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New Results on Difference Paracompactness in Topological Spaces

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Abstract. An interesting area of research in topology is $D-$paracompact spaces. It is a significant type of topological spaces that retain compactness while benefiting from paracompactness, which is considered a generalisation of compact spaces. The concept of $D-$paracompactness was introduced, and its basic characteristics were examined by the author in [18]. In this research, we introduce and improve this concept further by using a special type of covering and the difference sets (called $D-$sets), which contain new and impact properties. As a result, we obtained several new properties and results. We discuss the concept, characteristics, and theorems that related of $D-$paracompact space. We also studied different characterizations of $D-$paracompact spaces and discussed how they relate to other topological characteristics. We also give numerous instances of $D-$paracompact spaces along with highlighting their applicability in different topological spaces.

2020 Mathematics Subject Classifications: 54B05, 54B10, 54C05, 54D05, 54D10, 54D30, 54E55

Key Words and Phrases: Topological spaces, Paracompact spaces, $D-$paracompact spaces, Countably $D-$paracompact spaces

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1. Overview of the Historical Developments and Progress

Soon after the notion of paracompact spaces was introduced, several articles were published that examine the idea from various angles and tools. In the year 1944, Dieudonnè [7] first suggested the notion of paracompact space as a concept that is the extension of compactness. Later, in 1969, Singal and Arya [20] provided concept a new type of paracompactness named nearly paracompactness, weaker than the usual paracompactness, that determines its basic topological properties. Based on that, in 1978, Steen and Seebach [21] introduced the notion of metacompact in the topological space (X, τ). In 1982, Tong [22] introduced the notion of difference sets, often known as $D-$sets. Also, in 1984, Pareek [16], contributed the concept of $D-$paracompact and studied their properties and relations with other topological spaces. In 1998, Mukherjee and Debray [15] used paracompact to give, the same definition. It has been investigated in terms of a certain type of cover, called regular even cover. While, in 2006, the topologists Al-Ghour [13] defined the concepts of $\omega-$paracompactness and countable $\omega-$paracompactness as generalizations of paracompactness. He explained characterized each of them, which he studied dealt with subspaces, products, and mappings of each. Also, in 2006, Al-Zoubi [2] as a generalization of paracompact spaces, he uses the semi$-$open sets to developed the class of $S-$paracompact space and define $S-$paracompact spaces, he also investigated their fundamental characteristics. The investigations are made into the connections between $S-$paracompact spaces and other well-known spaces. Later, in 2007, Al-Zoubi and Al-Ghour [1] provided and studied P_3-paracompact, a weaker variant of paracompactness. With it and its interactions with other spaces, they obtained a variety of characterizations, properties, instances, and counterexamples. In 2013, Demir and Ozbakir [6] defined the concepts $\beta-$paracompact spaces and $\beta-$expandable spaces as a weak variant of paracompact and expandable spaces, respectively. These spaces' basic characteristics were also provided. Additionally, it is established that every $\beta-$paracompact space is a $\beta-$expandable space, and the connections between these spaces and a few previously researched spaces are investigated. In 2019, Turanli and Ozbakir [23], introduced and investigated $\beta_1 - L-$paracompact space, which is a stronger variant of $L-$paracompact space established on an ideal space. After that, they looked into connections between $\beta_1 - L-$paracompact spaces and other paracompactnesses. Additionally, they discovered numerous characteristics, instances, and counterexamples of $\beta_1 - L-$paracompactness. In 2021, Al Ghour [12], introduced both concepts of $\sigma - \omega-$paracompactness and feebly $\omega-$paracompactness, with $\omega-$paracompactness being a weaker version of $\sigma - \omega-$paracompactness. Furthermore, In 2021, Oudetallah, et al. [16] introduced the notions of $D-$metacompact spaces and studied their properties. However, for more studies, you can see [14], [3], and [10]. The present research is mainly a continuation of the study of $D-$paracompact spaces from new viewpoints. We have aimed to eventually arrive at new concepts as well as a develop several characteristic theorems and instances, all of which we present in this research.

2. Preliminary and Basic Notions

Within this section, we include some essential symbols, such as $\mathbb{R}$ to denote the collection of real numbers. Also, ϑ_u, ϑ_{cof}, $\vartheta_{l.r}$, ϑ_{ind}, and ϑ_{dis} will denote the standard, co-finite, left-ray, indiscreet, and discrete topologies, respectively. Now, we provide some basic definitions as well as the major results that are required.

