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On the Periodicity Solutions of Five Systems of Rational Systems of Difference Equations of Order Five

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Abstract. The primary aim of this paper is to explore the behavior of several nonlinear systems of difference equations following

$$T_{n+1} = \frac{E_n E_{n-4}}{T_{n-3}(\pm 1 \pm E_n E_{n-4})}, \quad E_{n+1} = \frac{T_n T_{n-4}}{E_{n-3}(\pm 1 \pm T_n T_{n-4})},$$

and obtain solution expressions for them. Moreover, we utilize MATLAB programming to simulate the dynamics and validate our results.

2020 Mathematics Subject Classifications: 39A10

Key Words and Phrases: Solutions of difference equations, periodic solution, recursive sequences

1. Introduction

The final 25 years of the 20th century saw significant advancements in the theory of discrete dynamical systems (DDSs) and difference equations (DEs). Recently, a wide variety of fields, including biology, economics, physics, resource management, and others, have seen the use of DDSs and DEs.A key role in practical analysis is played by the theory of DIFEs. It is improbable that the theory of DEs won't keep playing a significant part in mathematics as a whole. In applications, nonlinear difference equations (NDEs) of order greater than one are crucial. These equations also naturally arise as discrete analogs and numerical solutions of differential and delay differential equations, which model a wide range of diverse phenomena in biology, ecology, psychology, engineering, physics, probability theory, economics, genetics. Finding out how a system of higher-order rational difference equations (RDEs) behaves and talking about how stable its equilibrium points are locally asymptotically is quite interesting. Many articles cover the DEs system [1–22] . For example, In [7] El-Metwally discussed how various systems of third order RDEs with initial conditions involving non-zero real numbers should be solved. Additionally,

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he showed additional numerical examples and checked into some of the properties of the obtained solutions

$$T_{n+1} = \frac{T_{n-1}R_n}{\pm T_{n-1} \pm R_{n-2}}, \quad R_{n+1} = \frac{T_n R_{n-1}}{\pm R_{n-1} \pm T_{n-2}}.$$

El-Dessoky [6] studied the presence of solutions in the case of four dimensions for a class of rational systems of differential equations (SDEs) of order four.

$$X_{n+1} = \frac{X_{n-3}}{\pm 1 \pm T_n Z_{n-1} Y_{n-2} X_{n-3}}, \quad Y_{n+1} = \frac{Y_{n-3}}{\pm 1 \pm X_n T_{n-1} Z_{n-2} Y_{n-3}}.$$
$$Z_{n+1} = \frac{Z_{n-3}}{\pm 1 \pm Y_n X_{n-1} T_{n-2} Z_{n-3}}, \quad T_{n+1} = \frac{T_{n-3}}{\pm 1 \pm Z_n Y_{n-1} X_{n-2} T_{n-3}},$$

Elsayed and Alof [11] obtained formulas expressions for solutions of the following fractional SDEs

$$X_{n+1} = \frac{X_{n-3}Y_{n-4}}{Y_n(\pm 1 - X_{n-3}Y_{n-4}R_{n-1}T_{n-2})}, \quad Y_{n+1} = \frac{Y_{n-3}R_{n-4}}{R_n(\pm 1 - Y_{n-3}R_{n-4}T_{n-1}X_{n-2})},$$
$$R_{n+1} = \frac{R_{n-3}T_{n-4}}{T_n(\pm 1 \pm R_{n-3}T_{n-4}X_{n-1}Y_{n-2})}, \quad T_{n+1} = \frac{T_{n-3}X_{n-4}}{X_n(\pm 1 \pm T_{n-3}X_{n-4}Y_{n-1}R_{n-2})}.$$

Mansour et al. [20] checked the behavior of solutions of the SDEs

$$W_{n+1} = \frac{W_{n-5}}{-1 + W_{n-5}R_{n-2}}, \quad R_{n+1} = \frac{R_{n-5}}{\pm 1 \pm R_{n-5}W_{n-2}}.$$

