (\overline{\omega}^p, \overline{\omega}^n)$ and $\overline{\mathcal{T}}_2 = (\overline{\varpi}^p, \overline{\varpi}^n)$ are $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideals of G. Let $e \in G$. Then

$$\begin{aligned}
(\overline{\omega}^p \sqcap \overline{\varpi}^p)(e) \vee \overline{\lambda}^p &\geq (\overline{\omega}^p \sqcap \overline{\varpi}^p)(e) \vee \overline{\lambda}^p \wedge \overline{\delta}^p = (\overline{\omega}^p(e) \wedge \overline{\varpi}^p(e)) \vee \overline{\lambda}^p \wedge \overline{\delta}^p \\
&= (\overline{\omega}^p(e) \vee \overline{\lambda}^p \wedge \overline{\delta}^p) \wedge (\overline{\varpi}^p(e) \vee \overline{\lambda}^p \wedge \overline{\delta}^p) \\
&\geq (\overline{\omega}^p \circ \overline{G}^p)_{\overline{\lambda}}^{\overline{\delta}}(e) \wedge (\overline{G}^p \circ \overline{\omega}^p)_{\overline{\lambda}}^{\overline{\delta}}(e) \wedge \\
&\quad (\overline{\varpi}^p \circ \overline{G}^p)_{\overline{\lambda}}^{\overline{\delta}}(e) \wedge (\overline{G}^p \circ \overline{\varpi}^p)_{\overline{\lambda}}^{\overline{\delta}^p}(e) \\
&= \Big(\bigvee_{(i,j)\in A_e} \{\overline{\omega}^p(i) \wedge \overline{G}^p(j)\} \wedge \overline{\delta}^p\Big) \vee \overline{\lambda}^p \wedge \\
&\quad \Big(\bigvee_{(k,o)\in A_e} \{\overline{G}^p(k) \wedge \overline{\omega}^p(o)\} \wedge \overline{\delta}^p\Big) \vee \overline{\lambda}^p \wedge \\
&\quad \Big(\bigvee_{(i,j)\in A_e} \{\overline{\varpi}^p(i) \wedge \overline{G}^p(j)\} \wedge \overline{\delta}\Big) \vee \overline{\varpi}^p \wedge \\
&\quad \Big(\bigvee_{(k,o)\in A_e} \{\overline{G}^p(k) \wedge \overline{\varpi}^p(o)\} \wedge \overline{\delta}^p\Big) \vee \overline{\lambda}^p \\
&= \Big(\bigvee_{(i,j)\in A_e} \{\overline{\omega}^p(i) \wedge \overline{\varpi}^p(i) \wedge \overline{G}^p(j) \wedge \overline{\delta} \vee \overline{\lambda}^p\} \wedge \\
&\quad \Big(\bigvee_{(k,o)\in A_e} \{\overline{G}^p(k) \wedge \overline{\mu}^p(o) \wedge \overline{\varpi}^p(o)\} \wedge \overline{\delta}\Big) \vee \overline{\lambda}^p \\
&= \Big(\bigvee_{(i,j)\in A_e} \{(\overline{\omega}^p \sqcap \overline{\varpi}^p)(i) \wedge \overline{G}^p(j)\} \wedge \overline{\delta}\Big) \vee \overline{\lambda}^p \wedge \\
&\quad \Big(\bigvee_{(k,o)\in A_e} \{\overline{G}^p(k) \wedge (\overline{\omega}^p \sqcap \overline{\varpi}^p)(o)\} \wedge \overline{\delta}^p\Big) \vee \overline{\lambda}^p \\
&= (\overline{\omega}^p \sqcap \overline{\varpi}^p \circ \overline{G}^p)_{\overline{\lambda}}^{\overline{\delta}}(e) \wedge (\overline{G}^p \circ \overline{\omega}^p \sqcap \overline{\varpi}^p)_{\overline{\lambda}}^{\overline{\delta}}(e)
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\omega}^n \sqcup \overline{\varpi}^n)(e) \wedge \overline{\lambda}^n &\leq (\overline{\omega}^n \sqcup \overline{\varpi}^n)(e) \wedge \overline{\lambda}^n \vee \overline{\delta}^n = (\overline{\omega}^n(e) \vee \overline{\varpi}^n(e)) \wedge \overline{\lambda}^n \vee \overline{\delta}^n \\
&= (\overline{\omega}^n(e) \vee \overline{\lambda}^n \vee \overline{\delta}^n) \vee (\overline{\varpi}^n(e) \vee \overline{\lambda}^n) \vee \overline{\delta}^n \\
&\leq (\overline{\omega}^n \circ \overline{G}^n)_{\overline{\delta}}^{\overline{\lambda}}(e) \vee (\overline{G}^n \circ \overline{\omega}^n)_{\overline{\delta}}^{\overline{\lambda}}(e) \vee \\
&\quad (\overline{\varpi}^n \circ \overline{G}^n)_{\overline{\delta}}^{\overline{\lambda}}(e) \vee (\overline{G}^n \circ \overline{\varpi}^n)_{\overline{\delta}}^{\overline{\lambda}}(e) \\
&= \Big(\bigwedge_{(i,j)\in A_e} \{\overline{\omega}^n(i) \wedge \overline{G}^n(j)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \wedge \\
&\quad \Big(\bigwedge_{(k,o)\in A_e} \{\overline{G}^n(k) \wedge \overline{\omega}^n(o)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \vee \\
&\quad \Big(\bigwedge_{(i,j)\in A_e} \{\overline{\varpi}^n(i) \wedge \overline{G}^n(j)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \vee \\
&\quad \Big(\bigwedge_{(k,o)\in A_e} \{\overline{G}^n(k) \wedge \overline{\varpi}^n(o)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \\
&= \Big\{\bigwedge_{(i,j)\in A_e} (\overline{\omega}^n(i) \vee \overline{\varpi}^n(i) \wedge \overline{G}^n(j) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n\Big\} \wedge \\
&\quad \Big(\bigwedge_{(k,o)\in A_e} \{\overline{G}^p(k) \wedge \overline{\mu}^n(o) \vee \overline{\varpi}^n(o)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \\
&= \Big(\bigwedge_{(i,j)\in A_e} \{(\overline{\omega}^n \sqcup \overline{\varpi}^n)(i) \wedge \overline{G}^n(j)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \wedge \\
&\quad \Big(\bigwedge_{(k,o)\in A_e} \{\overline{G}^n(k) \wedge (\overline{\omega}^n \sqcup \overline{\varpi}^p)(o)\} \wedge \overline{\lambda}^n\Big) \vee \overline{\delta}^n \\
&= (\overline{\omega}^n \sqcup \overline{\varpi}^n \circ \overline{G}^n)_{\overline{\delta}}^{\overline{\lambda}}(e) \wedge (\overline{G}^n \circ \overline{\omega}^n \sqcup \overline{\varpi}^n)_{\overline{\delta}}^{\overline{\lambda}}(e).
\end{aligned}$$

