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Pythagorean Fuzzy Soft Somewhat Continuous Functions

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Abstract. In this work, we introduce the concept of Pythagorean fuzzy soft somewhat open sets utilizing the Pythagorean fuzzy soft interior operator, extending its application to Pythagorean fuzzy soft topological spaces. This study aims to enhance decision-making processes in future-assisted economies by addressing the limitations of existing fuzzy set theories. We investigate the distinctive properties of Pythagorean fuzzy soft somewhat open sets as a subclass of Pythagorean fuzzy soft somewhere dense sets. Additionally, we explore Pythagorean fuzzy soft somewhat metamorphism's within the context of Pythagorean fuzzy soft somewhat continuous functions, offering new insights into their topological invariant. Through detailed analysis and examples, we demonstrate the applicability of these concepts in various scientific and engineering problems. This work provides a comprehensive framework for understanding and utilizing Pythagorean fuzzy soft sets in complex decision-making scenarios. Finally, we compare various relationships across some generalizations of Pythagorean fuzzy soft continuous functions.

2020 Mathematics Subject Classifications: 54C08, 03E99, 54C10, 03E72

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1. Introduction

Many researchers in the fields of economics, engineering, medicine, and other sciences face the daily challenge of lacking sufficient data to make decisions due to the emergence of new problems in our daily lives that did not previously exist and for which innovative and modern approaches are needed to find solutions. A topology is an important branch of mathematics called rubber geometry that helps solve these problems. As a result, scientists are trying to expand the topological space in order to help with

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everyday problems related to the environment, economy, health, and even human needs. To get beyond these obstacles, a number of theories have been put forward, similar to the 1999 introduction of the notion of soft sets (brevity Sss) by Molodtsov [23] and which has been used in a number of sectors. The character of parameter sets is central to the notion of Sss, it offers an extensive structure for modeling ambiguous data. In a short time, this essentially advances the topic of soft set (brevity Ss) theory. The theoretical basis of the Ss theory has been extensively examined by Maji et al. [22] and Azzam et al. [10–12]. In addition, Radwan and et al. [1] proposed soft ditopological spaces to achieve nearly soft β-open sets. To address this problem, Zadeh [32] developed the fuzzy set (brevity Fs) theory. Following Fs theory concept for various specific purposes, Higher order and nonclassical fuzzy sets (abbreviated Fss) have been presented. Atanassov [8] established the idea of the intuitionistic fuzzy set (abbreviated IFs). A growth of theory Fs that addresses both membership and non-membership values (abbreviated m-values and n-m-values consequently) [7, 17]. Yager invented the Pythagorean fuzzy set (abbreviated $PyFs$) in two thousand thirteen, which is an additional extension of Fs and IFs [30]. Numerous applications in the scientific and social sciences have been made possible by this set theory. For instance, the work by Akram and et al. [2] introduced significant advancements in fuzzy subsets. Similarly, Azzam [9] explored its applications in social sciences, while Cuong and et al. [15] and Garg [16] provided critical insights into its mathematical underpinnings. Further contributions by Garg [18] and Yager [30] expanded its practical applications, and Zadeh [32] developed foundational theories that underpin the current study. Olgun et al. [25] suggested and studied Pythagorean fuzzy topological spaces (abbreviated $PyFTSs$) in 2019. Independent definitions of soft (generic) topology were provided in two thousand eleven by $Ca\breve{g}$man et al. [13] and Shabir and Naz [27]. Nazmul and Samanta [24] provided a definition of soft continuity (abbreviated SC) of functions in 2013. Next, a number of SC and soft openness generalizations functions that were documented in the literature. $PyFTS$ was first described in [25] and Pythagorean fuzzy soft topological space ($PyFSTS$) [7, 26]. In recent years, the need for advanced fuzzy set theories has grown significantly due to the increasing complexity of problems in economics, engineering, and decision-making processes. The motivation behind this study is to address these challenges by developing a robust framework using Pythagorean fuzzy soft sets. Our main contribution lies in defining and exploring the properties of Pythagorean fuzzy soft somewhat open sets and their applications in topological spaces. This approach provides a more nuanced understanding of fuzzy environments, enabling more precise modeling of uncertainty and imprecision in real-world scenarios. Following this quick introduction, we will review some preliminary principles in Part 2. Part 3 then introduces the idea that Pythagorean fuzzy soft somewhat open (brevity $PyFSsw$-open) sets and looks at how it relates to a few soft open set assumptions. The objectives of Part 4 is to examine $PyFSsw$-C functions, which are stronger than $PyFSsw$-dense C, but weaker than soft semicontinuous. We wrap up and offer some suggestions for next works in Part 5.

2. Preliminaries

Various fundamental ideas and symbols will be used in the sequel are contained in this part. We will henceforth refer to an original universe X, a collection of parameters η, an exponential set of X ($\wp(X)$), soft topology ST, a soft topological space STS, picture fuzzy set PFs, positive membership function pmf, negative membership function nmf, and continuous C.

Definition 1. [32] *A membership function $\xi_D(x)$ that assigns a real number in the range [0, 1] to each point in X characterizes a fuzzy set D in X. The "grade of m" of x in D is indicated by the value of $\xi_D(x)$ at x.*

Definition 2. [29] *Let X represent the universe, then the set $D = \{(x, \xi_D(x), \psi_D(x)) : x \in X\}$ is referred to as IFs of X, $\xi_D : X \to [0, 1]$ and $\psi_D : X \to [0, 1]$ are referred to as x's pmf in X, and within X, x has a nmf effectively under the circumstances $0 \leq \xi_D(x) + \psi_D(x) \leq 1$, $\forall x \in X$.*

Definition 3. [15] *Assume X is the universe setting, then the set $D = \{(x, \xi_D(x), \upsilon_D(x), \psi_D(x)) : x \in X\}$ is referred to as PFs of X, $\xi_D : X \to [0, 1]$, $\psi_D : X \to [0, 1]$ and $\psi_D : \Omega \to [0, 1]$ the degrees of positive, neutral, and negative m of x in X, as well as their respective conditions $0 \leq \xi_D(x) + \upsilon_D(x) + \psi_D(x)) \leq 1$, $\forall x \in X$, are designated accordingly.*

Definition 4. [29] *Let X represent the cosmos, then the set $D = \{(x, \xi(x), \psi(x)) : x \in X\}$ is named $PyFs$ of X, $\xi : X \to [0, 1]$ and $\psi : X \to [0, 1]$ are referred to the degree of pmf of x in X and nmf degree of x in X effectively under the circumstances $0 \leq \xi^2 + \psi^2 \leq 1$, $\forall x \in X$.*

Definition 5. [29] *Let $D_1 = \{(x, \xi_{D_1}(x), \psi_{D_1}(x)) : x \in X\}$ and $D_2 = \{(x, \xi_{D_2}(x), \psi_{D_2}(x)) : x \in X\}$ are two $FyFs$ on X, then*
i) $D_1 \sqcap D_2 = \{(x, \xi_{D_1}(x) \wedge \xi_{D_2}(x), \xi_{D_1}(x) \vee \xi_{D_2}(x) : x \in X\}$,
ii) $D_1 \sqcup D_2 = \{(x, \xi_{D_1}(x) \vee \xi_{D_2}(x), \xi_{D_1}(x) \wedge \xi_{D_2}(x) : x \in X\}$,
iii) $D_1 \sqsubseteq D_2$ if and only if $\xi_{D_1}(x) \leq \xi_{D_2}(x), \psi_{D_1}(x) \geq \psi_{D_2}(x) : x \in X$.

Definition 6. [25] *$PyFTS$ is the PyF family τ that subsets of a non-empty set X if*
i) 0_X, and 1_X belong to τ,
ii) We have $x_1 \sqcap x_2$ belong to τ for any pair $x_1, x_2 \in \tau$,
iii) We have $\sqcup_i x_i$ belong to τ for any $x_i \in \tau$.