DEs are also suitable models to describe circumstances where overlapping generations and seasonal population growth occur. The generalized Beverton-Holt stock recruitment mode has been studied by researchers in [3]

$$T_{n+1} = \alpha T_n + \frac{\beta T_{n-1}}{1 + \gamma T_{n-1} + \eta T_n}.$$

The following system of discrete-time two-predators and the one-prey Lot was explored dynamically by Khaliq et al.[14]

$$X_{n+1} = \frac{aX_n - cX_nY_n - eX_nZ_n}{1 + dX_n}, \quad Y_{n+1} = \frac{bY_n + tX_nY_n - pY_nZ_n}{1 + wY_n}, \quad Z_{n+1} = \frac{fZ_n + rX_nZ_n - hY_nZ_n}{1 + kZ_n}.$$

Din and Elsayed [5] evaluated the two-directional interacting and invasive species model's boundedness nature, persistence, local and global behavior

$$R_{n+1} = \eta + \alpha R_n + \beta R_{n-1} e^{-T_n}, \quad T_{n+1} = \gamma + cT_n + wT_{n-1} e^{-R_n}.$$

In a discrete-time COVID-19 epidemic model,The authors [18] used chaos management, bifurcation analysis, and topological classifications to study local dynamics. See also [19, 21, 22].

Motivated by the aforementioned researches, the objective of this work is to ascertain whether solutions to the SDEs exist in the two-dimensional instances

$$T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(\pm 1 \pm E_nE_{n-4})}, \quad E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(\pm 1 \pm T_nT_{n-4})},$$

the initial conditions are arbitrary nonzero real numbers.

2. The System $T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(1+E_nE_{n-4})}$, $E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(-1+T_nT_{n-4})}$

In this section, we give a specific form the solutions of the SDE in the form:

$$T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(1+E_nE_{n-4})}, \quad E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(-1+T_nT_{n-4})}. \tag{1}$$

Theorem 1. Assume $\{T_n, E_n\}$ are solutions of Eq.(1). Then, for all $n = 0, 1, 2, ...$, the following formulas yield periodic solutions of Eq. (1) with period eight.

$$T_{8n-4} = s, \qquad T_{8n-3} = d,$$

$$T_{8n-2} = c, \qquad T_{8n-1} = b,$$

$$T_{8n} = a, \qquad T_{8n+1} = \frac{fw}{d(1+fw)},$$

$$T_{8n+2} = \frac{as}{c(-1+2as)}, \qquad T_{8n+3} = \frac{-fw}{b(1-fw)},$$

$$E_{8n-4} = w, \qquad E_{8n-3} = m,$$

$$E_{8n-2} = g, \qquad E_{8n-1} = k,$$

$$E_{8n} = f, \qquad E_{8n+1} = \frac{as}{m(-1+as)},$$

$$E_{8n+2} = \frac{-fw}{g}, \qquad E_{8n+3} = \frac{as}{k(1-as)},$$

where $T_0 = a,\ T_{-1} = b,\ T_{-2} = c,\ T_{-3} = d,\ T_{-4} = s,\ E_0 = f,\ E_{-1} = k,\ E_{-2} = g,\ E_{-3} = m,$ and $E_{-4} = w$. Also, where $E_0E_{-4} \neq \pm 1$ and $T_0T_{-4} \neq \pm 1$.

Proof. For $n = 0$, the outcome is valid. Let us now assume that $n > 0$ and that $n - 1$ agrees with our assumption. That's

$$T_{8n-12} = s, \qquad T_{8n-11} = d,$$

$$T_{8n-10} = c, \qquad T_{8n-9} = b,$$

$$T_{8n-8} = a, \qquad T_{8n-7} = \frac{fw}{d(1+fw)},$$

$$T_{8n-6} = \frac{as}{c(-1+2as)}, \qquad T_{8n-5} = \frac{-fw}{b(1-fw)},$$

$$E_{8n-12} = w, \qquad E_{8n-11} = m,$$

$$E_{8n-10} = g, \qquad E_{8n-9} = k,$$

$$E_{8n-8} = f, \qquad E_{8n-7} = \frac{as}{m(-1+as)},$$

$$E_{8n-6} = \frac{-fw}{g}, \qquad E_{8n-5} = \frac{as}{k(1-as)}.$$

Next, from system (1) we have

$$T_{8n} = \frac{E_{8n-1}E_{8n-5}}{T_{8n-4}(1+E_{8n-1}E_{8n-5})} = \frac{\frac{kas}{k(1-as)}}{s(1+\frac{kas}{k(1-as)})} = a.$$