Thus, $(\overline{\omega}^p \sqcap \overline{\varpi}^p)(e) \vee \overline{\lambda}^p \geq (\overline{\omega}^p \sqcap \overline{\varpi}^p \circ \overline{G}^p)_{\overline{\lambda}}^{\overline{\delta}}(e) \wedge (\overline{G}^p \circ \overline{\omega}^p \sqcap \overline{\varpi}^p)_{\overline{\lambda}}^{\overline{\delta}}(e)$ and $(\overline{\omega}^n \sqcup \overline{\varpi}^n)(e) \wedge \overline{\lambda}^n \leq (\overline{\omega}^n \sqcup \overline{\varpi}^n \circ \overline{G}^n)_{\overline{\delta}}^{\overline{\lambda}}(e) \wedge (\overline{G}^n \circ \overline{\omega}^n \sqcup \overline{\varpi}^n)_{\overline{\delta}}^{\overline{\lambda}}(e)$

Hence $\overline{\mathcal{T}}_1 \sqcap \overline{\mathcal{T}}_2$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G.

Theorem 6. *If M is a quasi-ideal of an ordered semigroup G, then $\overline{\chi}_M = (G; \overline{\chi}_M^p, \overline{\chi}_M^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G.*

Proof. Suppose that M is a quasi-ideal of G and $e_1, e_2 \in G$ with $e_1 \geq e_2$. Then $\overline{\omega}^p(e_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(e_2) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(e_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(e_2) \vee \overline{\delta}^n$. Let $e \in G$.

If $e \in M$ or $A_e = \emptyset$, then $(\overline{\chi}_M^p)_{\overline{\lambda}}^{\overline{\delta}}(e) \geq (\overline{\chi}_M^p \circ \overline{G}^p)_{\overline{\lambda}}^{\overline{\delta}}(e) \wedge (\overline{G}^p \circ \overline{\chi}_M^p)_{\overline{\delta}}^{\overline{\lambda}}(e)$, $(\overline{\chi}_M^n)_{\overline{\delta}}^{\overline{\lambda}}(e) \leq (\overline{\chi}_M^n \circ \overline{G}^n)(e) \vee (\overline{G}^n \circ \overline{\chi}_M^n)_{\overline{\delta}}^{\overline{\lambda}}(e)$.