Definition 7. [23] *When $\xi : \eta \to \wp(X)$ is a (crisp) map, then a Ss over X is a pair $(\xi, \eta) = \{(a, \eta(a)) : a \in \eta\}$. Instead of writing the soft set (ξ, η), we write ξ_η. $Ss_\eta(X)$ or for all Sss on X, the class is represented by just $Ss(X)$. When $A \sqsubseteq \eta$, then $Ss_A(X)$ will serve as its symbol.*

Definition 8. [6]

The term for a Ss ξ_η on X is:
(1) a soft element if $\xi(a) = \{x\}$ for every $a \in \eta$, for $x \in X$, $\{x\}_\eta$ is used to represent

it(maybe soon x).
(2) a soft point if for every $a \neq \acute{a}$, $\xi(a) = \{x\}$ and $\xi(\acute{a}) = \phi$ for every $a \in \eta$ and $x \in X$. It's indicated by p_a^x. If $x \in \xi(a)$, then the expression $p_a^x \in \xi(a)$.

Definition 9. [5]
A soft set $X_\eta - \xi_\eta$ (or simply ξ_η^c is the complement of ξ_η, $\xi^c : \eta \to \wp(X)$ is defined as $\xi^c(a) = X - \xi(a)$ for every $a \in \eta$.

Definition 10. [23]
The term for a soft subset ξ_η over X is null for any $a \in \eta$ if $\xi(a) = \phi$, and absolute if $\xi(a) = X$. Both empty and absolute SSs are denoted by ϕ_η and X_η, respectively. It is evident that $\phi_\eta^c = X_\eta$ and $X_\eta^c = \phi_\eta$.

Definition 11. [22]
Assume $C, D \sqsubseteq \eta$. If $C \sqsubseteq D$ and $\xi(a) \sqsubseteq G(a)$ for each $a \in C$, then G_C is a soft subset of H_D (written as $G_C \sqsubseteq H_D$). If $G_C \sqsubseteq H_D$ and $H_D \sqsubseteq G_C$, we refer to G_C soft equates to H_D.

Maji et al. [22] defined the soft union and soft overlap of two Sss with respect to arbitrary subsets of η. However, as noted by Ali et al. [5], these definitions are imprecise and ambiguous. Consequently, we adhere to the definitions provided by Ali et al. [5] and M. Terepeta [28].

Definition 12. [27] *A subfamily τ of Ss_η is said to be a ST on X if (i) X_η and ϕ_η elements in τ, (ii) τ owns the finite intersection of sets from τ, and (iii) τ owns any union of sets from τ.*

We refer to (X, τ, η) as a STS on X. τ's elements are known as soft open sets, while their complements are known as soft closed sets.

Definition 13. [27] *Suppose Z_η is a non-null soft subset of (X, τ, η). In that case, (Z, τ_Z, η) represents a soft subspace of (X, τ, η), A soft relative topology on Z is denoted by $\tau_Z = \{G_\eta \sqcap Z_\eta : G_\eta \in \tau\}$.*

Definition 14. [27]
ξ_η is a soft subset of (X, τ, η). Denoted by $int\xi_\eta$, the largest soft open set contained in ξ_η is the soft interior of ξ_η. The soft closure of ξ_η is $cl\xi_\eta$, which is the smallest soft closed set containing ξ_η.

Definition 15. *The terms "soft dense," "soft co-dense," "soft semiopen [14]," "soft β-C [31]," "soft somewhat open [4]," and "soft somewhere dense [3]," if "$cl(G_\eta) = X_\eta$," "$int(G_\eta) = \phi_\eta$," "$G_\eta \sqsubseteq cl(int(G_\eta))$," "$G_\eta \sqsubseteq cl(int(clG_\eta))$," "$int(G_\eta) \neq \phi_\eta$," "$int(cl(G_\eta)) \neq \phi_\eta$," respectively "referring to the different states of a soft subset GE of (X, τ, η). (We compel ϕ_η to be soft somewhere dense in order to improve the connectivity between these soft sets).*

Definition 16. *Suppose* (X, τ, η) *and* $(Z, \rho, \acute{\eta})$ *be STS. A soft function* $\xi : (X, \tau, \eta) \to (Z, \rho, \acute{\eta})$ *is called*
i) SC [24] (resp., soft semi-C [20], soft SD-C [4], soft β*-C [31]) if every soft open subset of* $(Z, \rho, \acute{\eta})$ *has a soft open as its inverse image (resp., soft semiopen, soft somewhere dense,* β*-open) subset of* (X, τ, η)*.*
ii) soft open [23] (resp., soft semiopen [20], soft SD-open [4], soft β*-open [31]) if the image of each soft open subset of* (X, τ, η) *is a soft open (resp., soft semiopen, soft somewhere dense,* β*-open) subset of* $(Z, \rho, \acute{\eta})$*.*
iii) If it is one to one soft open and SC from (X, τ, η) *onto* $(Z, \rho, \acute{\eta})$*, then it is a soft homeomorphism [24].*

The reader is referred to [19] for a definition of soft functions spanning collections of all Sss. From here on, we refer to "soft function" when we use the term "function."

Definition 17. [29]
The $PyFSS$ *may be expressed as a collection of ordered pairs*
$(\tilde{\xi}, \tilde{\eta}) = \{(a, \{(x, \xi_{\tilde{\xi}(a)}(x), \psi_{\tilde{\xi}(a)}(x)) : a \in \tilde{\eta}\}\}$ *because it is not a set but rather a specified unit of certain components of the set* $PyF(\tilde{X})$*, where* $\xi_{\tilde{\xi}(a)}(x)$ *and* $\psi_{\tilde{\xi}(a)}(x)$ *are the* pmf_s *and* nmf_s*, successively. If* $x \in \tilde{X}, 0 \leq \xi^2_{\tilde{\xi}(a)}(x) + \psi^2_{\tilde{\xi}(a)}(x) \leq 1$*.*

We introduced the idea of $PyFSTS$ and looked into its properties in more detail. Let $PyF(\tilde{X}, \tilde{\eta})$ and $\tilde{X}$ represent, respectively, the family of $PyFSs$ on $\tilde{X}$ and the origin of the universal set.

Definition 18. [7]
A void $PyFSSs$ *(or* $\tilde{0}$*) is defined as a* $PyFSSs(\tilde{\xi}, \tilde{\eta})$ *over* $\tilde{X}$ *if and only if* $\forall a \in \tilde{\eta}, (\tilde{\xi}, \tilde{\eta})(a) = (\tilde{0}, \tilde{1})$*, where* $\tilde{0}, \tilde{1}$ *are the* pmf *and the value of the* nmf_s*, the null and absolute, respectively* $PyFSs$ *Pythagorean over* $\tilde{X}$*.*

Definition 19. [7]
An absolute $PyFSSs$*, or*$(\tilde{1})$*, is a* $PyFSSs(\tilde{\xi}, \tilde{\eta})$ *over* $\tilde{X}$ *if and only if* $\forall a \in \tilde{\eta}, (\tilde{\xi}, \tilde{\eta})(a) = (\tilde{0}, \tilde{1})$*, where* $\tilde{0}, \tilde{1}$ *are the* pmf *and the value*
of the nmf_s*, the null and absolute, respectively, of the absolute and null function.*

Definition 20. [7] *Let* $\tilde{\Omega} \sqsubseteq PyF(\tilde{X}, \tilde{\eta})$*, at hence,* $\tilde{\Omega}$ *is claimed to be a* $PyFSTS$ *if*
i) $\tilde{\Omega}$ *includes* $\tilde{0}$ *and* $\tilde{1}$ *as members,*
ii) Any two $PyFSS$ *that intersect in* $\tilde{\Omega}$ *are related to* $\tilde{\Omega}$*,*
ii) Any number of $PyFSS$ *in* $\tilde{\Omega}$ *that is united belongs to* $\tilde{\Omega}$*,*
It is argued that the triple $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$ *is a* $PyFSTS$ *over* $\tilde{X}$*.*

$*$. All $\tilde{\Omega}$ members are considered to be $\tilde{\Omega}$-open $PyFSS$.
$**$. A $\tilde{\Omega}$-closed $PyFSS$ is considered to be the complement of a $\tilde{\Omega}$-open.