$$E_{8n} = \frac{T_{8n-1}T_{8n-5}}{E_{8n-4}(-1+T_{8n-1}T_{8n-5})} = \frac{\frac{-bfw}{b(1-fw)}}{w(-1-\frac{fwb}{b(1-fw)})} = f.$$

$$T_{8n-4} = \frac{E_{8n-5}E_{8n-9}}{T_{8n-8}(1+E_{8n-5}E_{8n-9})} = \frac{\frac{ask}{k(1-as)}}{a(1+\frac{kas}{k(1-as)})} = s.$$

$$E_{8n-4} = \frac{T_{8n-5}T_{8n-9}}{E_{8n-8}(-1+T_{8n-5}T_{8n-9})} = \frac{\frac{-bfw}{b(1-fw)}}{f(-1-\frac{fwb}{b(1-fw)})} = w.$$

Similarly, we can prove the remaining relations. The proof is complete.

Example 1. Figure (1) demonstrates the behavior of the solutions of the SDE Eq.(1) with $T_0 = 1$, $T_{-1} = 0.5$, $T_{-2} = 0.2$, $T_{-3} = 2$, $T_{-4} = 3$, $E_0 = 0.1$, $E_{-1} = 0.5$, $E_{-2} = 1$, $E_{-3} = 0.2$, and $E_{-4} = 0.3$.

3. The System $T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(1+E_nE_{n-4})}$, $E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(1-T_nT_{n-4})}$

In this section, For the aforementioned system, we provide the period eight periodic solutions and solution expression.

$$T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(1+E_nE_{n-4})}, \quad E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(1-T_nT_{n-4})}. \tag{2}$$

Theorem 2. Suppose that $\{T_n, E_n\}$ be solutions of the system of SDEs (2). Then for n = 0, 1, 2, .., the solutions of Eq.(2) can be formed as follows

$$T_{8n-4} = s, \qquad T_{8n-3} = d,$$

$$T_{8n-2} = c, \qquad T_{8n-1} = b,$$

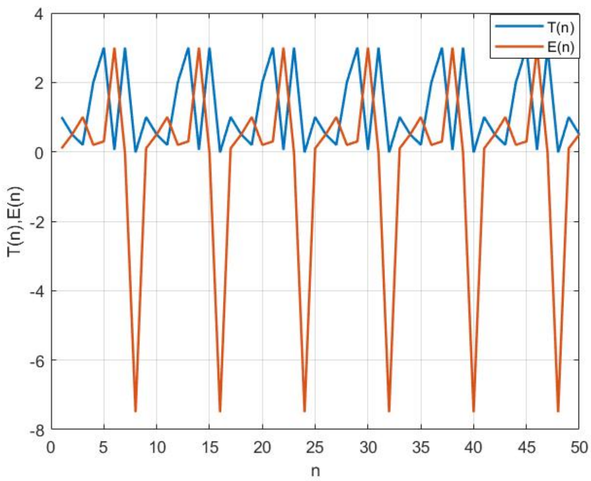

Figure 1: Periodic solutions of Eq.(1)

$$T_{8n} = a, \qquad T_{8n+1} = \frac{fw}{d(1+fw)},$$

$$T_{8n+2} = \frac{as}{c}, \qquad T_{8n+3} = \frac{fw}{b(1+fw)},$$

$$E_{8n-4} = w, \qquad E_{8n-3} = m,$$

$$E_{8n-2} = g, \qquad E_{8n-1} = k,$$

$$E_{8n} = f, \qquad E_{8n+1} = \frac{as}{m(1-as)},$$

$$E_{8n+2} = \frac{fw}{g}, \qquad E_{8n+3} = \frac{as}{k(1-as)},$$

where $E_0E_{-4} \neq -1$ and $T_0T_{-4} \neq 1$.