Assume that $e \notin M$ and $A_e \neq \emptyset$. Let $K = \{(i,j) \mid e = ij \text{ and } (i \notin M \text{ and } j \notin M)\}$. Thus $K \subseteq A_e$

On the other hand if $(i,j) \in A_e$, then $A_e = ij \notin M$ which it implies that $ij \notin MG \cap GM$. Thus $i \notin M$ or $j \notin M$ $e \in GM$ and $e \in MG$ and so $(i,j) \in M$. Hence $A_e \subseteq K$. Therefore, $A_e = K$. That is, $A_e = \{(i,j) \mid i \notin M \text{ or } j \notin M\}$. Thus,

$$
\begin{aligned}
(\overline{\chi}_M^p \circ \overline{G})_{\overline{\lambda}}^{\overline{\delta}} \wedge (\overline{G} \circ \overline{\chi}_M^p)_{\overline{\lambda}}^{\overline{\delta}} &= (\bigvee_{(i,j)\in A_e} \{\overline{\chi}_M^p(i) \wedge \overline{G}^p(j)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \wedge \\
&\quad (\bigvee_{(i,j)\in A_e} \{\overline{G}^p(i) \wedge \overline{\chi}_M^p(j)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\bigvee_{(i,j)\in A_e} \{\overline{\chi}_M^p(i) \wedge \overline{\chi}_M^p(j)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p = \overline{\lambda}^p
\end{aligned}
$$

and

$$
\begin{aligned}
(\overline{\chi}_M^n \circ \overline{G})_{\overline{\delta}}^{\overline{\lambda}} \vee (\overline{G} \circ \overline{\chi}_M^n)_{\overline{\delta}}^{\overline{\lambda}} &= (\bigwedge_{(i,j)\in A_e} \{\overline{\chi}_M^n(i) \vee \overline{G}^n(j)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \vee \\
&\quad (\bigwedge_{(i,j)\in A_e} \{\overline{G}^n(i) \vee \overline{\chi}_M^n(j)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\bigwedge_{(i,j)\in A_e} \{\overline{\chi}_M^n(i) \vee \overline{\chi}_M^n(j)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n = \overline{\delta}^n.
\end{aligned}
$$

Hence $\overline{\chi}_M = (G; \overline{\chi}_M^p, \overline{\chi}_M^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G.

Lemma 1. *If $\overline{\chi}_M = (G; \overline{\chi}_M^p, \overline{\chi}_M^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of an ordered semigroup G with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$, then M is a quasi-ideal of G.*

Proof. Suppose that $\overline{\chi}_M = (S; \overline{\chi}_M^p, \overline{\chi}_M^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$. Let $e \in MG \cap GM$. Then there exist $i, j \in G$ and $\mathfrak{y}, \mathfrak{z} \in G$ such that $e = i\mathfrak{y}$ and $e = j\mathfrak{z}$. Thus $(\overline{\chi}_M^p \circ \overline{G})_{\overline{\lambda}}^{\overline{\delta}}(e) = (\bigvee_{(i,\mathfrak{n})\in A_e} \{\overline{\chi}_M^p(i) \wedge \overline{G}^p(\mathfrak{n})\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \geq (\bigvee_{(j,\mathfrak{z})\in A_e} \{\overline{\chi}_M^p(j) \wedge \overline{G}^p(\mathfrak{z})\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p = \overline{\delta}^p$. Similarly $(\overline{G} \circ \overline{\chi}_M^p)_{\overline{\lambda}}^{\overline{\delta}} = \overline{\delta}^p$. And $(\overline{\chi}_M^n \circ \overline{G})_{\overline{\delta}}^{\overline{\lambda}} =$

$(\bigwedge_{(i,\mathfrak{n})\in A_e}\{\overline{\chi}^n_M(i)\vee\overline{G}^n(\mathfrak{n})\}\wedge\overline{\lambda}^n)\vee\overline{\delta}^n \leq (\bigvee_{(j,\mathfrak{z})\in A_e}\{\overline{\chi}^n_M(j)\vee\overline{G}^n(\mathfrak{z})\}\wedge\overline{\lambda}^n)\vee\overline{\delta}^n = \overline{\delta}^n$. By assumption,

$$(\overline{\chi}^p_M)^{\overline{\delta}}_{\overline{\lambda}}(e) \geq (\overline{\chi}^p_M \circ \overline{G}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \wedge (\overline{G}^p \circ \overline{\chi}^p_M)^{\overline{\lambda}}_{\overline{\delta}}(e) \text{ and } (\overline{\chi}^n_M)^{\overline{\lambda}}_{\overline{\delta}}(e) \leq (\overline{\chi}^n_M \circ \overline{G}^n)(e) \curlyvee (\overline{G}^n \circ \overline{\chi}^n_M)^{\overline{\lambda}}_{\overline{\delta}}(e). \quad (1)$$

If $e \notin M$, then by (1) $\overline{\lambda}^p \geq \overline{\delta}^p$and $\overline{\lambda}^n \leq \overline{\delta}^n$. It is a contradiction. Hence $e \in M$. Therefore M is a quasi-ideal of G.

