EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 3304-3335
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Intuitionistic Fuzzy Z-Contractions and Common Fixed Points with Applications

Maliha Rashid[1], Naeem Saleem[2,3,*], Quratulain Mumtaz[1], Maggie Aphane[3], Inayatur Rehman[4]

[1] *Department of Mathematics and Statistics,International Islamic University, Islamabad, Pakistan*
[2] *Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan*
[3] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Medunsa, Pretoria 0204, South Africa*
[4] *Department of Mathematics and Sciences, College of Arts and Applied Sciences Dhofar University, Salalah, Oman*

Abstract. In the context of b-metric spaces, this paper introduces two concepts: admissible hybrid intuitionistic fuzzy $\mathcal{Z}$-contractions and pairwise admissible hybrid intuitionistic fuzzy $\mathcal{Z}$-contractions, and establishes criteria for intuitionistic fuzzy fixed points under such contractions. It is demonstrated that a pair of set-valued maps possesses a common fixed point. Various illustrative examples are provided to validate these results. Moreover, the significant implications of our main theorem are explored and analyzed across different types of simulation functions. Furthermore, we derive several fixed point results in the context of partially ordered b-metric spaces, offering insights from an application-oriented perspective. These outcomes extend and generalize several prior results documented in the literature.

2020 Mathematics Subject Classifications: 54H25, 47H10, 03E72, 54E35, 06A06

Key Words and Phrases: b-metric space, Common intuitionistic fuzzy fixed point, Intuitionistic fuzzy set-valued map, Pairwise intuitionistic fuzzy $\mathcal{Z}$-contraction, Simulation function

---

1. Introduction

Fixed point theory is a pivotal area of mathematical analysis with broad implications across various disciplines, including functional analysis, topology, and applied mathematics. One of the foundational results in this field is the Banach Fixed Point Theorem, also known as the Contraction Mapping Principle, a cornerstone in the study of fixed points

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5431

Email addresses: maliha.rashid@iiu.edu.pk (M. Rashid),
quratulain.msma749@iiu.edu.pk (Q. Mumtaz), naeem.saleem2@gmail.com (N. Saleem),
maggie.aphane@smu.ac.za (M. Aphane), irehman@du.edu.om (I. Rehman)

that plays a crucial role in proving the existence and uniqueness of solutions across a broad spectrum of mathematical problems, including those in differential and integral equations. The Banach Contraction Theorem [13], a seminal result in this field, has profoundly influenced subsequent research and applications.

Fixed point theory is instrumental in addressing a variety of complex issues in nonlinear analysis. Specifically, in operator equations such as $Gx = 0$, the existence of a fixed point $fx = x$ can be effectively established for self-mappings within the relevant domain. Fixed point theory has evolved significantly, with numerous generalizations that extend beyond metric spaces to more abstract settings such as topological vector spaces, normed spaces, and ordered structures. These generalizations have been instrumental in solving complex problems in economics, computer science, and engineering, where finding equilibrium states or stable solutions is essential.

Fuzzy set theory, introduced by Lotfi A. Zadeh [2] in 1965, revolutionized classical set theory by allowing partial membership of elements in a set. This approach is particularly useful in modeling situations characterized by uncertainty, imprecision, or vagueness, as it provides a more flexible and realistic representation of real-world phenomena. Fuzzy sets have found extensive applications in various fields such as control systems, decision-making, pattern recognition, and artificial intelligence. Building upon the foundation of fuzzy sets, Krassimir Atanassov [23] introduced intuitionistic fuzzy sets (IFS) in 1986 as a further extension to better handle situations where there is hesitation or lack of complete information. In intuitionistic fuzzy sets, each element is described by two values: the degree of membership. Additionally, there exists a hesitation margin, often referred to as the degree of indeterminacy or uncertainty. This generalization has proven useful in many applications, particularly in decision-making scenarios where incomplete or ambiguous information exists. Notably, Gulzar [14] applied IFSs to group theory, while Kanwal and Akbar [27] used them to define intuitionistic fuzzy mappings and identify common fixed points [24, 26].

Incorporating fuzzy sets into traditional metric spaces has led to significant advancements. Deng [8] introduced the concept of fuzzy pseudo-metric spaces, which extend the classical metric framework to accommodate fuzzy topological and uniform structures. Erceg [17] further explored the equivalence of pseudo-quasi-metrics to distance functions in fuzzy set theory, proposing additional axioms for metrics and uniformity within this context. These developments underscore the ongoing effort to apply mathematical structures to fuzzy scenarios.

In mathematical analysis, the concept of metric spaces plays a fundamental role in understanding various aspects of distance and convergence. Extending the traditional notion of a metric, b-metric spaces provide a more generalized framework that retains many essential properties while relaxing some strict conditions. The development of b-metric spaces drew upon a long history of work on metric spaces and their generalizations. Several key contributions paved the way for the formalization of b-metric spaces: This line of inquiry was initiated by Bakhtin [5] and Bourbaki [7]. Czerwik [8] further contributed to this area by introducing a concept that relaxes the constraints of the traditional triangle inequality and formally defined b-MSs to enhance the Banach Fixed Point (FP) theorem. Since then, there has been a proliferation of research papers examining single-valued and multi-valued operators within b-MSs, often delving into FP theory or the variational principle (Refer to [1, 9, 10, 12, 22]).

The introduction of γ-admissibility has played a key role in extending fixed point theory to more complex spaces and functions. It has been particularly useful in the study of non linear analysis, multi valued mappings, dynamic system and generalized metric spaces. Several researchers have contributed to the development of fixed point theorems involving beta admissibility. Samet et al. (2012)[25] explored fixed point theorems for single and multi-valued mappings in spaces where γ-admissibility is assumed, contributing to the generalization of classical fixed point results. Rhoades [23] worked on extending fixed point theorems using γ-admissibility in partially ordered spaces, showing how this concept could be applied to more structured settings. The concept of γ-admissibility serves as a powerful generalization in fixed point theory, allowing for the study of mappings that do not necessarily satisfy the strict contractive conditions required in classical fixed point theorems. By introducing a function γ that controls the behavior of the mapping, γ-admissibility provides the flexibility needed to extend fixed point results to broader classes of spaces and functions. This has led to significant advances in nonlinear analysis, dynamic systems, and the study of generalized metric spaces. The foundational work on γ-admissibility continues to influence ongoing research in applied mathematics, offering new tools for solving complex problems in fields like optimization, economics, and mathematical modeling.

In a related vein, the exploration of fixed points arising from hybrid contractions represents a burgeoning area of research within fixed point theory. Recent contributions [18] have provided criteria for establishing the existence of intuitionistic fuzzy fixed points (IFFPs) within b-metric spaces, focusing on admissible hybrid intuitionistic fuzzy (AHIF) Z-contractions and HIF Z-contractions. This article introduces a modified form of an admissible hybrid intuitionistic fuzzy Z-contraction within the framework of IFS-valued mappings in extended b-metric spaces, providing sufficient conditions for intuitionistic fuzzy fixed point (IFFP) results. Several special cases of the main result are discussed through corollaries. The application of these findings pertains to the IFFP result in the context of an orderes b-metric space. Each result is supported by examples to validate the hypotheses. To the best of our knowledge, common fixed point theorems within the framework of IFSs utilizing simulation functions have not yet been explored, making the concepts presented here novel.

2. Preliminaries

The collections of natural, non-negative real and real numbers are indicated by $\mathbb{N}$, $\mathbb{R}_+$, and $\mathbb{R}$, respectively.

Definition 1. *[8] Assume a set X that is not empty and $h \geq 1$ is a constant. Suppose that the mapping $\delta : X \times X \to \mathbb{R}_+$ meets the criteria below for all $\sigma, \tau, \varpi \in X$;*

(i) $\delta(\sigma, \tau) = 0$ if and only if $\sigma = \tau$;

(ii) $\delta(\sigma,\tau)=\delta(\tau,\sigma)$;

(iii) $\delta(\sigma,\tau)\leq h[\delta(\sigma,\varpi)+\delta(\varpi,\tau)]$ *for all* $\sigma,\tau,\varpi\in X$.

Then, the triplet (X,δ,h) *is called a b-MS.*

Example 2. *Let* $X=l_p(\mathbb{R})$ *with* $0<p<1$, *where* $l_p(\mathbb{R})=\{\sigma=\{\sigma_n\}\subset\mathbb{R}:\sum^{+}\infty_{n=1}|\sigma_n|^p<+\infty\}$. *Then* $\delta(\sigma,\tau)=(\sum_{n=1}^{+\infty}|\sigma_n-y_n|^p)^{\frac{1}{p}}$ *is a b-MS on* X *with* $h=2^{\frac{1}{p}}$. *Notice that* (X,δ) *is not a MS.*

Definition 3. *[6] Let* (X,δ,h) *be a b-MS. Then, the subset* $\mathcal{A}$ *of* X *is called:*

(i) compact if and only if for every sequence of elements of $\mathcal{A}$, *there exists a subsequence that is convergent to an element of* $\mathcal{A}$.