Definition 1. [7] *If any open cover of the topological space (W, ϑ) has an open locally−finite refinement, then the space is called paracompact space.*

Definition 2. [8] *If any countable open cover of the topological space (W, ϑ) has an open locally−finite refinement, then the space is called countably paracompact.*

Definition 3. [23] *If any open cover of the topological space (W, ϑ) has an open locally-countable refinement, then the space is called paralindelöf space.*

Definition 4. [22] *A subset $W_1 \subseteq (W, \vartheta)$ is $D-$set if there are open sets A and B such that $A \neq W$ and $W_1 = A - B$. Furthermore, the subset W_1 is $D-$set that generated by A and B.*

Definition 5. [19] *A cover $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ of the space (W, ϑ) is called $D-$cover if any E_ρ is $D-$set, for every $\rho \in \Lambda$.*

Definition 6. *If any $D-$cover of the topological space (W, ϑ) has an open locally−countable refinement, then the space is called $D-$paralindelöf space.*

Definition 7. [19] *If any $D-$cover of the topological space (W, ϑ) has a countable subcover, then the space is called $D-$lindelöf.*

Definition 8. [19] *If any $D-$cover of the topological space (W, ϑ) has a finite−subcover, then the space is called $D-$compact.*

Definition 9. [11] *A subset $W_1 \subseteq (W, \vartheta)$ is called $D-$dense set, if for any $y \in W$, we have $D_y \cap W_1 \neq \phi$, and each $D-$set containing y.*

Definition 10. [11] *The space (W, ϑ) is called $D-$separable, if the space has a subset H, which is $D-$dense countable.*

Definition 11. [17] *The space (W, ϑ) is called locally−indiscreet if any set ϑ_n-open is ϑ_n-clopen, where $n = 1, 2$.*

Definition 12. [9] *If any cover of the topological space (W, ϑ) has a countable subcover, then the space is called lindelöf space.*

Definition 13. [11] *Let (W, ϑ) and (Q, ι) be any topological spaces. If the inverse image of any open subset in Q of $\varphi : W \longrightarrow Q$ is open in W, then the function φ is called continuous function. That is, if $x \in \iota$, then $\varphi^{-1}(x) \in \vartheta$ is its inverse image.*

Definition 14. [5] *Let (W, ϑ) and (Q, ι) be any topological spaces. If the image $\varphi(A)$ is open in Q for each open set A in W, then the function $\varphi : W \longrightarrow Q$ is called open.*

Definition 15. [9] *Let (W, ϑ) and (Q, ι) be any topological spaces. If the image $\varphi(F)$ is closed in Q for each closed set F in W, then the function $\varphi : W \longrightarrow Q$ is called closed.*

Definition 16. [4] *If the function $\varphi : (W, \vartheta) \longrightarrow (Q, \iota)$ is continuous, $D-$compact and closed for each $q \in Q$, then the function φ is called $D-$perfect.*

Theorem 1. [19] *Any open cover is $D-$cover.*

3. New Results of $D-$Paracompact Spaces

This section presents the new notion of $D-$paracompact spaces, and explores their connections to other spaces,and introduces a new set of properties.

Definition 17. *A topological space (W, ϑ) is called $D-$paracompact if every $D-$cover of the space (W, ϑ) has an open locally-finite refinement.*

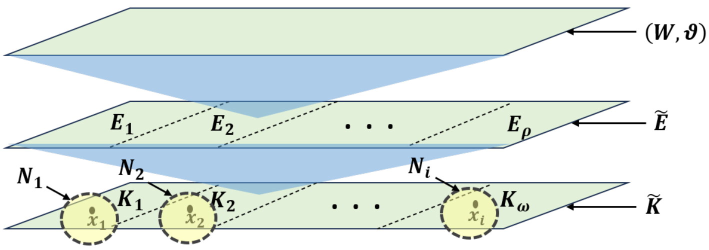

Figure 1: The space of Difference paracompact.

Figure 1 illustrates the space of $D-$paracompact such that $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is $D-$cover of the topological space (W, ϑ) has an open locally$-$finite refinement $\tilde{K} = \{K_\omega : \omega \in \Omega\}$. i.e. each E_ρ is a $D-$set for all $\rho \in \Lambda$ such that $W = \cup_{\rho\in\Lambda} E_\rho$ has an open locally$-$finite refinement $\tilde{K}$, which the open refinement $\tilde{K}$ is a locally$-$finite if every point x_i of the space W has a neighborhood N_i such that $E_\rho \cap N_i \neq \phi$ is finite for all ρ,$i \in \Lambda$.