Proof. For $n = 0$, the outcome is valid. Assume for the moment that $n > 0$ and that $n-1$ falls under our hypothesis. That's

$$T_{8n-12} = s, \qquad T_{8n-11} = d,$$

$$T_{8n-10} = c, \qquad T_{8n-9} = b,$$

$$T_{8n-8} = a, \qquad T_{8n-7} = \frac{fw}{d(1+fw)},$$

$$T_{8n-6} = \frac{as}{c}, \qquad T_{8n-5} = \frac{fw}{b(1+fw)},$$

$$E_{8n-12} = w, \qquad E_{8n-11} = m,$$

$$E_{8n-10} = g, \qquad E_{8n-9} = k,$$

$$E_{8n-8} = f, \qquad E_{8n-7} = \frac{as}{m(-1+as)},$$

$$E_{8n-6} = \frac{fw}{g}, \qquad E_{8n-5} = \frac{as}{k(1-as)}.$$

Next, from system (2) we have

$$T_{8n+1} = \frac{E_{8n}E_{8n-4}}{T_{8n-3}(1+E_{8n}E_{8n-4})} = \frac{fw}{d(1+fw)}.$$

$$E_{8n+1} = \frac{T_{8n}T_{8n-4}}{E_{8n-3}(1-T_{8n}T_{8n-4})} = \frac{as}{m(1-as)}.$$

$$T_{8n} = \frac{E_{8n-1}E_{8n-5}}{T_{8n-4}(1+E_{8n-1}E_{8n-5})} = \frac{\frac{ask}{k(1-as)}}{s(1+\frac{kas}{k(1-as)})} = a.$$

$$E_{8n} = \frac{T_{8n-1}T_{8n-5}}{E_{8n-4}(-1+T_{8n-1}T_{8n-5})} = \frac{\frac{bfw}{b(1+fw)}}{w(1-\frac{fwb}{b(1+fw)})} = f.$$

We can confirm the other forms using the same method. The evidence is finished.

Example 2. Figure (2) illustrates the behavior of the solutions of the SDEs (2) with $T_0 = 2,\ T_{-1} = 1.5,\ T_{-2} = 0.2,\ T_{-3} = 0.4,\ T_{-4} = 1,\ E_0 = 0.1,\ E_{-1} = 0.6,\ E_{-2} = 3,\ E_{-3} = 1,$ and $E_{-4} = 0.3$.

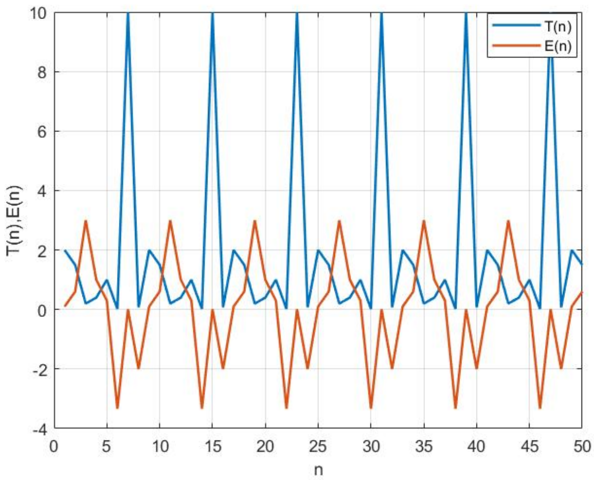

Figure 2: Chart the behavior of solutions of the SDEs(2)

