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# Quadri-Polar Fuzzy Fantastic Ideals in BCI-Algebras: A TOPSIS Framework and Application

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**Abstract.** A quadri-polar fuzzy ($q\mathcal{P}$-$\mathcal{F}$) set is an extension of a traditional fuzzy set that uses four degrees of membership to represent different aspects of belonging to provide a more detailed framework for handling uncertainty and vagueness. In this paper, we propose the notion of quadri-polar-$(\varpi, \vartheta)$-fuzzy fantastic ideals ($q\mathcal{P}$-$(\varpi, \vartheta)$-$\mathcal{FFI}(s)$) in BCI-algebras based on $q\mathcal{P}$-$\mathcal{F}$ set. Also, the notion of quadri-polar-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-fuzzy fantastic ideals ($q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}(s)$) is introduced, and the characterizations for an $\in_{\tilde{\sigma}}$-$q\mathcal{P}$-$\mathcal{F}$ set and $q_{\tilde{\tau}}$-$q\mathcal{P}$-$\mathcal{F}$ set to be quadri-polar fuzzy ideals ($q\mathcal{P}$-$\mathcal{FI}$) in BCI-algebras are established. Furthermore, we present the $q\mathcal{P}$-$\mathcal{F}$ TOPSIS technique for multi-criteria Group decision-making (MCGDM), which is a natural extension of the TOPSIS method and used to rank and choose the best alternatives under $q\mathcal{P}$-$\mathcal{F}$ positive and negative ideal solutions. Finally, practical examples interpreting the applicability of our proposed $q\mathcal{P}$-$\mathcal{F}$-TOPSIS are solved.

**2020 Mathematics Subject Classifications**: 03B47, 03E72, 08A72

**Key Words and Phrases**: BCK/BCI-algebras, $q$-polar fuzzy fantastic ideal, $q$-polar-$(\omega, \vartheta)$-fuzzy fantastic ideal, $q$-polar-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-fuzzy fantastic ideal

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## 1. Introduction

Axiom systems, developed by Imai et al. [20, 21] and used in propositional calculi, are collections of axioms and inference guidelines used to derive theorems and prove the correctness of logical arguments. Theorem logic and propositional calculus are other names for propositional logic, which deals with the manipulation and analysis of statements using logical operators like OR, AND, and NOT. Various mathematical systems, including propositional logic, can be modeled and analyzed using algebraic structures, particularly Boolean algebra, which is closely related to propositional logic. Iseki [22, 23] introduced the concept of BCK/BCI-algebras. BCI-algebras, also known as BCK-algebras, generalize Boolean algebras and other related algebraic structures. The idea of fantastic ideals in BCI-algebras is a significant algebraic substructure presented and discussed by Saeid [1].

Zadeh [39] proposed the concept of fuzzy (uncertainty) sets, which address ambiguity and vagueness in real world circumstances. A membership function with a range of [0,1] is used to illustrate an uncertainty structure. Throughout the history of uncertainty set, there are many kinds of uncertainty set extensions, for example bipolar [13] and multipolar [3] uncertainty sets, etc. The bipolar and multipolar uncertainty sets are in fact a generalization of an uncertainty set with a membership degree range $[-1, 1]$ and $[0, 1]^q$, respectively. In [5, 17], the few aspects of the bipolar fuzzy concept are applied to algebraic structures. The $q\mathcal{P}$-$\mathcal{FS}$ has an extensive range of implementations to address ambiguity and vagueness in real world issues related to the quadri-polar data, quadri-index and quadri-attributes information. Researchers in a lot of different areas are very interested in the multi-polar uncertainty set theory. These areas include Lie algebras [4], ordered semihypergroups [30], subgroups [16] and BCK/BCI-algebras [7, 36].

Rosenfeld [38] introduced fuzzy groups, while Bhakat et al [12] developed a specific type denoted as $(\in, \in \vee q)$, based on point fuzzy sets within group theory. Jun [27, 28] and Muhiuddin et al. [35] extended this concept to $(\alpha, \beta)$-fuzzy subalgebra. Ibrara et al. [19], Dudek et al. [14], and Narayanan et al. [37] furthered this idea with extensions to semigroups, hemirings, and near-rings, respectively. Al-Masarwah et al. [6, 8] explored $(\alpha, \beta)$ type subalgebras using m-F points within BCK-algebras. Ma et al. [33] introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals, while Jana et al. [24] proposed $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy soft BCI-algebras. Zulfiqar et al. [42, 43] introduced the idea of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subcommutative ideals and fuzzy fantastic ideals in BCI/BCH-algebras. Zhan [41] contributed with $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft $\Gamma$-hyper ideals. Abuhijleh et al. [2] introduced the complex fuzzy groups. Fallath et al. [15] introduced cosets and normals of $(\gamma, \delta)$-fuzzy HX-subgroups. Balamurugan et al. [10, 18] introduced anti-intuitionistic fuzzy soft ideals in BCK/BCI/BG-algebras. Balamurugan et al. [11, 34] introduced tripolar picture fuzzy ideals and bipolar intuitionistic fuzzy soft ideals in BCK/BCI-algebras. Moin et al. [9] introduced and studied a graph associated to UP-algebras. Fuzzy bi-ideals in ternary semirings are studied and explored by Kavikumar [29].

In this work, we combine $q\mathcal{P}$-$\mathcal{F}$ sets with BCI-algebras to extend fuzzy set theory and provide new approaches for studying quadri-polar fuzzy BCI-algebras. We introduce a new class of generalized $q\mathcal{P}$-$(\varpi, \vartheta)$-$\mathcal{FFI}$. The properties of $q\mathcal{P}$-$(\varpi, \vartheta)$-$\mathcal{FFI}(s)$ are highlighted.

We then discuss $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}(s)$ and explore their properties. Characterization theorems for $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}(s)$ are also established. Finally, we present a $q$-$\mathcal{PF}$ TOPSIS methodology, discuss potential applications, compare it with existing TOPSIS methods, and propose future directions.

To explain the novelty of this structure, some contributions by several researchers towards $q\mathcal{P}$-$\mathcal{FFI}(s)$, $q\mathcal{P}$-$(\varpi, \vartheta)$-$\mathcal{FFI}s$ and $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}(s)$ in BCI-algebras are presented in Table 1.

Table 1: Contributions of several researchers toward ceratin generalizations of $q\mathcal{P}$-$\mathcal{FFI}(s)$.

| **Authors** | **Year** | **Contributions** |
|---|---|---|
| Rosenfeld [38] | 1971 | Creation of fuzzy subgroups. |
| Xi [40] | 1991 | Creation of fuzzy ideals. |
| Bhakat and Das [12] | 1996 | Certain extensions of fuzzy subgroups. |
| Jun [26] | 2004 | Creation of $(\alpha, \beta)$-fuzzy ideals. |
| Lee [32] | 2009 | Creation of bipolar fuzzy ideals. |
| Jana et al. [25] | 2017 | Extensions of bipolar fuzzy ideals. |
| Al-Masarwah and Ahmad [6–8] | 2018 | Creation of multi $\mathcal{P}$-$\mathcal{FI}s$. |
| Alqahtani et al. | Present | Creation of $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}(s)$. |

## 2. Preliminaries

BCI-algebras are types of algebraic structures used in the study of non-classical logics, particularly in the context of certain types of implication algebras. These algebras generalize certain aspects of set theory, logic and have applications in some areas, such as theoretical computer science and mathematical logic.

A BCI-algebra is a structure $(\tilde{\aleph}; \between, 0)$ consisting of a non-void set $\tilde{\aleph}$, a binary operation $\between$ on $\tilde{\aleph}$, and a constant $0 \in \tilde{\aleph}$, satisfying the following axioms: $\forall \dot{\varsigma},\ \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$
$(I_1)$ $((\dot{\varsigma} \between \dot{\varrho}) \between (\dot{\varsigma} \between \dot{\kappa})) \between (\dot{\kappa} \between \dot{\varrho}) = 0,$
$(I_2)$ $(\dot{\varsigma} \between (\dot{\varsigma} \between \dot{\varrho})) \between \dot{\varrho} = 0,$
$(I_3)$ $\dot{\varsigma} \between \dot{\varsigma} = 0,$
$(I_4)$ $\dot{\varsigma} \between \dot{\varrho} = 0,\ \dot{\varrho} \between \dot{\varsigma} = 0 \Rightarrow \dot{\varsigma} = \dot{\varrho}.$

A subset $\mathcal{I}$ of $\tilde{\aleph}$ is referred to an *ideal* of $\tilde{\aleph}$ (see [22, 23]) if it meets:

$$0 \in \mathcal{I} \text{ and } (\forall \dot{\varsigma},\ \dot{\varrho} \in \mathcal{I})\ (\dot{\varsigma} \between \dot{\varrho} \in \mathcal{I}, \dot{\varrho} \in \mathcal{I} \Rightarrow \dot{\varsigma} \in \mathcal{I}). \tag{1}$$

A subset $\mathcal{I}$ of $\tilde{\aleph}$ is referred to a *fantastic ideal* of $\tilde{\aleph}$ (see [1]) if it meets:

$$0 \in \mathcal{I} \text{ and } (\forall \dot{\varsigma},\ \dot{\varrho}, \dot{\kappa} \in \mathcal{I})((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa} \in \mathcal{I}, \dot{\kappa} \in \mathcal{I} \Rightarrow \dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \mathcal{I}). \tag{2}$$

**Definition 1.** *[31] A mapping* $\tilde{A} : \tilde{\aleph} \to [0, 1]$ *is a fuzzy set* $\mathcal{FS}$ *for the set* $\tilde{\aleph}$.

A $\mathcal{FS}$ $\tilde{Ą}$ of $\tilde{\aleph}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$ if it meets:

$$(\forall \dot{\varsigma},\ \dot{\varrho} \in \tilde{\aleph}, \tilde{Ą}(0) \geq \tilde{Ą}(\dot{\varsigma}) \text{ and } \tilde{Ą}(\dot{\varsigma}) \geq \tilde{Ą}(\dot{\varsigma} \between \dot{\varrho}) \wedge \tilde{Ą}(\dot{\varrho})). \tag{3}$$

A $\mathcal{FS}$ $\tilde{Ą}$ of $\tilde{\aleph}$ is a $\mathcal{FFI}$ of $\tilde{\aleph}$ if it meets:

$$(\forall \dot{\varsigma},\ \dot{\varrho} \in \tilde{\aleph}, \tilde{Ą}(0) \geq \tilde{Ą}(\dot{\varsigma}) \text{ and } \tilde{Ą}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{Ą}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\varkappa}) \wedge \tilde{Ą}(\dot{\varkappa})). \tag{4}$$

## 3. Quadri-Polar Fuzzy Fantastic Ideals

**Definition 2.** *A mapping $\tilde{\eth} : \tilde{\aleph} \to [0,1]^4$ is a $q\mathcal{P}$-$\mathcal{F}$ for the set $\tilde{\aleph}$, where for any $\dot{\varsigma} \in \tilde{\aleph}$,*

$$\tilde{\eth}(\dot{\varsigma}) = (\tilde{\eth}^1(\dot{\varsigma}), \tilde{\eth}^2(\dot{\varsigma}), \tilde{\eth}^3(\dot{\varsigma}), \tilde{\eth}^4(\dot{\varsigma})) \text{ and } \tilde{\eth}^q(\dot{\varsigma}) \in [0,1],$$

*for $q = 1, 2, 3, 4$.*

**Definition 3.** *A $q\mathcal{P}$-$\mathcal{F}$ set $\tilde{\eth}$ of $\tilde{\aleph}$ is a $q\mathcal{P}$-$\mathcal{FFI}$ if, $\forall \dot{\varsigma}, \dot{\varrho}, \dot{\varkappa} \in \tilde{\aleph}$ and $q = 1, 2, 3, 4$,*

$$\tilde{\eth}(0) \geq \tilde{\eth}(\dot{\varsigma}) \text{ and } \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\varkappa}) \wedge \tilde{\eth}(\dot{\varkappa}).$$