(ii) closed if and only if for every sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ *of elements of* $\mathcal{A}$ *converging to an element* σ, *we have* $\sigma\in\mathcal{A}$.

Definition 4. *[19] A set* $\mathcal{A}$, *containing one or more elements, in* X *is called proximal if, for some* $\sigma\in X$, *there is* $k\in\mathcal{A}$ *such that* $\delta(\sigma,k)=\delta(\sigma,\mathcal{A})$.

Denote by $\mathcal{N}(X)$, $CB(X)$, $\mathcal{P}^r(X)$ and $\mathcal{Q}(X)$, the families of nonempty subsets of X, all nonempty closed and bounded subsets of X, all nonempty proximal subsets of X and all nonempty compact subsets of X, respectively.
Let (X,δ) be a b-MS. For $\mathcal{A},\mathcal{B}\in\mathcal{Q}(X)$, the function $\aleph:\mathcal{Q}(X)\times\mathcal{Q}(X)\to\mathbb{R}_+$, defined by,

$$\aleph(\mathcal{A},\mathcal{B})=\begin{cases}\max\{\sup_{\sigma\in\mathcal{A}}\delta(\sigma,\mathcal{B}),\sup_{x\in\mathcal{B}}\delta(\sigma,\mathcal{A})\}, & \text{if it exists}\\ +\infty, & \text{otherwise },\end{cases}$$

is named as generalized Hausdorff distance on X and $\mathcal{Q}(X,\delta,h)$ is Hausdorff b-M, where $\delta(\sigma,\mathcal{B})=\inf_{\tau\in\mathcal{B}}\delta(\sigma,\tau)$.

Lemma 5. *[27] Let* (X,δ,h) *be a b-MS. For* $\mathcal{A},\mathcal{B}\in\mathcal{Q}(X)$ *and* $\sigma,\tau\in X$, *the specifications listed below are true:*

(i) $\delta(\sigma,\mathcal{B})\leq\aleph(\mathcal{A},\mathcal{B})$ *for every* $\sigma\in\mathcal{A}$.

(ii) $\delta(\sigma,\mathcal{B})\leq\delta(\sigma,\mathfrak{b})$ *for any* $\mathfrak{b}\in\mathcal{B}$.

(iii) $\delta(\sigma,\mathcal{A})\leq h[\delta(\sigma,\tau)+\delta(\tau,\mathcal{A})]$.

(iv) $\delta(\sigma,\mathcal{A})=0\Leftrightarrow\sigma\in\mathcal{A}$.

(v) $\aleph(\mathcal{A},\mathcal{B})=0\Leftrightarrow\mathcal{A}=\mathcal{B}$.

(vi) $\aleph(\mathcal{A},\mathcal{B})=\aleph(\mathcal{B},\mathcal{A})$.

Khojasteh *et al.* [17] introduced a family of auxiliary functions known as SFs (simulation functions) to unify distinct types of contraction.

Definition 6. *[17] A SF (simulation function) is an operator $\wp : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ that meets the requirements listed below:*

(i) $\wp(0,0) = 0$;

(ii) $\wp(\mathfrak{a}, \mathfrak{b}) < \mathfrak{b} - \mathfrak{a}$ *for all* $\mathfrak{a}, \mathfrak{b} > 0$;

(iii) if $\{\mathfrak{a}_n\}_{n\geq 1}$ *and* $\{\mathfrak{b}_n\}_{n\geq 1}$ *are sequences in* $(0, +\infty)$ *such that* $\lim_{n\to+\infty} \mathfrak{a}_n = \lim_{n\to+\infty} \mathfrak{b}_n > 0$, *then* $\lim_{n\to+\infty} \sup \wp(\mathfrak{a}_n, \mathfrak{b}_n) < 0$.

The collection of SFs is denoted by $\mathcal{Z}$.

Example 7. *Let o and ψ be two altering distance functions such that $\psi(\mathfrak{t}) < \mathfrak{t} \leq \phi(\mathfrak{t})$ for all $\mathfrak{t} > 0$, also ϕ and ψ are continuous functions such that $\phi(\mathfrak{t}) = \psi(\mathfrak{t}) = 0$ if and only if $\mathfrak{t} = 0$. Then the mapping $\wp(\mathfrak{a}, \mathfrak{b}) = \psi(\mathfrak{b}) - \phi(\mathfrak{a})$ for all $\mathfrak{a}, \mathfrak{b} \in [0, +\infty)$ is a SF.*

To study even more details and examples of SF see [2, 11, 17]

Definition 8. *[17] Let (X, δ) be a MS. A correspondence $\mathcal{T} : X \to X$ is named as $\mathcal{Z}$-contraction with regard to $\wp \in \mathcal{Z}$, if*

$$\wp(\delta(\mathcal{T}\sigma, \mathcal{T}\tau), \delta(\sigma, \tau)) \geq 0 \text{ for all } \sigma, \tau \in X$$

Remark 9. *The description of the SF makes it evident that $\wp(\mathfrak{a}, \mathfrak{b}) < 0$ for all $\mathfrak{a} \geq \mathfrak{b} > 0$. Therefore, if $\mathcal{T}$ is a $\mathcal{Z}$-contraction with respect to $\wp \in \mathcal{Z}$ then $\delta(\mathcal{T}\sigma, \mathcal{T}\tau) < \delta(\sigma, \tau)$ for all distinct $\sigma, \tau \in X$. As a result, it can be seen that every $\mathcal{Z}$ contraction mapping is contractive and continuous.*

Theorem 10. *[17] Every $\mathcal{Z}$-contraction on a complete MS has a unique FP. Let X represent a universal set. A function with a domain of X and values in $[0, 1] = I$ is referred to as a FS in X. If $\mathcal{A}$ is a fuzzy set in X, then the function value $\mathcal{A}(X)$ is called the grade of membership of σ in $\mathcal{A}$.*

Definition 11. *[4] Let X be a universal set. An IFS (intuitionistic fuzzy set) $\mathcal{A}$ is described as:*

$$\mathcal{A} = \{\sigma \in X; (\mu_{\mathcal{A}}(\sigma), \nu_{\mathcal{A}}(\sigma))\}$$

with $\mu_{\mathcal{A}} : X \to [0, 1]$ and $\nu_{\mathcal{A}} : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership of each element σ to set $\mathcal{A}$ respectively, such that $0 \leq \mu_{\mathcal{A}}(\sigma) + \nu_{\mathcal{A}}(\sigma) \leq 1$ for all $\sigma \in X$.

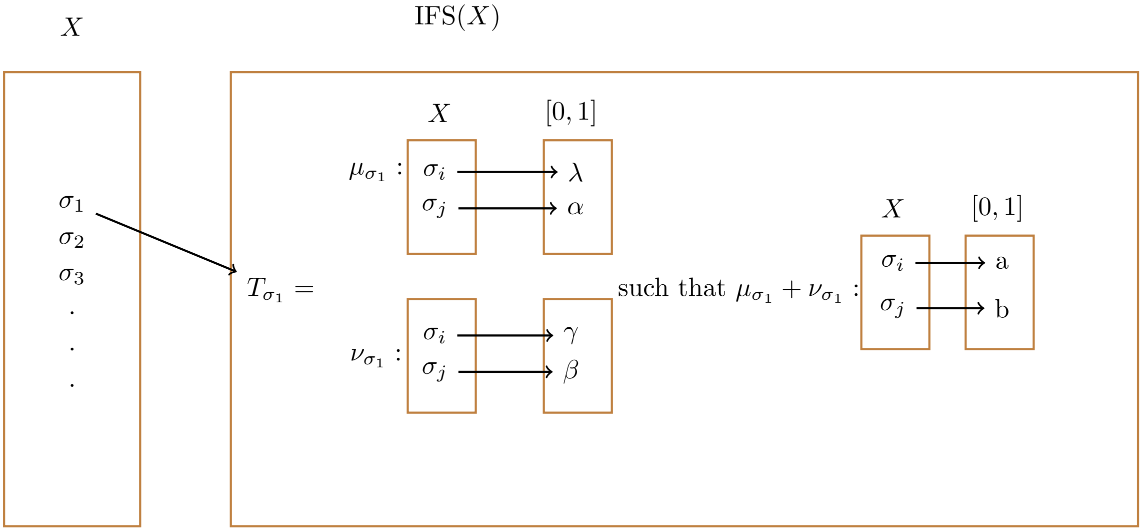

Figure 1: Pictorial representation of intuitionistic fuzzy set-valued mappings.