4. Characterizing ordered regular and intra-regular semigroups by using Generalized Interval Valued Bipolar Fuzzy Quasi-Ideals.

In this topic, we will use knowledge of the characteristics of interval valued fuzzy set and bipolar fuzzy sets to characterize regular and intra-regular semigroups by using generalized interval valued bipolar fuzzy quasi-ideals in ordered semigroups.

Theorem 7. *[14] Let I and K be a non-empty subsets of G. Then*

(1) $(\overline{\chi}_I \circ \overline{\chi}_K)^{\overline{\lambda}}_{\overline{\delta}} = (\overline{\chi}_{IK})^{\overline{\lambda}}_{\overline{\delta}}$ *i.e.* $\langle(\overline{\chi}^p_I \circ \overline{\chi}^p_K)^{\overline{\delta}}_{\overline{\lambda}}, (\overline{\chi}^n_I \circ \overline{\chi}^n_K)^{\overline{\lambda}}_{\overline{\delta}}\rangle = \langle(\overline{\chi}^p_{IK})^{\overline{\delta}}_{\overline{\lambda}}, (\overline{\chi}^n_{IK})^{\overline{\lambda}}_{\overline{\delta}}\rangle$

(2) $(\overline{\chi}_I \sqcap \overline{\chi}_K)^{\overline{\lambda}}_{\overline{\delta}} = (\overline{\chi}^p_I \sqcap \overline{\chi}_K)^{\overline{\lambda}}_{\overline{\delta}}$ *i.e.* $\langle(\overline{\chi}^p_I \cap \overline{\chi}^p_K)^{\overline{\delta}}_{\overline{\lambda}}, (\overline{\chi}^n_I \cup \overline{\chi}^n_I)^{\overline{\lambda}}_{\overline{\delta}}\rangle = \langle(\overline{\chi}^p_{I\cap K})^{\overline{\delta}}_{\overline{\lambda}}, (\overline{\chi}^n_{I\cup K})^{\overline{\lambda}}_{\overline{\delta}}\rangle$, *where* $\overline{\chi}_I = (G; \overline{\chi}^p_I, \overline{\chi}^n_I)$ *and* $\overline{\chi}_K = (G; \overline{\chi}^p_K, \overline{\chi}^n_K)$.

Remark 4. *Since $\overline{\chi}_{\mathcal{I}}$ is an interval valued characteristic function we have*

$$(\overline{\chi}^p_I)^{\overline{\lambda}}_{\overline{\delta}}(e) = \begin{cases} \overline{\lambda}^p & \text{if } \mathfrak{k} \in \mathcal{I}, \\ \overline{\delta}^p & \text{if } \mathfrak{k} \notin \mathcal{I} \end{cases}$$

and

$$(\overline{\chi}^n_I)^{\overline{\lambda}}_{\overline{\delta}}(e) = \begin{cases} \overline{\delta}^n & \text{if } \mathfrak{k} \in \mathcal{I}, \\ \overline{\lambda}^n & \text{if } \mathfrak{k} \notin \mathcal{I} \end{cases}$$

Lemma 2. *[4] For an ordered semigroup G, the following statements are equivalent.*

(1) *G is a regular*

(2) *$Q \cap L \subseteq (QL]$ for every quasi-ideal Q and every left ideal L of G.*

(3) *$R \cap Q \subseteq (RQ]$ for every right ideal R every quasi-ideal Q and of G.*

Theorem 8. *For an ordered semigroup G, the following conditions are equivalent.*

(1) *G is a regular,*

(2) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$, *for every $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ of G and every $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal $\overline{\mathcal{J}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ of G,*

(3) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$, *for every* $(\overline{\lambda}, \overline{\delta})$*-IVBF right ideal* $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *of* G *and every* $(\overline{\lambda}, \overline{\delta})$*-IVBF quasi-ideal* $\overline{\mathcal{J}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ *of* G.