4. The System $T_{n+1}=\frac{E_nE_{n-4}}{T_{n-3}(1+E_nE_{n-4})},\ E_{n+1}=\frac{T_nT_{n-4}}{E_{n-3}(-1-T_nT_{n-4})}$

In this section, we find form the solutions of the SDEs in the form:

$$T_{n+1}=\frac{E_nE_{n-4}}{T_{n-3}(1+E_nE_{n-4})},\quad E_{n+1}=\frac{T_nT_{n-4}}{E_{n-3}(-1-T_nT_{n-4})}. \tag{3}$$

Theorem 3. Suppose that $\{T_n,E_n\}$ are solutions of Eq.(3). After that, all of solutions of Eq.(3) are periodic with period eight and given by the following formulas for n=0,1,2,...
,

$$T_{8n-4}=s,\qquad T_{8n-3}=d,$$
$$T_{8n-2}=c,\qquad T_{8n-1}=b,$$
$$T_{8n}=a,\qquad T_{8n+1}=\frac{fw}{d(1+fw)},$$
$$T_{8n+2}=\frac{-as}{c},\qquad T_{8n+3}=\frac{fw}{b(-1-fw)},$$

$$E_{8n-4}=w,\qquad E_{8n-3}=m,$$
$$E_{8n-2}=g,\qquad E_{8n-1}=k,$$
$$E_{8n}=f,\qquad E_{8n+1}=\frac{as}{m(-1-as)},$$
$$E_{8n+2}=\frac{fw}{g(-1-2fw)},\qquad E_{8n+3}=\frac{-as}{k(-1+as)},$$

where $E_0E_{-4}\neq\pm1$ and $T_0T_{-4}\neq\pm1$.

Proof. The outcome is valid for $n=0$. Let us now assume that $n>0$ and that $n-1$ is consistent with our assumption. That is

$$T_{8n-12}=s,\qquad T_{8n-11}=d,$$
$$T_{8n-10}=c,\qquad T_{8n-9}=b,$$
$$T_{8n-8}=a,\qquad T_{8n-7}=\frac{fw}{d(1+fw)},$$
$$T_{8n-6}=\frac{-as}{c},\qquad T_{8n-5}=\frac{fw}{b(-1-fw)},$$

$$E_{8n-12}=w,\qquad E_{8n-11}=m,$$
$$E_{8n-10}=g,\qquad E_{8n-9}=k,$$
$$E_{8n-8}=f,\qquad E_{8n-7}=\frac{as}{m(-1-as)},$$
$$E_{8n-6}=\frac{fw}{g(-1-2fw)},\qquad E_{8n-5}=\frac{-as}{k(-1+as)}.$$

Next, from system (3) we have

$$T_{8n+2} = \frac{E_{8n+1}E_{8n-3}}{T_{8n-2}(1+E_{8n+1}E_{8n-3})} = \frac{\frac{mas}{m(-1-as)}}{c(1+\frac{mas}{m(-1-as)})} = \frac{-as}{c}.$$

$$E_{8n+2} = \frac{T_{8n+1}T_{8n-3}}{E_{8n-2}(-1-T_{8n+1}T_{8n-3})} = \frac{\frac{fwd}{d(1+fw)}}{g(-1-\frac{fwd}{d(1+fw)})} = \frac{fw}{g(-1-2fw)}.$$

$$T_{8n+3} = \frac{E_{8n+2}E_{8n-2}}{T_{8n-1}(1+E_{8n+2}E_{8n-2})} = \frac{\frac{fwg}{g(-1-2fw)}}{b(1+\frac{fwg}{g(-1-2fw)})} = \frac{fw}{b(-1-fw)}.$$

$$E_{8n+3} = \frac{T_{8n+2}T_{8n-2}}{E_{8n-1}(-1-T_{8n+2}T_{8n-2})} = \frac{\frac{-asc}{c}}{k(-1+\frac{asc}{c}} = \frac{-as}{k(-1+as)}.$$

We can validate the other forms by using the same procedure. The proof is complete.
Example 3. The solution is periodic of period eight when Theorem 3 is met and the initial values are $T_0 = 3$, $T_{-1} = 1$, $T_{-2} = 0.1$, $T_{-3} = 2$, $T_{-4} = 0.2$, $E_0 = 1$, $E_{-1} = 0.5$, $E_{-2} = 0.3$, $E_{-3} = 2$, and $E_{-4} = 0.9$, as shown in Figure (3).