Theorem 2. *Every $D-$paracompact space (W, ϑ) is paracompact.*

Proof. *Suppose that (W, ϑ) is a $D-$paracompact space and $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is an open-cover of (W, ϑ). By theorem 1, the cover $\tilde{E}$ is a $D-$cover, and it has an open locally$-$finite refinement.*

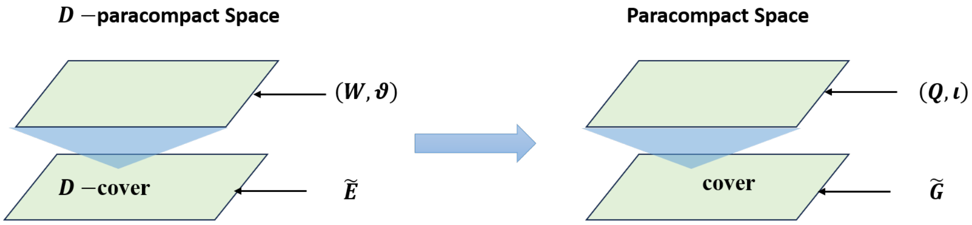

Figure 2: The relation of $D-$paracompact with paracompact spaces.

Figure 2 presents the basic relation between $D-$paracompact and paracompact spaces, which represents a crucial fact that every $D-$paracompact space must be paracompact such that $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is $D-$cover of the topological space (W, ϑ) has an open locally$-$finite refinement and we have $\tilde{G} = \{G_\omega : \omega \in \Omega\}$ is cover of the topological space (Q, ι) has an open locally$-$finite refinement.

Example 1. *Let* $(\mathbb{R}, \vartheta_u)$ *be a* $D-$*paracompact topological space. Then by using the Theorem 2 we get that the space* $(\mathbb{R}, \vartheta_u)$ *is paracompact.*

The explanation for why the previous theorem's converse may not be true is illustrated by the example that follows.

Example 2. *The topological space* $(\mathbb{R}, \vartheta_{cof})$ *is paracompact but not a D-paracompact. This is because for all* $x \in \mathbb{R}$*, each set of the form* $\mathbb{R} - x$ *is open in a topological space* $(\mathbb{R}, \vartheta_{cof})$*. Now, for all* $a, b \in \mathbb{R}$*, if* $A = \mathbb{R} - \{a\}$ *and* $B = \mathbb{R} - \{b\}$ *be any two open sets in* $(\mathbb{R}, \vartheta_{cof})$*, then* $D = A - B = \{b\}$ *is a* $D-$*set but not an open set. This is because a* $D-$*cover* $\tilde{D} = \{\{b\} : b \in \mathbb{R}\}$ *has not an open locally-finite refinement. Because, if* $\tilde{D}$ *has an open locally-finite refinement* $\{\{b_1, b_2, \ldots, b_n\}$*, then we have* $\mathbb{R} \subseteq \cup_{i=1}^{n} b_i$*, which implies that* $\mathbb{R}$ *is a finite set. Which is a contradiction.*

The following example illustrates the contrapositive of the above theorem.

Example 3. *The space* $(\mathbb{R}, \vartheta_{l.r})$ *is not paracompact, which is not* $D-$*paracompact space.*

The next theorem has the purpose to indicate that, in under conditions, the converses of the theorem above could be held.

Theorem 3. *Every topological space* (W, ϑ) *that is a locally-indiscreet paracompact is* $D-$*paracompact.*

Proof. *Let* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *be any* $D-$*cover of* (W, ϑ)*. Then* $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ *is open cover, which has an open locally-finite refinement. Hence, the result.*

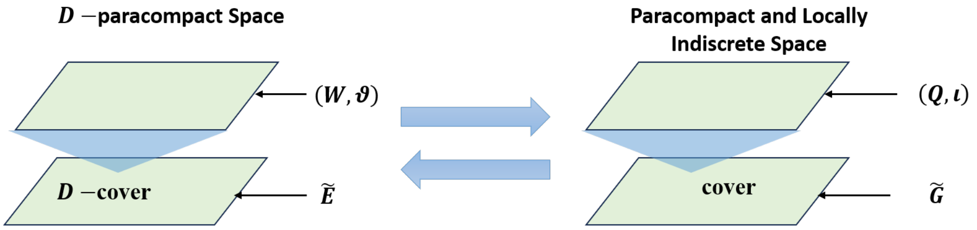

Figure 3: The relation of $D-$paracompact and paracompact spaces with locally indiscreet condition.