3. Pythagorean fuzzy soft somewhat open sets

We create key properties and introduce the concept of $PyFSsw$-open sets in this section. We provide examples to show the relationships between $PyFS$ semiopen and $PyFS$ somewhere dense sets, as well as various generalizations of $PyFSsw$-open sets.

Definition 21. *A subset $G_{\tilde{\eta}}$ of a $PyFSTS$ $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$ is claimed to be $PyFSsw$-open if $int(G_{\tilde{\eta}}) \neq \phi_{\tilde{\eta}}$ or $G_{\tilde{\eta}}$ is null. $PyFSsw$-closed is the complement of $PyFSsw$-open set. That is, a set $\xi_{\tilde{\eta}}$ is $PyFSsw$-closed if $cl(\xi_{\tilde{\eta}}) \neq H_{\tilde{\eta}}$ or $\xi_{\tilde{\eta}} = \tilde{X}_{\tilde{\eta}}$.*

Remark 1. *Let $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$ be a $PyFSTS$.*
i) If and only if there is a $PyFS$-open set $U_{\tilde{\eta}}$ that $\phi_{\tilde{\eta}} \neq U_{\tilde{\eta}} \sqsubseteq G_{\tilde{\eta}}$, The non-null set $G_{\tilde{\eta}}$ over $\tilde{X}$ is $PyFSsw$-open.
ii) If $\xi_{\tilde{\eta}}$ is a $PyFSsw$-closed set that $H_{\tilde{\eta}} \sqsubseteq \xi_{\tilde{\eta}} \neq \tilde{X}_{\tilde{\eta}}$, then a valid set $H_{\tilde{\eta}}$ over $\tilde{X}$ is $PyFSsw$-closed.

Proposition 1. *i) Each superset of a $PyFSsw$-open set is $PyFSsw$-open.*
ii) Each subset of a $PyFSsw$-closed set is $PyFSsw$-closed.

Proof. Obvious.

Proposition 2. *A non-null $PyFSs$ is $PyFSsw$-open if and only if it is a $PyFS$ neighborhood of a $PyFS$ point.*

Proof. Let $G_{\tilde{\eta}}$ be a $PyFSsw$-open set that isn't null. Next, there exists a $PyFS$ open set $U_{\tilde{\eta}}$, where $\phi_{\tilde{\eta}} \neq U_{\tilde{\eta}} \sqsubseteq G_{\tilde{\eta}}$. As a result, $G_{\tilde{\eta}}$ is every soft point in $U_{\tilde{\eta}}$'s soft neighborhood. Let $G_{\tilde{\eta}}$, on the other hand, be the $PyFS$ neighborhood of a PyF soft point p_a^x. After that, $U_{\tilde{\eta}}$ is PyF softly opened so that $p_a^x \in U_{\tilde{\eta}} \sqsubseteq G_{\tilde{\eta}}$. As a result, we get $intG_{\tilde{\eta}} \neq \phi_{\tilde{\eta}}$, as needed.

Proposition 3. *A union of $PyFSsw$-open sets is $PyFSsw$-open.*

Proof. Suppose that $\{G_{\tilde{\eta}}^{\beta} : \beta \in \Lambda\}$ is the collection of $PyFSsw$-open subsets of a $PyFSTS$ $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$. At hence, $int(\cup_{\beta\in\Lambda} G_{\tilde{\eta}}^{\beta}) \sqsupseteq \cup_{\beta\in\Lambda} int(G_{\tilde{\eta}}^{\beta}) \neq \phi_{\tilde{\eta}}$. Thus $\cup_{\beta\in\Lambda} G_{\tilde{\eta}}^{\beta}$ is $PyFSsw$-open.

Corollary 1. *The intersection of $PyFSsw$-closed sets is $PyFSsw$-closed.*

As demonstrated by the example that follows, the intersection of two $PyFSsw$-open sets need not be $PyFSsw$-open.

Example 1. *Let $\tilde{\eta} = \{a_1, a_2, a_3, a_4\}$ be the parameters or characteristics set and As the reference set, let $\tilde{X} = \{x_1, x_2, x_3\}$ represent the applicants who have been recommended for promotion, in which a_1 denotes intelligence, a_2 experience, a_3 attitude, and a_4 competence. Let $D_1 = \{a_1, a_2\} \sqsubseteq \tilde{\eta}$, $D_2 = \{a_2\} \sqsubseteq \tilde{\eta}$. Next, two $PyFSsw(\tilde{\xi_1}, D_1)$ and $(\tilde{\xi_2}, D_2)$ are examined. These are represented as follows:*

$(\tilde{\xi_1}, D_1) = \{(a_1, \tilde{\xi_1}(a_1)), (a_2, \tilde{\xi_1}(a_2))\}$, *and* $(\tilde{\xi_2}, D_2) = \{(a_2, \tilde{\xi_2}(a_2))\}$, *where* $\tilde{\xi_1}(a_1)) = \{x_1 = (0.5, 0.6), x_2 = (0.4, 0.7), x_3 = (0.1, 0.7)\}$,
$\tilde{\xi_1}(a_2)) = \{x_1 = (0.3, 0.2), x_2 = (0.6, 0.5), x_3 = (0.2, 0.7)\}$,
$\tilde{\xi_2}(a_2)) = \{x_1 = (0.8, 0.4), x_2 = (0.8, 0.3), x_3 = (0.4, 0.3)\}$
$\tilde{\Omega}_1 = \{\tilde{1}, \tilde{0}, (\tilde{\xi_1}, D_1)\}$ *and* $\tilde{\Omega}_2 = \{\tilde{1}, \tilde{0}, (\tilde{\xi_1}, D_1), (\tilde{\xi_2}, D_2)\}\}$ *are two* $PyFSTSs$ *and* $\tilde{\Omega} = \{\tilde{1}, \tilde{0}, (\tilde{\xi_1}, D_1), (\tilde{\xi_2}, D_2)\}$ *is a* $PyFST$ *over* $\tilde{X}$, $(\tilde{\xi_1}, D_1) \sqcap (\tilde{\xi_2}, D_2) \neq \phi_{\tilde{\eta}}$ *but* $int((\tilde{\xi_1}, D_1) \sqcap (\tilde{\xi_2}, D_2)) = \phi_{\tilde{\eta}}$.

There are several examples when the intersection of a $PyFSsw$-open set with another $PyFS$ open, $PyFS$ closed, or $PyFS$ dense set is not a $PyFSsw$-open set.

The following result shows when the intersection of $PyFSsw$-open and $PyFS$ open sets is a $PyFSsw$-open set.

Definition 22. *A* $PyFSTS$ $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$ *is named*
i) $PyFS$ *separable if it has a countable* $PyFS$ *dense subset.*
ii) $PyFS$ *hyperconnected if any pair of non-null* $PyFS$ *open subsets intersect.*

Proposition 4. *In a* $PyFS$ *hyperconnected space* $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$*, a* $PyFSsw$*-open set is the intersection of two* $PyFSsw$*-open sets.*

Proof.
The evidence is easy to understand if one of the two $PyFSsw$-open sets is null. Assume that there are two $PyFSsw$-open sets, $G_{\tilde{\eta}}$ and $H_{\tilde{\eta}}$. Next, $int(G_{\tilde{\eta}}) = U_{\tilde{\eta}} \neq \phi_{\tilde{\eta}}$ and $int(H_{\tilde{\eta}}) = V_{\tilde{\eta}} \neq \phi_{\tilde{\eta}}$ are obtained. Now, $int(G_{\tilde{\eta}} \sqcap H_{\tilde{\eta}}) = int(G_{\tilde{\eta}}) \sqcap int(H_{\tilde{\eta}}) = U_{\tilde{\eta}} \sqcap V_{\tilde{\eta}}$. Then, $U_{\tilde{\eta}} \sqcap V_{\tilde{\eta}} \neq \phi_{\tilde{\eta}}$ since $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$ is a $PyFS$ hyperconnected. Hence, $int(G_{\tilde{\eta}} \sqcap H_{\tilde{\eta}}) \neq \phi_{\tilde{\eta}}$, and we achieve the intended outcome.