Proof. (1) ⇒ (3) Let $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ and $\overline{\mathcal{J}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ be an $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal and an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G respectively and let $e \in G$. Since G is regular, there exists $g \in G$ such that $e \le ege$. Thus

$$\begin{aligned}
(\overline{\omega}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) &= (\bigvee_{(k,o)\in A_e} \{\overline{\omega}^p(k) \wedge \overline{\varpi}^p(o)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\bigvee_{(k,o)\in A_{ege}} \{\overline{\omega}^p(k) \wedge \overline{\varpi}^p(o)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&\ge ((\overline{\omega}^p(e) \wedge \overline{\varpi}^p(gr)) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\overline{\omega}^p(e) \wedge \overline{\varpi}^p(gr) \vee \overline{\lambda}^p) \wedge \overline{\delta}^p \vee \overline{\lambda}^p \\
&\ge (\overline{\omega}^p(e) \wedge \overline{\varpi}^p(e) \wedge \overline{\delta}^p) \wedge \overline{\delta}^p \vee \overline{\lambda}^p \\
&= ((\overline{\omega}^p(e) \wedge \overline{\varpi}^p(e)) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p = (\overline{\omega}^p \cap \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e)
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\omega}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) &= (\bigwedge_{(\mathfrak{k},o)\in A_e} \{\overline{\omega}^n(k) \vee \overline{\varpi}^n(o)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\bigwedge_{(k,o)\in A_{ege}} \{\overline{\omega}^n(k) \vee \overline{\varpi}^n(o)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&\le ((\overline{\omega}^n(e) \vee \overline{\varpi}^n(ge)) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\overline{\omega}^n(e) \vee \overline{\varpi}^n(ge) \wedge \overline{\lambda}^n) \wedge \overline{\lambda}^n \vee \overline{\delta}^n \\
&\le (\overline{\omega}^n(e) \vee \overline{\varpi}^n(e) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n \vee \overline{\delta}^n \\
&= ((\overline{\omega}^p(n) \vee \overline{\varpi}^n(e)) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n = (\overline{\omega}^n \cup \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e),
\end{aligned}$$

Thus, $(\overline{\omega}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \ge (\overline{\omega}^p \cap \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e)$ and $(\overline{\omega}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \le (\overline{\omega}^n \cup \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e)$
Hence, $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$.

(3) ⇒ (1) Let R and Q be a right ideal and quasi-ideal of $\mathfrak{F}$ respectively. Then by Theorem 1 and 6, $\overline{\chi}_R = (G; \overline{\chi}^p_R, \overline{\chi}^n_R)$ and $\overline{\chi}_Q = (G; \overline{\chi}^p_Q, \overline{\chi}^n_Q)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal and an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G respectively. By supposition and Thoerem7, we have

$$(\overline{\chi}^p_{(RQ]})^{\overline{\delta}}_{\overline{\lambda}}(e) = (\overline{\chi}^p_R \circ \overline{\chi}^p_Q)^{\overline{\delta}}_{\overline{\lambda}}(e) \sqsubseteq (\overline{\chi}^p_R \cap \overline{\chi}^p_Q)^{\overline{\delta}}_{\overline{\lambda}}(e) = (\overline{\chi}^p_{R\cap Q})^{\overline{\delta}}_{\overline{\lambda}}(e) = \overline{\delta}^p,$$

and

$$(\overline{\chi}^n_{(RQ]})^{\overline{\lambda}}_{\overline{\delta}}(e) = (\overline{\chi}^n_R \circ \overline{\chi}^p_Q)^{\overline{\lambda}}_{\overline{\delta}}(e) \sqsubseteq (\overline{\chi}^n_R \cup \overline{\chi}^p_Q)^{\overline{\lambda}}_{\overline{\delta}}(e) = (\overline{\chi}^n_{R\cup Q})^{\overline{\lambda}}_{\overline{\delta}}(e) = \overline{\delta}^n.$$

Thus, $e \in (RQ]$. Hence, $R \cap Q \subseteq (RL]$. Therefore by Lemma 2, G is regular.

In a similar manner, it may be shown that (1) ⇔ (2).

Corollary 1. *For an ordered semigroup* G *and let* $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *and* $\overline{\mathcal{J}} = (\overline{\varpi}^p, \overline{\varpi}^n)$*, the following conditions are equivalent.*

(1) *G is a regular,*

(2) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$, *for every* $(\overline{\lambda}, \overline{\delta})$*-IVBF quasi-ideal* $\overline{\mathcal{T}}$ *and every* $(\overline{\lambda}, \overline{\delta})$*-IVBF left ideal* $\overline{\mathcal{J}}$ *of* $\mathfrak{F}$,

(3) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$, *for every* $(\overline{\lambda}, \overline{\delta})$*-IVBF bi-ideal* $\overline{\mathcal{T}}$ *and every* $(\overline{\lambda}, \overline{\delta})$*-IVBF left ideal* $\overline{\mathcal{J}}$ *of* $\mathfrak{F}$,

(4) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$, *for every* $(\overline{\lambda}, \overline{\delta})$*-IVBF right ideal* $\overline{\mathcal{T}}$ *and every* $(\overline{\lambda}, \overline{\delta})$*-IVBF quasi-ideal* $\overline{\mathcal{J}}$ *of* $\mathfrak{F}$,

(5) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$, *for every* $(\overline{\lambda}, \overline{\delta})$*-IVBF right ideal* $\overline{\mathcal{T}}$ *and every* $(\overline{\lambda}, \overline{\delta})$*-IVBF bi-ideal* $\overline{\mathcal{J}}$ *of* $\mathfrak{F}$.