Figure 3 presents the complex relation between $D-$paracompact and paracompact spaces under extra condition, which represents a significant fact that every $D-$paracompact space must be paracompact, while the converse is true if the paracompact space is locally indiscreet such that $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is $D-$cover of the topological space (W, ϑ) has an open locally−finite refinement and we have $\tilde{G} = \{G_\omega : \omega \in \Omega\}$ is cover of the topological space (Q, ι) has an open locally−finite refinement.

Example 4. *(i) Notice that the space $(\mathbb{R}, \vartheta_{ind})$ is locally-indiscreet and paracompact, then it is $D-$paracompact space.*
(ii) Let $W = \mathbb{R}$ and $\vartheta = \{\phi, \mathbb{R}, \mathbb{R}-\{5\}, \{5\}\}$. Then (W, ϑ) is locally-indiscreet paracompact space. Thus, it must be $D-$paracompact.

Theorem 4. *Given that $A \subseteq W$ of the topological space (W, ϑ), then (A, ϑ_A) is $D-$paracompact if and only if any $D-$cover of A by $D-$sets in W has an open locally-finite refinement.*

Proof. $\Rightarrow$) *Let (A, ϑ_A) be $D-$paracompact and $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ be any $D-$cover of A in W. Let $E^*_\rho = E_\rho \cap A$, for all $\rho \in \Lambda$ is a $D-$set in A. Then, $\tilde{E}^* = \{E^*_\rho : \rho \in \Lambda\}$ must be $D-$cover of A by $D-$sets in A. Since (A, ϑ_A) is $D-$paracompact space, then $\tilde{E}^*$ has an open locally-finite refinement $\{E^*_{\rho 1}, E^*_{\rho 2}, \ldots, E^*_{\rho n}\}$. Thus, the family $\{E_{\rho 1}, E_{\rho 2}, \ldots, E_{\rho n}\}$ must be an open locally-finite refinement of $\tilde{E}$ in W for A, which $E^* = E \cap A$. Hence, the result.*
$\Leftarrow$) *Suppose that any $D-$cover of A by $D-$sets in W has an open locally-finite refinement. Let $\tilde{A} = \{A_\rho : \rho \in \Lambda\}$ be a $D-$cover of A by $D-$sets in W. In consequence, there is $D-$set of E_ρ in W such as $A_\rho = E_\rho \cap A$ for all $\rho \in \Lambda$. So, $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ where $\tilde{E}$ is a $D-$cover of A by $D-$sets in W. Based on the assumption that $\tilde{E}$ has an open locally-finite refinement $\{E_{\rho 1}, E_{\rho 2}, \ldots, E_{\rho n}\}$. Since $A_\rho \subseteq E_\rho$, for all $\rho \in \Lambda$, then the family $\{E_{\rho 1}, E_{\rho 2}, \ldots, E_{\rho n}\}$ must be an open locally-finite refinement of $\tilde{A}$ for A. Hence, the result.*

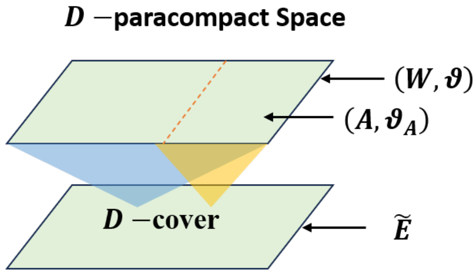

Figure 4: The subspace topology of $D-$paracompact space.

Figure 4 demonstrates a significant fact between the topological space and its subspace, where the topological subspace (A, ϑ_A) of the $D-$paracompact space (W, ϑ) must be $D-$paracompact space such that $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is $D-$cover of the topological subspace (A, ϑ_A) has an open locally$-$finite refinement, for all $D-$sets in the space (W, ϑ).

Observe the next corollaries such that each open cover is a $D-$cover.

Corollary 1. *If the space (A, ϑ_A) is $D-$paracompact, then any open cover of A by open sets in W has an open locally-finite refinement.*

Corollary 2. *A space (A, ϑ_A) is paracompact if any $D-$cover of A by $D-$sets in W has an open locally-finite refinement.*

Proof. *Since each $D-$paracompact space is paracompact, then the second part of Theorem 4 is this corollary's direct cause.*

Theorem 5. *Let (W, ϑ_1) and (W, ϑ_2) be any topological spaces. If $\vartheta_1 \subseteq \vartheta_2$ and (W, ϑ_2) is $D-$paracompact, then (W, ϑ_1) is a $D-$paracompact.*

Proof. *Let $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ be any $D-$cover of (W, ϑ_1). Since $\vartheta_1 \subseteq \vartheta_2$, we have $\tilde{E}$ is $D-$cover of (W, ϑ_2), which it is has an open locally-finite refinement. Hence, the result.*