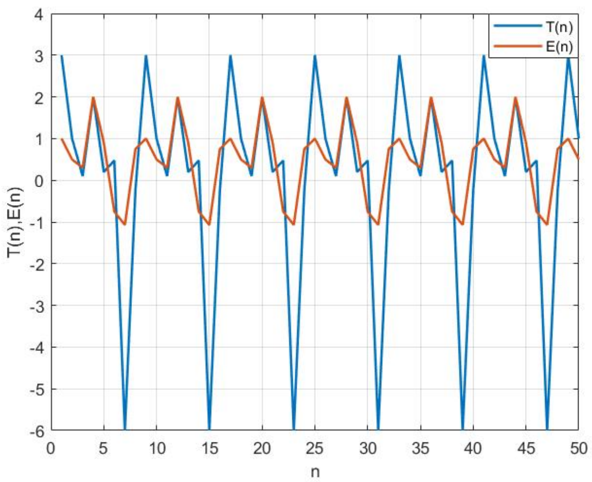

Figure 3: Draw a graph displaying periodicity of the solutions of SDEs (3)

5. The System $T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(-1+E_nE_{n-4})}$, $E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(-1+T_nT_{n-4})}$

In this section, we investigate the solutions of the SDEs in the form:

$$T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(-1+E_nE_{n-4})}, \quad E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(-1+T_nT_{n-4})}. \tag{4}$$

Theorem 4. Consider $\{T_n, E_n\}$ are solutions of Eq.(4). Afterwards, for all n = 0, 1, 2,..., the following formulas obtain periodic solutions of Eq. (4) with period eight.

$$T_{8n-4} = s, \qquad T_{8n-3} = d,$$
$$T_{8n-2} = c, \qquad T_{8n-1} = b,$$
$$T_{8n} = a, \qquad T_{8n+1} = \frac{fw}{d(-1+fw)},$$
$$T_{8n+2} = \frac{as}{c}, \qquad T_{8n+3} = \frac{fw}{b(-1+fw)},$$

$$E_{8n-4} = w, \qquad E_{8n-3} = m,$$
$$E_{8n-2} = g, \qquad E_{8n-1} = k,$$
$$E_{8n} = f, \qquad E_{8n+1} = \frac{as}{m(-1+as)},$$
$$E_{8n+2} = \frac{fw}{g}, \qquad E_{8n+3} = \frac{as}{k(-1+as)},$$

where $E_0 E_{-4} \neq 1$ and $T_0 T_{-4} \neq 1$.

Proof. The result is valid for $n = 0$. Let us now assume that $n > 0$ and that $n - 1$ agrees with our assumption. That is

$$T_{8n-12} = s, \qquad T_{8n-11} = d,$$
$$T_{8n-10} = c, \qquad T_{8n-9} = b,$$
$$T_{8n-8} = a, \qquad T_{8n-7} = \frac{fw}{d(-1+fw)},$$
$$T_{8n-6} = \frac{as}{c}, \qquad T_{8n-5} = \frac{fw}{b(-1+fw)},$$

$$E_{8n-12} = w, \qquad E_{8n-11} = m,$$
$$E_{8n-10} = g, \qquad E_{8n-9} = k,$$
$$E_{8n-8} = f, \qquad E_{8n-7} = \frac{as}{m(-1+as)},$$
$$E_{8n-6} = \frac{fw}{g}, \qquad E_{8n-5} = \frac{as}{k(-1+as)}.$$