Some equivalent conditions are important properties for $(\overline{\lambda}, \overline{\delta})$-IVBF-subsemigroups of semigroups.

Theorem 9. *An* $(\overline{\lambda}, \overline{\delta})$*-IVBF set* $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ *is an* $(\overline{\lambda}, \overline{\delta})$*-IVBF subsemigroup of an ordered semigroup* G *if and only if* $(\overline{\mathcal{T}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$.

Proof. ($\Rightarrow$) Assume that $\overline{\mathcal{T}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of an ordered semigroup G and let $e \in G$.

If $A_e = \emptyset$, then it is easy to verify that, $(\overline{\varpi}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \leq (\overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e)$ and $(\overline{\mu}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \geq (\overline{\mu}^n)^{\overline{\lambda}}_{\overline{\delta}}(e)$.

If $A_e \neq \emptyset$, then

$$\begin{aligned}
(\overline{\varpi}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) &= (\bigvee_{(k,o)\in A_e} \{\overline{\varpi}^p(k) \wedge \overline{\varpi}^p(o)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\bigvee_{(k,o)\in A_e} \{\overline{\varpi}^p(k) \wedge \overline{\varpi}^p(o) \wedge \overline{\delta}^p\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&\leq (\bigvee_{(k,o)\in A_e} \{\overline{\varpi}^p(ko) \vee \overline{\lambda}^p\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\overline{\varpi}^p(e) \vee \overline{\lambda}^p \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\overline{\varpi}^p(e) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p = (\overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e)
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\varpi}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) &= (\bigwedge_{(k,o)\in A_e} \{\overline{\varpi}^n(k) \vee \overline{\mu}^n(o)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\bigwedge_{(k,o)\in A_e} \{\overline{\varpi}^n(k) \vee \overline{\mu}^n(o) \vee \overline{\delta}^n\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&\leq (\bigwedge_{(k,o)\in A_e} \{\overline{\varpi}^n(ko) \wedge \overline{\lambda}^n\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\overline{\varpi}^n(e) \wedge \overline{\lambda}^n \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\
&= (\overline{\varpi}^p(e) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n = (\overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e)
\end{aligned}$$

Thus, $(\overline{\varpi}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \leq (\overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e)$ and $(\overline{\varpi}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \geq (\overline{\mu}^n)^{\overline{\lambda}}_{\overline{\delta}}(e)$. Hence, $(\overline{\mathcal{T}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$.

(⇐) Suppose $(\overline{\mathcal{T}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$ and let $e_1, e_2 \in G$. Then $(\overline{\omega}^p \circ \overline{\omega}^p)^{\overline{\delta}}_{\overline{\lambda}}(e_1e_2) \leq (\overline{\omega}^p)^{\overline{\delta}}_{\overline{\lambda}}(e_1e_2)$ and $(\overline{\omega}^n \circ \overline{\omega}^n)^{\overline{\lambda}}_{\overline{\delta}}(e_1e_2) \geq (\overline{\mu}^n)^{\overline{\lambda}}_{\overline{\delta}}(r_1r_2)$. Thus

$$\begin{aligned}\overline{\omega}^p(e_1e_2) \vee \overline{\lambda}^p &\geq (\overline{\omega}^p(e_1e_2) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p = (\overline{\omega}^p)^{\overline{\delta}}_{\overline{\lambda}}(e_1e_2) \geq (\overline{\omega}^p \circ \overline{\omega}^p)^{\overline{\delta}}_{\overline{\lambda}}(e_1e_2) \\ &= (\bigvee_{(k,o)\in A_{e_1e_2}} \{\overline{\mu}^p(k) \wedge \overline{\mu}^p(o)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \geq (\overline{\mu}^p(e_1) \wedge \overline{\mu}^p(e_2) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\ &\geq \overline{\mu}^p(e_1) \wedge \overline{\mu}^p(e_2) \wedge \overline{\delta}^p\end{aligned}$$