Theorem 6. *If the topological space (W, ϑ) is $D-$paracompact, then there are closed, paracompact subspaces in (W, ϑ).*

Proof. *Let A be any closed subset of W and W be $D-$paracompact space. If $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is an open cover of A by open sets in W, so $\tilde{E} \cup \{W - A\}$ must be open cover of W. Since W is a $D-$paracompact space, it has an open locally-finite refinement of $\tilde{E}^*$. Moreover, $\tilde{E}^* - \{W - A\}$ is an open locally-finite refinement of $\tilde{E}$ for A. Hence, we get the result.*

Theorem 7. *Any $D-$separable, $D-$paracompact (W, ϑ) must be $D-$lindelöf.*

Proof. *Let $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ be any $D-$cover of W. Believe that $\tilde{E}$ has no a countable subcover. Let $\tilde{K} = \{K_\varepsilon : \varepsilon \in \Upsilon\}$ is an open uncountable locally-finite refinement subcover of $\tilde{E}$. Now, if D is a countable $D-$dense subsets of W, then $K_\varepsilon \cap D \neq \phi$, for all $\varepsilon \in \Upsilon$. Thus, the set D is uncountable because $\tilde{K}$ is uncountable, that is a contradiction. Hence, the result.*

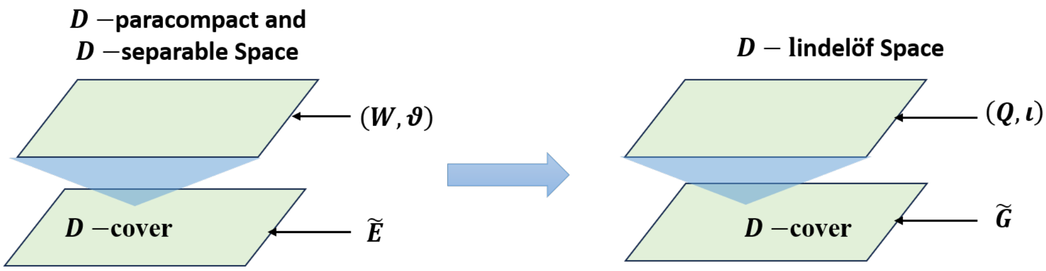

Figure 5: The relation of $D-$paracompact and $D-$lindelöf spaces with $D-$separable condition.

Figure 5 presents the basic relation between $D-$paracompact and $D-$lindelöf spaces under extra condition, which represents a significant fact that if the space of $D-$paracompact is $D-$separable then must be $D-$lindelöf such that $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is $D-$cover of the topological space (W, ϑ) has an open locally$-$finite refinement and we have $\tilde{G} = \{G_\omega : \omega \in \Omega\}$ is cover of the topological space (Q, ι) has an open locally$-$finite refinement.

With the same work, we can achieve the next corollary.

Corollary 3. *Any $D-$separable, $D-$paracompact (W, ϑ) must be lindelöf.*

Definition 18. *The topological space (W, ϑ) is called countably $D-$paracompact if every countably $D-$cover of (W, ϑ) has an open locally-finite refinement.*

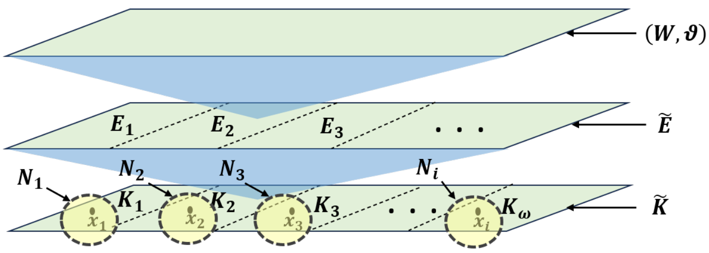

Figure 6: The space of countably $D-$paracompact.

Figure 6 illustrates the space of countably $D-$paracompact such that $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is countably $D-$cover of the topological space (W, ϑ) has an open locally$-$finite refinement $\tilde{K} = \{K_\omega : \omega \in \Omega\}$. i.e. each countably of E_ρ is $D-$set for all $\rho \in \Lambda$ such that $W = \cup_{\rho\in\Lambda} E_\rho$ has an open locally$-$finite refinement $\tilde{K}$, which the open refinement $\tilde{K}$ is a locally$-$finite if every point x_i of the space W has a neighborhood N_i such that $E_\rho \cap N_i \neq \phi$ is finite for all ρ,$i \in \Lambda$.