Corollary 2. *In a* $PyFS$ *hyperconnected space* $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$*, the intersection of* $PyFSsw$*-open and* $PyFS$ *open sets is a* $PyFSsw$*-open.*

Corollary 3. *A* $PyFS$ *topology is formed by the family of* $PyFSsw$*-open subsets of a* $PyFS$ *hyperconnected space* $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$.

Lemma 1. *Suppose* $G_{\tilde{\eta}}$ *and* $H_{\tilde{\eta}}$ *are subsets of* $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$*. If* $G_{\tilde{\eta}}$ *is sw-open and* $H_{\tilde{\eta}}$ *is a* $PyFS$ *dense over* $\tilde{X}$*, then* $G_{\tilde{\eta}} \sqcap H_{\tilde{\eta}}$ *is* $PyFSsw$*-open over* $\tilde{X}$.

Proof.
Since $int_H(G_{\tilde{\eta}} \sqcap H_{\tilde{\eta}}) = int_H(G_{\tilde{\eta}}) \sqcap H_{\tilde{\eta}} \sqsupseteq int(G_{\tilde{\eta}}) \sqcap H_{\tilde{\eta}} \neq \phi_{\tilde{\eta}}$, hence $G_{\tilde{\eta}} \sqcap H_{\tilde{\eta}}$ is $PyFSsw$-open over $\tilde{X}$.

Lemma 2. *Assume that* $G_{\tilde{\eta}} \sqsubseteq Y_{\tilde{\eta}}$ *and that* $(\tilde{Y}, \tilde{\Omega}_Y, \tilde{\eta})$ *is a* $PyFS$ *open subspace of* $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$*. If and only if* $G_{\tilde{\eta}}$ *is* $PyFSsw$*-open over* $\tilde{X}$*, then it is also* $PyFSsw$*-open over* $\tilde{Y}$.

Proof. Let's say that $G_{\tilde{\eta}}$ is $PyFSsw$-open over $\tilde{Y}$. It is possible to have a $PyFS$ open set $U_{\tilde{\eta}}$ over $\tilde{Y}$ such that $\phi_{\tilde{\eta}} \neq U_{\tilde{\eta}} \sqsubseteq G_{\tilde{\eta}}$. Because $Y_{\tilde{\eta}}$ is $PyFS$ open over $\tilde{X}$, $U_{\tilde{\eta}}$ is also $PyFS$ open over $\tilde{X}$. As a result, $G_{\tilde{\eta}}$ is $PyFSsw$-open over $\tilde{X}$.
In contrast, let's say that $G_{\tilde{\eta}}$ is $PyFSsw$-open over $\tilde{X}$. This is equivalent to $int_{\tilde{X}}(G_{\tilde{\eta}}) \neq \phi_{\tilde{\eta}}$. According to Theorem 2 in [19], and Remark 3.2, $int_{\tilde{X}}(G_{\tilde{\eta}}) \sqsubseteq int_{\tilde{Y}}(G_{\tilde{\eta}})$, hence $G_{\tilde{\eta}}$ is $PyFSsw$-open over $\tilde{Y}$.

If $Y_{\tilde{\eta}}$ is $PyFS$ dense in $\tilde{X}$, as the following example demonstrates, then the previous result is not valid.

Example 2. *Suppose* $\tilde{X} = \{x_1, x_2, x_3, x_4\}$, $\eta = \{a_1, a_2\}$, *and* $\tilde{\Omega} = \{\tilde{0}, F_{\tilde{\eta}}, G_{\tilde{\eta}}, H_{\tilde{\eta}}, \tilde{1}\}$, *where*
$F_{\tilde{\eta}} = \{(a_1, \{x_2, x_4\}), (a_2, \{x_1, x_2\})\}$
$G_{\tilde{\eta}} = \{(a_1, \tilde{X}), (a_2, \{x_3, x_4\})\}$
$H_{\tilde{\eta}} = \{(a_1, \{x_2, x_4\}), (a_2, \phi_{\tilde{\eta}})\}$
Let $\tilde{Y} = \{x_2, x_3\}$ *at hence,* $\tilde{\Omega}_Y = \{\tilde{0}, I_{\tilde{\eta}}, J_{\tilde{\eta}}, K_{\tilde{\eta}}, \tilde{1}\}$, *where*
$I_{\tilde{\eta}} = \{(a_1, \{x_2\}), (a_2, \{x_2\})\}$
$J_{\tilde{\eta}} = \{(a_1, \tilde{Y}), (a_2, \{x_3\})\}$
$K_{\tilde{\eta}} = \{(a_1, \{x_2\}), (a_2, \phi_{\tilde{\eta}})\}$
$\tilde{Y}_{\tilde{\eta}} = \{(a_1, \{x_2, x_4\}), (a_2, \{x_2, x_4\})\}$.
Over the $PyFS$ *dense set* $\tilde{Y}$, *the set* $I_{\tilde{\eta}}$ *is* $PyFSsw$*-open, but not over* $\tilde{X}$.

Lemma 3. *Suppose* $G_{\tilde{\eta}}$ *that a subset of* $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$. *Hence,* $G_{\tilde{\eta}}$ *is* $PyFS$ *semiopen if and only if* $cl(G_{\tilde{\eta}}) = cl(int(G_{\tilde{\eta}}))$.

Proof. Suppose $G_{\tilde{\eta}}$ is $PyFS$ semiopen, that $G_{\tilde{\eta}} \sqsubseteq cl(int(G_{\tilde{\eta}}))$, and then $cl(G_{\tilde{\eta}}) \sqsubseteq cl(int(G_{\tilde{\eta}}))$. For the opposite side of inclusion, there is always $int(G_{\tilde{\eta}}) \sqsubseteq G_{\tilde{\eta}}$. So, $cl(G_{\tilde{\eta}}) = cl(int(G_{\tilde{\eta}}))$.
In contrast, let's say that $cl(G_{\tilde{\eta}}) = cl(int(G_{\tilde{\eta}}))$, but $G_{\tilde{\eta}} \sqsubseteq cl(G_{\tilde{\eta}})$ always, at hence $G_{\tilde{\eta}} \sqsubseteq cl(int(G_{\tilde{\eta}}))$. So, $G_{\tilde{\eta}}$ is $PyFS$ semiopen.

Lemma 4. *Consider* $G_{\tilde{\eta}}$ *as a non-null subset of* $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$. *Hence,* $G_{\tilde{\eta}}$ *is* $PyFS$ *semiopen if* $int(G_{\tilde{\eta}}) \neq \phi_{\tilde{\eta}}$.

Proof. Suppose otherwise that, if $G_{\tilde{\eta}}$ is a non-null soft semiopen set with $int(G_{\tilde{\eta}}) = \phi_{\tilde{\eta}}$, then $G_{\tilde{\eta}} = \phi_{\tilde{\eta}}$ is implied by Lemma 3.14 since $cl(G_{\tilde{\eta}}) = \phi_{\tilde{\eta}}$. Inconsistency.

Remark 2. *Since* $int(G_{\tilde{\eta}}) = int(cl(G_{\tilde{\eta}}))$ *for each* $PyFS$ $G_{\tilde{\eta}}$ *in a* $PyFSTs$ $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$, *so each* $PyFSsw$*-open set is* $PyFS$ *somewhere dense.*

The following figure depicts the relation between different extensions of $PyFS$ open sets.

As demonstrated below, none of these implications can, in general, be replaced by equivalency.