*That is,*

$$\tilde{\eth}^q(0) \geq \tilde{\eth}^q(\dot{\varsigma}) \text{ and } \tilde{\eth}^q(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\eth}^q((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\varkappa}) \wedge \tilde{\eth}^q(\dot{\varkappa}).$$

**Example 1.** *Consider $\tilde{\aleph} = \{0, \dot{\varsigma}, \dot{\varrho}, \dot{\varkappa}\}$ with the binary operation $\between$ defined by Table 2:*

**Table 2.** *Cayley table representing by " $\between$ "*

| $\between$ | 0 | $\dot{\varsigma}$ | $\dot{\varrho}$ | $\dot{\varkappa}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $\dot{\varsigma}$ | $\dot{\varsigma}$ | 0 | 0 | $\dot{\varsigma}$ |
| $\dot{\varrho}$ | $\dot{\varrho}$ | $\dot{\varsigma}$ | 0 | $\dot{\varrho}$ |
| $\dot{\varkappa}$ | $\dot{\varkappa}$ | $\dot{\varkappa}$ | $\dot{\varkappa}$ | 0 |

*Thus, $(\tilde{\aleph}; \between, 0)$ forms a BCI-algebra.*
*Consider a $q\mathcal{P}$-$\mathcal{F}$ set $\tilde{\eth}$ defined on $\tilde{\aleph}$ as follows:*

$$\tilde{\eth}(\dot{\varsigma}) = \left\{ \begin{array}{ll} \langle 0, & (.58, .65, .75, .54)\rangle, \\ \langle \dot{\varsigma}, & (.48, .21, .45, .30)\rangle, \\ \langle \dot{\varrho}, & (.28, .52, .54, .30)\rangle, \\ \langle \dot{\varkappa}, & (.28, .41, .36, .54)\rangle. \end{array} \right\}$$

*Thus, $\tilde{\eth}$ is a $q\mathcal{P}$-$\mathcal{FFI}$ of $\tilde{\aleph}$.*

**Theorem 1.** *A $q\mathcal{P}$-$\mathcal{F}$ set $\tilde{\eth}$ is a $q\mathcal{P}$-$\mathcal{FFI}$ of $\tilde{\aleph}$ $\Leftrightarrow$ for any $\tilde{\rho} \in (0,1]^4$, the $\tilde{\rho}$-cut subset $\tilde{\eth}_{\tilde{\rho}} = \{\dot{\varsigma} \in \tilde{\aleph} \mid \tilde{\eth}(\dot{\varsigma}) \geq \tilde{\rho}\}$ is a fantastic ideal of $\tilde{\aleph}$.*

*Proof.* Let $\tilde{\eth}$ be a $q\mathcal{P}$-$\mathcal{FFI}$ of $\tilde{\aleph}$ and $\tilde{\rho} \in (0,1]^4$ be such that $\tilde{\eth}_{\tilde{\rho}} = \{\dot{\varsigma} \in \tilde{\aleph} \mid \tilde{\eth}(\dot{\varsigma}) \geq \tilde{\rho}\}$. Let $\dot{\varsigma}, \dot{\varrho}, \dot{\varkappa} \in \tilde{\eth}_{\tilde{\rho}}$. Then, $\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\varkappa}) \geq \tilde{\rho}$ and $\tilde{\eth}(\dot{\varkappa}) \geq \tilde{\rho}$. It follows from Definition 3.2 that,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) = \tilde{\rho} \wedge \tilde{\rho} = \tilde{\rho}.$$

Therefore, $\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \tilde{\eth}_{\tilde{\rho}}$. Hence, $\tilde{\eth}_{\tilde{\rho}}$ is a fantastic ideal of $\tilde{\aleph}$.

Conversely, assume $\tilde{\eth}_{\tilde{\rho}}$ is a fantastic ideal of $\tilde{\aleph}$. Suppose that $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) < \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa})$. Then $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) < \tilde{\rho} \leq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa})$. But $\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \geq \tilde{\rho}$ and $\tilde{\eth}(\dot{\kappa}) \geq \tilde{\rho}$. So, $\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \notin \tilde{\eth}_{\tilde{\rho}}$, a contradiction. Therefore,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}).$$

Hence, $\tilde{\eth}_{\tilde{\rho}}$ is a $\mathcal{FFI}$ of $\tilde{\aleph}$.

Consider a $q\mathcal{P}$-$\mathcal{F}$ set $\tilde{\eth}$ defined on $\tilde{\aleph}$, where

$$\tilde{\eth}(\dot{\varsigma}) = \begin{cases} \tilde{\rho} \in (0,1]^q, & \text{if } \dot{\varsigma} \in \tilde{\aleph} \\ \tilde{0}, & \text{if } \dot{\varsigma} \notin \tilde{\aleph}, \end{cases}$$

then $\tilde{\eth}(\dot{\varsigma}$ is a $q\mathcal{P}$-$\mathcal{F}$ point with support $\tilde{\aleph}$ and the value $\tilde{\rho}$, and it is symbolized by $\dot{\varsigma}_{\tilde{\rho}}$.

**Theorem 2.** *Every fantastic ideal of* $\tilde{\aleph}$ *is a* $q\mathcal{P}$-$\mathcal{FFI}$ *of* $\tilde{\aleph}$.

*Proof.* Suppose $\tilde{\eth}_{\tilde{\rho}}$ is a fantastic ideal of $\tilde{\aleph}$ and let $\tilde{\eth}$ be an $q\mathcal{P}$-$\mathcal{FS}$ in $\tilde{\aleph}$ defined by

$$\tilde{\eth}(\dot{\varsigma}) = \begin{cases} \tilde{\rho} \in (0,1]^q, & \text{if } \dot{\varsigma} \in \tilde{\aleph} \\ \tilde{0}, & \text{if } \dot{\varsigma} \notin \tilde{\aleph} \end{cases}$$

Let $\dot{\varsigma}, \dot{\varrho} \in \tilde{\aleph}$. To verify that $\tilde{\eth}$ is a $q\mathcal{P}$-$\mathcal{FFI}$ of $\tilde{\aleph}$.

Case 1: If $(\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa} \in \tilde{\eth}$ and $\dot{\kappa} \in \tilde{\eth}$, then $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \in \tilde{\eth}$. Thus $\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) = \tilde{\eth}(\dot{\kappa}) = \tilde{\rho}$. Hence by Definition 3.2, we have

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) = \tilde{\rho} \wedge \tilde{\rho} = \tilde{\rho}.$$

Case 2: If $(\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa} \notin \tilde{\eth}$ and $\dot{\kappa} \notin \tilde{\eth}$, then $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \notin \tilde{\eth}$. Thus, $\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) = \tilde{\eth}(\dot{\kappa}) = \tilde{0}$. Hence by Definition 3.2, we have

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) = \tilde{0} \wedge \tilde{0} = \tilde{0}.$$

Case 3: If either $(\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa} \in \tilde{\eth}$ or $\dot{\kappa} \in \tilde{\eth}$, then either $\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) = \tilde{0}$ or $\tilde{\eth}(\dot{\kappa}) = \tilde{0}$. So,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}).$$

Hence, $\tilde{\eth}$ is a $q\mathcal{P}$-$\mathcal{FFI}$ of $\tilde{\aleph}$.

## 4. Quadri-Polar $(\varpi, \vartheta)$-Fuzzy Fantastic Ideals

In this section, we introduce the concept of a $q\mathcal{P}$-$(\varpi, \vartheta)\mathcal{FFI}(s)$ in BCI-algebras and explore various properties associated with it. Here, we use $\varpi$ and $\vartheta$ to represent symbols such as $\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}},\ q_{\tilde{\tau}}$ or $\in_{\tilde{\sigma}} \wedge q_{\tilde{\tau}}$, unless specified otherwise.

Consider a $q\mathcal{P}$-$\mathcal{F}$ point denoted as $\dot{\varsigma}_{\tilde{\rho}}$ and a $q\mathcal{P}$-$\mathcal{F}$ set $\tilde{\eth}$ defined on $\tilde{\aleph}$. Then
(1) $\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$ if $\tilde{\eth}(\dot{\varsigma}) \geq \tilde{\rho} > \tilde{\sigma}$.
(2) $\dot{\varsigma}_{\tilde{\rho}} q_{\tilde{\tau}} \tilde{\eth}$ if $\tilde{\eth}(\dot{\varsigma}) + \tilde{\rho} > 2\tilde{\tau}$.
(3) $\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}$ if $\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$ or $\dot{\varsigma}_{\tilde{\rho}} q_{\tilde{\tau}} \tilde{\eth}$.
(4) $\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \wedge q_{\tilde{\tau}} \tilde{\eth}$ if $\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$ and $\dot{\varsigma}_{\tilde{\rho}} q_{\tilde{\tau}} \tilde{\eth}$.
(5) $\dot{\varsigma}_{\tilde{\rho}} \varpi \tilde{\eth}$ does not hold for $\varpi = \{\in_{\tilde{\sigma}},\ q_{\tilde{\tau}},\ \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}},\ \in_{\tilde{\sigma}} \wedge q_{\tilde{\tau}}\}, \forall \tilde{\sigma}, \tilde{\tau} \in [0,1]^4$, where $\tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\sigma}_4) < \tilde{\tau} = (\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4)$.

A $q\mathcal{P}$-$\mathcal{F}$ point $\dot{\varsigma}_{\tilde{\rho}} \in \tilde{\eth}$ if $\tilde{\eth}(\dot{\varsigma}) \geq \tilde{\rho}$. That is $\tilde{\eth}^q(\dot{\varsigma}) \geq \tilde{\rho}^q, \forall q = 1,2,3,4$. Also, $\dot{\varsigma}_{\tilde{\rho}} q \tilde{\eth}$ if $\tilde{\eth}(\dot{\varsigma}) + \tilde{\rho} > \hat{1}$. That is, $\tilde{\eth}^q(\dot{\varsigma}) + \tilde{\rho}^q > \hat{1}, \forall q = 1,2,3,4$.

By $\dot{\varsigma}_{\tilde{\rho}} \in \vee q \tilde{\eth}(resp., \dot{\varsigma}_{\tilde{\rho}} \in \wedge q \tilde{\eth}) \Rightarrow \dot{\varsigma}_{\tilde{\rho}} \in \tilde{\eth}$ or $\dot{\varsigma}_{\tilde{\rho}} q \tilde{\eth}$(resp., $\dot{\varsigma}_{\tilde{\rho}} \in \tilde{\eth}$ and $\dot{\varsigma}_{\tilde{\rho}} q \tilde{\eth}$). If $\phi \neq \tilde{C} \subseteq \tilde{\aleph}$, then the quadri-polar characteristic fuzzy set ($q\mathcal{P}$-$\mathcal{CF}$) of $\tilde{C}$, say $\hat{\chi}_{\tilde{C}}$, where

$$\hat{\chi}_{\tilde{C}} = \begin{cases} \hat{1} = (1,1,1,1), \text{ if } \dot{\varsigma} \in \tilde{C} \\ \tilde{0} = (0,0,0,0), \text{ if } \dot{\varsigma} \notin \tilde{C} \end{cases}$$

Clearly, a $q\mathcal{P}$-$\mathcal{CF}$ is a $q\mathcal{P}$-$\mathcal{F}$ subset of $\tilde{\aleph}$.