Theorem 8. *If any topological space (W, ϑ) is $D-$lindelöf and countably $D-$paracompact, then it must be $D-$paracompact.*

Proof. *Let $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ be a $D-$cover for W and let W is a $D-$lindelöf. Then $\tilde{E}$ has a countable subcover $\tilde{K} = \{K\}_{i=1}^{\infty}$. Since W is countably $D-$paracompact, then $\tilde{K}$ has an open locally-finite refinement $\tilde{S}$ of $\tilde{E}$.Thus, the space (W, ϑ) must be $D-$paracompact.*

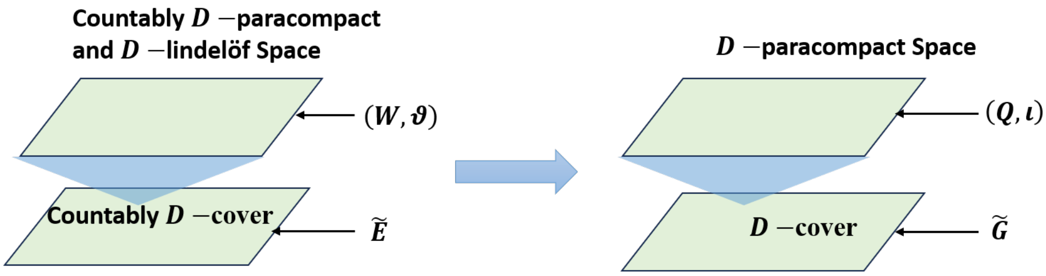

Figure 7: The relation of $D-$paracompact and countably $D-$paracompact spaces with $D-$lindelöf condition.

Figure 7 presents the relation between $D-$paracompact spaces countably $D-$paracompact under extra condition, which represents a significant fact that if the space of countably $D-$paracompact is $D-$lindelöf then must be $D-$paracompact such that $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is the countably $D-$cover of the topological space (W, ϑ) has an open locally$-$finite refinement and we have $\tilde{G} = \{G_\omega : \omega \in \Omega\}$ is $D-$cover of the topological space (Q, ι) has an open locally$-$finite refinement.

With the same work, we can achieve the next corollaries.

Corollary 4. *If any topological space (W, ϑ) is $D-$lindelöf and countably $D-$paracompact, then it must be paracompact.*

Corollary 5. *If any topological space (W, ϑ) is lindelöf and countably $D-$paracompact, then it must be paracompact.*

Example 5. *Let the space (W, ϑ_{dis}) be a $D-$lindelöf and a countably $D-$paracompact space. Then by using the Theorem 3.16, we get that (W, ϑ_{dis}) must be $D-$paracompact.*

Theorem 9. *If any topological space (W, ϑ) is $D-$paralindelöf and countably $D-$paracompact, then it must be a $D-$paracompact space.*

Proof. *Let $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ be any $D-$cover for W and let W is $D-$paralindelöf space. Then $\tilde{E}$ has an open locally-countable refinement $\tilde{K} = \{K_{\rho i}\}_{i=1}^{\infty}$, where is also a $D-$cover of (W, ϑ) . Now, since W is countably $D-$paracompact, then $\tilde{K}$ has an open locally-finite refinement $\tilde{S}$ of $\tilde{E}$. Thus, the space (W, ϑ) must be $D-$paracompact.*

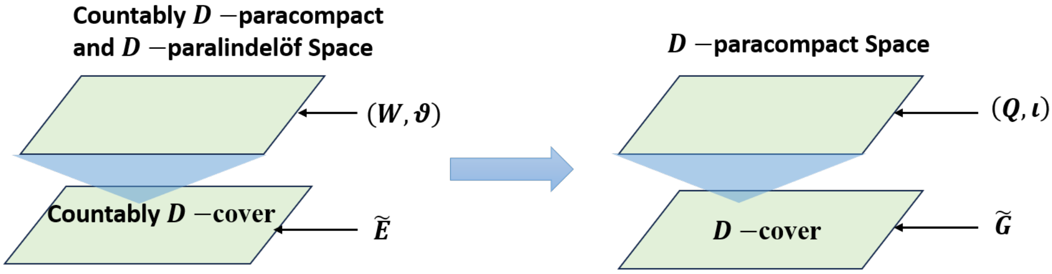

Figure 8: The relation of $D-$paracompact and countably $D-$paracompact spaces with $D-$paralindelöf condition.

