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Some Geraghty type Inequalities in b-Fuzzy Metric Spaces with an Application

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Abstract. In this note, we introduce novel Geraghty-type inequalities within the framework of a b-fuzzy metric space and develop new fixed point theorems for such mappings in a G-complete b-fuzzy metric space. To substantiate our findings, we present several illustrative examples using graphical methods. Additionally, we demonstrate the application of our introduced theorems by solving a non-linear integral equation, showing the practical utility of our results.

2020 Mathematics Subject Classifications: 54H25, 47H10

Key Words and Phrases: b-fuzzy metric space, Geraghty type mapping, α-Suzuki Geraghty type contraction

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1. Introduction and Preliminaries

For the first time, the traditional metric space framework was extended by incorporating fuzzy logic to address uncertainties in distance measurements by Kramosil and Michálek [7]. In a classical metric space, the distance between two points is precisely defined by a real number, adhering to strict metric properties. However, in a fuzzy metric space, distances are represented by fuzzy sets, allowing for a range of values that reflect varying degrees of proximity. Later, George and Veermani [4] modified the definition of fuzzy metric space given by Kramosil and Michálek [7] and proved some fixed point results. Inspired by this, concept of fuzzy b-metric space was introduced by Sedghi et al. [15], where the triangle inequality is replaced by a weaker one by involving $b > 1$, with this weaker inequality, The researchers introduced many contractive inequalities to obtain fixed point, see([1], [3], [6], [8]). In the line of this, We introduce the concepts of Geraghty type inequalities in this b-fuzzy metric spaces and we introduce the notion of fuzzy α-Geraghty type mapping within the context of b-fuzzy metric space. Additionally, we

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introduce the idea of α-Suzuki Geraghty type mapping in the framework of G-complete b-fuzzy metric space, and we investigate specific fixed point problems associated with this generalizations. We offer several illustrative examples with the graphical approach in the support of our findings. In last, as an application, we discuss a solution to a non-linear integral equation via fixed point tools. We must need the following:

Definition 1. *([14]). A function* $\diamond : [0,1]^2 \to [0,1]$ *is called a continuous triangular-norm if*

- $\diamond$ *is commutative and associative;*
- $\diamond$ *is continuous;*
- $1 \diamond a = a$*;*
- $a \diamond b \geq c \diamond d$*, whenever* $a \geq c$ *and* $b \geq d$.

for all $a, b, c, d \in [0,1]$.

Some of the l-norms are $a \diamond_m b = \min\{a, b\}$ (minimum), $a \diamond_p b = ab$ (product), $a \diamond_L b = \max\{a + b - 1, 0\}$.

Definition 2. *[6]. A b-fuzzy metric space is an ordered triple* $(\bar{\mathcal{Y}} \neq \phi, \mathcal{M}_z, \diamond)$*, where* $\mathcal{M}_z : \bar{\mathcal{Y}}^2 \times (0, +\infty) \to [0,1]$ *satisfying*

(i) $\mathcal{M}_z(\delta, \gamma, l) > 0$*;*

(ii) $\mathcal{M}_z(\delta, \gamma, l) = 1$ *if and only if* $\delta = \gamma$*;*

(iii) $\mathcal{M}_z(\delta, \gamma, l) = \mathcal{M}_z(\gamma, \delta, l)$*;*

(iv) $\mathcal{M}_z(\delta, \gamma, b(l + r)) \geq \mathcal{M}_z(\delta, \eta, l) \diamond \mathcal{M}_z(\eta, \gamma, r)$*, where* $b \geq 1$*;*

(v) $\mathcal{M}_z(\delta, \gamma, .) : (0, +\infty) \to (0,1]$ *is continuous from left and* $\lim_{l \to +\infty} \mathcal{M}_z(\delta, \gamma, l) = 1$.

for all $\delta, \gamma, \eta \in \bar{\mathcal{Y}}$ *and* $l, r > 0$.

Note: If $b = 1$ then definition (2) will become a fuzzy metric space.

Example 1. *Let* $\mathcal{M}_z(\delta, \gamma, l) = e^{-\frac{|\delta - \gamma|^p}{l}}$*, where* $p > 1$ *is a real number,* $\mathcal{M}_z$ *is a b-fuzzy metric with* $b = 2^{p-1}$ *but not fuzzy metric space.*

Definition 3. *[8] Suppose* $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ *is a b-fuzzy metric space. Then*

(i) $\{\delta_n\}$ *is called G-convergent sequence if there exists* $\delta \in \bar{\mathcal{Y}}$ *such that*

$$\lim_{n \to +\infty} \mathcal{M}_z(\delta_n, \delta, l) = 1.$$

(ii) $\{\delta_n\}$ *in* $\bar{\mathcal{Y}}$ *is called a G-Cauchy sequence if*

$$\lim_{n,m\to+\infty} \mathcal{M}_z(\delta_n, \delta_m, l) = 1$$

for all $m, n \in \mathbb{N}$ *and* $l > 0$.

(iii) The space is called complete if every Cauchy sequence is convergent in $\bar{\mathcal{Y}}$.

Geraghty ([5]) introduced a category denoted as $\mathcal{B}$ which is a collection of maps defined as $\beta : [0, +\infty) \to [0, 1)$ satisfying

$$\beta(t_n) \to 1 \quad \text{as} \quad n \to +\infty \Rightarrow t_n \to 0 \quad \text{as} \quad n \to +\infty.$$

Researchers introduced many contractive inequalities to obtain fixed points in this fuzzy space. In the line of this, we introduce the concepts of Geraghty type inequalities in this b-fuzzy metric space and we inject the notion of fuzzy α-Geraghty type mapping within the context of b-fuzzy metric space. Additionally, we introduce the idea of α-Suzuki-Geraghty type mapping in G-complete b-fuzzy metric space, and we investigate specific fixed point problems associated with these generalizations. We offer several illustrative examples with the graphical approach in support of our findings. In last, as an application, we discuss a solution to a non-linear integral equation via fixed point tools.

2. Main Results

We must require to introduce the following definitions.

Definition 4. *A b-fuzzy metric* $\mathcal{M}_z$ *is said to be C-triangular, if for all* $\delta, \gamma, \eta \in \bar{\mathcal{Y}}$ *and* $l > 0$,

$$\mathcal{M}_z(\delta, \gamma, l) \geq \mathcal{M}_z(\delta, \eta, l) + \mathcal{M}_z(\eta, \gamma, l) - 1 \tag{1}$$

holds.

Definition 5. *A self map* $\mathcal{L}$ *defined on a G-complete b-fuzzy metric space* $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ *is called a Geragthy type-I contractive if*

$$1 - \mathcal{M}_z(\mathcal{L}\delta, \mathcal{L}\gamma, l) \leq (1 - \mathcal{M}_z(\delta, \gamma, l)) \cdot \beta(1 - \mathcal{M}_z(\delta, \gamma, l)) \tag{2}$$

where $\beta \in \mathcal{B}$, *for all* $\delta, \gamma \in \bar{\mathcal{Y}}$ *and* $l > 0$.

Now we write a theorem for such introduced Geraghty type-I contractive mapping using C-triangular property.

Theorem 1. *Suppose* $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ *is a G-complete b-fuzzy metric space with C-triangular fuzzy metric and a self map* $\mathcal{L}$ *defined on* $\bar{\mathcal{Y}}$ *is a Geraghty type-I contractive map. Then* $\mathcal{L}$ *has a unique fixed point in* $\bar{\mathcal{Y}}$.