**Definition 4.** *A $q\mathcal{P}$-$\mathcal{F}$ set $\tilde{\eth}$ is a $q\mathcal{P}$-$(\varpi, \vartheta)$-$\mathcal{FFI}$ of $\tilde{\aleph}$, if*

$$((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \varpi \tilde{\eth},\ \dot{\kappa}_{\tilde{\eta}} \varpi \tilde{\eth} \Rightarrow (\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \vartheta \tilde{\eth},$$

*where $\varpi \neq \in_{\tilde{\sigma}} \wedge q_{\tilde{\tau}}, \forall \tilde{\sigma} < \tilde{\rho}, \tilde{\eta} \leq \hat{1}$ and $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}), \dot{\kappa} \in \tilde{\aleph}$.*

Consider a $q\mathcal{P}$-$\mathcal{F}$ set $\tilde{\eth}$ defined on $\tilde{\aleph}$ such that $\tilde{\eth}(\dot{\varsigma}) \leq \tilde{\tau}, \forall \dot{\varsigma} \in \tilde{\aleph}$. Let $\dot{\varsigma} \in \tilde{\aleph}$ and $\tilde{\sigma} < \tilde{\rho} \leq \hat{1}$ be such that $\dot{\varsigma}_{\tilde{\rho}} \in \wedge q_{\tilde{\tau}} \tilde{\eth}$. Then, $\tilde{\eth}(\dot{\varsigma}) \geq \tilde{\rho} > \tilde{\sigma}$ and $\tilde{\eth}(\dot{\varsigma}) + \tilde{\rho} > 2\tilde{\tau}$. Thus,

$$2\tilde{\tau} < \tilde{\eth}(\dot{\varsigma}) + \tilde{\rho} \leq \tilde{\eth}(\dot{\varsigma}) + \tilde{\eth}(\dot{\varsigma}) = 2\tilde{\eth}(\dot{\varsigma}) \Rightarrow \tilde{\eth}(\dot{\varsigma}) > \tilde{\tau}.$$

Hence, $\{\dot{\varsigma}_{\tilde{\rho}} \mid \dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \wedge q_{\tilde{\tau}} \tilde{\eth}\} = \phi$. Therefore, we exclude the case $\varpi = \in_{\tilde{\sigma}} \wedge q_{\tilde{\tau}}$ in Definition 4.1 is neglected.

**Theorem 3.** *Let $\tilde{\eth}$ be a $q\mathcal{P}$-$(\varpi, \vartheta)$-$\mathcal{FFI}$ and $\tilde{\sigma} + \hat{1} = 2\tilde{\tau}$ of $\tilde{\aleph}$. Then, the set*

$$\tilde{\eth}_{\tilde{\sigma}} = \{\dot{\varsigma} \in \tilde{\aleph} \mid \tilde{\eth}(\dot{\varsigma}) > \tilde{\sigma}\}$$

*is a $\mathcal{FI}$ of $\tilde{\aleph}$.*

*Proof.* Let $\varsigma, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$ be such that $\varsigma, \dot{\varrho}, \dot{\kappa} \in \tilde{\eth}_{\tilde{\sigma}}$. Then, $\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) > \tilde{\sigma}$ and $\tilde{\eth}(\dot{\kappa}) > \tilde{\sigma}$. Assume $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \leq \tilde{\sigma}$.
If $\varpi \in \{\in_{\tilde{\sigma}},\ \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}\}$, then

$$((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\eth}(\dot{\varsigma})}\varpi\tilde{\eth} \text{ and } \dot{\kappa}_{\tilde{\eth}(\dot{\varrho})}\varpi\tilde{\eth}.$$

But

$$\tilde{\eth}(\varsigma \between (\dot{\varrho} \between (\varrho \between \dot{\varsigma}))) \leq \tilde{\sigma} < \tilde{\eth}((\dot{\varsigma} \between \varrho) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa})$$

and

$$\tilde{\eth}(\dot{\varsigma} \between (\varrho \between (\varrho \between \dot{\varsigma}))) + \tilde{\eth}((\dot{\varsigma} \between \varrho) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \leq \tilde{\sigma} + \hat{1} = 2\tilde{\tau}.$$

So,

$$(\dot{\varsigma} \between (\varrho \between (\varrho \between \dot{\varsigma})))_{\tilde{\tilde{\eth}}(\dot{\varsigma}) \wedge \tilde{\eth}(\dot{\eta})}\vartheta\tilde{\eth}, \forall \vartheta \in \{\in_{\tilde{\sigma}}, q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \wedge q_{\tilde{\tau}}\}, \text{ a contradiction.}$$

Hence, $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\varrho \between \dot{\varsigma}))) > \tilde{\sigma} \Rightarrow \dot{\varsigma} \between (\dot{\varrho} \between (\varrho \between \dot{\varsigma})) \in \tilde{\eth}_{\tilde{\sigma}}$.
Also,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) + \hat{1} > \tilde{\sigma} + \hat{1} = 2\tilde{\tau} \Rightarrow (\dot{\varsigma} \between (\varrho \between (\varrho \between \dot{\varsigma})))_{\hat{1}} q_{\tilde{\tau}}\tilde{\eth}.$$

But

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \leq \tilde{\sigma} \Rightarrow (\dot{\varsigma} \between (\varrho \between (\varrho \between \dot{\varsigma})))_{\hat{1}}\overline{\in_{\tilde{\sigma}}}\tilde{\eth}$$

and

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) + \hat{1} \leq \tilde{\sigma} + \hat{1} = 2\tilde{\tau} \Rightarrow (\dot{\varsigma} \between (\varrho \between (\varrho \between \dot{\varsigma})))_{\hat{1}}\overline{q_{\tilde{\tau}}}\tilde{\eth}, \text{ a contradiction.}$$

Thus,

$$\tilde{\eth}(\dot{\varsigma} \between (\varrho \between (\varrho \between \dot{\varsigma}))) > \tilde{\sigma} \Rightarrow \dot{\varsigma} \between (\varrho \between (\varrho \between \dot{\varsigma})) \in \tilde{\eth}_{\tilde{\sigma}}.$$

Therefore, $\tilde{\eth}_{\tilde{\sigma}}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$.

**Theorem 4.** *Let $\phi \neq \tilde{C} \subseteq \tilde{\aleph}$ and $\tilde{\sigma} + \hat{1} = 2\tilde{\tau}$. Then $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$ if and only if the $q\mathcal{P}$-$\mathcal{F}$ subset $\tilde{\eth}$ of $\tilde{\aleph}$, which is defined as follows:*
(1) $\tilde{\eth}(\dot{\varsigma}) \geq \tilde{\tau}, \forall \dot{\varsigma} \in \tilde{C}$,
(2) $\tilde{\eth}(\dot{\varsigma}) \leq \tilde{\sigma}, \forall \dot{\varsigma} \notin \tilde{C}$ *is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.*

*Proof.* Let $\tilde{C}$ be a $\mathcal{FI}$ of $\tilde{\aleph}$, $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$ and let $\tilde{\sigma} < \tilde{\rho}, \tilde{\eta} \leq \hat{1}$ be such that

$$((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}\tilde{\eth} \text{ and } \dot{\kappa}_{\tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth}.$$

Then

$$\tilde{\eth}(((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})) \geq \tilde{\rho} > \tilde{\sigma} \text{ and } \tilde{\eth}(\dot{\kappa}) \geq \tilde{\eta} > \tilde{\sigma}.$$

Thus,

$$\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \tilde{C} \Rightarrow \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\tau}.$$

If $\tilde{\rho} \wedge \tilde{\eta} \leq \tilde{\tau}$, then

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\tau} \geq \tilde{\rho} \wedge \tilde{\eta} > \tilde{\sigma} \Rightarrow (\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth}.$$

If $\tilde{\rho} \wedge \tilde{\eta} > \tilde{\tau}$, then

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) + \tilde{\rho} \wedge \tilde{\eta} > \tilde{\tau} + \tilde{\tau} = 2\tilde{\tau} \Rightarrow (\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} q_{\tilde{\tau}} \tilde{\eth}.$$

Thus,

$$(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}.$$

Hence, $\tilde{\eth}$ is an $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}$ of $\tilde{\aleph}$.

On the contrary, assume $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}$ of $\tilde{\aleph}$. Then $\tilde{C}$ is equal to $\tilde{\eth}_{\tilde{\sigma}}$. Consequently, according to Theorem 4.1, $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$.

**Corollary 1.** *Let $\tilde{\sigma} + \hat{1} = 2\tilde{\tau}$ and $\phi \neq \tilde{C} \subseteq \tilde{\aleph}$. Then, $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$ if and only if the characteristic function $\hat{\chi}_{\tilde{C}}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}$ of $\tilde{\aleph}$.*

**Theorem 5.** *Let $\phi \neq \tilde{C} \subseteq \tilde{\aleph}$ and $\tilde{\sigma} + \hat{1} = 2\tilde{\tau}$. Then, $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$ if and only if the $q\mathcal{P}$-$\mathcal{F}$ subset $\tilde{\eth}$ of $\tilde{\aleph}$ defined by the following conditions:*
(1) $\tilde{\eth}(\dot{\varsigma}) \geq \tilde{\tau}, \forall \dot{\varsigma} \in \tilde{C}$,
(2) $\tilde{\eth}(\dot{\varsigma}) \leq \tilde{\sigma}, \forall \dot{\varsigma} \notin \tilde{C}$
*is a $q\mathcal{P}$-$(q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.*

*Proof.* Let $\tilde{C}$ be a $\mathcal{FI}$ of $\tilde{\aleph}$, $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$ and let $\tilde{\sigma} < \tilde{\rho}, \tilde{\eta} \leq \hat{1}$ be such that $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} q_{\tilde{\tau}} \tilde{\eth}$ and $\dot{\kappa}_{\tilde{\eta}} q_{\tilde{\tau}} \tilde{\eth}$.
Then

$$\tilde{\eth}(((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})) + \tilde{\rho} > 2\tilde{\tau} \Rightarrow \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) > 2\tilde{\tau} - \tilde{\rho} \geq 2\tilde{\tau} - \hat{1} = \tilde{\sigma}$$

and

$$\tilde{\eth}(\dot{\kappa}) + \tilde{\eta} > 2\tilde{\tau} \Rightarrow \tilde{\eth}(\dot{\kappa}) > 2\tilde{\tau} - \tilde{\eta} \geq 2\tilde{\tau} - \hat{1} = \tilde{\sigma}.$$

Thus,

$$\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \tilde{C} \Rightarrow \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\tau}.$$

Now, if $\tilde{\rho} \wedge \tilde{\eta} \leq \tilde{\tau}$, then

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\tau} \geq \tilde{\rho} \wedge \tilde{\eta} > \tilde{\sigma}.$$

Hence, $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth}$.
If $\tilde{\rho} \wedge \tilde{\eta} > \tilde{\tau}$, then

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) + \tilde{\rho} \wedge \tilde{\varpi} > \tilde{\tau} + \tilde{\tau} = 2\tilde{\tau} \Rightarrow (\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} q_{\tilde{\tau}} \tilde{\eth}.$$

Therefore, $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in \vee q_{\tilde{\tau}} \tilde{\eth}$. Thus, $\tilde{\eth}$ is an $q\mathcal{P}$-$(q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.

On the contrary, assume $\tilde{\eth}$ is a $q\mathcal{P}$-$(q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$. Then, $\tilde{C}$ is equal to $\tilde{\eth}_{\tilde{\sigma}}$. Consequently, according to Theorem 4.1, $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$.

**Corollary 2.** *Let $\phi \neq \tilde{C} \subseteq \tilde{\aleph}$ and $\tilde{\sigma} + \hat{1} = 2\tilde{\tau}$. Then, $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$ if and only if the characteristic function $\hat{\chi}_{\tilde{C}}$ is a $q\mathcal{P}$- $(q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.*

**Theorem 6.** *Let $\phi \neq \tilde{C} \subseteq \tilde{\aleph}$ and $\tilde{\sigma} + \hat{1} = 2\tilde{\tau}$. Then, $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$ if and only if $q\mathcal{P}$-$\mathcal{F}$ subset $\tilde{\eth}$ of $\tilde{\aleph}$ defined by the following conditions:*
*(1) $\tilde{\eth}(\dot{\varsigma}) \geq \tilde{\tau}, \forall \dot{\varsigma} \in \tilde{C}$,*
*(2) $\tilde{\eth}(\dot{\varsigma}) \leq \tilde{\sigma}, \forall \dot{\varsigma} \notin \tilde{C}$*
*is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}$ of $\tilde{\aleph}$.*

*Proof.* Let $\tilde{C}$ be a $\mathcal{FI}$ of $\tilde{\aleph}$, $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$ and let $\tilde{\sigma} < \tilde{\rho}, \tilde{\eta} \leq \hat{1}$ be such that

$$((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth} \Rightarrow ((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth} \text{ or } ((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} q_{\tilde{\tau}} \tilde{\eth}$$

and

$$\dot{\kappa}_{\tilde{\eta}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth} \Rightarrow \dot{\kappa}_{\tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth} \text{ or } \dot{\kappa}_{\tilde{\eta}} q_{\tilde{\tau}} \tilde{\eth}.$$

If $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$, and $\dot{\kappa}_{\tilde{\eta}} q_{\tilde{\tau}} \tilde{\eth}$, then

$$\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \geq \tilde{\rho} > \tilde{\sigma} \text{ and } \tilde{\eth}(\dot{\kappa}) + \tilde{\eta} > 2\tilde{\tau} \Rightarrow \tilde{\eth}(\dot{\kappa}) > 2\tilde{\tau} - \tilde{\eta} \geq 2\tilde{\tau} - \hat{1} = \tilde{\sigma}.$$

Thus, $\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \tilde{C}$. Analogous as in Theorems 4.3 and 4.5,

$$(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}.$$

Hence, $\tilde{\eth}$ is a *q*-$\mathcal{P}$-$(\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$. The other scenarios can be approached in a similar manner to this one.