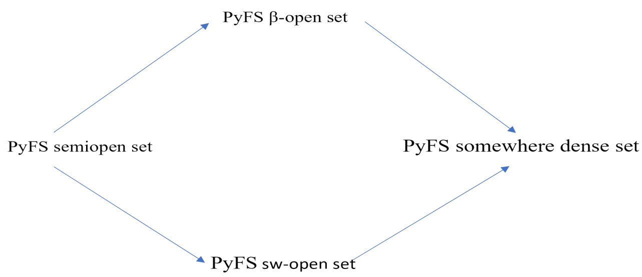

Figure 1: The relationships between some generalizations of $PyFS$ open sets.

Example 3. *Think about $PyFST$ over $\tilde{X}$ that Example 3.7. The $PyFSs$ over $\tilde{X}$ is not $PyFSsw$-open, meaning it is not $PyFS$ semiopen, but is $PyFS\beta$-open, meaning it is $PyFS$ somewhere dense. However, it is evident that the set $\{(a_1, \tilde{\xi_1}(a_1)), (a_2, \tilde{\xi_1}(a_2))\}$ is not $PyFS$ semiopen, but rather $PyFSsw$-open.*

Lemma 5. *Suppose $G_{\tilde{\eta}}$ that a non-null subset of $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$. Then $cl(G_{\tilde{\eta}}) \sqcap H_{\tilde{\eta}} \sqsubseteq cl(G_{\tilde{\eta}} \sqcap H_{\tilde{\eta}})$ for all $PyFS$ open set $H_{\tilde{\eta}}$ on $\tilde{X}$.*

Lemma 6. *Assume that $G_{\tilde{\eta}}, H_{\tilde{\eta}}$ is a subset of $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$. $G_{\tilde{\eta}} \sqcap H_{\tilde{\eta}}$ is $PyFS$ semiopen over $\tilde{X}$ if $G_{\tilde{\eta}}$ is $PyFS$ open and $H_{\tilde{\eta}}$ is $PyFS$ semiopen.*

Proof. Suppose that $G_{\tilde{\eta}}$ is $PyFS$ open and $H_{\tilde{\eta}}$ is $PyFS$ semiopen. Then there's a $PyFS$ open set. $U_{\tilde{\eta}}$ over $\tilde{X}$ with $U_{\tilde{\eta}} \sqsubseteq H_{\tilde{\eta}} \sqsubseteq cl(U_{\tilde{\eta}})$. Now $U_{\tilde{\eta}} \sqcap G_{\tilde{\eta}} \sqsubseteq H_{\tilde{\eta}} \sqcap G_{\tilde{\eta}} \sqsubseteq cl(U_{\tilde{\eta}}) \sqcap G_{\tilde{\eta}}$. By Lemma 3.18, $U_{\tilde{\eta}} \sqcap G_{\tilde{\eta}} \sqsubseteq H_{\tilde{\eta}} \sqcap G_{\tilde{\eta}} \sqsubseteq cl(U_{\tilde{\eta}} \sqcap G_{\tilde{\eta}})$ and since $U_{\tilde{\eta}} \sqcap G_{\tilde{\eta}}$ is $PyFS$ open, then $H_{\tilde{\eta}} \sqcap G_{\tilde{\eta}}$ is $PyFS$ semiopen over $\tilde{X}$.

Lemma 7. *Assume that $G_{\tilde{\eta}}, H_{\tilde{\eta}}$ is a subset of $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$. $G_{\tilde{\eta}} \sqcap H_{\tilde{\eta}}$ is $PyFS$ semiopen over $G_{\tilde{\eta}}$ if $G_{\tilde{\eta}}$ is $PyFS$ open and $H_{\tilde{\eta}}$ is $PyFS$ semiopen.*

Proof. Utilizing the same procedures as in the lemma proof above, apply the assertion that $cl(U_{\tilde{\eta}}) \sqcap G_{\tilde{\eta}} = cl_{G_{\tilde{\eta}}}(U_{\tilde{\eta}})$.

Lemma 8. *A subset $G_{\tilde{\eta}}$ of $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$ is $PyFS$ semiopen if and only if $G_{\tilde{\eta}} \sqcap U_{\tilde{\eta}}$ is $PyFSsw$ open for each $PyFS$ open set $U_{\tilde{\eta}}$ over $\tilde{X}$.*

Proof. The first part follows because each $PyFS$ semiopen set is $PyFSsw$ open and because the intersection of a $PyFS$ semiopen set with a $PyFS$ open set is semiopen according to Lemma 3.19. On the other hand, suppose that $p_a^x \in G_{\tilde{\eta}}$ and that for any $PyFS$ open set $U_{\tilde{\eta}}$ over $\tilde{X}$, $G_{\tilde{\eta}} \sqcap U_{\tilde{\eta}}$ is $PyFSsw$ open. That is $int(G_{\tilde{\eta}} \sqcap U_{\tilde{\eta}}) \neq \phi_{\tilde{\eta}}$. But $\phi_{\tilde{\eta}} \neq int(G_{\tilde{\eta}} \sqcap U_{\tilde{\eta}}) = int(G_{\tilde{\eta}}) \sqcap int(U_{\tilde{\eta}}) = int(G_{\tilde{\eta}}) \sqcap U_{\tilde{\eta}}$, that is $p_a^x \in cl(int(G_{\tilde{\eta}}))$ and then $G_{\tilde{\eta}} \sqsubseteq cl(int(G_{\tilde{\eta}}))$. This demonstrates $G_{\tilde{\eta}}$'s $PyFS$ semiopenness.

Lemma 9. *Consider $F_{\tilde{\eta}}$. is a subset of $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$. If $F_{\tilde{\eta}}$ is $PyFS$ semiclosed and $PyFS$ somewhere dense, it is $PyFSsw$ open.*

Proof. It may be inferred directly from Lemma 3.15 that $F_{\tilde{\eta}}$ is semiclosed if and only if
$int(cl(F_{\tilde{\eta}}))$=$int(F_{\tilde{\eta}})$.

4. $PyFSsw$-continuous functions

This part focuses on outlining the ideas behind $PyFSsw$ C functions, also known as $PyFSsw$ C, and providing several characterizations of them. Furthermore, we demonstrate its connections to various forms of $PyFS$ continuity. In conclusion, we obtain certain findings about hyperconnected and $PyFS$ separable spaces.

Definition 23. *Consider $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ and $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ are a $PyFSTSs$.*
If every $PyFS$ open set over $\tilde{Y}$ has an inverse image that is also $PyFSsw$ open over $\tilde{X}$, then the function $f : (\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ is considered $PyFSsw$-C.

Remark 3. *A function $f : (\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ is $PyFSsw$-C if each $p_a^x \in \tilde{X}$ and each $PyFS$ open set $V_{\tilde{\eta}_2}$ over $\tilde{Y} \sqsupseteq f(p_a^x)$, there is a $PyFSsw$ open set $U_{\tilde{\eta}}$ on $\tilde{X} \sqsupseteq p_a^x$ that $f(U_{\tilde{\eta}}) \sqsubseteq V_{\tilde{\eta}_2}$.*

Based on Figure 1, we deduce that

The ramifications shown in the above graphic are all irreversible.

Example 4. *Let $\tilde{X} = \{x_1, x_2, x_3\}$, $\eta = \{a_1, a_2\}$, and $\tilde{\Omega} = \{\tilde{0}, F_{\tilde{\eta}}, G_{\tilde{\eta}}, \tilde{1}\}$, where*
$F_{\tilde{\eta}} = \{(a_1, \{x_2\}), (a_2, \{x_2\})\}$
$G_{\tilde{\eta}} = \{(a_1, \{x_1, x_3\}), (a_2, \{x_1, x_3\})\}$ *and* $\tilde{\Omega}_1 = \{\tilde{0}, H_{\tilde{\eta}}, \tilde{1}\}$ *where*
$H_{\tilde{\eta}} = \{(a_1, \tilde{X}\}), (a_2, \{x_1, x_2\})\}$.
Let $f : (\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}) \to (\tilde{X}, \tilde{\Omega}_2, \tilde{\eta})$ be the $PyFS$ identity function. At hence, f is $PyFSsw$-C but not $PyFSsw$-semicontinuous.