Figure 8 presents the relation between $D-$paracompact spaces countably $D-$paracompact under extra condition, which represents a significant fact that if the space of countably $D-$paracompact is $D-$paralindelöf then must be $D-$paracompact such that $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ is the countably $D-$cover of the topological space (W, ϑ) has an open locally$-$finite refinement and we have $\tilde{G} = \{G_\omega : \omega \in \Omega\}$ is $D-$cover of the topological space (Q, ι) has an open locally$-$finite refinement.

The following corollaries can be obtained with the same work.

Corollary 6. *If any topological space* (W, ϑ) *is* $D-$*paralindelöf and countably* $D-$*paracompact, then it must be a paracompact space.*

Corollary 7. *If any topological space* (W, ϑ) *is paralindelöf and countably* $D-$*paracompact, then it must be a paracompact space.*

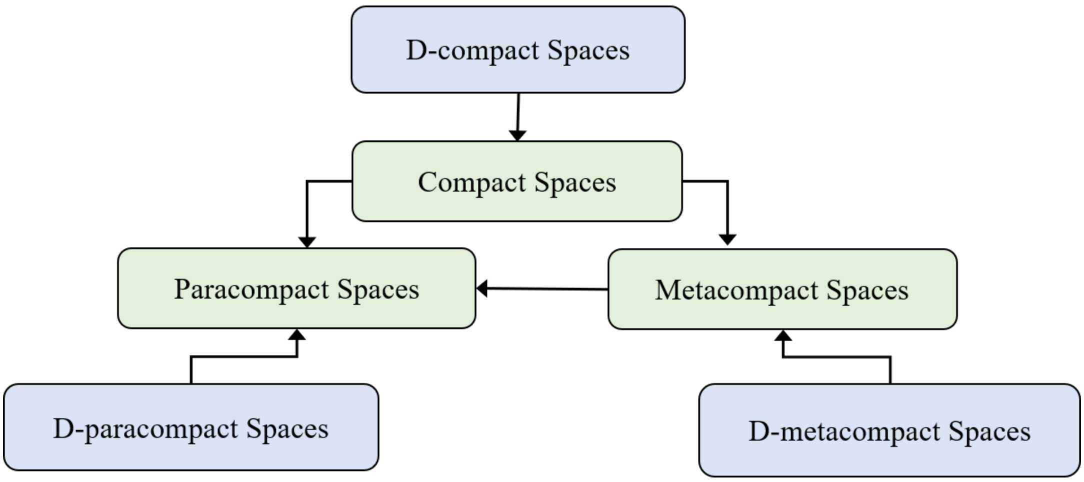

Figure 9: The flow chart of $D-$paracompact with some spaces.

Figure 9 presents the study's results flow chart, which represents the relation of Difference paracompact spaces with common topological spaces, in which the space of Difference paracompact represents a basic class characterized by special coverings.

4. Product of D-Paracompact Topological Spaces

This section presents several main theoretical results the concepts of maps with the product for two $D-$paracompact spaces.

Theorem 10. *Let (W, ϑ) and (Q, ι) be any topological spaces. If $\varphi : W \longrightarrow Q$ is a $D-$perfect function, and W is locally-indiscreet space, then W must be $D-$paracompact if the space Q is so*

Proof. *Let $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ be any $D-$cover of W. Since φ is $D-$perfect, then for any $q \in Q$, we have $\varphi^{-1}(q)$ is $D-$compact subsets of W. So, there is a finite subset ι_q of Λ, such that $\varphi^{-1}(q) \subseteq \cup_{\rho\in\iota_q} H_\rho$, and $\tilde{E}$ is an open cover of W. That is, $P_q = Q - \varphi(W - \cup_{\rho\in\iota_q} H_\rho)$ is $D-$open subsets of Q and $\varphi^{-1}(P_q) \cup_{\rho\in\iota_q} H_\rho$, for any $q \in P_q$, therefore $\tilde{P} = \{P_q : q \in Q\}$ is open $D-$cover of Q. Since Q is $D-$paracompact, then we get that $\tilde{P}$ has an open locally-finite refinement $\tilde{P}^* = \{P^*_q : q \in Q\}$. Thus, the set P^*_q is $D-$open subsets of W. Since the function φ is $D-$perfect, that is means $\{\varphi^{-1}(P^*_q) : q \in Q\}$ is an open locally-finite refinement of W. Hence, the space W is $D-$paracompact space.*

Theorem 11. *Let $\varphi : W \longrightarrow Q$ as $D-$perfect function. Then, W is paracompact space if the space Q is $D-$paracompact*

Proof. *As the proof of the above Theorem 10, this theory is also simply to prove.*

Theorem 12. *If $\varphi : W \longrightarrow Q$ be $D-$perfect, where Q is a countable and W is locally-indiscreet, then W must be countably $D-$paracompact, if the space Q is so.*