On the contrary, assume $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$. Then, $\tilde{C}$ is equal to $\tilde{\eth}_{\tilde{\sigma}}$. Consequently, according to Theorem 4.1, $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$.

**Corollary 3.** *Let $\phi \neq \tilde{C} \subseteq \tilde{\aleph}$ and $\tilde{\sigma} + \hat{1} = 2\tilde{\tau}$. Then, $\tilde{C}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$ if and only if the characteristic function $\hat{\chi}_{\tilde{C}}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.*

**Theorem 7.** *Every $q\mathcal{P}$-$(q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.*

*Proof.* Let $\tilde{\eth}$ be a $q\mathcal{P}$-$(q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$, $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$ and let $\tilde{\sigma} < \tilde{\rho}, \tilde{\eta} \leq \hat{1}$ be such that $\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$ and $\dot{\kappa}_{\tilde{\eta}} \in_{\tilde{\tau}} \tilde{\eth}$. Then,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\rho} > \tilde{\sigma} \text{ and } \tilde{\eth}(\dot{\kappa}) \geq \tilde{\eta} > \tilde{\sigma}.$$

Suppose that $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \overline{\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}} \tilde{\eth}$. Then

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) + \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) < \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) + \tilde{\rho} \wedge \tilde{\varpi} \leq 2\tilde{\tau}.$$

Therefore, $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) < \tilde{\tau}$.
Now,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} < \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Choose $\tilde{\sigma} < \hat{r} \leq \hat{1}$ Then

$$2\tilde{\tau} - \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq \hat{r} > 2\tilde{\tau} - \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Therefore,

$$2\tilde{\tau} - \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \wedge (2\tilde{\tau} - \tilde{\sigma}) \geq (2\tilde{\tau} - \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho})) \vee (2\tilde{\tau} - \tilde{\eth}(\dot{\kappa})) \vee \tilde{\tau}.$$

Thus,

$$\hat{r} > 2\tilde{\tau} - \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}), \hat{r} > 2\tilde{\tau} - \tilde{\eth}(\dot{\kappa}) \text{ and } 2\tilde{\tau} - \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) > \hat{r}$$

So,

$$\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) + \hat{r} > 2\tilde{\tau}, \tilde{\eth}(\dot{\kappa}) + \hat{r} > 2\tilde{\tau} \text{ and } \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) + \hat{r} < 2\tilde{\tau}.$$

Thus, $\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))_{\hat{r}} q_{\tilde{\tau}} \tilde{\eth}, \dot{\kappa}_{\hat{r}} q_{\tilde{\tau}} \tilde{\eth}$ but $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\hat{r}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}$, a contradiction. Hence, $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.

**Theorem 8.** *Every* $q\mathcal{P}$-$(\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ *of* $\tilde{\aleph}$ *is a* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ *of* $\tilde{\aleph}$.

*Proof.* The proof is based on the observation that if $\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$, then $\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}$.

**Theorem 9.** *Every* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}})\mathcal{FFI}$ *is a* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \ \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}$ *of* $\tilde{\aleph}$.

*Proof.* Let $\tilde{\eth}$ be a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$, $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$ and let $\tilde{\sigma} < \tilde{\rho}, \tilde{\eta} \leq \hat{1}$. So the $q\mathcal{P}$-$\mathcal{F}$ points $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$ and $\dot{\kappa}_{\tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth}$. Then

$$((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth} \text{ and } \dot{\kappa}_{\tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth} \Rightarrow (\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}.$$

Thus, $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.

## 5. Quadri-Polar $(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-Fuzzy Fantastic Ideals

In this section, we present the notion of a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}(s)$ and explore several of its essential characteristics and properties.

**Definition 5.** *A* $q\mathcal{P}$-$\mathcal{F}$ *set* $\tilde{\eth}$ *of* $\tilde{\aleph}$ *is a* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ *of* $\tilde{\aleph}$ *if it satisfies condition (1), as follows:*
*(1)* $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})\tilde{\rho} \in_{\tilde{\sigma}} \tilde{\eth}, \dot{\kappa}_{\tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth} \Rightarrow (\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}, \forall \tilde{\sigma} < \tilde{\rho}, \tilde{\eta} \leq \hat{1}$ *and* $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$.

**Example 2.** *Consider the BCK-algebra* $(\tilde{\aleph}; \between, 0)$ *and a* $q\mathcal{P}$-$\mathcal{F}$ *set* $\tilde{\eth}$ *as illustrated in Example* 3.1. *It is evident from Definition* 5.1, $\tilde{\eth}$ *is a* $q\mathcal{P}$-$(\in_{(0.2,0.1,0.3,0.2)}, \in_{(0.2,0.1,0.3,0.2)} \vee q_{(0.61,0.68,0.78,0.57)})$-$\mathcal{FFI}$ *of* $\tilde{\aleph}$.

**Theorem 10.** *For a* $q\mathcal{P}$-$\mathcal{F}$ *set* $\tilde{\eth}$ *of* $\tilde{\aleph}$, *condition (1) in Definition* 5.1 *is similar with condition (2), as follows:*
*(2)* $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}), \tilde{\tau}, \forall \dot{\varsigma}, \dot{\varrho} \in \tilde{\aleph}$.

*Proof.* (1) $\Rightarrow$ (2). Assume that (2) does not hold. Then, $\exists \dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$ such that

$$\tilde{\eth}(\varsigma \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} < \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Then,

$$\tilde{\eth}(\varsigma \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} < \tilde{\rho} \leq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Thus,

$$((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth} \text{ and } \dot{\kappa}_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}.$$

But $(\varsigma \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho}} \overline{\in_{\tilde{\sigma}} \vee \in_{\tilde{\sigma}}} \tilde{\eth}$, a contradiction.

$$\tilde{\eth}(\varsigma \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

(2) $\Rightarrow$ (1) Let $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}, \dot{\kappa}_{\tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth}$. Then, $\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \geq \tilde{\rho} > \tilde{\sigma}$ and $\tilde{\eth}(\dot{\kappa}) \geq \tilde{\eta} > \tilde{\sigma}$. If $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}}$, then (1) is hold.
If $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}} \tilde{\eth}$, then $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) < \tilde{\rho} \wedge \tilde{\eta}$. Since

$$\begin{aligned}\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} &\geq \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\eth}(\dot{\varrho}) \wedge \tilde{\tau} \\ &\geq \tilde{\rho} \wedge \tilde{\varpi} \wedge \tilde{\tau}.\end{aligned}$$

Therefore,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\tau} \text{ and } \tilde{\rho} \wedge \tilde{\varpi} > \tilde{\tau}.$$

Thus,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) + \tilde{\rho} \wedge \tilde{\eta} > \tilde{\tau} + \tilde{\tau} = 2\tilde{\tau} \Rightarrow (\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} q_{\tilde{\tau}} \tilde{\eth}.$$

Hence, $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho} \wedge \tilde{\eta}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}$.

**Corollary 4.** *A* $q\mathcal{P}$-$\mathcal{F}$ *set* $\tilde{\eth}$ *of* $\tilde{\aleph}$ *is a* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ *of* $\tilde{\aleph}$ *if it satisfies* $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}), \tilde{\tau}, \forall \dot{\varsigma}, \dot{\varrho} \in \tilde{\aleph}$.

**Theorem 11.** *The intersection of any collection of* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}s$ *of* $\tilde{\aleph}$ *is a* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ *of* $\tilde{\aleph}$.

*Proof.* Let $\{\tilde{\eth}_i\}_{i \in I}$ be a collection of $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}s$ of $\tilde{\aleph}$ and $(\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}, \dot{\kappa} \in \tilde{\aleph}$. Then,

$$\tilde{\eth}_i(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq \tilde{\eth}_i((\dot{\varsigma} \between \dot{\varrho})\dot{\kappa}) \wedge \tilde{\eth}_i(\dot{\kappa}) \wedge \tilde{\tau}.$$

Thus,

$$\begin{aligned}(\wedge_{i \in I}\tilde{\eth}_i)(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} &= \wedge_{i \in I}\tilde{\eth}_i(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \\ &\geq \wedge_{i \in I}(\tilde{\eth}_i((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}_i(\dot{\kappa}) \wedge \tilde{\tau}) \\ &\geq (\wedge_{i \in I}\tilde{\eth}_j)((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge (\wedge_{i \in I}\tilde{\eth}_i)(\dot{\kappa}) \wedge \tilde{\tau}.\end{aligned}$$

Therefore, $(\wedge_{i\in I}\tilde{\eth}_i)(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq (\wedge_{i\in I}\tilde{\eth}_i)((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge (\wedge_{i\in I}\tilde{\eth}_i)(\dot{\kappa}) \wedge \tilde{\tau}$. Hence, $\wedge_{i\in I}\tilde{\eth}_i$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}$ of $\tilde{\aleph}$.

For any $q\mathcal{P}$-$\mathcal{F}$ set $\tilde{\eth}$ of $\tilde{\aleph}$ and $\tilde{\rho} \in [0,1]^q$, we define:
(1) $\tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}} = \{\dot{\varsigma} \in \tilde{\aleph} \mid \dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}\}$,
(2) $\langle\tilde{\eth}\rangle^{\tilde{\tau}}_{\tilde{\rho}} = \{\dot{\varsigma} \in \tilde{\aleph} \mid \dot{\varsigma}_{\tilde{\rho}} q_{\tilde{\tau}} \tilde{\eth}\}$,
(3) $[\tilde{\eth}]^{\tilde{\tau}}_{\tilde{\rho}} = \{\dot{\varsigma} \in \tilde{\aleph} \mid \dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}\}$.
It is clear that

$$[\tilde{\eth}]^{\tilde{\tau}}_{\tilde{\rho}} = \tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}} \cup \langle\tilde{\eth}\rangle^{\tilde{\tau}}_{\tilde{\rho}}.$$

The ensuing theorems elucidate the connection between $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}s$ and the crisp $\mathcal{FI}$s in $\tilde{\aleph}$.

**Theorem 12.** *Let $\tilde{\eth}$ be a $q\mathcal{P}$-$\mathcal{F}$ set of $\tilde{\aleph}$. Then, $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph} \Leftrightarrow \tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}} \neq \phi$ is a $\mathcal{FI}$ of $\tilde{\aleph}$, $\forall \tilde{\sigma} < \tilde{\rho} \leq \tilde{\tau}$.*

*Proof.* Let $\tilde{\eth}$ be a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$ and let $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}$ for $\tilde{\sigma} < \tilde{\rho} \leq \tilde{\tau}$. Then $\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \geq \tilde{\rho} > \tilde{\sigma}$ and $\tilde{\eth}(\dot{\kappa}) \geq \tilde{\rho} > \tilde{\sigma}$. Thus, we have

$$\begin{aligned}
\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} &\geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau} \\
&\geq \tilde{\rho} \wedge \tilde{\rho} \wedge \tilde{\tau} \\
&= \tilde{\rho} \wedge \tilde{\tau} \\
&= \tilde{\rho} > \tilde{\sigma}.
\end{aligned}$$

Therefore,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\rho} \Rightarrow \dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}.$$

Thus, $\tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}$ is a $\mathcal{FI}$ of $U$.