Example 12. *Consider the following real-world example of an IFS representing the idea of "customer satisfaction" for a restaurant. We will design eight customer satisfaction elements and give membership and non-membership values to every single factor.*

Table 1: Intuitionistic Fuzzy Set Data

Factors	**Membership**	**Non Membership**
Food quality	0.8	0.1
Service speed	0.7	0.2
Cleanliness	0.6	0.2
Atmosphere	0.6	0.3
Prices	0.4	0.5
Menu variety	0.8	0.2
Staff friendliness	0.6	0.4
Noise level	0.5	0.3

Definition 13. *Let* $L = \{(\ltimes, \beta); \ltimes + \beta \leq 1, (\ltimes, \beta) \in (0, 1] \times [0, 1)\}$ *and* $\mathcal{A}$ *is an IFS, then* $(\ltimes, \beta)$*-cut set of* $\mathcal{A}$ *is defined as:*

$$[\mathcal{A}]_{(\ltimes,\beta)} = \{\sigma \in X : \mu_{\mathcal{A}}(\sigma) \geq \ltimes \text{ and } \nu_{\mathcal{A}}(\sigma) \leq \beta\}$$

Let X *be a nonempty set and* Υ *be a MS. A mapping* $\mathcal{T} : X \to IFS(X)$ *is called IFS-valued map. An IFS-valued map* $\mathcal{T}$ *is an IF subset of* X *with membership function* $\mu_{\mathcal{T}}(\sigma)(\tau)$ *and nonmembership function* $\nu_{\mathcal{T}}(\sigma)(\tau)$.

An IFS $\mathcal{A}$ in a metric linear space V is known to be an approximate quantity if and only if $[\mathcal{A}]_{(\ltimes,\beta)}$ is compact and convex in V for each $(\ltimes, \beta) \in (0, 1] \times [0, 1)$ with $\sup_{\sigma\in V} \mu_{\mathcal{A}}(\sigma) = 1$ and $\inf_{\sigma\in V} \nu_{\mathcal{A}}(\sigma) = 0$. The compilation of all approximate quantities in V is denoted by

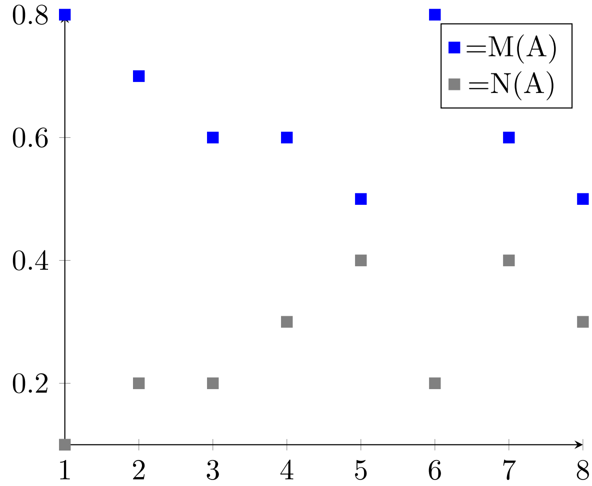

Figure 2: Scattered Plot for Membership and Non-Membership of an IFS A

$W(V)$.
If there is $(\ltimes, \beta) \in (0,1] \times [0,1)$ in such a way that $[\mathcal{A}]_{(\ltimes,\beta)}, [B]_{(\ltimes,\beta)} \in \mathcal{Q}(X)$, then establish

$$D_{(\ltimes,\beta)}(\mathcal{A}, \mathcal{B}) = \aleph([\mathcal{A}]_{(\ltimes,\beta)}, [B]_{(\ltimes,\beta)})$$

$$\delta_{+}\infty(\mathcal{A}, \mathcal{B}) = \sup_{(\ltimes,\beta)} D_{(\ltimes,\beta)}(\mathcal{A}, \mathcal{B}).$$

Definition 14. *[4] Let X be an arbitrary set, a point $\varsigma \in X$ is called an IFFP of an IFM $S : X \to IFS(X)$, if there exists $(\ltimes, \beta) \in (0,1] \times [0,1)$ so that $\varsigma \in [S\varsigma]_{(\ltimes,\beta)}$.*
A point $\varsigma \in X$ is referred to as FP of an IFM $S : X \to IFS(X)$, if $\mu_{(S\varsigma)}(\varsigma) \geq \mu_{(S\varsigma)}(\sigma)$ and $\nu_{(S\varsigma)}(\varsigma) \leq \nu_{(S\varsigma)}(\sigma)$ for all $\sigma \in X$.

Give the set of all FPs of $\mathcal{T}$, the symbol $Fix(\mathcal{T})$.

Rus [24] introduced the idea of comparison function (CF), which a lot of authors have thoroughly reviewed to expand more generic contraction type mappings.

Definition 15. *[24] A function $\varphi : \mathbb{R}_{+} \to \mathbb{R}_{+}$ is known as CF (comparison function) if it does not decrease and $\varphi^{n}(\mathfrak{t}) \to 0$ as $n \to +\infty$ for all $\mathfrak{t} \geq 0$.*

Definition 16. *[24] A nondecreasing function $\varphi : \mathbb{R}_{+} \to \mathbb{R}_{+}$ is called:*

(i) a c-CF if it fulfils the criteria that $\sum_{n=0}^{+\infty} \varphi^{n}(\mathfrak{t})$ converges for all $t > 0$.

(ii) a b-CF if it meets the requirement that $\sum_{n=0}^{+\infty} S^{n}\varphi^{n}(\mathfrak{t})$ converges for all $\mathfrak{t} \in \mathbb{R}_{+}$ where (X, δ) be a b-MS with $S \geq 1$.

Example 17. *Let (X, δ) be a b-MS with coefficient $h \geq 1$. Then $\varphi(\mathfrak{t}) = \wp\mathfrak{t}$; $\mathfrak{t} \in \mathbb{R}_{+}$ with $0 < \wp < \frac{1}{h}$ is a b-CF. The definition of b-CF becomes equivalent to c-CF when $h = 1$.*

Indicate by Λ_{b}, the family of all functions $\varphi : \mathbb{R}_{+} \times \mathbb{R}_{+}$ fulfilling:

(i) φ is a b-CF;

(ii) $\varphi(\mathfrak{t}) = 0$ if and only if $\mathfrak{t} = 0$;

(iii) φ is continuous.

Lemma 18. *[24] For a CF $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, the characteristics, defined below, are true:*

(i) each iterate φ^n, $n \in \mathbb{N}$ is also a CF;

(ii) $\varphi(\mathfrak{t}) < \mathfrak{t}$ for all $\mathfrak{t} > 0$.

Lemma 19. *[24] Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a b-CF. Then, the series $\sum^+ \infty_{k=0} h^k \varphi^k(\mathfrak{t})$ converges for each $\mathfrak{t} \in \mathbb{R}_+$.*

3. Main Results

Within this part, we'll discuss the idea of γ-admissibility of IFS-valued maps, which is inspired by the idea of γ-admissibility raised by Samet *et al.* [25].

Definition 20. *Let (X, δ) be a metric linear space. A mapping $\mathcal{T} : X \to IFS(X)$ is called an IF λ-contraction, if there is $\lambda \in (0, 1)$ in a way that for each $\sigma, \tau \in X$, $\delta_{+\infty}(\mathcal{T}(\sigma), \mathcal{T}(\tau)) \leq \lambda \delta(\sigma, \tau)$.*

Definition 21. *For a b-MS (X, δ, h), $\gamma : X \times X \to \mathbb{R}_+$ and IFSs S and $\mathcal{T}$, the ordered pair $(S, \mathcal{T})$ is known as γ-admissible if the criteria bellow are met:*

(i) for each $\sigma \in X$ and $\tau \in [S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$, where $(\ltimes(\sigma), \beta(\sigma)) \in (0, 1] \times [0, 1)$, with $\gamma(\sigma, \tau) \geq 1$, we have $\gamma(\tau, \varpi) \geq 1$ for all $\varpi \in [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))} \neq \varphi$ where $(\ltimes(\tau), \beta(\tau)) \in (0, 1] \times [0, 1)$

(ii) for each $\sigma \in X$ and $\tau \in [\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$, where $(\ltimes(\sigma), \beta(\sigma)) \in (0, 1] \times [0, 1)$ with $\gamma(\sigma, \tau) \geq 1$, we have $\gamma(\tau, \varpi) \geq 1$ for all $\varpi \in [S\tau]_{(\ltimes(\tau),\beta(\tau))} \neq \varphi$, where $(\ltimes(\tau), \beta(\tau)) \in (0, 1] \times [0, 1)$.

If $S = \mathcal{T}$, then $\mathcal{T}$ is called γ-admissible.