Proof. Consider a Picard sequence $\{\delta_n\}$ such that $\delta_{n+1} = \mathcal{L}\delta_n$. We assume that $\delta_n \neq \delta_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, otherwise we will get fixed point. Now

$$1 - \mathcal{M}_z(\delta_{n+1}, \delta_{n+2}, l) \leq \beta(1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l))(1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l)) < 1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l) \tag{3}$$

Thus, we conclude that $\mathcal{M}_z(\delta_{n+1}, \delta_{n+2}, l) \geq \mathcal{M}_z(\delta_n, \delta_{n+1}, l)$ for all $n \in \mathbb{N}$. Hence $\{\mathcal{M}_z(\delta_{n+1}, \delta_n, l)\}$ is an increasing sequence of positive real numbers in $(0, 1]$. So, there exists $s(l) \in (0, 1]$ such that $\lim_{n\to+\infty} \mathcal{M}_z(\delta_n, \delta_{n+1}, l) = s(l)$ for all $l > 0$. Now, we need to prove $s(l) = 1$. Suppose $s(l_0) < 1$, for any $l_0 > 0$. By (3), we obtain

$$\lim_{n\to+\infty} \beta(1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l)) = 1.$$

Since $\beta \in \mathcal{B}$. This implies that

$$\lim_{n\to+\infty} \mathcal{M}_z(\delta_n, \delta_{n+1}, l) = s(l) = 1,$$

a contradiction to our assumption. Hence, we conclude that

$$\lim_{n\to+\infty} \mathcal{M}_z(\delta_n, \delta_{n+1}, l) = 1, \tag{4}$$

for all $l > 0$. Next, we need to show $\{\delta_n\}$ is a Cauchy sequence. Consider a contrary,

$$\lambda = \lim_{n,m\to+\infty} \mathcal{M}_z(\delta_n, \delta_m, l) < 1. \tag{5}$$

By (2),

$$1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l) \leq \beta(1 - \mathcal{M}_z(\delta_n, \delta_m, l))(1 - \mathcal{M}_z(\delta_n, \delta_m, l)).$$

Taking the limit as $n, m \to +\infty$ in the above inequality where $n > m$, then we get

$$\lim_{n,m\to+\infty} (1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l)) \leq \lim_{n,m\to+\infty} \beta(1 - \mathcal{M}_z(\delta_n, \delta_m, l))(1 - \mathcal{M}_z(\delta_n, \delta_m, l)).$$

By using (5), we get

$$\lim_{n,m\to+\infty} (1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l)) \leq \lim_{n,m\to+\infty} \beta(1 - \mathcal{M}_z(\delta_n, \delta_m, l))(1 - \lambda). \tag{6}$$

On the flip side, using C-traingular property

$$\begin{aligned} 1 - \mathcal{M}_z(\delta_n, \delta_m, l) &\leq 1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l) + 1 - \mathcal{M}_z(\delta_{n+1}, \delta_m, l) \\ &\leq 1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l) + 1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l) + 1 - \mathcal{M}_z(\delta_{m+1}, \delta_m, l). \end{aligned}$$

Putting limit as $n, m \to +\infty$ and using (4) and (6), we get

$$(1 - \lambda) \leq \lim_{n,m\to+\infty} (1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l)) \leq \lim_{n,m\to+\infty} \beta(1 - \mathcal{M}_z(\delta_n, \delta_m, l))(1 - \lambda)$$

this implies

$$\lim_{n,m\to+\infty} \beta(1 - \mathcal{M}_z(\delta_n, \delta_m, l)) = 1$$

which implies

$$\lim_{n,m\to+\infty} (1 - \mathcal{M}_z(\delta_n, \delta_m, l)) = 0.$$

This yields that $\lim_{n,m\to+\infty} \mathcal{M}_z(\delta_n, \delta_m, l) = \lambda = 1$, a contradiction with the assumption (5). Therefore, sequence $\{\delta_n\}$ is a G-Cauchy in $\bar{\mathcal{Y}}$. Since the space $\bar{\mathcal{Y}}$ is complete then there exists $u \in \bar{\mathcal{Y}}$ such that sequence $\{\delta_n\}$ converges to u,

$$\lim_{n\to+\infty} \mathcal{M}_z(\delta_n, u, l) = 1, \tag{7}$$

for all $l > 0$. Next we need to show u is a fixed point of $\mathcal{L}$.

$$1 - \mathcal{M}_z(\delta_{n+1}, \mathcal{L}u, l) \le (1 - \mathcal{M}_z(\delta_n, u, l))\beta(1 - \mathcal{M}_z(\delta_n, u, l)),$$

consider limit as $n \to +\infty$, this implies

$$\lim_{n\to+\infty} \mathcal{M}_z(\delta_{n+1}, \mathcal{L}u, l) = 1, \tag{8}$$

for all $l > 0$. Using triangle inequality, we write

$$\mathcal{M}_z(u, \mathcal{L}u, l) \ge \mathcal{M}_z(u, \delta_{n+1}, l) \diamond \mathcal{M}_z(\delta_{n+1}, \mathcal{L}u, l).$$

considering limit as $n \to +\infty$ and with (4) and (8), we obtain $\mathcal{M}_z(u, \mathcal{L}u, l) = 1$ for all $l > 0$. Consider v is another fixed point of $\mathcal{L}$ such that $u \neq v$, $\mathcal{M}_z(u, v, l) < 1$. Thus

$$1 - \mathcal{M}_z(u, v, l) = 1 - \mathcal{M}_z(\mathcal{L}u, \mathcal{L}v, l) \le (1 - \mathcal{M}_z(u, v, l))\beta(1 - \mathcal{M}_z(u, v, l)) < 1 - \mathcal{M}_z(u, v, l).$$

We get a contrary, thus fixed point is unique.

Example 2. *Consider $\bar{\mathcal{Y}} = [0, 1]$ and let $\mathcal{M}_z : \bar{\mathcal{Y}} \times \bar{\mathcal{Y}} \to [0, 1]$ defined by $\mathcal{M}_z(\delta, \gamma, l) = e^{-\frac{|\delta-\gamma|^2}{l+0.5}}$ and $\mathcal{M}_z$ be a C-triangular for all $\delta, \gamma \in \bar{\mathcal{Y}}$ and $l > 0$. Then $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ is a G-complete b-fuzzy metric space. Consider the mapping $\mathcal{L} : \bar{\mathcal{Y}} \to \bar{\mathcal{Y}}$ defined by*

$$\mathcal{L}(\delta) = \begin{cases} \frac{1}{3}\delta^2, & \textit{if } \delta \in [0, 1) \\ \frac{1}{4}, & \textit{if } \delta = 1, \end{cases}$$

for all $\delta, \gamma \in \bar{\mathcal{Y}}$ and $l > 0$.
Now, in the following three cases will be formed for which Geraghty type-I contraction is to be verified for $\beta(t_1) = 1 - t_1$.
Case 1. If $\delta, \gamma \in [0, 1)$ then

$$\mathcal{M}_z(\mathcal{L}\delta, \mathcal{L}\gamma, l) = e^{-\frac{|\mathcal{L}\delta-\mathcal{L}\gamma|^2}{l+0.5}}$$

$$
\begin{aligned}
&= e^{-\frac{\frac{1}{9}|\delta^2-\gamma^2|^2}{l+0.5}} \\
1-\mathcal{M}_z(\mathcal{L}\delta,\mathcal{L}\gamma,l) &= (1-e^{-\frac{|\mathcal{L}\delta-\mathcal{L}\gamma|^2}{l+0.5}}) \\
&= (1-e^{-\frac{\frac{1}{9}|\delta^2-\gamma^2|^2}{l+0.5}}) \\
&\leq e^{-\frac{|\delta-\gamma|^2}{l+0.5}}(1-e^{-\frac{|\delta-\gamma|^2}{l+0.5}}) \qquad (9)\\
&= \beta(1-\mathcal{M}_z(\delta,\gamma,l))(1-\mathcal{M}_z(\delta,\gamma,l)),
\end{aligned}
$$

Case 2. If $\delta \in [0,1), \gamma = 1$ *then*

$$
\begin{aligned}
\mathcal{M}_z(\mathcal{L}\delta,\mathcal{L}\gamma,l) &= e^{-\frac{|\mathcal{L}\delta-\mathcal{L}\gamma|^2}{l+0.5}} \\
&= e^{-\frac{|\frac{\delta^2}{3}-\frac{1}{4}|^2}{l+0.5}} \\
1-\mathcal{M}_z(\mathcal{L}\delta,\mathcal{L}\gamma,l) &= 1-e^{-\frac{|\frac{\delta^2}{3}-\frac{1}{4}|^2}{l+0.5}} \\
&\leq e^{-\frac{|\delta-1|^2}{l+0.5}}(1-e^{-\frac{|\delta-1|^2}{l+0.5}}) \qquad (10)\\
&= \beta(1-\mathcal{M}_z(\delta,\gamma,l))(1-\mathcal{M}_z(\delta,\gamma,l)),
\end{aligned}
$$

Case 3. If $\delta = \gamma = 0$ *then the Geragthy type-I contraction trivially holds. Now, the graphical representation of cases 1 and 2;*

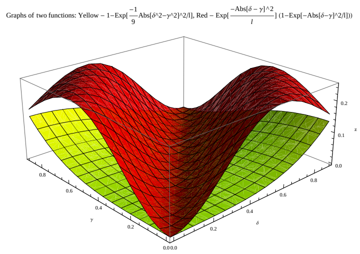

Figure 1: $(1-e^{-\frac{\frac{1}{9}|\delta^2-\gamma^2|^2}{l+0.5}}) \leq e^{-\frac{|\delta-\gamma|^2}{l+0.5}}(1-e^{-\frac{|\delta-\gamma|^2}{l+0.5}})$

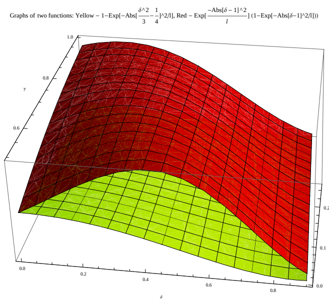

Figure 2: $1-e^{-\frac{|\frac{\delta^2}{3}-\frac{1}{4}|^2}{l+0.5}} \leq e^{-\frac{|\delta-1|^2}{l+0.5}}(1-e^{-\frac{|\delta-1|^2}{l+0.5}})$

In Figure 1, the yellow colour represents the L.H.S. and the red colour represents the R.H.S. of equation (9), and in Figure 2, the yellow colour represents the L.H.S. The red colour represents the R.H.S. of equation (10).