On the other hand, suppose that $\tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}$ is a $\mathcal{FI}$ of $U, \forall \tilde{\sigma} < \tilde{\rho} \leq \tilde{\tau}$. Assume $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$ such that

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} < \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Select $\tilde{\sigma} < \tilde{\rho} \leq \tilde{\tau}$ such that

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} < \hat{r} = \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Then, $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}, \dot{\kappa}_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$, but $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))_{\hat{r}} \overline{\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}} \tilde{\eth}$.
Since $\tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$,

$$\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}, \text{ a contradiction.}$$

Hence, $\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}$. Therefore, $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}$ of $\tilde{\aleph}$.

By setting $\tilde{\sigma} = \tilde{0}$ and $\tilde{\tau} = \widehat{0.5}$ in Theorem 5.3, we can derive the subsequent corollary.

**Corollary 5.** *Let* $\tilde{\eth}$ *be a* $q\mathcal{P}$-$\mathcal{F}$ *set of* $\tilde{\aleph}$. *Then* $\tilde{\eth}$ *is a* $q\mathcal{P}$-$(\in, \in \vee q)\mathcal{FFI}$ *of* $\tilde{\aleph} \Leftrightarrow \tilde{\eth}_{\tilde{\rho}} = \{\dot{\varsigma} \in \tilde{\aleph} \mid \dot{\varsigma}_{\tilde{\rho}} \in \tilde{\eth}\} \neq \phi$ *is a* $\mathcal{FI}$ *of* $\tilde{\aleph}, \forall \tilde{\rho} \in (0, 0.5]^q$.

**Theorem 13.** *Let* $\tilde{\eth}$ *be a* $q\mathcal{P}$-$\mathcal{F}$ *set of* $\tilde{\aleph}$. *Then*
(1) $\tilde{\eth}$ *is a* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}s$ *of* $\tilde{\aleph} \Leftrightarrow \tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}} \neq \phi$ *is a* $\mathcal{FI}$ *of* $\tilde{\aleph}, \forall \tilde{\sigma} < \tilde{\rho} \leq \tilde{\tau}$.
(2) *If* $\hat{1} + \tilde{\sigma} = 2\tilde{\tau}$, *then* $\tilde{\eth}$ *is a* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ *of* $\tilde{\aleph} \Leftrightarrow \langle\tilde{\eth}\rangle^{\tilde{\tau}}_{\tilde{\rho}} \neq \phi$ *is a* $\mathcal{FI}$ *of* $\tilde{\aleph}, \forall \tilde{\tau} < \tilde{\rho} \leq \hat{1}$.
(3) *If* $\hat{1} + \tilde{\sigma} = 2\tilde{\tau}$, *then* $\tilde{\eth}$ *is a* $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ *of* $\tilde{\aleph} \Leftrightarrow [\tilde{\eth}]^{\tilde{\tau}}_{\tilde{\rho}} \neq \phi$ *is a* $\mathcal{FI}$ *of* $\tilde{\aleph}, \forall \tilde{\sigma} < \tilde{\rho} \leq \hat{1}$.

*Proof.* (1) Let $\tilde{\eth}$ be a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$. Let $\dot{\varsigma} \in \tilde{\eth}^{\tilde{\tau}}_{\tilde{\rho}}$. Then

$$\tilde{\eth}(0) \vee \tilde{\sigma} \geq \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\tau} \geq \tilde{\rho} \wedge \tilde{\tau} > \tilde{\sigma}.$$

Hence, $\tilde{\eth}(0) \geq \tilde{\rho} \Rightarrow 0 \in \tilde{\eth}^{\tilde{\tau}}_{\tilde{\rho}}$.
Let $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\eth}^{\tilde{\tau}}_{\tilde{\rho}}$. Then,

$$\tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \geq \tilde{\rho} > \tilde{\sigma} \text{ and } \tilde{\eth}(\dot{\kappa}) \geq \tilde{\rho} > \tilde{\sigma}.$$

By Theorem 5.1 (2), we have

$$\begin{aligned}
\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} &\geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\tau} \\
&\geq \tilde{\rho} \wedge \tilde{\rho} \wedge \tilde{\tau} \\
&= \tilde{\rho} \wedge \tilde{\tau} \\
&= \tilde{\rho} > \tilde{\sigma}.
\end{aligned}$$

Therefore,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\rho} \Rightarrow \dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}.$$

Hence, $\tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$.

Conversely, assume $\tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$, $\forall \tilde{\rho} \in (\sigma, \tau]$. Let $\dot{\varsigma} \in \tilde{\aleph}$ be such that $\tilde{\eth}(0) \vee \sigma < \tilde{\rho} = \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\tau}$. Then $\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$, but $0_{\tilde{\rho}} \overline{\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}} \tilde{\eth}$, a contradiction. Suppose $\dot{\varsigma}, \dot{\varrho}, \dot{\kappa} \in \tilde{\aleph}$. Then

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} < \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Select some $\tilde{\rho} \in (\tilde{\sigma}, \tilde{\tau}]$ such that

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} < \tilde{\rho} = \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Then $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}, \dot{\kappa}_{\tilde{\rho}} \in_{\sigma} \tilde{\eth}$, but $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho}} \overline{\in_{\tilde{\sigma}} \vee q_{\tilde{\tau}}} \tilde{\eth}$.
Since $\tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$, we have $\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})) \in \tilde{\eth}^{\tilde{\sigma}}_{\tilde{\rho}}$, a contradiction.
Hence

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Therefore $\tilde{\eth}$ be a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$.

(2) The proof follows a similar pattern as in (1), and therefore, we omit it for brevity.

(3) Let $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})\mathcal{FFI}$ of $\tilde{\aleph}$ and $\tilde{\rho} \in (\tilde{\sigma}, \hat{1}]$. Then $\forall \dot{\varsigma} \in [\tilde{\eth}]^{\tilde{\tau}}_{\tilde{\rho}}$,

$$\dot{\varsigma}_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth} \Rightarrow \tilde{\eth}(\dot{\varsigma}) \geq \tilde{\rho} > \tilde{\sigma} \text{ or } \tilde{\eth}(\dot{\varsigma}) > 2\tilde{\tau} - \tilde{\rho} > 2\tilde{\tau} - \hat{1} = \tilde{\sigma}.$$

Since $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in \tilde{\sigma}, \in \tilde{\sigma} \vee q_{\tilde{\tau}})\mathcal{FFI}$ of $\tilde{\aleph}$,

$$\tilde{\eth}(0) \vee \tilde{\sigma} \geq \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\tau} > \tilde{\sigma} \wedge \tilde{\tau} = \tilde{\sigma},$$

and so

$$\tilde{\eth}(0) \geq \tilde{\sigma} \Rightarrow \tilde{\eth}(0) \geq \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\tau}.$$

Case 1: Let $\tilde{\rho} \in (\tilde{\sigma}, \tilde{\tau}]$. Then $2\tilde{\tau} - \tilde{\rho} \geq \tilde{\tau} \geq \tilde{\rho}$,

$$\tilde{\eth}(0) \geq \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\tau} \geq \tilde{\rho} \wedge \tilde{\tau} = \tilde{\rho} \text{ or } \tilde{\eth}(0) \geq \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\tau} > (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} = \tilde{\rho} \wedge \tilde{\tau} = \tilde{\rho}.$$

Thus, $0_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$.

Case 2: Let $\tilde{\rho} \in (\tilde{\tau}, \hat{1}]$. Then $2\tilde{\tau} - \tilde{\rho} < \tilde{\tau} < \tilde{\rho}$,

$$\tilde{\eth}(0) \geq \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\tau} = \tilde{\rho} \wedge \tilde{\tau} = \tilde{\tau} > 2\tilde{\tau} - \tilde{\rho} \text{ or } \tilde{\eth}(0) \geq \tilde{\eth}(\dot{\varsigma}) \wedge \tilde{\tau} > (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} = 2\tilde{\tau} - \tilde{\rho}.$$

Hence, $0_{\tilde{\rho}} q_{\tilde{\tau}} \tilde{\eth} \Rightarrow 0_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}$.
Let $(\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}, \dot{\kappa} \in [\tilde{\eth}]^{\tilde{\tau}}_{\tilde{\rho}}$. Then $((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa})_{\tilde{\rho}}, \dot{\kappa}_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth}$,

$$\tilde{\eth}((\dot{\varsigma}^{\between} \dot{\varrho}) \between \dot{\kappa}) \geq \tilde{\rho} > \tilde{\sigma} \text{ or } \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) > 2\tau - \tilde{\rho} > 2\tilde{\tau} - \hat{1} = \tilde{\sigma}$$

and

$$\tilde{\eth}(\dot{\kappa}) \geq \tilde{\rho} > \tilde{\sigma} \text{ or } \geq (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} \tilde{\eth}(\dot{\kappa}) > 2\tilde{\tau} - \tilde{\rho} = 2\tilde{\tau} - \tilde{\rho} > 2\tilde{\tau} - \hat{1} = \tilde{\sigma}.$$

Since $\tilde{\eth}$ is a $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}$ of $\tilde{\aleph}$,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \vee \tilde{\sigma} \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau} > \tilde{\sigma} \wedge \tilde{\sigma} \wedge \tilde{\tau} > \tilde{\sigma} \wedge \tilde{\tau} = \tilde{\sigma}.$$

Therefore,

$$\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\sigma} \Rightarrow \tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau}.$$

Case 1: Let $\tilde{\rho} \in (\tilde{\sigma}, \tilde{\tau}]$. Then $2\tilde{\tau} - \tilde{\rho} \geq \tilde{\tau} \geq \tilde{\rho}$,

$$\begin{aligned}
\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) &\geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau} \\
&\geq \tilde{\rho} \wedge \tilde{\rho} \wedge \tilde{\tau} \\
&= \tilde{\rho} \wedge \tilde{\tau} \\
&= \tilde{\rho}
\end{aligned}$$

or

$$\begin{aligned}\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) &\geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau} \\ &\geq \tilde{\rho} \wedge (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} \\ &= \tilde{\rho} \wedge \tilde{\tau} \wedge \tilde{\tau} \\ &= \tilde{\rho} \wedge \tilde{\tau} \\ &= \tilde{\rho}\end{aligned}$$

or

$$\begin{aligned}\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) &\geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau} \\ &\geq (2\tilde{\tau} - \tilde{\rho}) \wedge (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} \\ &= \tilde{\tau} \wedge \tilde{\tau} \wedge \tilde{\tau} \\ &= \tilde{\tau} > \tilde{\rho}.\end{aligned}$$

Hence, $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho}} \in_{\tilde{\sigma}} \tilde{\eth}$.

Case 2: Let $\tilde{\rho} \in (\tilde{\tau}, \hat{1}]$. Then $2\tilde{\tau} - \tilde{\rho} < \tilde{\tau} < \tilde{\rho}$,

$$\begin{aligned}\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) &\geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau} \\ &\geq \tilde{\rho} \wedge \tilde{\rho} \wedge \tilde{\tau} \\ &= \tilde{\rho} \wedge \tilde{\tau} \\ &= \tilde{\tau} > 2\tilde{\tau} - \tilde{\rho}\end{aligned}$$

or

$$\begin{aligned}\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) &\geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau} \\ &\geq \tilde{\rho} \wedge (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} \\ &\geq \tilde{\tau} \wedge (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} \\ &\geq \tilde{\tau} \wedge (2\tilde{\tau} - \tilde{\rho}) \\ &= (2\tilde{\tau} - \tilde{\rho})\end{aligned}$$

or

$$\begin{aligned}\tilde{\eth}(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) &\geq \tilde{\eth}((\dot{\varsigma} \between \dot{\varrho}) \between \dot{\kappa}) \wedge \tilde{\eth}(\dot{\kappa}) \wedge \tilde{\tau} \\ &\geq (2\tilde{\tau} - \tilde{\rho}) \wedge (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} \\ &\geq (2\tilde{\tau} - \tilde{\rho}) \wedge \tilde{\tau} > (2\tilde{\tau} - \tilde{\rho}).\end{aligned}$$

Thus, $(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho}} q_{\tilde{\tau}} \tilde{\eth}$.
Hence,

$$(\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma})))_{\tilde{\rho}} \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}} \tilde{\eth} \Rightarrow (\dot{\varsigma} \between (\dot{\varrho} \between (\dot{\varrho} \between \dot{\varsigma}))) \in [\tilde{\eth}]^{\tilde{\tau}}_{\tilde{\rho}}.$$

Therefore $[\tilde{\eth}]_{\tilde{\rho}}^{\tilde{\tau}}$ is a $\mathcal{FI}$ of $\tilde{\aleph}$.