Example 5. *Let $\tilde{X} = \Re$ be the set of real numbers and $\eta = \{a\}$ be a collection of parameters. Let $\tilde{\Omega}$ be the $PyFST$ on $\Re$ generated by $\{(a, \xi(a)) : (x_1, x_2) \in \Re; x_1 < x_2\}$. Define a $PyFS$ function $f : (\tilde{X}, \tilde{\Omega}, \tilde{\eta}) \to (\tilde{X}, \tilde{\Omega}, \tilde{\eta})$ by*

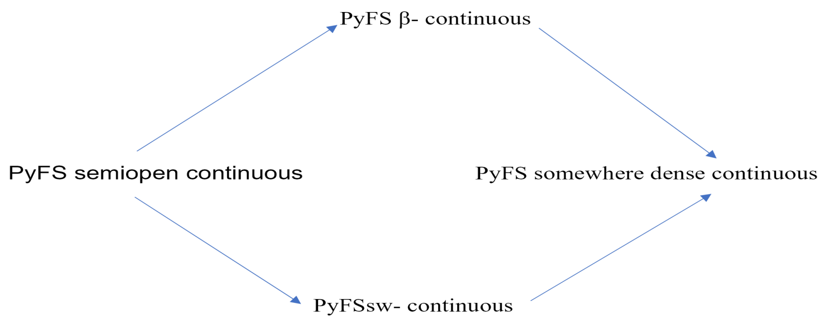

Figure 2: The relationships between some generalizations of $PyFS$ continuous.

$$f(x)=\begin{cases} x & \text{if } x\notin\{\tilde{0},\tilde{1}\}_{\tilde{\eta}},\\ 0 & \text{if } x=1,\\ 1 & \text{if } x=0.\end{cases}$$

Given that every $PyFS$ basic open set has an inverse image that also contains another $PyFS$ basic open, one can simply demonstrate that f is $PyFSsw$-C (and hence, $PyFS$ SD-C), since its $PyFS$ interior cannot be null. However, f is not $PyFS\beta$-C. Let $G_{\tilde{\eta}}=\{(a,(-\varepsilon,\varepsilon))\}$ be the $PyFS$ open set, with $\varepsilon<1$. Therefore
$f^{-1}(G_{\tilde{\eta}})=\{(a,(-\varepsilon,0))\}\sqcup\{(a,(0,\varepsilon))\}\sqcup\{(a,\{1\})\}$.
But $cl(int((cl(f^{-1}(G_{\tilde{\eta}})))=\{(a,[-\varepsilon,\varepsilon])\}$ and so $f^{-1}(G_{\tilde{\eta}})\not\sqsubseteq cl(int((cl(f^{-1}(G_{\tilde{\eta}})))$. As a result, f is not $PyFS$ semicontinuous and cannot be $PyFS\beta$-C.

Example 6. *Consider the $PyFSTS$ $(\tilde{X},\tilde{\Omega},\tilde{\eta})$ as described in Example 4.4. Define $f:(\tilde{X},\tilde{\Omega},\tilde{\eta})\to(\tilde{X},\tilde{\Omega},\tilde{\eta})$ as follows:*

$$f(x)=\begin{cases} 0 & \text{if } x\notin Q_{\tilde{\eta}},\\ 1 & \text{if } x\in Q_{\tilde{\eta}}.\end{cases}$$

In such case, f is not $PyFSsw$-continuous but soft SD-continuous. Any $PyFS$ open set with only one element is its inverse image, and $Q_{\tilde{\eta}}$ is not a $PyFSsw$-open set over $\tilde{X}$.

Definition 24. *We introduce the following for a subset $G_{\tilde{\eta}}$ of a PyFSTS $(\tilde{X}, \tilde{\Omega}, \tilde{\eta})$:*
1-$cl_{sw}(G_{\tilde{\eta}}) = \sqcap\{F_{\tilde{\eta}} : F_{\tilde{\eta}}$ is PyFSsw-closed over $\tilde{X}$ and $G_{\tilde{\eta}} \sqsubseteq F_{\tilde{\eta}}\}$.
2-$int_{sw}(G_{\tilde{\eta}}) = \sqcup\{O_{\tilde{\eta}} : O_{\tilde{\eta}}$ is PyFSsw-open over $\tilde{X}$ and $O_{\tilde{\eta}} \sqsubseteq G_{\tilde{\eta}}\}$.

Proposition 5. *Consider $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ and $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ that a PyFSTSs. The function $f : (\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ can be represented by the following functions:*
1- f is PyFSsw-C,
2- $f^{-1}(F_{\tilde{\eta}_2})$ is PyFSsw-closed set over $\tilde{X}$, for every PyFS closed set $F_{\tilde{\eta}_2}$ over $\tilde{Y}$,
3- $f(cl_{sw}(G_{\tilde{\eta}})) \sqsubseteq cl(f(G_{\tilde{\eta}}))$ for every set $G_{\tilde{\eta}}$ on $\tilde{X}$,
4- $cl_{sw}(f^{-1}(H_{\tilde{\eta}_2})) \sqsubseteq f^{-1}(cl(H_{\tilde{\eta}_2}))$, for every set $H_{\tilde{\eta}_2}$ on $\tilde{Y}$,
5- $f^{-1}(int(H_{\tilde{\eta}_2})) \sqsubseteq int_{sw}(f^{-1}(H_{\tilde{\eta}_2}))$, for every set $H_{\tilde{\eta}_2}$ on $\tilde{Y}$.

Proof. Straightforward.

Definition 25. *Assume that $PyF(\tilde{X}, \tilde{\eta}_1)$ and $PyF(\tilde{Y}, \tilde{\eta}_2)$ be PyFSSs and let $D_{\tilde{\eta}_1} \in (\tilde{X}, \tilde{\eta}_1)$. The restriction of $f : PyF(\tilde{X}, \tilde{\eta}_1) \to PyF(\tilde{Y}, \tilde{\eta}_2)$ is the FyFS function $f_{D_{\tilde{\eta}_1}} : PyF(\tilde{X}, \tilde{\eta}_1) \to PyF(\tilde{Y}, \tilde{\eta}_2)$ defined by $f_{D_{\tilde{\eta}_1}}(P_a^x) = f(P_a^x)$ for all $P_a^x \in D_{\tilde{\eta}_1}$. a PyFS function's expansion f is a PyFS function of g, meaning that f restricts g.*

Theorem 1. *Consider $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ and $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ that a PyFSTSs, and let $d_{\tilde{\eta}_1}$ be a PyFS dense subspace over $\tilde{X}$. If $f : (\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ is PyFSsw-C over $\tilde{X}$, then $f \mid d_{\tilde{\eta}_1}$ is PyFSsw-C over d.*

Proof. Straightforward (with the aid of Lemma 3.12).

Theorem 2. *Let $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ and $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ be a PyFSTSs, and let $f : (\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ be a function and $\{G^{\beta}_{\tilde{\eta}_1} : \beta \in \Lambda\}$ be a PyFS open cover of $\tilde{X}$. At hence, f is PyFSsw-C, if $f \mid G^{\beta}_{\tilde{\eta}_1}$ is PyFSsw-C for each $\beta \in \Lambda$.*

Proof. Suppose $V_{\tilde{\eta}_2}$ is a $PyFS$ open set across $\tilde{Y}$. By presumption, $(f \mid G^{\beta}_{\tilde{\eta}_1})^{-1}(V_{\tilde{\eta}_2})$ is $PyFSsw$ open over $G^{\beta}_{\tilde{\eta}_1}$. By Lemma 3.13, $(f \mid G^{\beta}_{\tilde{\eta}_1})^{-1}(V_{\tilde{\eta}_2})$ is $PyFSsw$ open over $\tilde{X}$ foe all $\beta \in \Lambda$. But $f^{-1}(V_{\tilde{\eta}_2}) = \sqcup_{\beta\in\Lambda}[(f \mid G^{\beta}_{\tilde{\eta}_1})^{-1}(V_{\tilde{\eta}_2})]$, this is the union of $PyFSsw$ open sets, and $f^{-1}(V_{\tilde{\eta}_2})$ is $PyFSsw$ open over $\tilde{X}$. f is hence $PyFSsw$-C.