From the graphical representations it is clearly visible that the contraction (2) is satisfied.
Hence, the inequality holds in these cases for $l > 0$ and $\beta(t_1) = 1 - t_1$. Now, for all $\delta, \gamma, \eta \in [0, 1]$, then it is easy to check that $\mathcal{M}_z$ is C-triangular. Hence, all assumptions of theorem (1) are satisfied for $l > 0$ and $\beta(t_1) = 1 - t_1$, and 0 is a unique fixed point of $\mathcal{L}$.

Now we recall the following definitions from [2].

Definition 6. *[2] A self map $\mathcal{L}$ is said to be a triangular α-admissible if there exists $\alpha : \bar{\mathcal{Y}}^2 \times (0, +\infty) \to \mathbb{R}$ such that*

(i) $\alpha(\delta, \gamma, l) \geq 1 \Rightarrow \alpha(\mathcal{L}\delta, \mathcal{L}\gamma, l) \geq 1$

(ii) $\alpha(\delta, \eta, l) \geq 1$ *and* $\alpha(\eta, \gamma, l) \geq 1 \Rightarrow \alpha(\delta, \gamma, l) \geq 1$

for all $\delta, \gamma, \eta \in \bar{\mathcal{Y}}$ and any $l > 0$.

Lemma 1. *[2] Consider a fuzzy metric space denoted by $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ and let $\mathcal{L} : \bar{\mathcal{Y}} \to \bar{\mathcal{Y}}$ be a triangular α-admissible mapping. Suppose there exists an element $\delta_0 \in \bar{\mathcal{Y}}$ such that $\alpha(\delta_0, \mathcal{L}\delta_0, l) \geq 1$. Let us define a $\{\delta_n\}$ recursively by setting $\delta_{n+1} = \mathcal{L}\delta_n$. Then*

$$\alpha(\delta_m, \delta_n, l) \geq 1,$$

for all $m, n \in \mathbb{N}$ with $m < n$ and $l > 0$.

Next we define a new contractive inequality in Suzuki view.

Definition 7. *A triangular α-admissible self mapping $\mathcal{L}$ defined on a b-fuzzy metric space $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ is called a α-Suzuki-Geraghty type-I if there exists a $\beta \in \mathcal{B}$ such that*

$$\begin{aligned} &\mathcal{M}_z(\delta, \mathcal{L}\delta, l) > q \cdot \mathcal{M}_z(\delta, \gamma, l) \\ &\Rightarrow \alpha(\delta, \gamma, l)(1 - \mathcal{M}_z(\mathcal{L}\delta, \mathcal{L}\gamma, l)) \leq \beta(1 - \mathcal{M}_z(\delta, \gamma, l))(1 - \mathcal{M}_z(\delta, \gamma, l)) \end{aligned} \tag{11}$$

where $q \in (0, 1)$ and $\delta, \gamma \in \bar{\mathcal{Y}}$.

Now we write a few definitions which are essential for our next result.

Definition 8. *A triangular α-admissible self mapping $\mathcal{L}$ is defined on a b-fuzzy metric space $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ is said to have property G_1 if for any two sequences $\{\delta_n\}$, $\{\delta_m\}$ in $\bar{\mathcal{Y}}$ such that*

$$\lim_{n,m \to +\infty} \mathcal{M}_z(\delta_n, \delta_m, l) = a(l) \in (0, 1],$$

where $n > m$ and $n, m \in \mathbb{N}$, $q \in (0, 1)$, then

$$\mathcal{M}_z(\delta_n, \delta_{n+1}, l) > q \cdot \mathcal{M}_z(\delta_n, \delta_m, l),$$

for all $l > 0$.

Definition 9. *A triangular α-admissible self mapping $\mathcal{L}$ is defined on a b-fuzzy metric space $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ is said to have property G_2 if for any convergent sequence $\{\delta_n\}$ in $\bar{\mathcal{Y}}$ converging to u,*

$$\mathcal{M}_z(\delta_n, \mathcal{L}\delta_n, l) > q \cdot \mathcal{M}_z(\delta_n, u, l),$$

where $n \in \mathbb{N}$ and $q \in (0,1)$.

Example 3. *Let $\bar{\mathcal{Y}} = [0,1]$ and define $\mathcal{M}_z(\delta, \gamma, l) = \frac{l}{l+|\delta-\gamma|^2}$. Let $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ is a G-complete b-fuzzy metric space. Let a self-map $\mathcal{L} : \bar{\mathcal{Y}} \to \bar{\mathcal{Y}}$ defined by*

$$\mathcal{L}(\delta) = \begin{cases} 1, & \text{if } \delta \in (0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\delta_n = 1 - \frac{1}{n}$ and $\delta_m = 1 - \frac{1}{m}$, with $n, m \in \mathbb{N}$. Since

$$\lim_{n\to+\infty} \mathcal{M}_z(\delta_n, \delta_{n+1}, l) = \lim_{n\to+\infty} \mathcal{M}_z\left(1 - \frac{1}{n}, 1 - \frac{1}{n+1}, l\right) \in (0,1],$$

Then, $\mathcal{M}_z(\delta_n, \delta_{n+1}, l) > q \cdot \mathcal{M}_z(\delta_n, \delta_m, l)$ where $q = \frac{1}{2}$, Definition (8) satisfied.
We have $\mathcal{M}_z(\delta_n, \mathcal{L}\delta_n, l) = \mathcal{M}_z(1-\frac{1}{n}, 1, l) > q \cdot \mathcal{M}_z(1-\frac{1}{n}, 1, l)$ for all $n \in \mathbb{N}$ and $q \in (0,1)$. Hence, Definition 9 holds.

Theorem 2. *Consider a self map $\mathcal{L}$ defined on a G-complete b-fuzzy metric space $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ where fuzzy metric is C- triangular satisfying:*

(i) map $\mathcal{L}$ is b-fuzzy α-Suzuki-Geraghty type-I;

(ii) $\mathcal{L}$ has property G_1 and G_2;

(iii) there exists $\delta_0 \in \bar{\mathcal{Y}}$ such that $\alpha(\delta_0, \mathcal{L}\delta_0, l) \geq 1$ for all $l > 0$;

(iv) if $\alpha(\delta_n, \delta_{n+1}, l) \geq 1$ and $\delta_n \to u$ as $n \to +\infty$, then $\alpha(\delta_n, u, l) \geq 1$ for all $n \in \mathbb{N}$.

Then $\mathcal{L}$ has a fixed point.