Definition 22. *Let (X, δ, h) be a b-MS and $S, \mathcal{T} : X \to IFS(X)$ are IFS-valued maps. The ordered pair $(S, \mathcal{T})$ is said to be $\aleph$-continuous at $\varsigma \in X$ if for each sequence $\{\sigma_n\}_{n\geq 1}$ in X,*

(i) $\lim_{n\to+\infty} \delta(\sigma_n, \varsigma) = 0$ *implies that* $\lim_{n\to+\infty} \aleph([S\sigma_n]_{(\ltimes(\sigma_n),\beta(\sigma_n))}, [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) = 0$ *where* $(\ltimes(\sigma_n), \beta(\sigma_n)), (\ltimes(\varsigma), \beta(\varsigma)) \in (0, 1] \times [0, 1)$.

(ii) $\lim_{n\to+\infty} \delta(\sigma_n, \varsigma) = 0$ *implies that* $\lim_{n\to+\infty} \aleph([\mathcal{T}\sigma_n]_{(\ltimes(\sigma_n),\beta(\sigma_n))}, [S\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) = 0$ *where* $(\ltimes(\sigma_n), \beta(\sigma_n)), (\ltimes(\varsigma), \beta(\varsigma)) \in (0, 1] \times [0, 1)$.

Definition 23. *Let (X, δ, h) be a b-MS and S and $\mathcal{T}$ are IFS-valued maps. The mappings S and $\mathcal{T}$ are called pairwise AHIF $\mathcal{Z}$-contraction with respect to $\wp \in \mathcal{Z}$, if there exists $(\ltimes, \beta) \in (0,1] \times [0,1)$ a function $\gamma : X \times X \to \mathbb{R}_+$ and a b-CF $\varphi : \mathbb{R}_+ \times \mathbb{R}_+$ in a manner that the following prerequisites are fulfilled:*

(i) $\wp(\gamma(\sigma,\tau)\aleph([S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}, [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}), \varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau))) \geq 0$ *for all $\sigma, \tau \in X$, where*

$$M^r_{(S,\mathcal{T})}(\sigma,\tau) = \begin{cases} [\mathcal{A}(\sigma,\tau)]^{\frac{1}{r}}, & \text{for } r > 0,\ \sigma,\tau \in X, \\ \mathcal{B}(\sigma,\tau), & \text{for } r = 0,\ \sigma,\tau \in X. \end{cases}$$

$$\begin{aligned} \mathcal{A}(\sigma,\tau) &= k_1(\delta(\sigma,\tau))^r + k_2(\delta(\sigma,[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}))^r + k_3(\delta(y,[\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}))^r \\ &\quad + k_4\left(\frac{\delta(\tau,[\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))})(1+\delta(\sigma,[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}))}{1+\delta(\sigma,\tau)}\right)^r \\ &\quad + k_5\left(\frac{\delta(\tau,[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))})(1+\delta(\sigma,[\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}))}{1+\delta(\sigma,\tau)}\right)^r \end{aligned}$$

$$\begin{aligned} \mathcal{B}(\sigma,\tau) &= (\delta(\sigma,\tau))^{k_1} \times (\delta(\sigma,[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}))^{k_2} \times (\delta(\tau,[\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}))^{k_3} \\ &\quad \times \left(\frac{\delta(\tau,[\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))})(1+\delta(\sigma,[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}))}{1+\delta(\sigma,\tau)}\right)^{k_4} \\ &\quad \times \left(\frac{\delta(\sigma,[\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))})+\delta(\tau,[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))})}{2h}\right)^{k_5} \end{aligned}$$

(ii) $\wp(\gamma(\sigma,\tau)\aleph([\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}, [S\tau]_{(\ltimes(\tau),\beta(\tau))}), \varphi(M^r_{(\mathcal{T},S)}(\sigma,\tau))) \geq 0$ *for all $\sigma, \tau \in X$, where*

$$M^r_{(\mathcal{T},S)}(\sigma,\tau) = \begin{cases} [\mathcal{A}(\sigma,\tau)]^{\frac{1}{r}}, & \text{for } r > 0,\ \sigma,\tau \in X, \\ \mathcal{B}(\sigma,\tau), & \text{for } r = 0,\ \sigma,\tau \in X. \end{cases}$$

$$\begin{aligned} \mathcal{A}(\sigma,\tau) &= k_1(\delta(\sigma,\tau))^r + k_2(\delta(\sigma,[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}))^r + k_3(\delta(\tau,[S\tau]_{(\ltimes(\tau),\beta(\tau))}))^r \\ &\quad + k_4\left(\frac{\delta(\tau,[S\tau]_{(\ltimes(\tau),\beta(\tau))})(1+\delta(\sigma,[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}))}{1+\delta(\sigma,\tau)}\right)^r \\ &\quad + k_5\left(\frac{\delta(\tau,[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\tau))})(1+\delta(\sigma,[S\tau]_{(\ltimes(\tau),\beta(\tau))}))}{1+\delta(\sigma,\tau)}\right)^r \end{aligned}$$

$$\begin{aligned} \mathcal{B}(\sigma,\tau) &= (\delta(\sigma,\tau))^{k_1} \times (\delta(\sigma,[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}))^{k_2} \times (\delta(\tau,[S\tau]_{(\ltimes(\tau),\beta(\tau))}))^{k_3} \\ &\quad \times \left(\frac{\delta(\tau,[S\tau]_{(\ltimes(\tau),\beta(\tau))})(1+\delta(\sigma,[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}))}{1+\delta(\sigma,\tau)}\right)^{k_4} \end{aligned}$$

$$\times\left(\frac{\delta(\sigma,[S\tau]_{(\ltimes(\tau),\beta(\tau))})+\delta(\tau,[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))})}{2h}\right)^{k_5}$$

where $(\ltimes(\sigma),\beta(\sigma)),(\ltimes(\tau),\beta(\tau))\in(0,1]\times[0,1)$ *with* $\gamma\geq 0$ *and* $k_i\geq 0$ *such as* $\sum_{i=1}^{5}k_i=1$.

Remark 24.

(i) If the pair $(S,\mathcal{T})$ *is* $\aleph$*-continuous, then* $(\mathcal{T},S)$ *is also* $\aleph$*-continuous.*

(ii) If $(S,\mathcal{T})$ *is pairwise AHIF* $\mathcal{Z}$*-contraction then so is* $(\mathcal{T},S)$.

Let $IFS_s(X)$ be a subset of $IFS(X)$ defined by:

$$IFS_s(X)=\{\mathcal{A}\in IFS(X):[\mathcal{A}]_{(\ltimes,\beta)}\in\mathcal{Q}(X),\ (\ltimes,\beta)\in(0,1]\times[0,1)\}$$

Theorem 25. *Let* (X,δ,h) *be a complete b-MS and* $S,\mathcal{T}:X\to IFS(X)$ *be pairwise AHIF* $\mathcal{Z}$*-contraction pertaining to* $\wp\in\mathcal{Z}$.
Furthermore, suppose that:

(i) $(S,\mathcal{T})$ *is a* γ*-admissible pair;*

(ii) there is $\sigma_0\in X$ *and*

  (a) $\sigma_1\in[S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}$ *such that* $\gamma(\sigma_0,\sigma_1)\geq 1$;

  (b) $\sigma_1\in[\mathcal{T}\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}$ *such that* $\gamma(\sigma_0,\sigma_1)\geq 1$;

(iii) $(S,\mathcal{T})$ *is* $\aleph$*-continuous;*

(iv) The sets $[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$ *and* $[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$ *are proximal for each* $\sigma\in X$.

Then S *and* $\mathcal{T}$ *have at least single common IFFP in* X.