By substituting $\tilde{\sigma} = \tilde{0}$ and $\tilde{\tau} = \widehat{0.5}$ into Theorem 5.4, we can derive the ensuing corollary.

**Corollary 6.** *Let* $\tilde{\eth}$ *be a* $q\mathcal{P}$-$\mathcal{F}$ *set of* $\tilde{\aleph}$. *Then*
(1) $\tilde{\eth}$ *is a* $q\mathcal{P}$-$(\in, \in \vee q)$-$\mathcal{FFI}$ *of* $\tilde{\aleph} \Leftrightarrow \tilde{\eth}_{\tilde{\rho}}^{\tilde{\sigma}}(\neq \phi)$ *is a* $\mathcal{FI}$ *of* $\tilde{\aleph}, \forall \tilde{\rho} \in (0, 0.5]^q$.
(2) $\tilde{\eth}$ *is a* $q\mathcal{P}$-$(\in, \in \vee q)$-$\mathcal{FFI}$ *of* $\tilde{\aleph} \Leftrightarrow \langle\tilde{\eth}\rangle_{\tilde{\rho}}^{\tilde{\tau}}(\neq \phi)$ *is a* $\mathcal{FI}$ *of* $\tilde{\aleph}, \forall \tilde{\rho} \in (0.5, 1]^q$.
(3) $\tilde{\eth}$ *is a* $q\mathcal{P}$-$(\in, \in \vee q)$-$\mathcal{FFI}$ *of* $\tilde{\aleph} \Leftrightarrow [\tilde{\eth}]_{\tilde{\rho}}^{\tilde{\tau}}(\neq \phi)$ *is a* $\mathcal{FI}$ *of* $\tilde{\aleph}, \forall \tilde{\rho} \in (0, 1]^q$.

## 6. Quadri-Polar Fuzzy TOPSIS Approach

In this section, we present a $q$-$\mathcal{PF}$ TOPSIS approach for multi-criteria group decision-making (MCGDM) problems. For these problems, we use a TOPSIS method based on $q\mathcal{PF}$-sets to address a set of alternatives $\c{A} = \{\dot{\varsigma}_1, \dot{\varsigma}_2, \dot{\varsigma}_3, \dot{\varsigma}_4\}$ and a set $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$ classified by $q$. Decision-makers must evaluate the four possibilities based on the $q$-$\mathcal{PF}$ criteria. The possible ratings of alternatives are evaluated in terms of q different attributes among four membership values, represented as $(i = 1, 2, 3, 4)$.

**Step 1:** The degree of each alternative $\dot{\varsigma}_j \in \c{A}, j = 1, 2, 3, 4)$ over all the criteria $(c_k \in \mathcal{C}, k = 1, 2, 3, 4)$ may be expressed as $q$-$\mathcal{PF}\mathcal{E}s$.

$$\tilde{\eth}^{jk}(\dot{\varsigma}) = (\rho_1 \ o \ \tilde{\eth}^{jk}(\dot{\varsigma}), \rho_2 \ o \ \tilde{\eth}^{jk}(\dot{\varsigma}), \rho_3 \ o \ \tilde{\eth}^{jk}(\dot{\varsigma}), \rho_4 \ o \ \tilde{\eth}^{jk}(\dot{\varsigma})),$$

where $= (\rho_1 \ o \ \tilde{\eth}^{jk}(\dot{\varsigma}) \mid i = 1, 2, ..., q)$. The tabular representation of the $q$-$\mathcal{PF}$ decision matrix is given by Table 2, which describes the ratings of alternatives.

**Table 3**. Tablular representation of $q$-$\mathcal{PF}$ decision matrix.

| Alternatives | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $\dot{\varsigma}_1$ | $\tilde{\eth}^{11}(\dot{\varsigma}_1)$ | $\tilde{\eth}^{12}(\dot{\varsigma}_1)$ | $\tilde{\eth}^{13}(\dot{\varsigma}_1)$ | $\tilde{\eth}^{14}(\dot{\varsigma}_1)$ |
| $\dot{\varsigma}_2$ | $\tilde{\eth}^{21}(\dot{\varsigma}_2)$ | $\tilde{\eth}^{22}(\dot{\varsigma}_2)$ | $\tilde{\eth}^{23}(\dot{\varsigma}_2)$ | $\tilde{\eth}^{24}(\dot{\varsigma}_2)$ |
| $\dot{\varsigma}_3$ | $\tilde{\eth}^{31}(\dot{\varsigma}_3)$ | $\tilde{\eth}^{32}(\dot{\varsigma}_3)$ | $\tilde{\eth}^{33}(\dot{\varsigma}_3)$ | $\tilde{\eth}^{34}(\dot{\varsigma}_3)$ |
| $\dot{\varsigma}_4$ | $\tilde{\eth}^{41}(\dot{\varsigma}_4)$ | $\tilde{\eth}^{42}(\dot{\varsigma}_4)$ | $\tilde{\eth}^{43}(\dot{\varsigma}_4)$ | $\tilde{\eth}^{44}(\dot{\varsigma}_4)$ |

**Step 2:** We build the optimistic or pessimistic $q$-$\mathcal{PF}$ decision matrix by adding the maximal and smallest values to equalize the length of all $q$-$\mathcal{PF}\mathcal{E}$s.

**Step 3:** Weights can be assigned to each $q$-$\mathcal{PF}$ criteria of alternatives by decision-makers based on their choice and importance of each criterion. We assume that the weights assigned by the decision-makers are

$$\c{W} = (w_1, w_2, w_3, w_q) \in (0, 1],$$

satisfying the normalized condition

$$\sum_{k=1}^{q} w_k = 1, q = 1, 2, 3, 4..$$

**Step 4:** The weighted $q$-$\mathcal{PF}$ decision matrix is calculated in Table 4.

**Table 4.** Tabular representation of a weighted $q$-$\mathcal{PF}$ decision matrix.

| Alternatives | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $\dot{\varsigma}_1$ | $\tilde{\eth}(\dot{\varsigma}_1)^{11'}$ | $\tilde{\eth}(\dot{\varsigma}_1)^{12'}$ | $\tilde{\eth}(\dot{\varsigma}_1)^{13'}$ | $\tilde{\eth}(\dot{\varsigma}_1)^{14'}$ |
| $\dot{\varsigma}_2$ | $\tilde{\eth}(\dot{\varsigma}_2)^{21'}$ | $\tilde{\eth}(\dot{\varsigma}_2)^{22'}$ | $\tilde{\eth}(\dot{\varsigma}_2)^{23'}$ | $\tilde{\eth}(\dot{\varsigma}_2)^{24'}$ |
| $\dot{\varsigma}_3$ | $\tilde{\eth}(\dot{\varsigma}_3)^{31'}$ | $\tilde{\eth}(\dot{\varsigma}_3)^{32'}$ | $\tilde{\eth}(\dot{\varsigma}_3)^{33'}$ | $\tilde{\eth}(\dot{\varsigma}_3)^{34'}$ |
| $\dot{\varsigma}_4$ | $\tilde{\eth}(\dot{\varsigma}_4)^{41'}$ | $\tilde{\eth}(\dot{\varsigma}_4)^{42'}$ | $\tilde{\eth}(\dot{\varsigma}_4)^{43'}$ | $\tilde{\eth}(\dot{\varsigma}_4)^{44'}$ |

For each possible j and k,

$$\tilde{\eth}^{jk'}(\dot{\varsigma}) = (\rho_1 \ o \ \tilde{\eth}^{jk'}(\varsigma), \rho_2 \ o \ \tilde{\eth}^{jk'}(\varsigma), \rho_3 \ o \ \tilde{\eth}^{jk'}(\varsigma), \rho_4 \ o \ \tilde{\eth}^{jk'}(\dot{\varsigma})),$$

**Step 5:** The $q\mathcal{P}$-$\mathcal{F}$ positive ideal solution ($q\mathcal{P}$-$\mathcal{FPIS}$) and $q\mathcal{P}$-$\mathcal{F}$ negative ideal solution ($q\mathcal{P}$-$\mathcal{FNIS}$) of alternatives under the $q\mathcal{P}$-$\mathcal{F}$ environment can be calculated by Equations (5) and (6) as

$$q\mathcal{P} - \mathcal{FPIS} = \{(\tilde{\eth}^{1'}(\varsigma))^+, (\tilde{\eth}^{2'}(\varsigma))^+, (\tilde{\eth}^{3'}(\varsigma))^+, (\tilde{\eth}^{4'}(\varsigma))^+\}, \tag{5}$$

$$q\mathcal{P} - \mathcal{FNIS} = \{(\tilde{\eth}^{1'}(\varsigma))^-, (\tilde{\eth}^{2'}(\varsigma))^-, (\tilde{\eth}^{3'}(\varsigma))^-, (\tilde{\eth}^{4'}(\varsigma))^-\}, \tag{6}$$

where

$$\begin{aligned}(\tilde{\eth}^{k'}(\varsigma))^+ &= \sup_j (\tilde{\eth}^{k'}(\varsigma))\\ &= \sup_j (\rho_1 \ o \ \tilde{\eth}^{jk'}(\dot{\varsigma}), \rho_2 \ o \ \tilde{\eth}^{jk'}(\varsigma), \rho_3 \ o \ \tilde{\eth}^{jk'}(\varsigma), \rho_4 \ o \ \tilde{\eth}^{jk'}(\varsigma))\\ &= ((\rho_1 \ o \ \tilde{\eth}^{k'}(\varsigma))^+, (\rho_2 \ o \ \tilde{\eth}^{k'}(\varsigma))^+, (\rho_3 \ o \ \tilde{\eth}^{k'}(\varsigma))^+, (\rho_4 \ o \ \tilde{\eth}^{k'}(\varsigma))^+),\end{aligned}$$

and

$$\begin{aligned}(\tilde{\eth}^{k'}(\varsigma))^- &= \inf_j (\tilde{\eth}^{k'}(\varsigma))\\ &= \inf_j (\rho_1 \ o \ \tilde{\eth}^{jk'}(\varsigma), \rho_2 \ o \ \tilde{\eth}^{jk'}(\varsigma), \rho_3 \ o \ \tilde{\eth}^{jk'}(\varsigma), \rho_4 \ o \ \tilde{\eth}^{jk'}(\varsigma))\\ &= ((\rho_1 \ o \ \tilde{\eth}^{k'}(\varsigma))^-, (\rho_2 \ o \ \tilde{\eth}^{k'}(\varsigma))^-, (\rho_3 \ o \ \tilde{\eth}^{k'}(\varsigma))^-, (\rho_4 \ o \ \tilde{\eth}^{k'}(\varsigma))^-).\end{aligned}$$

**Step 6:** The $q\mathcal{P}$-$\mathcal{F}$ Euclidean distance of each alternative $\dot{\varsigma}_j$ from $q\mathcal{P}$-$\mathcal{FPIS}$ and $q\mathcal{P}$-$\mathcal{FNIS}$ can be calculated by Equations (7) and (8).