Proof. By assumption (3), there exists $\delta_0 \in \bar{\mathcal{Y}}$ such that $\alpha(\delta_0, \delta_1, l) \geq 1$ for all $l > 0$ and define a sequence $\{\delta_n\}$ in $\bar{\mathcal{Y}}$ by $\delta_{n+1} = \mathcal{L}\delta_n$ for all $n \in \mathbb{N}$. Suppose that $\delta_n = \delta_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$ no need to prove anything automatically completed. Suppose $\delta_n \neq \delta_{n+1}$ for all $n \in \mathbb{N}$.
By lemma 1, we have

$$\alpha(\delta_n, \delta_{n+1}, l) \geq 1, \tag{12}$$

for all $n \in \mathbb{N}$ and $l > 0$. By (11),

$$\mathcal{M}_z(\delta_n, \mathcal{L}\delta_n, l) > q \cdot \mathcal{M}_z(\delta_n, \delta_{n+1}, l) \quad \text{implies}$$

$$\alpha(\delta_n, \delta_{n+1}, l)(1 - \mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_{n+1}, l)) \leq \beta(1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l))(1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l)).$$

Now

$$\begin{aligned}(1 - \mathcal{M}_z(\delta_{n+1}, \delta_{n+2}, l)) &= (1 - \mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_{n+1}, l)) \leq \alpha(\delta_n, \delta_{n+1}, l)(1 - \mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_{n+1}, l)) \\ &\leq \beta(1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l))(1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l)) < (1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l)).\end{aligned} \tag{13}$$

This concludes that $\{\mathcal{M}_z(\delta_n, \delta_{n+1}, l)\}$ is non-decreasing sequence of positive real number in $(0, 1]$. So there exists $s(l) \in (0, 1]$ such that

$$\lim_{n\to+\infty} \mathcal{M}_z(\delta_n, \delta_{n+1}, l) = s(l).$$

Suppose to the contrary, $s(l_0) < 1$ for any $l_0 > 0$. Now put limit as $n \to +\infty$

$$\lim_{n\to+\infty} \beta(1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l)) = 1.$$

By the characteristic of $\mathcal{B}$, we have

$$\lim_{n\to+\infty} \mathcal{M}_z(\delta_n, \delta_{n+1}, l) = 1, \tag{14}$$

a contradiction. Hence, we need to show $\{\delta_n\}$ is a G-Cauchy sequence. Suppose

$$\lambda = \mathcal{M}_z(\delta_n, \delta_m, l) < 1,$$

By using property (G_1),

$$\mathcal{M}_z(\delta_n, \delta_{n+1}, l) > q \cdot \mathcal{M}_z(\delta_n, \delta_m, l) \text{ implies}$$
$$\alpha(\delta_n, \delta_m, l)(1 - \mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_m, l)) \leq \beta(1 - \mathcal{M}_z(\delta_n, \delta_m, l))(1 - \mathcal{M}_z(\delta_n, \delta_m, l)).$$

We have

$$\begin{aligned}(1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l)) &= (1 - \mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_m, l)) \leq \alpha(\delta_n, \delta_m, l)(1 - \mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_m, l)) \\ &\leq \beta(1 - \mathcal{M}_z(\delta_n, \delta_m, l))(1 - \mathcal{M}_z(\delta_n, \delta_m, l)).\end{aligned}$$

Taking the limit as $n, m \to +\infty$ and lemma 1,

$$\begin{aligned}\lim_{n,m\to+\infty}(1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l)) &\leq \lim_{n,m\to+\infty} \alpha(\delta_n, \delta_m, l)(1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l)) \\ &\leq \lim_{n,m\to+\infty} \beta(1 - \mathcal{M}_z(\delta_n, \delta_m, l))(1 - \lambda).\end{aligned} \tag{15}$$

On the flip side,

$$\begin{aligned}(1 - \mathcal{M}_z(\delta_n, \delta_m, l)) &\leq (1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l)) + (1 - \mathcal{M}_z(\delta_{n+1}, \delta_m, l)) \\ &\leq (1 - \mathcal{M}_z(\delta_n, \delta_{n+1}, l)) + (1 - \mathcal{M}_z(\delta_{n+1}, \delta_{m+1}, l)) + (1 - \mathcal{M}_z(\delta_{m+1}, \delta_m, l)).\end{aligned}$$

Putting limit as $n, m \to +\infty$ and using (14) and (15),

$$(1-\lambda) \leq \lim_{n,m\to+\infty}(1-\mathcal{M}_z(\delta_{n+1},\delta_{m+1},l)) \leq \lim_{n,m\to+\infty}\beta(1-\mathcal{M}_z(\delta_n,\delta_m,l))(1-\lambda).$$

which gives

$$\lim_{n,m\to+\infty}\beta(1-\mathcal{M}_z(\delta_m,\delta_n,l)) = 1,$$

$$\lim_{n,m\to+\infty}\mathcal{M}_z(\delta_m,\delta_n,l) = 1.$$

Which is a contradiction with λ. Thus, $\{\delta_n\}$ is a G-Cauchy sequence. Since $\bar{\mathcal{Y}}$ is a G-complete, there exists $u \in \bar{\mathcal{Y}}$ such that

$$\lim_{n\to+\infty}\mathcal{M}_z(\delta_n,u,l) = 1. \tag{16}$$

By the property of (G_2),

$$\begin{aligned}
&\mathcal{M}_z(\delta_{n-1},\mathcal{L}\delta_{n-1},l) > q\cdot\mathcal{M}_z(\delta_{n-1},u,l)\\
&\Rightarrow \alpha(\delta_{n-1},u,l)(1-\mathcal{M}_z(\mathcal{L}\delta_{n-1},\mathcal{L}u,l)) \leq \beta(1-\mathcal{M}_z(\delta_{n-1},u,l))(1-\mathcal{M}_z(\delta_{n-1},u,l))\\
&\qquad (1-\mathcal{M}_z(\mathcal{L}\delta_{n-1},\mathcal{L}u,l)) \leq \alpha(\delta_{n-1},u,l)(1-\mathcal{M}_z(\mathcal{L}\delta_{n-1},\mathcal{L}u,l))\\
&\qquad\qquad \leq \beta(1-\mathcal{M}_z(\delta_{n-1},u,l))(1-\mathcal{M}(\delta_{n-1},u,l))\\
&\qquad\qquad < (1-M(\delta_{n-1},u,l))\\
&\qquad 1-\mathcal{M}_z(\delta_n,\mathcal{L}u,l) < 1-\mathcal{M}_z(\delta_{n-1},u,l),
\end{aligned}$$

Put limit as $n \to +\infty$, we get $\mathcal{M}_z(u,\mathcal{L}u,l) = 1$. that is, $\mathcal{L}u = u$.
Next, assume v is another fixed point of $\mathcal{L}$ such that $u \neq v$ that is $\mathcal{M}_z(u,v,l) < 1$. By the property of G_2, we know that

$$\begin{aligned}
\mathcal{M}_z(u,u,l) = \mathcal{M}_z(u,\mathcal{L}u,l) &> q\cdot\mathcal{M}_z(u,v,l) \Rightarrow\\
(1-\mathcal{M}_z(u,v,l)) = (1-\mathcal{M}_z(\mathcal{L}u,\mathcal{L}v,l)) &\leq \alpha(u,v,l)(1-\mathcal{M}_z(\mathcal{L}u,\mathcal{L}v,l))\\
&\leq \beta(1-\mathcal{M}_z(u,v,l))(1-\mathcal{M}_z(u,v,l))\\
&< (1-\mathcal{M}_z(u,v,l)),
\end{aligned}$$

a contradiction with the assumption, so fixed point u is unique.

Now we write an example which is α-Suzuki Geraghty type-I contractive mapping but not a Geraghty type-I mapping.

Example 4. *Define a fuzzy metric* $\mathcal{M}_z(\delta,\gamma,l) = e^{-\frac{|\delta-\gamma|^2}{l}}$ *for all* $l > 0$ *on* $[0,1]$ *with standard triangular norm and space is* G*-complete and Define a self map like*

$$\mathcal{L}(\delta) = \begin{cases} \frac{\delta}{2}, & \text{if} \quad \delta \in [0,1) \\ 0, & \delta = 1. \end{cases}$$

Also define function α *by*

$$\alpha(\delta, \gamma, l) = \begin{cases} 1, & if \quad \delta, \gamma \in [0, 1) \\ 0, otherwise. \end{cases}$$

for all $l > 0$ *and* $\beta(t_1) = e^{-t_1}$.
Suppose $q = \frac{1}{2}$. *Put the following cases to verify the* α*-Suzuki-Geraghty type-I contraction mapping:*

Case 1. If $\delta, \gamma \in [0, 1)$ *then* $\alpha(\delta, \gamma, l) = 1$, $\mathcal{M}_z(\delta, \frac{\delta}{2}, l) > q \cdot \mathcal{M}_z(\delta, \gamma, l)$, *that is,* $e^{-\frac{|\delta - \frac{\delta}{2}|^2}{l}} > q \cdot e^{-\frac{|\delta-\gamma|^2}{l}}$ *implies* $\alpha(\delta, \gamma, l)[1 - e^{-\frac{|\frac{\delta}{2} - \frac{\gamma}{2}|^2}{l}}] \leq e^{-(1-e^{-\frac{|\delta-\gamma|^2}{l}})}[1 - e^{-\frac{|\delta-\gamma|^2}{l}}]$.