Proof. *Let $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ be countable $D-$cover of W and φ is $D-$perfect function. Then for any $q \in Q$, $\varphi^{-1}(q)$ is $D-$compact space subsets of W. It therefore obtains in a finite subset of Λ such as $\varphi^{-1}(q) \subseteq \cup_{i=1}^{n} A_i$. Since W is locally indiscreet, then A_i is $D-$open subset of W, for any $i \in \Lambda$. Presently, $B_q = Q - \varphi(W - \cup_{i=1}^{n} A_i)$ is $D-$set containing q. Moreover, $\varphi^{-1}(B_q) \subseteq \cup_{i=1}^{n} A_i$. Thus, $\tilde{B} = \{B_q : q \in Q\}$ presents countable $D-$cover for Q. Now, since Q is countably $D-$paracompact, then $\tilde{B}$ has an open locally-finite refinement $\tilde{B}^* = \{B_{q1}, B_{q2}, \ldots, B_{qn}\}$ and so B^*_q is $D-$open subset of W. Also, since φ is $D-$perfect, so $\{\varphi^{-1}(B^*_q) : q \in Q\}$ is an open locally-finite refinement of W. Thus, the space W must be countably $D-$paracompact.*

Theorem 13. *Let (W, ϑ) and (Q, ι) be topological spaces, such that W is D-compact and Q is $D-$paracompact spaces. Then $W \times Q$ must be $D-$paracompact space.*

Proof. *Given the truth that the projection function $T : W \times Q \longrightarrow Q$ is continuous and $T^{-1}\{q\} = W \times \{q\} \simeq W$ is $D-$compact, for any $q \in Q$. Then $T : W \times Q \longrightarrow Q$ is $D-$perfect function. So, since Q is $D-$paracompact space, then $W \times Q$ must be also $D-$paracompact.*

Theorem 14. *Let (W, ϑ) and (Q, ι) be topological spaces. If W is paracompact, and Q is $D-$paracompact spaces, then the projection function $T : (W \times Q, \vartheta \times \iota) \longrightarrow (Q, \iota)$ must be closed.*

Proof. *Let (W, ϑ) be paracompact and (Q, ι) be a $D-$paracompact. Then $(W \times Q, \vartheta \times \iota)$ is $D-$paracompact, therefore the projection function $T : (W \times Q, \vartheta \times \iota) \longrightarrow (Q, \iota)$ must be closed function.*

Theorem 15. *Let $\varphi : (W, \vartheta) \longrightarrow (Q, \iota)$ be a closed, continuous, onto function such that Q is locally-indiscreet space. Then Q is $D-$paracompact, if the space W is so.*

Proof. *Let $\tilde{E} = \{E_\rho : \rho \in \Lambda\}$ be any $D-$cover of Q. Since φ is onto function and continuous, that means $\tilde{E} = \{\varphi^{-1}(E_\rho) : \rho \in \Lambda\}$ is an open cover of W. Since W is $D-$paracompact, then there is an open locally-finite refinement of $\tilde{E}$ as $\tilde{E}^* = \{\varphi^{-1}(E^*_\rho) : \rho \in \Lambda\}$. Thus, Q must be a $D-$paracompact space.*

With the same work, the corollary that follows can be achieved.

Corollary 8. *Let $\varphi : (W, \vartheta) \longrightarrow (Q, \iota)$ be a closed, onto, continuous function such that Q is locally-indiscreet space. Then, Q is $D-$paracompact, if the space W is paracompact.*

5. Conclusions

This research highlights that $D-$paracompact topological spaces have a key topological characteristic. Their flexibility to provide $D-$covers with locally-finite refinements is demonstrated by their conclusion. The research introduced several new properties and explained examples that relate to such superimposed spaces. The study's additional objective was to draw attention to some of the new notions and characteristics of $D-$paracompact spaces, as well as some properties of the Cartesian multiplication of such spaces under special conditions. Furthermore, certain illustrative examples and the main characteristics of these concepts were carefully studied. We identified their principal characteristics along with illustrated figures. We talked about their main traits and demonstrated how they work together. The study lies at the interface of topology and other branches of mathematics. As in fuzzy sets, researchers can generalize the fuzzy paracompact spaces to be fuzzy $D-$paracompact spaces. Also, in the algebra field, there is a compact (topological) group, which we can generalize to a $D-$paracompact group and provide mathematicians with a way to expand their knowledge about both topological and group algebra. This also acts as an effective basis for future research. In general topology, these spaces might be generalized to bitopological and tritopological spaces.

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