Theorem 3. *Let $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ and $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ be a PyFSTSs, and let $U_{\tilde{\eta}_1}$ be a PyFS open set over $\tilde{X}$. If $f : (\tilde{U}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ is a PyFSsw-C function that $f(U_{\tilde{\eta}_1})$ is FyS dense over $\tilde{Y}$, then PyFSsw-C is the extension function of each f over $\tilde{X}$.*

Proof. Let $V_{\tilde{\eta}_2}$ be a (non-null) $PyFS$ open set on $\tilde{Y}$ and let g be an extension of f. If $g^{-1}(V_{\tilde{\eta}_2}) = \phi_{\tilde{\eta}_1}$, then g is simply $PyFSsw$-C. Let $g^{-1}(V_{\tilde{\eta}_2}) \neq \phi_{\tilde{\eta}_1}$. By density of $f(U_{\tilde{\eta}_1})$, $f(U_{\tilde{\eta}_1}) \sqcap V_{\tilde{\eta}_2} \neq \phi_{\tilde{\eta}_2}$ it suggests that $U_{\tilde{\eta}_1} \sqcap f^{-1}(V_{\tilde{\eta}_2}) \neq \phi_{\tilde{\eta}_1}$. Therefore $f^{-1}(V_{\tilde{\eta}_2}) \neq \phi_{\tilde{\eta}_1}$. Presumably, a non-null $PyFS$ open set $W_{\tilde{\eta}_1}$ on U exists such that
$W_{\tilde{\eta}_1} = W_{\tilde{\eta}_1} \sqcap U_{\tilde{\eta}_1} \sqsubseteq f^{-1}(V_{\tilde{\eta}_2}) \sqcap U_{\tilde{\eta}_1} = g^{-1}(V_{\tilde{\eta}_2}) \sqcap U_{\tilde{\eta}_1} \sqsubseteq g^{-1}(V_{\tilde{\eta}_2})$. Since $W_{\tilde{\eta}_1}$ is a $PyFS$ open set over $\tilde{X}$ according to Lemma 3.13, $\phi_{\tilde{\eta}_1} \neq W_{\tilde{\eta}_1} \sqsubseteq g^{-1}(V_{\tilde{\eta}_2})$. Consequently, across $\tilde{X}, g$ is $PyFSsw$-C.

Theorem 4. *Consider* $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ *and* $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ *be a PyFSTSs. A function* $f : (\tilde{U}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ *is a PyFS-semicontinuous if and only if* $f \mid W_{\tilde{\eta}_1}$ *is sw-C for all PyFS open set* $W_{\tilde{\eta}_1}$ *over* $\tilde{X}$*.*

Proof. Let f be a $PyFS$-semicontinuous, $W_{\tilde{\eta}_1}$ is any $PyFS$ open set on $\tilde{X}$. Let $G_{\tilde{\eta}_2}$ be a $PyFS$ open set on $\tilde{Y}$. then $f^{-1}(G_{\tilde{\eta}_2})$ is $PyFS$ semiopen and from Lemma 3.19, $(f \mid W_{\tilde{\eta}_1})^{-1}(G_{\tilde{\eta}_2}) = f^{-1}(G_{\tilde{\eta}_2}) \sqcap W_{\tilde{\eta}_1}$ is $PyFS$ semiopen over W. Then $f \mid W_{\tilde{\eta}_1}$ is $PyFS$-semicontinuous and hence $PyFSsw$-C.
Conversely, Let $f \mid W_{\tilde{\eta}_1}$ is sw-C for all $PyFS$ open set $W_{\tilde{\eta}_1}$ over $\tilde{X}$, and $H_{\tilde{\eta}_2}$ be $PyFS$ open set over $\tilde{Y}$. Then $(f \mid W_{\tilde{\eta}_1})^{-1}(H_{\tilde{\eta}_2}) = f^{-1}(H_{\tilde{\eta}_2}) \sqcap W_{\tilde{\eta}_1}$ is $PyFSsw$-open over W. Since $W_{\tilde{\eta}_1}$ is a $PyFSsw$-open over $\tilde{X}$ by Lemma 3.12, $f^{-1}(H_{\tilde{\eta}_2}) \sqcap W_{\tilde{\eta}_1}$ is a $PyFSsw$-open over $\tilde{X}$ and so, by Lemma 3.22, $f^{-1}(H_{\tilde{\eta}_2})$ is $PyFS$ semiopen over $\tilde{X}$. Thus f is $PyFS$-semicontinuous.

Theorem 5. *Consider* $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ *and* $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ *be a PyFSTS. The function* $f : (\tilde{U}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ *can be represented by the following function:*
1- f is PyFSsw-continuous,
2-There is a non-null PyFS open set $W_{\tilde{\eta}_1}$ *on* $\tilde{X}$ *that* $W_{\tilde{\eta}_1} \sqsubseteq f^{-1}(V_{\tilde{\eta}_2})$*, for any PyFS open set* $f^{-1}(V_{\tilde{\eta}_2})$ *on Y with* $f^{-1}(V_{\tilde{\eta}_2}) \neq \phi_{\tilde{\eta}_1}$*,*
3-There is a proper PyFS closed $K_{\tilde{\eta}_1}$ *on* $\tilde{X}$ *that* $f^{-1}(F_{\tilde{\eta}_2}) \sqsubseteq K_{\tilde{\eta}_1}$*, for any PyFS closed set* $F_{\tilde{\eta}_2}$ *on Y with* $f^{-1}(F_{\tilde{\eta}_2}) \neq \tilde{X}_{\tilde{\eta}_1}$*,*
4- $f(d_{\tilde{\eta}_1})$ *is PyFS dense over* $f(\tilde{X})$ *for any PyFS dense set* $d_{\tilde{\eta}_1}$ *over* $\tilde{X}$*.*