$$D_E'(\dot{\varsigma}_j, q\mathcal{P} - \mathcal{FPIS}) = \sqrt{\frac{1}{16}\sum_{k=1}^{4}[\sum_{l=1}^{4}\{\sum_{i=1}^{4}(\rho_i \ o \ \tilde{\eth}^{jk'}(\varsigma) - \rho_i \ o \ \tilde{\eth}^{k'}(\dot{\varsigma})^+\}]}, \tag{7}$$

and

$$D_E^{'}(\dot{\varsigma}_j, q\mathcal{P} - \mathcal{FNIS}) = \sqrt{\frac{1}{16}\sum_{k=1}^{4}[\sum_{l=1}^{4}\{\sum_{i=1}^{4}(\rho_i \ o \ \tilde{\eth}^{jk'}(\dot{\varsigma}) - \rho_i \ o \ \tilde{\eth}^{k'}(\dot{\varsigma})^{-}\}]}, \quad (8)$$

**Step 7:** The relative $q\mathcal{P}$-$\mathcal{F}$ closeness coefficient of each alternative $\dot{\varsigma}_j$ using the following formula as described by (9),

$$E_j^{'} = \frac{D_E^{'}(\dot{\varsigma}_j, q\mathcal{P} - \mathcal{FNIS})}{D_E^{'}(\dot{\varsigma}_j, q\mathcal{P} - \mathcal{FPIS}) + D_E^{'}(\dot{\varsigma}_j, q\mathcal{P} - \mathcal{FNIS})}, j = 1, 2, 3, 4. \quad (9)$$

The alternative with the highest $q\mathcal{P}$-$\mathcal{F}$ closeness coefficient is the best one, and we can rank each alternative in order.
We present our proposed decision-making method in Algorithm 1.
In Section 6.1, we examine the practical usage of our suggested model. Specifically, we

**Algorithm 1.** The algorithm of the proposed approaches for dealing MCGDM problems.

**Step 1.** Input.
**Step 2.** Determine the optimistic or pessimistic $q$-$\mathcal{PF}$ decision matrix.
**Step 3.** Calculate the normalized weights.
**Step 4.** Calculate the weight for the pessimistic $q\mathcal{P}$-$\mathcal{F}$ decision matrix.
**Step 5.** Compute the $q\mathcal{P}$-$\mathcal{FPIS}$, and $q\mathcal{P}$-$\mathcal{FNIS}$.
**Step 6.** $q\mathcal{P}$-$\mathcal{F}$ Euclidean distance of each alternative $\dot{\varsigma}_j$ from $q\mathcal{FPIS}$ and $q\mathcal{P}$-$\mathcal{FPIS}$.
**Step 7.** Calculate the relative $q\mathcal{P}$-$\mathcal{F}$ closeness coefficients $E_{j'}$.
**Step 8.** Output.
Rank the possibilities for the final decision and choose the best one.

show how $q\mathcal{P}$-$\mathcal{F}$ is useful in the selection of solar power plant stations in a rural region.

## Selection of Solar Power Plant Station in a Rural Area

Assume that the government want to build a solar power plant station in a rural area. The government has four options for the location of a new solar power plant. Each site was appraised by a group of decision makers based on the following criteria as follows:

1. "Solar Irradiance ($T_1$)", which could have the characteristics shown below:
• Solar Irradiance Density: The amount of solar power received per unit area.
• Sun Light Duration: The number of sunlight hours per day.
• Solar Tracking: The technology used to follow the sun's path to maximize energy capture.
• Temporal Variability: The fluctuation of solar irradiance over time, including daily and seasonal changes.

2. "Land Availability ($T_2$)", which could have the characteristics shown below:

• Land Size: The total area available for the solar power plant.

• Land Ownership: The ownership status of the land, such as government-owned, privately-owned, or leased.

• Land Access: The ease of access to the land for construction and maintenance.

•Land Cost: The cost of acquiring or leasing the land for the project.

3. "Environmental Impacts ($T_3$)", which could have the characteristics shown below:

• Land Use: The current and previous use of the land and the impact of converting it to a solar power plant.

• Water Consumption: The amount of water required for cleaning solar panels and other operations.

• Biodiversity: The effect of the solar power plant on local wildlife and plant species.

• Materials and Waste: The environmental impact of materials used in the construction and the waste generated.

4. "Proximity to Grid Connections ($T_4$)",which could have the characteristics shown below:

• Grid Connection Cost: The expense associated with connecting the solar power plant to the nearest grid infrastructure.

• Electricity Demand: The local demand for electricity and how the new plant will meet or exceed this demand.

• Grid Stability: The ability of the existing grid to handle the additional load from the solar power plant.

• Grid Capacity: The existing capacity of the grid to integrate the new power supply without significant upgrades.

5. "Local Regulations ($T_5$)", which could have the characteristics shown below:

• Zoning Laws: Regulations that dictate land use in the area.

• Permitting Process: The complexity and duration of obtaining the necessary permits for construction and operation.

• Incentives and Subsidies: Availability of government incentives, tax credits, and subsidies for renewable energy projects.

• Compliance Requirements: Environmental and operational regulations that need to be met for the project to proceed.

The five criteria and their attributes are shown below:

**1.** The $q\mathcal{P}$-$\mathcal{F}$ initial decision matrix is represented in Table 5.

**Table 5.** $q\mathcal{P}$-$\mathcal{F}$ initial decision matrix.

| Alternatives | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_1$-Solar Radiance |
|---|---|
| $\dot{\varsigma}_1$ | {(.18,.73,.43,.67),(.32,.64,.49,.72),(.38,.66,.54,.64)} |
| $\dot{\varsigma}_2$ | {(.46,.76,.45,.27),(.56,.91,.36,.48)} |
| $\dot{\varsigma}_3$ | {(.85,.37,.45,.59),(.72,.48,.72,.58),(.51,.64,.55,.32)} |
| $\dot{\varsigma}_4$ | {(.21,.52,.34,.77),(.41,.61,.43,.78),(.42,.66,.39,.87)} |
| $\dot{\varsigma}_5$ | {(.11,.33,.56,.61),(.31,.41,.6,.73)} |

| Alternatives | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_2$-Local Availability |
|---|---|
| $\dot{\varsigma}_1$ | {(.75,.45,.67,.69),(.79,.37,.57,.69) } |
| $\dot{\varsigma}_2$ | {(.45,.70,.49,.86),(.56,.72,.66,.74),(.47,.62,.58,.72)}' |
| $\dot{\varsigma}_3$ | {(.46,.66,.71,.17),(.41,.77,.78,.19),(.48,.80,.83,.15)} |
| $\dot{\varsigma}_4$ | {(.57,.64,.38,.57),(.51,.56,.59,.47)} |
| $\dot{\varsigma}_5$ | {(.24,.12,.81,.77),(.31,.25,.86,.73),(.23,.31,.80,.75)} |

| Alternatives | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_3$-Environmental Impacts |
|---|---|
| $\dot{\varsigma}_1$ | {(.82,.51,.67,.7),(.8,.64,.69,.8),(.7,.59,.68,.78)} |
| $\dot{\varsigma}_2$ | {(.70,.56,.45,.29),(.73,.51,.37,.43),(.83,.63,.65,.54)} |
| $\dot{\varsigma}_3$ | {(.74,.46,.61,.8),(.81,.52,.59,.91)} |
| $\dot{\varsigma}_4$ | {(.66,.55,.46,.88),(.61,.58,.41,.86),(.71,.61,.48,.93)} |
| $\dot{\varsigma}_5$ | {(.11,.76,.42,.61),(.31,.79,.46,.72),(.23,.8,.57,.84)} |

| Alternatives | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_4$-Local regulations |
|---|---|
| $\dot{\varsigma}_1$ | {(.72,.91,.67,.75),(.69,.84,.69,.87),(.73,.87,.60,.81)} |
| $\dot{\varsigma}_2$ | {(.60,.44,.76,.80),(.56,.51,.75,.77),(.63 .41,.68,.85)} |
| $\dot{\varsigma}_3$ | {(.74,.56,.47,.80),(.81,.56,.53,.91),(.61,.60,.65,.89)} |
| $\dot{\varsigma}_4$ | {(.43,.66,.77,.41),(.51,.68,.76,.36),(.53,.60,.87,.27)} |
| $\dot{\varsigma}_5$ | {(.21,.65,.52,.61),(.27,.67,.41,.77)} |

**2.** The pessimistic decision matrix $q\mathcal{P}$-$\mathcal{F}$ in Table 6.

**3.** The criteria's normalised weights are shown below,

$$w_1 = .2412, w_2 = .2424, w_3 = .2548, w_4 = .2616,$$

where $\sum_{i=1}^{4} w_i = 1$.

**4.** The weight pessimistic of the $q\mathcal{P}$-$\mathcal{F}$ decision matrix is calculated in Table 7.

**Table 6.** The pessimistic $q\mathcal{P}$-$\mathcal{F}$ decision matrix.

| Alternatives | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_1$-Solar Radiance |
|---|---|
| $\dot{\varsigma}_1$ | {(.18,.73,.43,.67),(.32,.64,.49,.72),(.38,.66,.54,.64)} |
| $\dot{\varsigma}_2$ | {(.46,.76,.45,.27),(.46,.76,.45,.27),(.56,.91,.36,.48)} |
| $\dot{\varsigma}_3$ | {(.85,.37,.45,.59),(.72,.48,.72,.58),(.51,.64,.55,.32)} |
| $\dot{\varsigma}_4$ | {(.21,.52,.34,.77),(.41,.61,.43,.78),(.42,.66,.39,.87)} |
| $\dot{\varsigma}_5$ | {(.11,.33,.56,.61),(.11,.33,.56,.61),(.31,.41,.6,.73)} |

| Alternatives | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_2$-Local Availability |
|---|---|
| $\dot{\varsigma}_1$ | {(.75,.45,.67,.69),(.79,.37,.57,.69),(.79,.37,.57,.69) } |
| $\dot{\varsigma}_2$ | {(.45,.7,.49,.86),(.56,.72,.66,.74),(.47,.62,.58,.72)} |
| $\dot{\varsigma}_3$ | {(.46,.66,.71,.17),(.41,.77,.78,.19),(.48,.80,.83,.15)} |
| $\dot{\varsigma}_4$ | {(.57,.64,.38,.57),(.51,.56,.59,.47),(.51,.56,.59,.47)} |
| $\dot{\varsigma}_5$ | {(.24,.12,.81,.77),(.31,.25,.86,.73),(.23,.31,.8,.75)} |

| Alternatives | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_3$-Environmental Impacts |
|---|---|
| $\dot{\varsigma}_1$ | {(.82,.51,.67,.7),(.8,.64,.69,.8),(.7,.59,.68,.78)} |
| $\dot{\varsigma}_2$ | {(.7,.56,.45,.29),(.73,.51,.37,.43),(.83,.63,.65,.54)} |
| $\dot{\varsigma}_3$ | {(.74,.46,.61,.8),(.74,.46,.61,.8),(.81,.52,.59,.91)} |
| $\dot{\varsigma}_4$ | {(.66,.55,.46,.88),(.61,.58,.41,.86),(.71,.61,.48,.93)} |
| $\dot{\varsigma}_5$ | {(.11,.76,.42,.61),(.31,.79,.46,.72),(.23,.8,.57,.84)} |

| Alternatives | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_4$-Local regulations |
|---|---|
| $\dot{\varsigma}_1$ | {(.72,.91,.67,.75),(.69,.84,.69,.87),(.73,.87,.6,.81)} |
| $\dot{\varsigma}_2$ | {(.60,.44,.76,.80),(.56,.51,.75,.77),(.63 .41,.68,.85)} |
| $\dot{\varsigma}_3$ | {(.74,.56,.47,.80),(.81,.56,.53,.91),(.61,.60,.65,.89)} |
| $\dot{\varsigma}_4$ | {(.43,.66,.77,.41),(.51,.68,.76,.36),(.53,.60,.87,.27)} |
| $\dot{\varsigma}_5$ | {(.21,.65,.52,.61),(.21,.65,.54,.61),(.27,.67,.41,.77)} |