Proof. Using $(ii)(\mathfrak{a})$, we have $(\ltimes(\sigma_0),\beta(\sigma_0))\in(0,1]\times[0,1)$, $\sigma_0\in X$ and $\sigma_1\in[S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}$ such that $\gamma(\sigma_0,\sigma_1)\geq 1$. If $\sigma_1=\sigma_0$ and $\mathcal{T}=S$ then, from condition (i) in definition (23), we have

$$\begin{aligned}0&\leq\wp(\gamma(\sigma,\tau)\aleph([S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}),\varphi(M^r_{(S,\mathcal{T})}(\sigma_0,\sigma_1)))\\&<\varphi(M^r_{(S,\mathcal{T})}(\sigma_0,\sigma_1))-\gamma(\sigma_0,\sigma_1)\aleph([S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))})\end{aligned}$$

which is analogous to

$$\gamma(\sigma_0,\sigma_1)\aleph([S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))})\leq\varphi(M^r_{(S,\mathcal{T})}(\sigma_0,\sigma_1))\tag{1}$$

Then for $r>0$, we get

$$M^r_{(S,\mathcal{T})}(\sigma_0,\sigma_1)=[\mathcal{A}(\sigma_0,\sigma_1)]^{\frac{1}{r}}$$

$$= \Bigg[k_1(\delta(\sigma_0,\sigma_1))^r + k_2(\delta(\sigma_0,[S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}))^r + k_3(\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))^r + k_4\left(\frac{\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))})(1+\delta(\sigma_0,[S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}))}{1+\delta(\sigma_0,\sigma_1)}\right)^r + k_5\left(\frac{\delta(\sigma_1,[S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))})(1+\delta(\sigma_0,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))}{1+\delta(\sigma_0,\sigma_1)}\right)^r\Bigg]^{\frac{1}{r}}$$

using proximality of S, $\sigma_1 \in [S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}$

$$\begin{aligned} M^r_{(S,\mathcal{T})}(\sigma_0,\sigma_1) &= \Bigg[k_1(\delta(\sigma_0,\sigma_1))^r + k_2(\delta(\sigma_0,\sigma_1))^r + k_3(\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))^r \\ &\quad + k_4\left(\frac{\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))})(1+\delta(\sigma_0,\sigma_1))}{1+\delta(\sigma_0,\sigma_1)}\right)^r \\ &\quad + k_5\left(\frac{\delta(\sigma_1,\sigma_1)(1+\delta(\sigma_0,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))}{1+\delta(\sigma_0,\sigma_1)}\right)^r\Bigg]^{\frac{1}{r}} \\ &= [(k_1+k_2)(\delta(\sigma_0,\sigma_1))^r + (k_3+k_4)(\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))^r]^{\frac{1}{r}} \end{aligned}$$

for $\sigma_1 = \sigma_0$ and $\mathcal{T} = S$, we get

$$\begin{aligned} M^r_{(S,\mathcal{T})}(\sigma_0,\sigma_1) &= [(k_1+k_2)(\delta(\sigma_0,\sigma_0))^r + (k_3+k_4)(\delta(\sigma_1,[S\sigma_0]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))^r]^{\frac{1}{r}} \\ &= 0 \end{aligned}$$

Similarly $\mathcal{B}(\sigma_0,\sigma_1) = 0$. Hence (1) becomes,

$$\gamma(\sigma_0,\sigma_1)\aleph([S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}) \leq \varphi(0) = 0$$

which implies that

$$\aleph([S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}) = 0$$
$$\text{that is, } \sigma_1 \in [S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))} = [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$$

which means σ_1 is IFFP of $\mathcal{T}$.
Since $\sigma_1 \in [S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}$ with $\gamma(\sigma_0,\sigma_1) \geq 1$,so by (i), $\gamma(\sigma_1,\varpi) \geq 1$ for all $\varpi \in [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$.
As $\sigma_1 \in [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$,we have $\gamma(\sigma_1,\sigma_1) \geq 1$ and

$$0 \leq \wp(\gamma(\sigma_1,\sigma_1)\aleph([S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}),\varphi(M^r_{(S,\mathcal{T})}(\sigma_1,\sigma_1)))$$

which is equivalent to

$$\gamma(\sigma_1,\sigma_1)\aleph([S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}) \leq \varphi(M^r_{(S,\mathcal{T})}(\sigma_1,\sigma_1)) \tag{2}$$

Then for $r > 0$, we get

$$\begin{aligned} M^r_{(S,\mathcal{T})}(\sigma_1,\sigma_1) &= [\mathcal{A}(\sigma_1,\sigma_1)]^{\frac{1}{r}} \\ &= \Bigg[k_1(\delta(\sigma_1,\sigma_1))^r + k_2(\delta(\sigma_1,[S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))^r + k_3(\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))^r \\ &\quad + k_4\left(\frac{\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))})(1+\delta(\sigma_1,[S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))}{1+\delta(\sigma_1,\sigma_1)}\right)^r \\ &\quad + k_5\left(\frac{\delta(\sigma_1,[S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))})(1+\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))}{1+\delta(\sigma_1,\sigma_1)}\right)^r\Bigg]^{\frac{1}{r}} \end{aligned}$$

Using the proximality of $\mathcal{T}$, we get

$$\begin{aligned} M^r_{(S,\mathcal{T})} &= [k_1(0) + k_2(\delta(\sigma_1,[S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))^r + k_3(0) + k_4(0) + k_5(\delta(\sigma_1,[S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}))^r]^{\frac{1}{r}} \\ &= 0 \end{aligned}$$

Similarly $\mathcal{B}(\sigma_1,\sigma_1) = 0$. Therefore (2) becomes,

$$\gamma(\sigma_1,\sigma_1)\aleph([S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}) \leq \varphi(0) = 0$$

which yields

$$\aleph([S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}) = 0$$
$$\text{implies that } [S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))} = [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$$

that is, σ_1 is an IFFP of S, which yields

$$\sigma_1 \in [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))} \cap [S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$$

Henceforth we presume that $\sigma_0 \neq \sigma_1$ and $S \neq \mathcal{T}$ so, $\sigma_1 \notin [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))} \cap [S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$ which implies that $\sigma_1 \notin [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$. So that

$$\delta(\sigma_1,[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}) > 0.$$

Since $[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))} \in \mathcal{Q}(X)$ and $\sigma_1 \in [S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}$, there is $\sigma_2 \in [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$ with $\sigma_1 \neq \sigma_2$ such that

$$\begin{aligned} \delta(\sigma_1,\sigma_2) &\leq \aleph([S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}) \\ &\leq \gamma(\sigma_0,\sigma_1)\aleph([S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))},[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}) \end{aligned} \quad (3)$$

using (2) and (3), we get

$$\delta(\sigma_1,\sigma_2) \leq \varphi(M^r_{(S,\mathcal{T})}(\sigma_0,\sigma_1))$$

Provided that $(S,\mathcal{T})$ is γ-admissible pair and $\sigma_2 \in [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$, we have $\gamma(\sigma_1,\sigma_2) \geq 1$.

If $\sigma_2 \in [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}$, then taking $\sigma_1 = \sigma_2$ and $\mathcal{T} = S$ consistent with earlier steps, we find directly that $\sigma_2 \in [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))} \cap [\mathcal{T}\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}$ so we suppose $\sigma_2 \neq [S\sigma_2]_{(\ltimes(\sigma_1),\beta(\sigma_1))} > 0$. So that

$$\delta(\sigma_2, [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}) > 0.$$

Since $[\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}, [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))} \in \mathcal{Q}(X)$ and $\sigma_2 \in [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$ there exists a point $\sigma_3 \in [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}$ with $\sigma_2 \neq \sigma_3$ such that

$$\begin{aligned} \delta(\sigma_2,\sigma_3) &\leq \aleph([\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}, [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}) \\ &\leq \gamma(\sigma_1,\sigma_2)\aleph([\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}, [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}) \end{aligned} \tag{4}$$

from condition (ii) in definition (23), we get

$$\begin{aligned} 0 &\leq \wp(\gamma(\sigma_1,\sigma_2)\aleph([\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}, [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}), \varphi(M^r_{(\mathcal{T},S)}(\sigma_1,\sigma_2))) \\ &< \varphi(M^r_{(\mathcal{T},S)}(\sigma_1,\sigma_2)) - \gamma(\sigma_1,\sigma_2)\aleph([\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}, [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}) \end{aligned}$$

which can be written as

$$\gamma(\sigma_1,\sigma_2)\aleph([\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}, [S\sigma_2]_{(\ltimes(\sigma_2),\beta(\sigma_2))}) \leq \varphi(M^r_{(\mathcal{T},S)}(\sigma_1,\sigma_2)) \tag{5}$$

combining (4) and (5) yields,

$$\delta(\sigma_2,\sigma_3) \leq \varphi(M^r_{(\mathcal{T},S)}(\sigma_1,\sigma_2))$$

Likewise, we generate a sequence $\{\sigma_n\}_{n\geq 1}$ with $\sigma_{2n+1} \in [S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))}$, $\sigma_{2n+2} \in [\mathcal{T}\sigma_{2n+1}]_{(\ltimes(\sigma_{2n+1}),\beta(\sigma_{2n+1}))}$, $\gamma(\sigma_{2n},\sigma_{2n+1}) \geq 1$ and $\gamma(\sigma_{2n+1},\sigma_{2n+2}) \geq 1$ such that

$$\delta(\sigma_{2n+1},\sigma_{2n+2}) \leq \varphi(M^r_{(S,\mathcal{T})}(\sigma_{2n},\sigma_{(2n+1)})) \tag{6}$$

$$\delta(\sigma_{2n+2},\sigma_{2n+3}) \leq \varphi(M^r_{(\mathcal{T},S)}(\sigma_{2n+1},\sigma_{2n+2})) \tag{7}$$