and

$$\begin{aligned}\overline{\omega}^n(e_1e_2) \wedge \overline{\lambda}^n &\leq (\overline{\omega}^n(e_1e_2) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n = (\overline{\omega}^n)^{\overline{\lambda}}_{\overline{\delta}}(e_1e_2) \geq (\overline{\omega}^n \circ \overline{\omega}^n)^{\overline{\lambda}}_{\overline{\delta}}(e_1e_2) \\ &= (\bigwedge_{(k,o)\in A_{e_1e_2}} \{\overline{\mu}^n(k) \vee \overline{\mu}^n(o)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^p \leq (\overline{\mu}^n(e_1) \vee \overline{\mu}^n(e_2) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\ &\leq \overline{\mu}^n(e_1) \vee \overline{\mu}^n(e_2) \vee \overline{\delta}^n\end{aligned}$$

Hence, $\overline{\omega}^p(e_1e_2) \vee \overline{\lambda}^p \geq \overline{\mu}^p(e_1) \wedge \overline{\mu}^p(e_2) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(e_1e_2) \wedge \overline{\lambda}^n \leq \overline{\mu}^n(e_1) \vee \overline{\mu}^n(e_2) \vee \overline{\delta}^n$. Therefore $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of G.

Lemma 3. *[9] For an ordered semigroup G, the following conditions are equivalent.*

(1) *G is regular and intra-regular.*

(2) *Every quasi-ideal of $\mathfrak{F}$ is idempotent.*

(3) *Every bi-ideal of $\mathfrak{F}$ is idempotent.*

Theorem 10. *Let $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ and $\overline{\mathcal{J}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ be IVBF sets of an ordered semigroup G. Then the followings are equivalent.*

(1) *G is both regular and intra-regular,*

(2) $(\overline{\mathcal{T}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} = \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$ *for every $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal $\overline{\mathcal{T}}$ of $\mathfrak{F}$,*

(3) $(\overline{\mathcal{T}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} = \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$ *for every $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal $\overline{\mathcal{T}}$ of $\mathfrak{F}$,*

(4) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$ *for every $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideals $\overline{\mathcal{T}}$ and $\overline{\mathcal{J}}$ of $\mathfrak{F}$,*

(5) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$ *for every $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal $\overline{\mathcal{T}}$ and every $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal $\overline{\mathcal{J}}$ of $\mathfrak{F}$,*

(6) $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$ *for every $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideals $\overline{\mathcal{T}}$ and $\overline{\mathcal{J}}$ of $\mathfrak{F}$.*

Proof. (1) ⇒ (6) Let $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ and $\overline{\mathcal{J}} = (\overline{\varpi}^p, \overline{\varpi}^n)$ be $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideals of G and let $e \in G$. Since G is both regular and intra-regular, there exist $\mathfrak{k}, \mathfrak{l}, \mathfrak{e} \in G$ such that

$e = rkr$ and $e = km^2e$. Thus $e = rkr = rkrkm = rk(lr^2e)kr = (rklr)(relr)$. It follows that

$$\begin{aligned}(\overline{\omega}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) &= \bigvee_{(i,j)\in A_e} \{\overline{\omega}^p(i) \wedge \overline{\lambda}^p(j)\} = (\bigvee_{(i,j)\in A_{(rklr)(relr)}} \{\overline{\omega}^p(i) \wedge \overline{\varpi}^p(j)\} \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\ &\geq (\overline{\omega}^p(rklr) \wedge \overline{\varpi}^p(relr) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\ &= (\overline{\omega}^p(r(kl)r) \vee \overline{\lambda}^p \wedge \overline{\varpi}^p(r(el)r) \vee \overline{\lambda}^p \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\ &\geq (\overline{\omega}^p(e) \wedge \overline{\omega}^p(e) \wedge \overline{\delta}^p \wedge \overline{\varpi}^p(e) \wedge \overline{\varpi}^p(e) \wedge \overline{\delta}^p \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\ &= (\overline{\omega}^p(e) \wedge \overline{\delta}^p \wedge \overline{\varpi}^p(e) \wedge \overline{\delta}^p \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\ &= ((\overline{\omega}^p(e) \wedge \overline{\varpi}^p(e)) \wedge \overline{\delta}^p \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\ &= ((\overline{\omega}^p(e) \wedge \overline{\varpi}^p(e)) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p = (\overline{\omega}^p \sqcap \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e),\end{aligned}$$