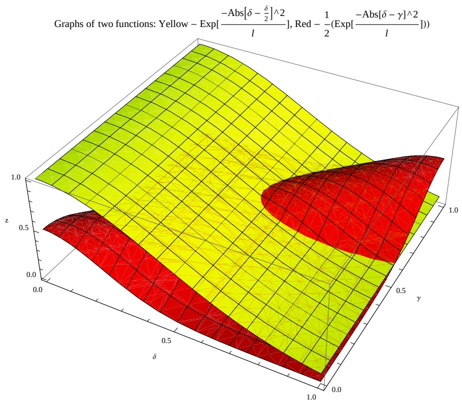

Figure 3: $e^{-\frac{|\delta - \frac{\delta}{2}|^2}{l}} > q \cdot e^{-\frac{|\delta-\gamma|^2}{l}}$

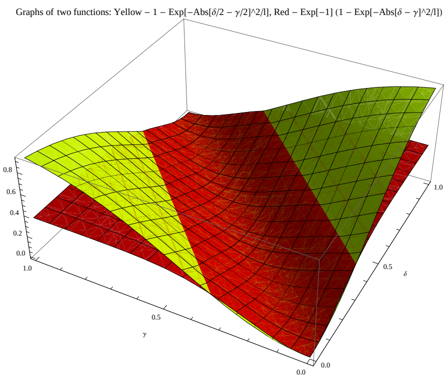

Figure 4: $\alpha(\delta, \gamma, l)[1 - e^{-\frac{|\frac{\delta}{2} - \frac{\gamma}{2}|^2}{l}}] \leq e^{-(1-e^{-\frac{|\delta-1|^2}{l}})}[1 - e^{-\frac{|\delta-\gamma|^2}{l}}]$

Figure 3. The yellow colour represents the value of $e^{-\frac{|\delta - \frac{\delta}{2}|^2}{l}}$, *and the red colour represents the value of* $q \cdot e^{-\frac{|\delta-\gamma|^2}{l}}$ *and Figure 4. The yellow colour represents the value of* $\alpha(\delta, \gamma, l)[1 - e^{-\frac{|\frac{\delta}{2} - \frac{\gamma}{2}|^2}{l}}]$, *and the red colour represents the value of* $e^{-(1-e^{-\frac{|\delta-\gamma|^2}{l}})}(1 -$

$e^{-\frac{|\delta-\gamma|^2}{l}}$). *Hence, it is clear that the hypothesis of inequality does not hold, and also the conclusion part does not hold. So, the inequality holds for this particular case.*
Case 2. If $\delta \in [0,1)$ *and* $\gamma = 1$, *then* $\alpha(\delta,\gamma,l) = 0$, $\mathcal{M}_z(\delta,1,l) > q \cdot \mathcal{M}_z(\delta,1,l)$, *that is,* $e^{-\frac{|\delta-\frac{\delta}{2}|^2}{l}} > q \cdot e^{-\frac{|\delta-1|^2}{l}}$. *Since the hypothesis inequality is not supportive, there is no need to continue for further calculations, but even then,* $0 \leq e^{-(1-e^{-\frac{|\delta-1|^2}{l}})}[1-e^{-\frac{|\delta-1|^2}{l}}]$. *Hence, the inequality holds for this case.*
Case 3. If $\delta = \gamma = 1$, *then* $\alpha(\delta,\gamma,l) = 0$. $\mathcal{M}_z(1,\frac{1}{2},l) > q \cdot \mathcal{M}_z(1,1,l)$, *that is,* $e^{-\frac{1}{4l}} > q \cdot 1$ *implies* $0 \leq 0$.
Case 4. If $\delta = 1$ *and* $\gamma \in [0,1)$, *then* $\alpha(\delta,\gamma,l) = 0$, $\mathcal{M}_z(1,0,l) > q \cdot \mathcal{M}_z(1,\gamma,l)$, *that is,* $e^{-\frac{1}{l}} > q \cdot e^{-\frac{|1-\gamma|^2}{l}}$. *the hypothesis is not supportive, there is no need to continue further calculations, but even then, we get conclusion part will be zero from both sides. Hence, the inequality holds in this case.*
This example demonstrate that the self-mapping $\mathcal{L}$ *is* α*-Suzuki Geraghty type-I contractive mapping but not a Geraghty type-I mapping.*
Let $\delta,\gamma \in \bar{\mathcal{Y}}$ *for all* $l > 0$ *such that* $\alpha(\delta,\gamma,l) \geq 1$ *this implies that* $\delta,\gamma \in [0,1)$, *then* $\mathcal{L}\delta, \mathcal{L}\gamma \in [0,1)$. *Thus,* $\alpha(\mathcal{L}\delta,\mathcal{L}\gamma,l) = 1$ *for all* $l > 0$. *Let* $\delta,\gamma,\eta \in [0,1]$ *such that* $\alpha(\delta,\eta,l) \geq 1$ *and* $\alpha(\eta,\gamma,l) \geq 1$ *for all* $l > 0$. *This implies that* $\delta,\gamma,\eta \in [0,1)$. *So,* $\alpha(\delta,\gamma,l) \geq 1$ *for all* $l > 0$. *Therefore,* $\mathcal{L}$ *is triangular* α*-admissible. Hence all the assumptions of the above theorem are gratified and also hold property* G_1, G_2 *and condition (3) for* $q = 0.5$, $\beta(t_1) = e^{-t_1}$. *So,* $\delta = 0$ *is a fixed point of* $\mathcal{L}$.

This is another supportive example of our results.

Example 5. *Let* $\bar{\mathcal{Y}} = \{0,\frac{1}{2},1,2\}$ *with* $\mathcal{M}_z(\delta,\gamma,l) = \frac{l}{l+|\delta-\gamma|^2}$ *for* $l > 0$ *is a G-complete b-fuzzy metric space. Define* $\mathcal{L}(0) = \mathcal{L}(\frac{1}{2}) = \mathcal{L}(1) = \frac{1}{2}$ *and* $\mathcal{L}(2) = 1$. *we can calculate* $\mathcal{L}$ *satisfies each assumptions Theorem 2 with unique fixed point* $\delta = \frac{1}{2}$ *for function* α *like*

$$\alpha(\delta,\gamma,l) = \begin{cases} 1, & \textit{if} \ \ \delta,\gamma \in \{0,\frac{1}{2},1\} \\ 0 & \textit{otherwise.} \end{cases}$$

for all $l > 0$, $\beta(t_1) = 1 - t_1$ *and* $q \in (0,1)$.

Now we introduce one more inequality.

Definition 10. *A triangular* α*-admissible self mapping* $\mathcal{L}$ *defined on a b-fuzzy metric space* $(\bar{\mathcal{Y}},\mathcal{M}_z,\diamond)$ *is called a* α*-Suzuki-Geraghty type-II if there exists a* $\beta \in \mathcal{B}$ *such that*

$$\begin{gathered}\mathcal{M}_z(\delta,\mathcal{L}\delta,l) > q \cdot \mathcal{M}_z(\delta,\gamma,l) \quad \textit{implies} \\ \alpha(\delta,\gamma,l)\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta,\mathcal{L}\gamma,l)} - 1\right) \leq \beta\left(\frac{1}{\mathcal{M}_z(\delta,\gamma,l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(\delta,\gamma,l)} - 1\right),\end{gathered} \tag{17}$$

for all $\delta,\gamma \in \bar{\mathcal{Y}}$ *and* $l > 0$, $q \in (0,1)$.

Theorem 3. *Consider a self map $\mathcal{L}$ defined on a G-complete b-fuzzy metric space $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ where fuzzy metric is triangular satisfying:*

(i) map $\mathcal{L}$ is b-fuzzy α-Suzuki-Geraghty type-II;

(ii) $\mathcal{L}$ has property G_1 and G_2;

(iii) there exists $\delta_0 \in \bar{\mathcal{Y}}$ such that $\alpha(\delta_0, \mathcal{L}\delta_0, l) \geq 1$ for all $l > 0$;

(iv) if $\alpha(\delta_n, \delta_{n+1}, l) \geq 1$ and $\delta_n \to u$ as $n \to +\infty$, then $\alpha(\delta_n, u, l) \geq 1$ for all $n \in \mathbb{N}$.

Then $\mathcal{L}$ has a fixed point.