Now we investigate (6) and (7) under the criteria below,
Case 1: $r > 0$
In this case, from condition (i) in definition (23) using the proximality of $\mathcal{T}$ and S in (6) we have

$$\begin{aligned} M^r_{(S,\mathcal{T})}(\sigma_{2n},\sigma_{2n+1}) &= [\mathcal{A}(\sigma_{2n},\sigma_{2n+1})]^{\frac{1}{r}} \\ &= \Bigg[k_1(\delta(\sigma_{2n},\sigma_{2n+1}))^r + k_2(\delta(\sigma_{2n},[S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))}))^r \\ &+ k_3(\delta(\sigma_{2n+1},[\mathcal{T}\sigma_{2n+1}]_{(\ltimes(\sigma_{2n+1}),\beta(\sigma_{2n+1}))}))^r \\ &+ k_4\left(\frac{\delta(\sigma_{2n+1},[\mathcal{T}\sigma_{2n+1}]_{(\ltimes(\sigma_{2n+1}),\beta(\sigma_{2n+1}))})(1+\delta(\sigma_{2n},[S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))}))}{1+\delta(\sigma_{2n},\sigma_{2n+1})}\right)^r \\ &+ k_5\left(\frac{\delta(\sigma_{2n+1},[S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))})(1+\delta(\sigma_{2n},[\mathcal{T}\sigma_{2n+1}]_{(\ltimes(\sigma_{2n+1}),\beta(\sigma_{2n+1}))}))}{1+\delta(\sigma_{2n},\sigma_{2n+1})}\right)^r\Bigg]^{\frac{1}{r}} \end{aligned}$$

$$= \Bigg[k_1(\delta(\sigma_{2n},\sigma_{2n+1}))^r + k_2(\delta(\sigma_{2n},\sigma_{2n+1}))^r + k_3(\delta(\sigma_{2n+1},\sigma_{2n+2}))^r$$
$$+ k_4\left(\frac{\delta(\sigma_{2n+1},\sigma_{2n+2})(1+\delta(\sigma_{2n},\sigma_{2n+1}))}{1+\delta(\sigma_{2n},\sigma_{2n+1})}\right)^r$$
$$+ k_5\left(\frac{\delta(\sigma_{2n+1},\sigma_{2n+1})(1+\delta(\sigma_{2n},\sigma_{2n+2}))}{1+\delta(\sigma_{2n},\sigma_{2n+1})}\right)^r\Bigg]^{\frac{1}{r}}$$

$$= [(k_1+k_2)(\delta(\sigma_{2n},\sigma_{2n+1}))^r + (k_3+k_4)(\delta(\sigma_{2n+1},\sigma_{2n+2}))^r]^{\frac{1}{r}} \tag{8}$$

using (8) in (6), we get

$$\delta(\sigma_{2n+1},\sigma_{2n+2}) \leq \varphi([(k_1+k_2)(\delta(\sigma_{2n},\sigma_{2n+1}))^r + (k_3+k_4)(\delta(\sigma_{2n+1},\sigma_{2n+2}))^r]^{\frac{1}{r}}) \tag{9}$$

Assume that $\delta(\sigma_{2n},\sigma_{2n+1}) \leq \delta(\sigma_{2n+1},\sigma_{2n+2})$, then since φ is non-decreasing, we get

$$\delta(\sigma_{2n+1},\sigma_{2n+2}) \leq \varphi([(k_1+k_2)(\delta(\sigma_{2n+1},\sigma_{2n+2}))^r + (k_3+k_4)(\delta(\sigma_{2n+1},\sigma_{2n+2}))^r]^{\frac{1}{r}})$$

noting that $k_1+k_2+k_3+k_4 \leq 1$,

$$\begin{aligned}\delta(\sigma_{2n+1},\sigma_{2n+2}) &\leq \varphi([(\delta(\sigma_{2n+1},\sigma_{2n+2}))^r]^{\frac{1}{r}}) \\ &= \varphi(\delta(\sigma_{2n+1},\sigma_{2n+2})) \\ &< \delta(\sigma_{2n+1},\sigma_{2n+2})\end{aligned}$$

a contradiction. Consequently, we have

$$\delta(\sigma_{2n+1},\sigma_{2n+2}) \leq \delta(\sigma_{2n},\sigma_{2n+1})$$

so, (9) becomes

$$\begin{aligned}\delta(\sigma_{2n+1},\sigma_{2n+2}) &\leq \varphi(\delta(\sigma_{2n},\sigma_{2n+1})) \\ &\leq \varphi^2(\delta(\sigma_{2n-1},\sigma_{2n})) \\ &\vdots \\ &\leq \varphi^{2n+1}(\delta(\sigma_0,\sigma_1))\end{aligned}$$

$$\delta(\sigma_{2n+1},\sigma_{2n+2}) \leq \varphi^{2n+1}(\delta(\sigma_0,\sigma_1))$$

Similarly, for (7) we are able to demonstrate that

$$\delta(\sigma_{2n+2},\sigma_{2n+3}) \leq \varphi^{2n+2}(\delta(\sigma_0,\sigma_1))$$

from above two equations, we conclude that

$$\delta(\sigma_n,\sigma_{n+1}) \leq \varphi^n(\delta(\sigma_0,\sigma_1)) \tag{10}$$

Take $m, n \in \mathbb{N}$ where $m > n$, then

$$\delta(\sigma_n, \sigma_m) \leq h\delta(\sigma_n, \sigma_{n+1}) + h^2\delta(\sigma_{n+1}, \sigma_{n+2}) + \cdots + h^{m-n}\delta(\sigma_{m-1}, \sigma_m)$$

using (10), we get

$$\begin{aligned}
\delta(\sigma_n, \sigma_m) &\leq h\varphi^n(\delta(\sigma_0, \sigma_1)) + h^2\varphi^{n+1}(\delta(\sigma_0, \sigma_1)) + \cdots + h^{m-n}\varphi^{m-1}(\delta(\sigma_0, \sigma_1)) \\
&= h^{n-n+1}\varphi^n(\delta(\sigma_0, \sigma_1)) + h^{n-n+2}\varphi^{n+1}(\delta(\sigma_0, \sigma_1)) + \cdots + h^{m-n+1-1}\varphi^{m-1}(\delta(\sigma_0, \sigma_1)) \\
&= \frac{1}{h^{n-1}}[h^n\varphi^n(\delta(\sigma_0, \sigma_1)) + h^{n+1}\varphi^{n+1}(\delta(\sigma_0, \sigma_1)) + \cdots + h^{m-1}\varphi^{m-1}(\delta(\sigma_0, \sigma_1))] \\
&= \frac{1}{h^{n-1}}\sum_{i=n}^{m-1} h^i\varphi^i(\delta(\sigma_0, \sigma_1)) \\
&\leq \frac{1}{h^{n-1}}\sum_{i=0}^{+\infty} h^i\varphi^i(\delta(\sigma_0, \sigma_1))
\end{aligned}$$

thus

$$\delta(\sigma_n, \sigma_m) \leq \frac{1}{h^{n-1}}\sum_{i=0}^{+\infty} h^i\varphi^i(\delta(\sigma_0, \sigma_1)) \tag{11}$$

Since φ is a b-CF, it follows that the series $\sum_{i=0}^{+\infty} h^i\varphi^i(\delta(\sigma_0, \sigma_1))$ is convergent. Setting,

$$S_k = \sum_{i=1}^{k} h^i\varphi^i(\delta(\sigma_0, \sigma_1))$$

then (11) becomes

$$\delta(\sigma_n, \sigma_m) \leq \frac{1}{h^{n-1}}(S_{m-1} - S_{n-1}) \tag{12}$$

applying limit as $n, m \to +\infty$ in (12) we attain $\delta(\sigma_n, \sigma_m) \to 0$, this indicates that $\{\sigma_n\}_{n\geq 1}$ is a Cauchy sequence in X. Completeness of X demonstrates that there is $\varsigma \in X$, so that

$$\lim_{n\to+\infty} \delta(\sigma_n, \varsigma) = 0.$$

Now we show that $\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}$ by using TI in X,

$$\begin{aligned}
\delta(\varsigma, [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) &\leq h(\delta(\varsigma, \sigma_{2n+1})) + \delta(\sigma_{2n+1}, [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) \\
&= h(\delta(\varsigma, \sigma_{2n+1})) + h(\delta(\sigma_{2n+1})).
\end{aligned}$$

using $\delta(\sigma, \mathcal{B}) \leq \aleph(\mathcal{A}, \mathcal{B})$ for $\sigma \in \mathcal{A}$,

$$\delta(\varsigma, [\mathcal{T}\varsigma]_{(\ltimes(\varsigma)+\beta(\varsigma))}) \leq h\delta(\varsigma, \sigma_{2n+1}) + h\aleph([S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))}, [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) \tag{13}$$