Proof. $1 \Rightarrow 2$ The definition of sw-continuity and Remark 3.2.
$2 \Rightarrow 3$ Given a $PyFS$ closed set $F_{\tilde{\eta}_2}$ over $\tilde{Y}$, $f^{-1}(F_{\tilde{\eta}_2}) \neq \tilde{X}_{\tilde{\eta}_1}$. $f^{-1}(\tilde{Y}_{\tilde{\eta}_2} \setminus F_{\tilde{\eta}_2}) \neq \phi_{\tilde{\eta}_1}$ indicates that $\tilde{Y}_{\tilde{\eta}_2} \setminus F_{\tilde{\eta}_2}$ is $PyFS$ open over $\tilde{Y}$. A $PyFS$ open set $W_{\tilde{\eta}_1}$ over $\tilde{X}$ exists according to (2) in such a way that $\phi_{\tilde{\eta}_1} \neq W_{\tilde{\eta}_1} \sqsubseteq f^{-1}(\tilde{Y}_{\tilde{\eta}_2} \setminus F_{\tilde{\eta}_2}) = \tilde{X}_{\tilde{\eta}_1} \setminus f^{-1}(F_{\tilde{\eta}_2})$. This suggests that $f^{-1}(F_{\tilde{\eta}_2}) \sqsubseteq \tilde{X}_{\tilde{\eta}_1} \setminus W_{\tilde{\eta}_1} \neq \tilde{X}_{\tilde{\eta}_1}$. $K_{\tilde{\eta}_1}$ is a proper $PyFS$ closed set that meets the necessary property if $K_{\tilde{\eta}_1} = \tilde{X}_{\tilde{\eta}_1} \mid W_{\tilde{\eta}_1}$.
$3 \Rightarrow 4$ Over $\tilde{X}$, let $d_{\tilde{\eta}_1}$ be $PyFS$ dense. The claim that $f(d_{\tilde{\eta}_1})$ is $PyFS$ dense over $f(\tilde{X})$ must be proven. Assume that over $f(\tilde{X})$, c is not $PyFS$ dense. A proper $PyFS$ closed set $F_{\tilde{\eta}_2}$, exists such that $f(d_{\tilde{\eta}_1}) \sqsubseteq F_{\tilde{\eta}_2} \sqsubset f(\tilde{X}_{\tilde{\eta}_1})$. So, $d_{\tilde{\eta}_1} \sqsubseteq f(F_{\tilde{\eta}_2})$. According to (3), there is a $PyFS$ closed set $K_{\tilde{\eta}_1}$ over $\tilde{X}$ such that $d_{\tilde{\eta}_1} \sqsubseteq f^{-1}(F_{\tilde{\eta}_2}) \sqsubseteq K_{\tilde{\eta}_1} \neq \tilde{X}_{\tilde{\eta}_1}$. That $d_{\tilde{\eta}_1}$ is $PyFS$ dense over $\tilde{X}$ is contradicted by this. Therefore, (4) is true.
$4 \Rightarrow 1$ Let $H_{\tilde{\eta}_2}$ be a $PyFS$ open set over $\tilde{Y}$ with $f^{-1}(H_{\tilde{\eta}_2}) \neq \phi_{\tilde{\eta}_1}$ without losing generality, since it is trivially $PyFSsw$-open if $f^{-1}(H_{\tilde{\eta}_2}) = \phi_{\tilde{\eta}_1}$. Assume that $f^{-1}(H_{\tilde{\eta}_2})$ is not $PyFSsw$-open, i.e. $int(f^{-1}(H_{\tilde{\eta}_2})) = \phi_{\tilde{\eta}_1}$. At hence, $cl(\tilde{X}_{\tilde{\eta}_1} \setminus f^{-1}(H_{\tilde{\eta}_2}) = \tilde{X}_{\tilde{\eta}_1}$. This suggests that on $\tilde{X}$, $\tilde{X}_{\tilde{\eta}_1} \setminus f^{-1}(H_{\tilde{\eta}_2})$ is $PyFS$ dense. From 4, $f(\tilde{X}_{\tilde{\eta}_1} \setminus f^{-1}(H_{\tilde{\eta}_2}))$ is $PyFS$ dense over $f(\tilde{X})$, this means that $cl(f(\tilde{X}_{\tilde{\eta}_1}) \setminus f^{-1}(H_{\tilde{\eta}_2})) = f(\tilde{X}_{\tilde{\eta}_1})$. This results in $cl(f(\tilde{X}_{\tilde{\eta}_1}) \setminus f^{-1}(H_{\tilde{\eta}_2}) = f(\tilde{X}_{\tilde{\eta}_1}) \setminus H_{\tilde{\eta}_2} = f(\tilde{X}_{\tilde{\eta}_1})$ and so $H_{\tilde{\eta}_2} = \phi_{\tilde{\eta}_2}$. In contrast to the selection of $H_{\tilde{\eta}_2}$. As a result, $int(f^{-1}(H))$ cannot be null. As a result, $f^{-1}(H_{\tilde{\eta}_2})$ is $PyFSsw$ on $\tilde{X}$.

Corollary 4. *Consider* $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ *and* $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ *be a PyFSTS. The corresponding values for a one-to-one function are as follows:* $f : (\tilde{U}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$*:*

a-f is $PyFSsw$-C,
b-$f(M_{\tilde{\eta}_1})$ is $PyFS$ co-dense over $\tilde{Y}$ for any soft co-dense set $M_{\tilde{\eta}_1}$ over $\tilde{X}$.

This section concludes with two results about soft separable and hyperconnected space.

Theorem 6. *Consider $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ and $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ be a $PyFSTSs$, and $f : (\tilde{U}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$. If f is $PyFSsw$-C and $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ is $PyFS$ separable, then $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ is $PyFS$ separable.*

Proof. Allow $d_{\tilde{\eta}_1}$ to be a countable $PyFS$ dense set on $\tilde{X}$. $f(d_{\tilde{\eta}_1})$ is clearly countable. According to $f(d_{\tilde{\eta}_1})$ is $PyFS$ dense over $f(\tilde{X}) = \tilde{Y}$. As a result, $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ is $PyFS$ separable.

Theorem 7. *Let $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ and $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ be a $PyFSTSs$, and $f : (\tilde{U}, \tilde{\Omega}_1, \tilde{\eta}_1) \to (\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$. If f is $PyFSsw$-C and $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ is $PyFS$ hyperconnected, then $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ is $PyFS$ hyperconnected.*

Proof. Allow $G_{\tilde{\eta}_2}, H_{\tilde{\eta}_2}$ be any two $PyFS$ open sets over $\tilde{Y}$ with $G_{\tilde{\eta}_2} \neq H_{\tilde{\eta}_2} \neq \phi_{\tilde{\eta}_2}$. Since f is $PyFSsw$-C, then $int(f^{-1}(G_{\tilde{\eta}_2}) \neq \phi_{\tilde{\eta}_1} \neq int(f^{-1}(H_{\tilde{\eta}_2})$. But $(\tilde{X}, \tilde{\Omega}_1, \tilde{\eta}_1)$ is $PyFS$ hyperconnected, then $int(f^{-1}(G_{\tilde{\eta}_2})) \sqcap int(f^{-1}(H_{\tilde{\eta}_2}) \neq \phi_{\tilde{\eta}_1}$.
If $x \in int(f^{-1}(G_{\tilde{\eta}_2})) \sqcap int(f^{-1}(H_{\tilde{\eta}_2})) \sqsubseteq f^{-1}(G_{\tilde{\eta}_2}) \sqcap f^{-1}(H_{\tilde{\eta}_2})$, at hence $f(x) \in G_{\tilde{\eta}_2} \sqcap H_{\tilde{\eta}_2}$. Thus $(\tilde{Y}, \tilde{\Omega}_2, \tilde{\eta}_2)$ is $PyFS$ hyperconnected.

5. Conclusion

Numerous aspects of everyday existence are uncertain. The $PyFSs$ theory is one theory developed to deal with uncertainty. This study is based on a novel mathematical structure called $PyFST$, which was initiated by typologists using $PyFSss$. In this work, we presented the idea of $PyFSsw$ open sets as a new extension of $PyFS$ open sets. On the one hand, the family of $PyFS$ open to some extent sets is located between the families of $PyFS$ semiopen sets and $PyFS$ somewhere dense sets. The families of $PyFSsw$ open sets and $PyFS\beta$-open sets, on the other hand, are independent of one another. With the help of examples, these linkages have been explained and main attributes established. Then, to define $PyFSsw$-continuous, we used $PyFSsw$ open sets. We defined these two functions and explored their key characteristics. Investigates some intriguing relationships in a certain $PyFST$ in [8]. The purpose of developing these categories was to analyze the distinctions between$PyFS$ homeomorphism and $PyFS$ partly homeomorphism in terms of preserving certain $PyFST$ features. In the following work, we intend to investigate some topological concepts such as $PyFS$ compactness, $PyFS$ Lindelofness, and $PyFS$ connectedness using $PyFSsw$ open sets. It is also planned to investigate certain applications of $PyFSsw$ homeomorphisms. In addition, we investigate $PyFSsw$ open sets in the context of supra $PyFSTS$. This study has laid the groundwork for further exploration of $PyFSs$ and their applications. Future research should investigate the potential of T-Bipolar Sss [3], spherical and T-spherical Fss [9], complex $PyFss$ [7, 16], and Bipolar complex Fss [21]. These directions offer promising avenues for developing more sophisticated models and decision-making tools in various scientific and engineering domains.

Declaration

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