Proof. Constructing of Picard sequence such that $\delta_n \neq \delta_{n+1}$ for all $n \in \mathbb{N}$. By lemma 1, we have

$$\alpha(\delta_n, \delta_{n+1}, l) \geq 1$$

for all $n \in \mathbb{N}$ and $l > 0$. By the α-Suzuki-Geraghty type -II contraction, we have

$$\mathcal{M}_z(\delta_n, \delta_{n+1}, l) > q \cdot \mathcal{M}_z(\delta_n, \delta_{n+1}, l)$$
$$\Rightarrow \alpha(\delta_n, \delta_{n+1}, l)\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_{n+1}, l)} - 1\right) \leq \beta\left(\frac{1}{\mathcal{M}_z(\delta_n, \delta_{n+1}, l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(\delta_n, \delta_{n+1}, l)} - 1\right)$$

for all $l > 0$.

$$\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_{n+1}, l)} - 1\right) \leq \alpha(\delta_n, \delta_{n+1}, l)\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_{n+1}, l)} - 1\right)$$
$$\leq \beta\left(\frac{1}{\mathcal{M}_z(\delta_n, \delta_{n+1}, l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(\delta_n, \delta_{n+1}, l)} - 1\right) < \left(\frac{1}{\mathcal{M}_z(\delta_n, \delta_{n+1}, l)} - 1\right). \quad (18)$$

we conclude that $\mathcal{M}_z(\delta_{n+1}, \delta_{n+2}, l) > \mathcal{M}_z(\delta_n, \delta_{n+1}, l)$ for all $n \in \mathbb{N}$ it means, it is non-decreasing sequence of positive real numbers. So there exists $s(l) \in (0, 1]$ such that

$$\lim_{n \to +\infty} \mathcal{M}_z(\delta_n, \delta_{n+1}, l) = s(l), \quad \text{for} \quad \text{all} \quad l > 0.$$

Next, we require to prove $s(l) = 1$. Taken a contrary, $s(l_0) < 1$ for all $l_0 > 0$. Now taking limit as $n \to +\infty$

$$\lim_{n \to +\infty} \beta\left(\frac{1}{\mathcal{M}_z(\delta_n, \delta_{n+1}, l)} - 1\right) = 1 \Rightarrow \lim_{n \to +\infty} \mathcal{M}_z(\delta_n, \delta_{n+1}, l) = 1, \quad (19)$$

a contradiction. Next we must prove that sequence is a G-Cauchy sequence. Suppose $\lambda = \mathcal{M}_z(\delta_n, \delta_m, l) < 1$. By (G_1) property,

$$\mathcal{M}_z(\delta_n, \delta_{n+1}, l) > q \cdot \mathcal{M}_z(\delta_n, \delta_m, l)$$
$$\textit{implies} \quad \alpha(\delta_n, \delta_m, l)\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n, \mathcal{L}\delta_m, l)} - 1\right) \leq \beta\left(\frac{1}{\mathcal{M}_z(\delta_n, \delta_m, l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(\delta_n, \delta_m, l)} - 1\right).$$

By (17) and lemma (1), we get

$$\begin{aligned}\left(\frac{1}{\mathcal{M}_z(\delta_{n+1},\delta_{m+1},l)}-1\right) = \left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n,\mathcal{L}\delta_m,l)}-1\right) &\leq \alpha(\delta_n,\delta_m,l)\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n,\mathcal{L}\delta_m,l)}-1\right)\\ &\leq \beta\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)\\ &< \left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)\end{aligned}$$

taking limit on both sides,

$$\begin{aligned}\lim_{n,m\to+\infty}\left(\frac{1}{\mathcal{M}_z(\delta_{n+1},\delta_{m+1},l)}-1\right) &\leq \lim_{n,m\to+\infty}\alpha(\delta_n,\delta_m,l)\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n,\mathcal{L}\delta_m,l)}-1\right)\\ &\leq \lim_{n,m\to+\infty}\beta\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)\left(\frac{1}{\lambda}-1\right) < \left(\frac{1}{\lambda}-1\right).\end{aligned} \tag{20}$$

On the flip side,

$$\begin{aligned}\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right) &\leq \left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_{n+1},l)}-1\right)+\left(\frac{1}{\mathcal{M}_z(\delta_{n+1},\delta_m,l)}-1\right)\\ &\leq \left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_{n+1},l)}-1\right)+\left(\frac{1}{\mathcal{M}_z(\delta_{n+1},\delta_{m+1},l)}-1\right)\\ &+\left(\frac{1}{\mathcal{M}_z(\delta_{m+1},\delta_m,l)}-1\right).\end{aligned}$$

taking limit as $n,m\to+\infty$ and using (19, 20),

$$\begin{aligned}\lim_{n,m\to+\infty}\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right) &\leq \lim_{n,m\to+\infty}\alpha(\delta_n,\delta_m,l)\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n,\mathcal{L}\delta_m,l)}-1\right)\\ &\leq \lim_{n,m\to+\infty}\beta\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)\lim_{n,m\to+\infty}\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)\\ &< \lim_{n,m\to+\infty}\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)\\ \frac{1}{\lambda}-1 &\leq \lim_{n,m\to+\infty}\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta_n,\mathcal{L}\delta_m,l)}-1\right)\\ &\leq \lim_{n,m\to+\infty}\beta\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)(\frac{1}{\lambda}-1) < \frac{1}{\lambda}-1.\end{aligned}$$

which suggest that

$$\lim_{n,m\to+\infty}\beta\left(\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}-1\right)=1 \Rightarrow \lim_{n,m\to+\infty}\frac{1}{\mathcal{M}_z(\delta_n,\delta_m,l)}=1,$$

a contradiction. Thus, $\{\delta_n\}$ is a G-Cauchy sequence. Since $\bar{\mathcal{Y}}$ is a G-complete then there exists $u \in \bar{\mathcal{Y}}$ such that

$$\lim_{n\to+\infty} \mathcal{M}_z(\delta_n, u, l) = 1. \tag{21}$$

Next to prove fixed point of map $\mathcal{L}$, we require property (G_2),

$$\begin{aligned}
&\mathcal{M}_z(\delta_n, \mathcal{L}\delta_n, l) > q \cdot \mathcal{M}_z(\delta_{n-1}, u, l)\\
&\textit{implies}\ \ \alpha(\delta_{n-1}, u, l)\left(\frac{1}{\mathcal{M}_z(\delta_n, \mathcal{L}u, l)} - 1\right) \leq \beta\left(\frac{1}{\mathcal{M}_z(\delta_{n-1}, u, l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(\delta_{n-1}, u, l)} - 1\right)\\
&\left(\frac{1}{\mathcal{M}_z(\delta_n, \mathcal{L}u, l)} - 1\right) \leq \alpha(\delta_{n-1}, u, l)\left(\frac{1}{\mathcal{M}_z(\delta_n, \mathcal{L}u, l)} - 1\right)\\
&\qquad \leq \beta\left(\frac{1}{\mathcal{M}_z(\delta_{n-1}, u, l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(\delta_{n-1}, u, l)} - 1\right).
\end{aligned}$$

Put limit as $n \to +\infty$, $\lim_{n\to+\infty}\left(\frac{1}{\mathcal{M}_z(\delta_{n-1}, \mathcal{L}u, l)} - 1\right) \leq 0$. So, $\mathcal{M}_z(u, \mathcal{L}u, l) = 1$ that is $\mathcal{L}u = u$.

Finally, we require to show uniqueness of the fixed point. Consider another fixed point v such that $u \neq v$, it means $\mathcal{M}_z(u, v, l) < 1$. Using (G_2) property,

$$\begin{aligned}
&\mathcal{M}_z(u, u, l) = \mathcal{M}_z(u, \mathcal{L}u, l) > q \cdot \mathcal{M}_z(u, v, l)\\
&\textit{implies}\ \ \alpha(u, v, l)\left(\frac{1}{\mathcal{M}_z(u, v, l)} - 1\right) \leq \beta\left(\frac{1}{\mathcal{M}_z(u, v, l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(u, v, l)} - 1\right)\\
&\frac{1}{\mathcal{M}_z(u, v, l)} - 1 \leq \alpha(u, v, l)\left(\frac{1}{\mathcal{M}_z(u, v, l)} - 1\right) \leq \beta\left(\frac{1}{\mathcal{M}_z(u, v, l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(u, v, l)} - 1\right)\\
&\qquad < \frac{1}{\mathcal{M}_z(u, v, l)} - 1,
\end{aligned}$$

a contradiction. Hence, u is a unique fixed point for self map $\mathcal{L}$.