Since $(S, \mathcal{T})$ pair is $\aleph$-continuous, by applying limit as $n \to +\infty$ in (13), we get

$\delta(\varsigma, [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) = 0$ this indicate that $\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}$

Analogously it can be shown that $\varsigma \in [S\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}$.
Thus

$$\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))} \cap [S\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}$$

that is, ς is a common IFFP of mappings S and $\mathcal{T}$.
Case 2: $r = 0$
For $r = 0$, taking inequality (6) with condition (i), we have

$$\begin{aligned}
M^r_{(S,\mathcal{T})}(\sigma_{2n}, \sigma_{2n+1}) &= \mathcal{B}(\sigma_{2n}, \sigma_{2n+1}) \\
&= (\delta(\sigma_{2n}, \sigma_{2n+1}))^{k_1} \times (\delta(\sigma_{2n}, [S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))}))^{k_2} \\
&\times (\delta(\sigma_{2n+1}, [\mathcal{T}\sigma_{2n+1}]_{(\ltimes(\sigma_{2n+1}),\beta(\sigma_{2n+1}))}))^{k_3} \\
&\times \left(\frac{\delta(\sigma_{2n+1}, [\mathcal{T}\sigma_{2n+1}]_{(\ltimes(\sigma_{2n+1}),\beta(\sigma_{2n+1}))})(1 + \delta(\sigma_{2n}, [S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))}))}{1 + \delta(\sigma_{2n}, \sigma_{2n+1})}\right)^{k_4} \\
&\times \left(\frac{\delta(\sigma_{2n}, [\mathcal{T}\sigma_{2n+1}]_{(\ltimes(\sigma_{2n+1}),\beta(\sigma_{2n+1}))}) + \delta(\sigma_{2n+1}, [S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))})}{2h}\right)^{k_5} \\
&= (\delta(\sigma_{2n}, \sigma_{2n+1}))^{k_1} \times (\delta(\sigma_{2n}, \sigma_{2n+1}))^{k_2} \times (\delta(\sigma_{2n+1}, \sigma_{2n+2}))^{k_3} \\
&\times \left(\frac{\delta(\sigma_{2n}, \sigma_{2n+2})(1 + \delta(\sigma_{2n}, \sigma_{2n+1}))}{1 + \delta(\sigma_{2n}, \sigma_{2n+1})}\right)^{k_4} \\
&\times \left(\frac{\delta(\sigma_{2n}, \sigma_{2n+2}) + \delta(\sigma_{2n+1}, \sigma_{2n+1})}{2h}\right)^{k_5} \\
&\le (\delta(\sigma_{2n}, \sigma_{2n+1}))^{k_1+k_2} \times (\delta(\sigma_{2n+1}, \sigma_{2n+2})^{k_3+k_4}) \\
&\times \left(\frac{h(\delta(\sigma_{2n}, \sigma_{2n+1}) + \delta(\sigma_{2n+1}, \sigma_{2n+2}))}{2h}\right)^{k_5} \\
&= (\delta(\sigma_{2n}, \sigma_{2n+1}))^{k_1+k_2} \times (\delta(\sigma_{2n+1}, \sigma_{2n+2}))^{k_3+k_4} \\
&\times \left(\frac{\delta(\sigma_{2n}, \sigma_{2n+1}) + \delta(\sigma_{2n+1}, \sigma_{2n+2})}{2}\right)^{k_5} \qquad (14)
\end{aligned}$$

It is widely acknowledged that, for any $p, q, l > 0$ we have

$$\left(\frac{p+q}{2}\right)^l \le \frac{p^l + q^l}{2} \qquad (15)$$

applying (15) to (14), we get

$$\begin{aligned}
M^r_{(S,\mathcal{T})}(\sigma_{2n}, \sigma_{2n+1}) \quad \le \quad & (\delta(\sigma_{2n}, \sigma_{2n+1}))^{k_1+k_2} \times (\delta(\sigma_{2n+1}, \sigma_{2n+2}))^{k_3+k_4} \\
& \times \left(\frac{(\delta(\sigma_{2n}, \sigma_{2n+1}))^{k_5}}{2} + \frac{(\delta(\sigma_{2n+1}, \sigma_{2n+2}))^{k_5}}{2}\right) \qquad (16)
\end{aligned}$$

using (16) in (6), we get

$$\begin{aligned}\delta(\sigma_{2n+1},\sigma_{2n+2}) &\leq \varphi(\delta(\sigma_{2n},\sigma_{2n+1}))^{k_1+k_2}\times(\delta(\sigma_{2n+1},\sigma_{2n+2}))^{k_3+k_4}\\ &\quad\times\left(\frac{(\delta(\sigma_{2n},\sigma_{2n+1}))^{k_5}}{2}+\frac{(\delta(\sigma_{2n+1},\sigma_{2n+2}))^{k_5}}{2}\right) \quad (17)\end{aligned}$$

Suppose $\delta(\sigma_{2n},\sigma_{2n+1})\leq\delta(\sigma_{2n+1},\sigma_{2n+2})$ and noting that φ is non-decreasing, (17) gives:

$$\begin{aligned}\delta(\sigma_{2n+1},\sigma_{2n+2}) &\leq \varphi((\delta(\sigma_{2n+1},\sigma_{2n+2}))^{k_1+k_2+k_3+k_4+k_5})\\ &= \varphi(\delta(\sigma_{2n+1},\sigma_{2n+2}))\\ &< \delta(\sigma_{2n+1},\sigma_{2n+2})\end{aligned}$$

a contradiction. So we have

$$\delta(\sigma_{2n+1},\sigma_{2n+2})\leq\delta(\sigma_{2n},\sigma_{2n+1})$$

therefore (17) becomes

$$\begin{aligned}\delta(\sigma_{2n+1},\sigma_{2n+2}) &\leq \varphi(\delta(\sigma_{2n},\sigma_{2n+1}))\\ &\leq \varphi^2(\delta(\sigma_{2n-1},\sigma_{2n}))\\ &\vdots\\ &\leq \varphi^{2n+1}(\delta(\sigma_0,\sigma_1))\end{aligned}$$

$$\delta(\sigma_{2n+1},\sigma_{2n+2})\leq\varphi^{2n+1}(\delta(\sigma_0,\sigma_1)).$$

Similarly, using (7) we can show that

$$\delta(\sigma_{2n+2},\sigma_{2n+3})\leq\varphi^{2n+2}(\delta(\sigma_0,\sigma_1))$$

combining above two equations, we can write

$$\delta(\sigma_n,\sigma_{n+1})\leq\varphi^n(\delta(\sigma_0,\sigma_1)). \quad (18)$$

Adhering to the same process as in case 1, this can be deduced from (18) that $\{\sigma_n\}_{n\geq1}$ is a Cauchy sequence in X. Completeness of X reveals that there is an element $\varsigma\in X$ such as,

$$\lim_{n\to+\infty}\delta(\sigma_n,\varsigma)=0. \quad (19)$$

Now, to illustrate that $\varsigma\in[\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}$, consider

$$\begin{aligned}\delta(\varsigma,[\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) &\leq h(\delta(\varsigma,\sigma_{2n+1})+\delta(\sigma_{2n+1},[\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}))\\ &\leq h(\delta(\varsigma,\sigma_{2n+1})+\aleph([S\sigma_{2n}]_{(\ltimes(\sigma_{2n}),\beta(\sigma_{2n}))},[\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))})) \quad (20)\end{aligned}$$

using $\aleph$-continuity of $(S,\mathcal{T})$ pair, by allowing $n\to+\infty$ in (20) and considering (19), we attain

$\delta(\varsigma, [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) = 0$ which implies that $\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}$.
Correspondingly we can prove that $\delta(\varsigma, [S\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}) = 0$ which means that $\varsigma \in [S\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}$.
Thus we have

$$\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))} \cap [S\varsigma]_{(\ltimes(\varsigma),\beta(\varsigma))}$$

that is, ς is common IFFP of S and $\mathcal{T}$.