and

$$\begin{aligned}(\overline{\omega}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) &= \bigwedge_{(i,j)\in A_e} \{\overline{\omega}^n(i) \vee \overline{\lambda}^n(j)\} = (\bigwedge_{(i,j)\in A_{(rklr)(relr}} \{\overline{\omega}^n(i) \vee \overline{\varpi}^n(j)\} \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\ &\leq (\overline{\omega}^n(rklr) \vee \overline{\varpi}^n(relr) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\ &= (\overline{\omega}^n(r(kl)r) \wedge \overline{\lambda}^n \vee \overline{\varpi}^n(r(el)r) \wedge \overline{\lambda}^n \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\ &\leq (\overline{\omega}^n(e) \vee \overline{\omega}^n(e) \vee \overline{\delta}^n \vee \overline{\varpi}^n(e) \vee \overline{\varpi}^n(e) \vee \overline{\delta}^n \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\ &= (\overline{\omega}^n(e) \vee \overline{\delta}^n \vee \overline{\varpi}^n(e) \vee \overline{\delta}^n \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\ &= ((\overline{\omega}^n(e) \vee \overline{\varpi}^n(e)) \vee \overline{\delta}^n \wedge \overline{\lambda}^n) \vee \overline{\delta}^n \\ &= ((\overline{\omega}^n(e) \vee \overline{\varpi}^n(e)) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n = (\overline{\omega}^n \sqcap \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e).\end{aligned}$$

Hence, $(\overline{\omega}^p \circ \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e) \geq (\overline{\omega}^p \sqcap \overline{\varpi}^p)^{\overline{\delta}}_{\overline{\lambda}}(e)$ and $(\overline{\omega}^n \circ \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e) \leq (\overline{\omega}^n \sqcap \overline{\varpi}^n)^{\overline{\lambda}}_{\overline{\delta}}(e)$.
Therefore, $(\overline{\mathcal{T}} \sqcap \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{J}})^{\overline{\lambda}}_{\overline{\delta}}$.

(6) ⇒ (5) ⇒ (4) and (3) ⇒ (2) This is obvious because every $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of G.

(4) ⇒ (2) Take $\overline{\mathcal{T}} = \overline{\mathcal{J}}$ in (4), we get $\overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}} = (\overline{\mathcal{T}} \sqcap \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq (\overline{\mathcal{T}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}}$. Since every $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of G and by Theorem 9, we have $(\overline{\mathcal{T}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} \sqsubseteq \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$. Thus, $(\overline{\mathcal{T}} \circ \overline{\mathcal{T}})^{\overline{\lambda}}_{\overline{\delta}} = \overline{\mathcal{T}}^{\overline{\lambda}}_{\overline{\delta}}$.

(2) ⇒ (1) Let Q be a quasi-ideal of G. Then by Theorem 6, $\overline{\chi}_Q = (G; \overline{\chi}^p_Q, \overline{\chi}^n_Q)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF quasi-ideal of G. By supposition and Thoerem7, we have

$$(\overline{\chi}^p_{(Q^2]})^{\overline{\delta}}_{\overline{\lambda}}(e) = (\overline{\chi}^p_Q \circ \overline{\chi}^p_Q)^{\overline{\delta}}_{\overline{\lambda}}(e) = (\overline{\chi}^p_Q)^{\overline{\delta}}_{\overline{\lambda}}(e) = \overline{\delta}^p,$$

and

$$(\overline{\chi}^n_{(Q^2]})^{\overline{\lambda}}_{\overline{\delta}}(e) = (\overline{\chi}^n_Q \circ \overline{\chi}^n_Q)^{\overline{\lambda}}_{\overline{\delta}}(e) = (\overline{\chi}^n_Q)^{\overline{\lambda}}_{\overline{\delta}}(e) = \overline{\delta}^n.$$

Thus, $(QQ] = Q$. An application of Lemma 3 shows us that G is both regular and intra-regular.

5. Conclusion

The theory of fuzzy sets, initially introduced by L. A. Zadeh, was later extended to interval-valued fuzzy sets. Building upon this foundation, K. Arulmozhi et al. explored interval-valued bipolar fuzzy sets in algebraic structures. In 2021, S. Lekkoksung advanced this area by developing the concept of interval-valued bipolar fuzzy ideals in ordered semigroups and characterized regular ordered semigroups in terms of generalized interval-valued bipolar fuzzy ideals and bi-ideals. In this paper, we introduce new definitions of generalized interval-valued bipolar fuzzy quasi-ideals and establish their properties. Using intra-regular ordered semigroups, we prove several properties of these generalized quasi-ideals. Furthermore, we provide a characterization of intra-regular ordered semigroups through the framework of generalized interval-valued bipolar fuzzy quasi-ideals. We hope that the study of intra-regular ordered semigroups in terms of generalized interval valued bipolar fuzzy quasi-ideal are useful mathematical tools. In the future, we study characterized semisimple ordered semigroups in terms of generalized interval valued bipolar fuzzy interior ideals.

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