Next, from system (4) we have

$$T_{8n-4} = \frac{E_{8n-5}E_{8n-9}}{T_{8n-8}(-1+E_{8n-5}E_{8n-9})} = \frac{\frac{ask}{k(-1+as)}}{a(-1+\frac{ask}{k(-1+as)})} = s.$$

$$E_{8n-4} = \frac{T_{8n-5}T_{8n-9}}{E_{8n-8}(-1+T_{8n-5}T_{8n-9})} = \frac{\frac{fwb}{b(-1+fw)}}{f(-1+\frac{fwb}{b(-1+fw)})} = w.$$

$$T_{8n} = \frac{E_{8n-1}E_{8n-5}}{T_{8n-4}(-1+E_{8n-1}E_{8n-5})} = \frac{\frac{ask}{k(-1+as)}}{s(-1+\frac{kas}{k(-1+as)})} = a.$$

$$E_{8n} = \frac{T_{8n-1}T_{8n-5}}{E_{8n-4}(-1+T_{8n-1}T_{8n-5})} = \frac{\frac{bfw}{b(-1+fw)}}{w(-1+\frac{fwb}{b(-1+fw)})} = f.$$

Also, we can prove the other relations. This completes the proof.

Example 4. We assume that $T_0 = 0.1,\ T_{-1} = 2,\ T_{-2} = 0.5,\ T_{-3} = 1,\ T_{-4} = 0.3,\ E_0 = 0.1,\ E_{-1} = 1,\ E_{-2} = 0.6,\ E_{-3} = 1.1,$ and $E_{-4} = 0.7$ for the Eq.(4). (See Fig. 4).

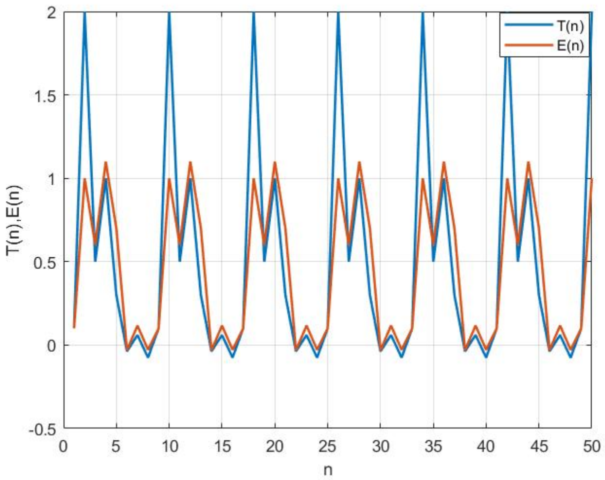

Figure 4: Sketch the periodicity of the solution of Eq.(4)

6. The System $T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(-1-E_nE_{n-4})},\ E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(1-T_nT_{n-4})}$

In this section, we obtain the form of the solutions of the SDEs

$$T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(-1-E_nE_{n-4})}, \quad E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(1-T_nT_{n-4})}. \tag{5}$$

Theorem 5. Let that $\{T_n, E_n\}$ are solutions of Eq. (5). Next, each of solutions Eq.(5) are periodic, having a period of eight, and is given by formulas for n = 0, 1, 2,...,

$$T_{8n-4} = s, \qquad T_{8n-3} = d,$$

$$T_{8n-2} = c, \qquad T_{8n-1} = b,$$

$$T_{8n} = a, \qquad T_{8n+1} = \frac{fw}{d(-1-fw)},$$

$$T_{8n+2} = \frac{-as}{c}, \qquad T_{8n+3} = \frac{fw}{b(1+fw)},$$

$$E_{8n-4} = w, \qquad E_{8n-3} = m,$$

$$E_{8n-2} = g, \qquad E_{8n-1} = k,$$

$$E_{8n} = f, \qquad E_{8n+1} = \frac{as}{m(1-as)},$$

$$E_{8n+2} = \frac{fw}{g(-1-2fw)}, \qquad E_{8n+3} = \frac{-as}{k(1+as)},$$

where $E_0E_{-4} \neq \pm 1$ and $T_0T_{-4} \neq \pm 1$.