Remark 1. *If $\alpha(\delta, \gamma, l) = 1$ in definition (10), then mapping $\mathcal{L}$ becomes a Suzuki Geraghty type-II contractive map.*

Corollary 1. *Suppose $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ is a G-complete b-fuzzy metric space with triangular fuzzy metric and a self map $\mathcal{L}$ defined on $\bar{\mathcal{Y}}$ is a Suzuki Geraghty type-II contractive map with properties (G_1) and (G_2). Then $\mathcal{L}$ has a unique fixed point.*

Now we present another definition which is not in view of Suzuki type.

Definition 11. *A triangular α-admissible self mapping $\mathcal{L}$ defined on a b-fuzzy metric space $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ is called a α-Geraghty type-II if there exists a $\beta \in \mathcal{B}$ such that*

$$\alpha(\delta, \gamma, l)\left(\frac{1}{\mathcal{M}_z(\mathcal{L}\delta, \mathcal{L}\gamma, l)} - 1\right) \leq \beta\left(\frac{1}{\mathcal{M}_z(\delta, \gamma, l)} - 1\right)\left(\frac{1}{\mathcal{M}_z(\delta, \gamma, l)} - 1\right), \tag{22}$$

$q \in (0, 1)$, for all $\delta, \gamma \in \bar{\mathcal{Y}}$ and $l > 0$.

Theorem 4. *Consider a self map $\mathcal{L}$ defined on a G-complete b-fuzzy metric space $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ where fuzzy metric is triangular satisfying:*

(i) map $\mathcal{L}$ is b-fuzzy α-Geraghty type-II;

(ii) there exists $\delta_0 \in \bar{\mathcal{Y}}$ such that $\alpha(\delta_0, \mathcal{L}\delta_0, l) \geq 1$ for all $l > 0$;

(iii) if $\alpha(\delta_n, \delta_{n+1}, l) \geq 1$ and $\delta_n \to u$ as $n \to +\infty$, then $\alpha(\delta_n, u, l) \geq 1$ for all $n \in \mathbb{N}$.

Then $\mathcal{L}$ has a fixed point.

Remark 2. $\alpha(\delta, \gamma, l) = 1$ *in definition (11) results in the following:*

Corollary 2. *Consider a self map $\mathcal{L}$ which is Geraghty type-II contractive defined on a G-complete b-fuzzy metric space $(\bar{\mathcal{Y}}, \mathcal{M}_z, \diamond)$ where fuzzy metric is triangular then the self-mapping $\mathcal{L}$ has a unique fixed point.*

In the support of Corollary 2, we have an example.

Example 6. *Consider a fuzzy metric $\mathcal{M}_z(\delta, \gamma, l) = \frac{l+0.3}{l+0.3+|\delta-\gamma|^2}$ for all $\delta, \gamma \in \bar{\mathcal{Y}} = [0, 1]$ and $l > 0$. We can check it is a G-complete b-fuzzy metric space with respect to standard triangular norm.*
Define a self map $\mathcal{L}$ such as

$$\mathcal{L}(\delta) = \begin{cases} \frac{\delta}{2}, & \text{if } \delta, \gamma \in (0, 1] \\ 0, & \text{if } \delta = 0. \end{cases}$$

To show mapping $\mathcal{L}$ is a Geraghty type-II contractive with $\beta(t_1) = \frac{1}{1+t_1}$.
Case 1. If $\delta, \gamma \in (0, 1]$ then

$$\begin{aligned} \frac{1}{\mathcal{M}_z(\mathcal{L}\delta, \mathcal{L}\gamma, l)} - 1 &= \frac{1}{\frac{l+0.3}{l+0.3+|\mathcal{L}\delta-\mathcal{L}\gamma|^2}} - 1 \\ &= \frac{1}{\frac{l+0.3}{l+0.3+\frac{1}{4}|\delta-\gamma|^2}} - 1 \\ &= \frac{1 - \frac{l+0.3}{l+0.3+\frac{1}{4}|\delta-\gamma|^2}}{\frac{l+0.3}{l+0.3+\frac{1}{4}|\delta-\gamma|^2}} \\ &= \frac{\frac{1}{4}|\delta-\gamma|^2}{l+0.3+\frac{1}{4}|\delta-\gamma|^2} \cdot \frac{l+0.3+\frac{1}{4}|\delta-\gamma|^2}{l+0.3} \\ &= \frac{\frac{1}{4}|\delta-\gamma|^2}{l+0.3} \\ &\leq \frac{|\delta-\gamma|^2}{l+0.3+|\delta-\gamma|^2} \\ &= 1 - \frac{l+0.3}{l+0.3+|\delta-\gamma|^2} \end{aligned}$$

$$= \beta \left(\frac{1}{\mathcal{M}_z(\delta, \gamma, l)} - 1 \right) \left(\frac{1}{\mathcal{M}_z(\delta, \gamma, l)} - 1 \right),$$

Case2. If $\delta = 0, \gamma \in (0, 1]$ *then*

$$\begin{aligned}
\frac{1}{\mathcal{M}_z(\mathcal{L}\delta, \mathcal{L}\gamma, l)} - 1 &= \frac{1}{\frac{l+0.3}{l+0.3+|\mathcal{L}\delta - \mathcal{L}\gamma|^2}} - 1 \\
&= \frac{1}{\frac{l+0.3}{l+0.3+\frac{1}{4}|\gamma|^2}} - 1 \\
&= \frac{\frac{1}{4}|\gamma|^2}{l+0.3} \\
&\leq \frac{|\gamma|^2}{l+0.3+|\gamma|^2} \\
&= 1 - \frac{l+0.3}{l+0.3+|\gamma|^2} \\
&= \beta \left(\frac{1}{\mathcal{M}_z(\delta, \gamma, l)} - 1 \right) \left(\frac{1}{\mathcal{M}_z(\delta, \gamma, l)} - 1 \right),
\end{aligned}$$

Case 3. If $\delta = \gamma = 0$ *then it is trivial. It is easy to check that* $\mathcal{M}_z$ *is triangular. Hence,* $\delta = 0$ *is a unique fixed point of* $\mathcal{L}$.

3. Application

In this section, we discuss the existence of a unique solution of a non-linear integral equation and need some specific conditions for the solution. A b-fuzzy metric space that resembles $C([a, b], \mathbb{R})$ is the space $\bar{\mathcal{Y}}$ of all continuous real valued functions defined on the interval $[a, b]$ with the b-fuzzy metric

$$\mathcal{M}_z(\delta, \gamma, l) = \frac{l}{l + \max\limits_{a \leq s_1 \leq b} |\delta(s_1) - \gamma(s_1)|^2}.$$

Consider an integral equation

$$\delta(l_1) = f(l_1) + \int_a^b h(l_1, s_1) F(l_1, s_1, \delta(s_1)) ds_1, \tag{23}$$

where $f : [a, b] \to \mathbb{R}$, $h : [a, b] \times [a, b] \to \mathbb{R}$ and $F : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 5. *Suppose*

(i) for all $l_1, s_1 \in [a, b]$, $\delta, \gamma \in \bar{\mathcal{Y}}$

$$\mathcal{M}_z(\delta(s_1), \mathcal{L}(\delta(s_1)), l) > q \cdot \mathcal{M}_z(\delta(s_1), \gamma(s_1), l)$$
$$\Rightarrow |F(l_1, s_1, \delta(s_1)) - F(l_1, s_1, \gamma(s_1))|^2 \leq e^{-\frac{\max_{a \leq s_1 \leq b} |\delta(s_1) - \gamma(s_1)|^2}{l}} |\delta(s_1) - \gamma(s_1)|^2,$$