**5.** The evaluation of the $q\mathcal{P}$-$\mathcal{F}$ positive ideal solution ($q\mathcal{P}$-$\mathcal{FPIS}$) and $q\mathcal{P}$-$\mathcal{F}$ negative ideal solution ($q\mathcal{P}$-$\mathcal{FNIS}$) is as follows:
$q\mathcal{P}$-$\mathcal{FPIS}$ =

$$\begin{aligned}
&\{\{(.2050, .1833, .1351, .1857), (.1737, .1833, .1737, .1881), (.1351, .2195, .1447, .2098)\},\\
&\{(.1818, .1697, .1963, .2085), (.1915, .1866, .2085, .1794), (.1915, .1939, .2012, .1818)\},\\
&\{(.2089, .1936, .1707, .2242), (.2038, .2013, .1758, .2191), (.2115, .2038, .1733, .2370)\},\\
&\{(.1936, .2381, .2014, .2093), (.2119, .2197, .1988, .2381), (.1910, .2276, .2276, .2328)\}\},
\end{aligned}$$

and

**Table 7.** The weighted pessimistic $q\mathcal{P}$-$\mathcal{F}$ decision matrix.

| $\dot{\varsigma}$ | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_1$-Solar Radiance |
|---|---|
| $\dot{\varsigma}_1$ | {(.0434,.1761,.1037,.1568),(.0724,.1544,.1182,.1688),(.0917,.1592,.1302,.1544)} |
| $\dot{\varsigma}_2$ | {(.1110,.1833,.1085,.0651),(.1110,.1833,.1085,.0651),(.1351,.2195,.0868,.1158)} |
| $\dot{\varsigma}_3$ | {(.2050,.0892,.1085,.1375),(.1737,.1158,.1737,.1399),(.1230,.1544,.1327,.0772)} |
| $\dot{\varsigma}_4$ | {(.0507,.1254,.0820,.1857),(.0989,.1471,.1037,.1881),(.1013,.1592,.0941,.2098)} |
| $\dot{\varsigma}_5$ | {(.0265,.0796,.1351,.1471),(.0265,.0796,.1351,.1471),(.0748,.0989,.1447,.1761)} |

| $\dot{\varsigma}$ | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_2$-Local Availability |
|---|---|
| $\dot{\varsigma}_1$ | {(.1818,.1091,.1624,.1673),(.1915,.0897,.1382,.1673),(.1915,.0897,.1382,.1673)} |
| $\dot{\varsigma}_2$ | {(.1091,.1697,.1188,.2085),(.1357,.1745,.1600,.1794),(.1139,.1503,.1406,.1745)} |
| $\dot{\varsigma}_3$ | {(.1115,.1600,.1721,.0412),(.0994,.1866,.1891,.0461),(.1164,.1939,.2012,.0364)} |
| $\dot{\varsigma}_4$ | {(.1382,.1551,.0921,.1382),(.1236,.1357,.1430,.1139),(.1236,.1357,.1430,.1139)} |
| $\dot{\varsigma}_5$ | {(.0582,.0291,.1963,.1866),(.0751,.0606,.2085,.1770),(.0558,.0751,.1939,.1818)} |

| $\dot{\varsigma}$ | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_3$-Environmental Impacts |
|---|---|
| $\dot{\varsigma}_1$ | {(.2089,.1299,.1707,.1784),(.2038,.1631,.1758,.2038),(.1784,.1503,.1733,.1987)} |
| $\dot{\varsigma}_2$ | {(.1784,.1427,.1147,.0739),(.1860,.1299,.0943,.1096),(.2115,.1605,.1656,.1376)} |
| $\dot{\varsigma}_3$ | {(.1886,.1172,.1554,.2038),(.1886,.1172,.1554,.2038),(.2064,.1325,.1503,.2319)} |
| $\dot{\varsigma}_4$ | {(.1682,.1401,.1172,.2242),(.1554,.1478,.1045,.2191),(.1809,.1554,.1223,.2370)} |
| $\dot{\varsigma}_5$ | {(.0280,.1936,.1070,.1554),(.0790,.2013,.1172,.1835),(.0586,.2038,.1452,.2140)} |

| $\dot{\varsigma}$ | $q\mathcal{P}$-$\mathcal{F}$ criteria $c_4$-Local regulations |
|---|---|
| $\dot{\varsigma}_1$ | {(.1884,.2381,.1753,.1962),(.1805,.2197,.1805,.2276),(.1910,.2276,.1570,.2119)} |
| $\dot{\varsigma}_2$ | {(.1570,.1151,.1988,.2093),(.1465,.1334,.1962,.2014),(.1648,.1073,.1779,.2224)} |
| $\dot{\varsigma}_3$ | {(.1936,.1465,.1230,.2093),(.2119,.1465,.1386,.2381),(.1596,.1570,.1700,.2328)} |
| $\dot{\varsigma}_4$ | {(.1125,.1727,.2014,.1073),(.1334,.1779,.1988,.0942),(.1386,.1570,.2276,.0706)} |
| $\dot{\varsigma}_5$ | {(.0549,.1700,.1360,.1596),(.0549,.1700,.1413,.1596),(.0706,.1753,.1073,.2014)} |

$q\mathcal{P}$-$\mathcal{FNIS}$ =

$$\{\{(.0265,.0796,.0820,.0651),(.0265,.0796,.1037,.0651),(.0748,.0989,.0868,.0772)\},$$
$$\{(.0582,.0291,.0921,.0412),(.0751,.0606,.1430,.0461),(.0558,.0751,.1406,.0364)\},$$
$$\{(.0280,.1172,.1070,.0739),(.0790,.1172,.0943,.1096),(.0586,.1325,.1223,.1376)\},$$
$$\{(.0549,.1151,.1230,.1073),(.0549,.1334,.1386,.0942),(.0706,.1073,.1073,.0706)\}\},$$

**6.** Using (7) and (8), the $q\mathcal{P}$-$\mathcal{F}$ Euclidean distance of each alternative $\dot{\varsigma}_j$ from $q\mathcal{FPIS}$ and $q\mathcal{P}$-$\mathcal{FPIS}$ are calculated as:

$$D'_E(\dot{\varsigma}_1, q\mathcal{P}\text{-}\mathcal{FPIS}) = .0951,$$
$$D'_E(\dot{\varsigma}_2, q\mathcal{P}\text{-}\mathcal{FPIS}) = .1293,$$
$$D'_E(\dot{\varsigma}_3, q\mathcal{P}\text{-}\mathcal{FPIS}) = .1229,$$
$$D'_E(\dot{\varsigma}_4, q\mathcal{P}\text{-}\mathcal{FPIS}) = .1311,$$

$$D'_E(\dot{\varsigma}_5, q\mathcal{P}\text{-}\mathcal{FPIS}) = .1786,$$

and

$$\begin{aligned} D'_E(\dot{\varsigma}_1, q\mathcal{P}\text{-}\mathcal{FNIS}) &= .1817, \\ D'_E(\dot{\varsigma}_2, q\mathcal{P}\text{-}\mathcal{FNIS}) &= .1627, \\ D'_E(\dot{\varsigma}_3, q\mathcal{P}\text{-}\mathcal{FNIS}) &= .1740, \\ D'_E(\dot{\varsigma}_4, q\mathcal{P}\text{-}\mathcal{FNIS}) &= .1461, \\ D'_E(\dot{\varsigma}_5, q\mathcal{P}\text{-}\mathcal{FNIS}) &= .1218. \end{aligned}$$

Using Equation (9), the relative $q\mathcal{P}$-$\mathcal{F}$ closeness coefficients $E_{j'}$ are calculated as:

$$\begin{aligned} E_{1'} &= .6565, \\ E_{2'} &= .5572, \\ E_{3'} &= .5860, \\ E_{4'} &= .5270, \\ E_{5'} &= .4055. \end{aligned}$$

According to the foregoing computations, the final ranking of power plant selection is as follows:

$$\dot{\varsigma}_1 > \dot{\varsigma}_3 > \dot{\varsigma}_2 > \dot{\varsigma}_4 > \dot{\varsigma}_5.$$

Hence, solar power plant station $\dot{\varsigma}_1$ is selected in the rural area.

As a result of the evaluation among the alternatives, the most risky structure is $\dot{\varsigma}_1$. The positive ideal solution, negative ideal solution and ranking of alternatives based on closeness coefficients is shown in Figure 1.

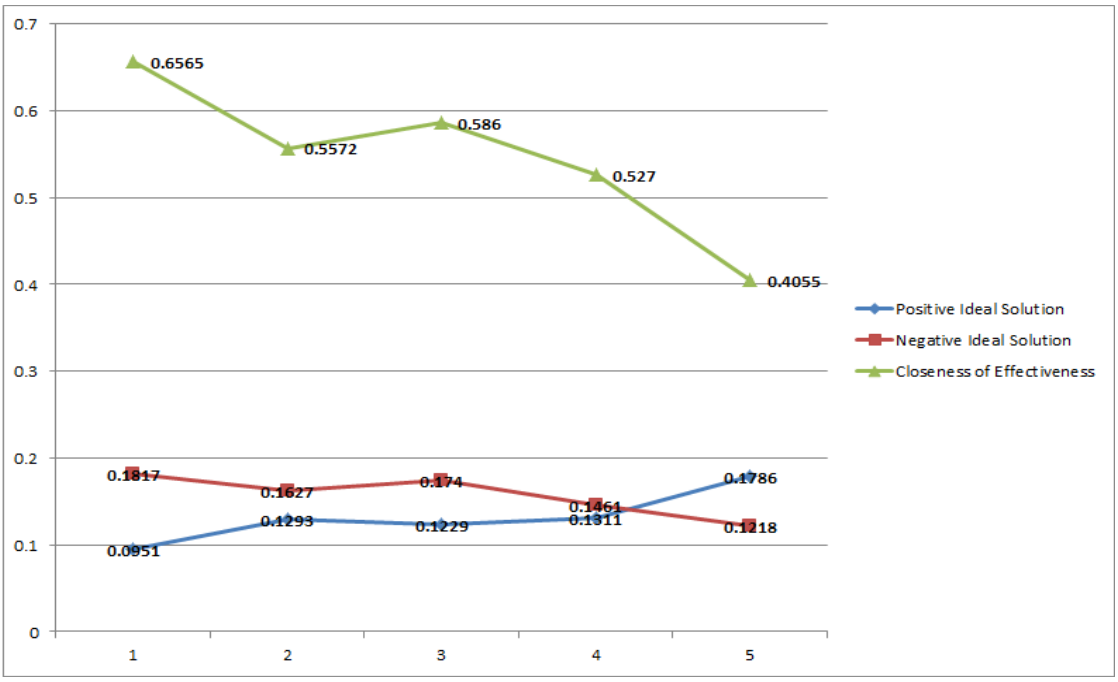


**Fig. 1:** Positive ideal solution, Negative ideal solution and Ranking of alternatives

## 7. Conclusions

Quadri-polar structures are often preferred over clear-cut circumstances. Various forms of information can be used to manipulate the membership degrees to facilitate the handling of quadri-polar fuzzy information. We have proposed a novel class of generalized $q\mathcal{P}$-$\mathcal{FFI}(s)$ of $\tilde{\aleph}$ called, a $q\mathcal{P}$-$(\varpi, \vartheta)$-$\mathcal{FFI}(s)$ and compared to the generalizations of present fuzzy sets, it is shown to be a more flexible approach that can be evaluated in quadri-polar ways based on practical interests and requirements. We defined and analyzed the concept of $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}(s)$. Moreover, we presented various characterizations of $q\mathcal{P}$-$(\in_{\tilde{\sigma}}, \in_{\tilde{\sigma}} \vee q_{\tilde{\tau}})$-$\mathcal{FFI}s$. It is used to manage data that includes quadri-polar information suggested by decision-makers. The final decision on the proposed approach is determined by the decision-maker's optimistic or pessimistic outlook. In practical perspective, we have devised a $q$-$\mathcal{PF}$ TOPSIS approach to address MCGDM problems. This method represents a natural extension of the TOPSIS method tailored to our specific model. Ultimately, we have implemented our methodology in addressing real-world issues. In the future, we will delve into additional decision-making methods associated with the proposed concept, such as $q$-polar fuzzy semi-hyper groups, $q$-polar fuzzy rough sets, $q$-polar fuzzy in different logical algebras environment.

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