Proof. For $n = 0$, the result is true. Now, let's suppose that $n > 0$ and that $n-1$ supports our hypothesis. That's

$$T_{8n-12} = s, \qquad T_{8n-11} = d,$$

$$T_{8n-10} = c, \qquad T_{8n-9} = b,$$

$$T_{8n-8} = a, \qquad T_{8n-7} = \frac{fw}{d(-1-fw)},$$

$$T_{8n-6} = \frac{-as}{c}, \qquad T_{8n-5} = \frac{fw}{b(1+fw)},$$

$$E_{8n-12} = w, \qquad E_{8n-11} = m,$$

$$E_{8n-10} = g, \qquad E_{8n-9} = k,$$

$$E_{8n-8} = f, \qquad E_{8n-7} = \frac{as}{m(1-as)},$$

$$E_{8n-6} = \frac{fw}{g(-1-2fw)}, \qquad E_{8n-5} = \frac{-as}{k(1+as)}.$$

Next, from system (5) we have

$$T_{8n+1} = \frac{E_{8n}E_{8n-4}}{T_{8n-3}(-1-E_{8n}E_{8n-4})} = \frac{fw}{d(-1-fw)}.$$

$$E_{8n+1} = \frac{T_{8n}T_{8n-4}}{E_{8n-3}(1-T_{8n}T_{8n-4})} = \frac{as}{m(1-as)}.$$

$$T_{8n+3} = \frac{E_{8n+2}E_{8n-2}}{T_{8n-1}(-1-E_{8n+2}E_{8n-2})} = \frac{\frac{fwg}{g(-1-2fw)}}{b(-1-\frac{fwg}{g(-1-2fw)})} = \frac{fw}{b(1+fw)}.$$

$$E_{8n+3} = \frac{T_{8n+2}T_{8n-2}}{E_{8n-1}(1 - T_{8n+2}T_{8n-2})} = \frac{\frac{-asc}{c}}{k(1 - \frac{-asc}{c})} = \frac{-as}{k(1 + as)}.$$

We can prove the other relations as well. This brings the proof to a close.

Example 5. See Figure (5) where we takes system (5) with $T_0 = 0.5,\ T_{-1} = 4,\ T_{-2} = 7,\ T_{-3} = 0.5,\ T_{-4} = 1,\ E_0 = 1.2,\ E_{-1} = 0.5,\ E_{-2} = 3,\ E_{-3} = 4,$ and $E_{-4} = 0.2$.

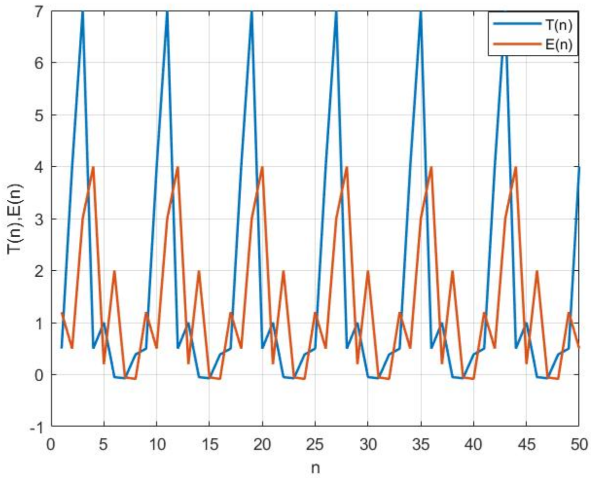

Figure 5: Chart the behavior of the solutions of system (5)

Remark 1 The solutions of the following systems can be also obtained .

$$T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(1 + E_nE_{n-4})}, \quad E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(1 + T_nT_{n-4})}.$$

$$T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(-1 + E_nE_{n-4})}, \quad E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(-1 - T_nT_{n-4})}.$$

$$T_{n+1} = \frac{E_nE_{n-4}}{T_{n-3}(-1 + E_nE_{n-4})}, \quad E_{n+1} = \frac{T_nT_{n-4}}{E_{n-3}(1 - T_nT_{n-4})}.$$

7. Conclusion

This study investigates the solutions of five SDEs. Across five sections, we derive periodic solutions with a period of eight for each system. To corroborate our theoretical findings, numerical examples are presented for each system, with Figures 1-5 providing visual confirmation of the results.

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