(ii) for all $l_1, s_1 \in [a,b]$

$$\left(\int_a^b h(l_1,s_1)ds_1\right)^2 \leq \frac{1}{b-a}.$$

(iii) if $\{\delta_n(l_1)\}$ *and* $\{\delta_m(l_1)\}$ *are the two sequences in* $\bar{\mathcal{Y}}$ *such that*

$$\begin{aligned}&\lim_{n,m\to+\infty}\max_{a\leq l_1\leq b}|\delta_n(l_1)-\delta_m(l_1)|^2 \to r(k)\\ &\Rightarrow q\cdot(l+\max_{a\leq l_1\leq b}|\delta_n(l_1)-\delta_{n+1}(l_1)|)^2 < (l+\max_{a\leq l_1\leq b}|\delta_n(l_1)-\delta_m(l_1)|^2),\end{aligned}$$

for all $n,m\in\mathbb{N}$ *such that* $n>m$*,* $q\in(0,1)$*.*

(iv) if $\{\delta_n(l_1)\}$ *is a sequence in* $C([a,b],\mathbb{R})$ *such that*

$$\delta_n(l_1)\to\delta(l_1)\Rightarrow q\cdot(l+\max_{a\leq l_1\leq b}|\delta_n(l_1)-\delta_{n+1}(l_1)|^2) < (l+\max_{a\leq l_1\leq b}|\delta_n(l_1)-\delta(l_1)|^2),$$

for all $n\in\mathbb{N}$ *and* $l>0$*,* $q\in(0,1)$*.*

Then the integral equation (23) has a solution in $\bar{\mathcal{Y}}$*.*

Proof. Suppose $\mathcal{L}:\bar{\mathcal{Y}}\to\bar{\mathcal{Y}}$ is an integral operator

$$\mathcal{L}\delta(l_1) = f(l_1)+\int_a^b h(l_1,s_1)F(l_1,s_1,\delta(s_1))ds_1,$$

for $\delta\in\bar{\mathcal{Y}}$. Now

$$\begin{aligned}\frac{1}{\mathcal{M}_z(\mathcal{L}\delta,\mathcal{L}\gamma,l)}-1 &= \frac{1}{\frac{l}{l+\max\limits_{a\leq s_1\leq b}|\mathcal{L}\delta(s_1)-\mathcal{L}\gamma(s_1)|^2}}-1\\ &= \frac{1-\frac{l}{l+\max\limits_{a\leq s_1\leq b}|\mathcal{L}\delta(s_1)-\mathcal{L}\gamma(s_1)|^2}}{\frac{l}{l+\max\limits_{a\leq s_1\leq b}|\mathcal{L}\delta(s_1)-\mathcal{L}\gamma(s_1)|^2}}\\ &= \frac{\max\limits_{a\leq s_1\leq b}|\mathcal{L}\delta(s_1)-\mathcal{L}\gamma(s_1)|^2}{l}\\ &= \frac{\max\limits_{a\leq s_1\leq b}|\int_a^b h(l_1,s_1)F(l_1,s_1,\delta(s_1))ds_1-\int_a^b h(l_1,s_1)F(l_1,s_1,y(s_1))ds_1|^2}{l}\\ &\leq \max_{a\leq s_1\leq b} e^{-\frac{|\delta(s_1)-\gamma(s_1)|^2}{l}}\cdot\frac{|\delta(s_1)-\gamma(s_1)|^2}{l}\\ &\leq \beta\left(\frac{1}{\mathcal{M}_z(\delta,\gamma,l)}-1\right)\left(\frac{1}{\mathcal{M}_z(\delta,\gamma,l)}-1\right).\end{aligned}$$

Therefore, $\mathcal{L}$ is a Suzuki Geraghty type-II contractive mapping for $\beta(t_1) = e^{-t_1}$ and $l_1 > 0$. For two sequences in $\bar{\mathcal{Y}}$ such that $n > m$ and $n, m \in \mathbb{N}$, by using the assumption (3)

$$\begin{aligned}\mathcal{M}_z\left(\delta_n(l_1), \delta_m(l_1), l\right) &= \frac{l}{l + \max_{a \leq l_1 \leq b} |\delta_n(l_1) - \delta_m(l_1)|^2} \\ &> q \cdot \frac{l}{l + r(k)} = r(l) \in (0, 1]\end{aligned}$$

implies

$$\begin{aligned}\mathcal{M}_z\left(\delta_n(l_1), \delta_{n+1}(l_1), l\right) &= \frac{l}{l + \max_{a \leq l_1 \leq b} |\delta_n(l_1) - \delta_{n+1}(l_1)|^2} \\ &> q \cdot \frac{l}{l + \max_{a \leq l_1 \leq b} |\delta_n(l_1) - \delta_m(l_1)|^2} \\ &= q \cdot \mathcal{M}_z\left(\delta_n(l_1), \delta_m(l_1), l\right).\end{aligned}$$

Hence, property-(G_1) holds.
If a sequence $\{\delta_n(l_1)\}$ in $\bar{\mathcal{Y}}$ such that $\delta_n(l_1) \to \delta(l_1)$ in $\bar{\mathcal{Y}}$ by using assumption (4),

$$\begin{aligned}\mathcal{M}_z\left(\delta_n(l_1), \delta_{n+1}(l_1), l\right) &= \frac{l}{l + \max_{a \leq l_1 \leq b} |\delta_n(l_1) - \delta_{n+1}(l_1)|^2} \\ &> q \cdot \frac{l}{l + \max_{a \leq l_1 \leq b} |\delta_n(l_1) - \delta(l_1)|^2} \\ &= q \cdot \mathcal{M}_z\left(\delta_n(l_1), \delta(l_1), l\right).\end{aligned}$$

Therefore, property (G_2) holds. Therefore every requirements of corollary (2) are gratified with the consideration of the function $\beta(t_1) = e^{-t_1}$. Hence, we conclude that there exists $\delta(l_1) \in C\left([a, b], \mathbb{R}\right)$ such that $\mathcal{L}\delta(l_1) = \delta(l_1)$ and the integral equations (23) has a solution. This way we complete the proof.

4. Conclusion and Future Work

In this study, we explore the concept of fuzzy α-Geraghty type mappings and α-Suzuki-Geraghty type mappings within the framework of b-fuzzy metric space. We extended the theory of fixed-point theorems by employing these mappings and demonstrated their utility through several illustrative examples, including a graphical approach for better visualization. Our findings provide a foundation for solving fixed-point problems in more generalized settings, such as G-complete b-fuzzy metric space. The application to non-linear integral equations highlights the practical significance of these results.

Future research could focus on expanding these concepts to more complex metric spaces or hybrid structures, exploring their applications in various fields such as optimization, dynamic systems, and network theory. Additionally, investigating the interplay between different types of fuzzy metrics and mappings could lead to new insights and fixed-point results with broader implications. These results can be expanded upon by readers with applications in fuzzy contexts, refer [9], [10], [12], [11], [13].

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References

[1] S Czerwik. Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica universitatis ostraviensis*, 1(1):5–11, 1993.

[2] M Dinarvand. Some fixed point results for admissible Geraghty contraction type mapping in fuzzy metric spaces. *Iranian Journal of fuzzy systems*, 14(3):161–177, 2017.

[3] H Faraji, D Savić, and S Radenović. Fixed point theorems for Geraghty contraction type mappings in b-metric spaces and applications. *Axioms*, 8(1):34, 2019.

[4] A George and P Veeramani. On some results in fuzzy metric spaces. *Fuzzy sets and systems*, 64(3):395–399, 1994.

[5] M A Geraghty. On contractive mappings. *Proceedings of the American Mathematical Society*, 40(2):604–608, 1973.

[6] N Hussain, P Salimi, and V Parvaneh. Fixed point results for various contractions in parametric and fuzzy b-metric spaces. *J. Nonlinear Sci. Appl*, 8(5):719–739, 2015.

[7] I Kramosil and J Michálek. Fuzzy metrics and statistical metric spaces. *Kybernetika*, 11(5):336–344, 1975.

[8] S Nădăban. Fuzzy b-metric spaces. *International Journal of Computers Communications and Control*, 11(2):273–281, 2016.

[9] U D Patel and S Radenović. Suzuki-type fuzzy contractive inequalities in 1-Z-complete fuzzy metric-like spaces with an application. *Nonlinear Analysis: Modelling and Control*, 28:1–17, 2023.

[10] M Rashid, N Saleem, R Bibi, and R George. Solution of integral equations using some multiple fixed point results in special kinds of distance spaces. *Mathematics*, 10(24):4707, 2022.

[11] N Saleem, U Ishtiaq, K Ahmad, and M De la Sen. Multivalued neutrosophic fractals and Hutchinson-Barnsley operator in neutrosophic metric space. *Chaos, Solitons & Fractals*, 172:113607, 2023.

[12] N Saleem, U Ishtiaq, F Uddin, S Sessa, K Ahmad, and F Di Martino. Graphical views of intuitionistic fuzzy double-controlled metric-like spaces and certain fixed-point results with application. *Symmetry*, 14(11):2364, 2022.

[13] N Saleem, M Zhou, and S Bashir. Solution of fractional integral equations via fixed point results. *Journal of Inequalities and Applications*, 2022(1):148, 2022.

[14] B Schweizer and A Sklar. *Probabilistic metric spaces.* North Holland series in probability and applied mathematics. Elsevier, New York, 1983.

[15] S Sedghi and N. Shobe. Common Fixed Point Theorems in b-Fuzzy Metric Spaces. *Nonlinear Functional Analysis and Applications*, 17(3):349–359, 2013.