Example 26. *Let $X = [1, +\infty)$ and $\delta(\sigma, \tau) = |\sigma - \tau|^2$ for all $\sigma, \tau \in X$. Then $(X, \delta, h = 2)$ is a complete b-MS. For each $\sigma \in X$, consider IFS-valued maps $S, \mathcal{T} : X \to IFS(X)$ and $S\sigma, \mathcal{T}\sigma$ are IFSs such that $\mu_{\mathcal{T}\sigma} : X \to [0, 1]$, $\mu_{S\sigma} : X \to [0, 1]$ are membership functions of $\mathcal{T}\sigma$ and $S\sigma$ respectively and $\nu_{\mathcal{T}\sigma} : X \to [0, 1]$ and $\nu_{S\sigma} : X \to [0, 1]$ are non-membership functions of $\mathcal{T}\sigma$ and $S\sigma$ respectively with $\mu_{\mathcal{T}\sigma}(\mathfrak{t}) + \nu_{\mathcal{T}\sigma}(\mathfrak{t}) \leq 1$ and $\mu_{S\sigma}(\mathfrak{t}) + \nu_{S\sigma}(\mathfrak{t}) \leq 1$ for all $\mathfrak{t} \in X$.*
If $\sigma = 1$:

$$\mu_{T_1}(\mathfrak{t}) = \begin{cases} \frac{1}{8}, & \text{if } \mathfrak{t} = 1 \\ \frac{2}{5}, & \text{if } \mathfrak{t} \neq 1 \end{cases}$$

$$\nu_{T_1}(\mathfrak{t}) = \begin{cases} \frac{3}{7}, & \text{if } \mathfrak{t} = 1 \\ \frac{4}{7}, & \text{if } \mathfrak{t} \neq 1 \end{cases}$$

$$\mu_{S_1}(\mathfrak{t}) = \begin{cases} 1, & \text{if } \mathfrak{t} = 1 \\ \frac{1}{5}, & \text{if } \mathfrak{t} \neq 1 \end{cases}$$

$$\nu_{S_1}(\mathfrak{t}) = \begin{cases} 0, & \text{if } \mathfrak{t} = 1 \\ \frac{4}{9}, & \text{if } \mathfrak{t} \neq 1. \end{cases}$$

If $\sigma \neq 1$;

$$\mu_{\mathcal{T}\sigma}(\mathfrak{t}) = \begin{cases} \ltimes, & \text{if } 1 \leq \mathfrak{t} \leq 3\sigma \\ 1 - \frac{2}{3}\ltimes, & \text{if } 3\sigma < \mathfrak{t} \leq 5\sigma \\ \frac{\ltimes}{6}, & \text{if } 5\sigma < \mathfrak{t} < +\infty \end{cases}$$

$$\nu_{\mathcal{T}\sigma}(\mathfrak{t}) = \begin{cases} \frac{\beta}{4}, & \text{if } 1 \leq \mathfrak{t} \leq 2\sigma \\ 0, & \text{if } 2\sigma < \mathfrak{t} \leq 5\sigma \\ \beta, & \text{if } 5\sigma < \mathfrak{t} < +\infty \end{cases}$$

$$\mu_{S\sigma}(\mathfrak{t}) = \begin{cases} \ltimes, & \text{if } 1 \leq \mathfrak{t} \leq 5\sigma \\ \frac{\ltimes}{3}, & \text{if } 5\sigma < \mathfrak{t} \leq 9\sigma \\ \frac{2\ltimes}{19}, & \text{if } 9\sigma < \mathfrak{t} < +\infty \end{cases}$$

$$\nu_{S\sigma}(\mathfrak{t}) = \begin{cases} \beta^2, & if \;\; 1 \leq \mathfrak{t} \leq 7\sigma \\ \frac{\beta^3}{5}, & if \;\; 7\sigma < \mathfrak{t} \leq 11\sigma \\ \beta, & if \;\; 11\sigma < \mathfrak{t} < +\infty. \end{cases}$$

Let $\ltimes = \frac{3}{8}$ *and* $\beta = \frac{1}{2}$. *Then*

$$[\mathcal{T}\sigma]_{(\ltimes,\beta)} = \begin{cases} \{1\}, & if \;\; \sigma = 1 \\ [1, 5\sigma], & if \; \sigma \neq 1 \end{cases}$$

and

$$[S\sigma]_{(\ltimes,\beta)} = \begin{cases} \{1\}, & if \;\; \sigma = 1 \\ [1, 5\sigma], & if \; \sigma \neq 1. \end{cases}$$

Clearly $S\sigma, \mathcal{T}\sigma \in IFS(X)$ *for each* $\sigma \in X$. *Define the functions* $\gamma : X \times X \to \mathbb{R}_+$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *by*

$$\gamma(\sigma, \tau) = \begin{cases} 6, & if \;\; \sigma = \tau = 1 \\ \frac{1}{430}, & if \; \sigma, \tau \in \{4, 5\} \\ 0, & elsewhere. \end{cases}$$

and $\varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{4}$ *for all* $t > 0$. *Let* $\wp(\mathfrak{a}, \mathfrak{b}) = \frac{1}{2}\mathfrak{b} - \mathfrak{a}$ *for all* $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$. *Obviously* $\wp \in \mathcal{Z}$ *and* $\varphi \in \Lambda_b$
Now we verify conditions

$$\wp(\gamma(\sigma, \tau)\aleph([S\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(S,\mathcal{T})}(\sigma, \tau))) \geq 0$$

and

$$\wp(\gamma(\sigma, \tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [S\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(\mathcal{T},S)}(\sigma, \tau))) \geq 0$$

for $r > 0$ *under the these cases;*

Case 1

if $\sigma = \tau = 1$, *Then* $[S\sigma]_{(\ltimes,\beta)} = \{1\} = [\mathcal{T}\tau]_{(\ltimes,\beta)}$, *this implies that*

$$\aleph([S\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) = 0$$

so,

$$\wp(6(0), \varphi(M^r_{(S,\mathcal{T})}(\sigma, \tau))) \geq 0.$$

Similarly

$$\wp(\gamma(\sigma, \tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [S\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(\mathcal{T},S)}(\sigma, \tau))) \geq 0.$$

Case 2

if $\sigma, \tau \in \{4, 5\}$ *such that* $\sigma \neq \tau$, *let* $\sigma = 4$ *and* $\tau = 5$. *Then* $[S\sigma]_{(\ltimes,\beta)} = [1, 20]$ *and* $[\mathcal{T}\tau]_{(\ltimes,\beta)} = [1, 25]$.

$$\aleph([S\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) = \aleph([1, 20], [1, 25])$$

$$= \delta(20,25) = 25$$

$$\begin{aligned} M^r_{(S,\mathcal{T})}(4,5) &= [\mathcal{A}(4,5)]^{\frac{1}{r}} \\ &= \Bigg[k_1(\delta(4,5))^r + k_2(\delta(4,[S4]_{(\ltimes,\beta)}))^r + k_3(\delta(5,[\mathcal{T}5]_{(\ltimes,\beta)}))^r \\ &\quad +k_4\left(\frac{\delta(5,[\mathcal{T}5]_{(\ltimes,\beta)})(1+\delta(4,[S4]_{(\ltimes,\beta)}))}{1+\delta(4,5)}\right)^r \\ &\quad +k_5\left(\frac{\delta(5,[S4]_{(\ltimes,\beta)})(1+\delta(4,[\mathcal{T}5]_{(\ltimes,\beta)}))}{1+\delta(4,5)}\right)^r\Bigg]^{\frac{1}{r}} \end{aligned}$$

taking $k_1 = k_2 = \frac{1}{2}$ *and* $k_3 = k_4 = k_5 = 0$,

$$\begin{aligned} M^r_{(S,\mathcal{T})}(4,5) &= [\frac{1}{2}(1)^r + \frac{1}{2}(0)^r]^{\frac{1}{r}} \\ &= \left[\frac{1}{2}\right]^{\frac{1}{r}} \to 1 \text{ as } r \to +\infty \end{aligned}$$

Therefore,

$$\wp\left(\frac{1}{430}(25), \varphi(1)\right) = \frac{1}{2}\left(\frac{1}{4}\right) - \frac{25}{430} \geq 0$$

Similarly,

$$\wp(\gamma(\sigma,\tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [S\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(\mathcal{T},S)}(\sigma,\tau))) \geq 0$$

Case 3

If $\sigma, \tau \in X - \{1,4,5\}$. *Then* $\gamma(\sigma,\tau) = 0$

$$\wp(0, \varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau)) = \frac{1}{2}\varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau))) \geq 0$$

Similarly,

$$\wp(0, \varphi(M^r_{(\mathcal{T},S)}(\sigma,\tau))) \geq 0.$$

Moreover, it is clear that the pair $(S,\mathcal{T})$ *is* γ*-admissible,* $\aleph$*-continuous and* $[S\sigma]_{(\ltimes,\beta)}$, $[\mathcal{T}\sigma]_{(\ltimes,\beta)}$ *are proximal for each* $\sigma \in X$. *As the conditions outlined in Theorem 2 are fulfilled, we see that* S *and* $\mathcal{T}$ *have many common IFFPs.*

Here is another example(non-trivial) for our main result,

Example 27. *Let* $X = l_p(\mathbb{R})$ *with* $0 < p < 1$, *where* $l_p(\mathbb{R}) = \{\sigma = \{\sigma_n\} \subset \mathbb{R} : \sum^{+}\infty_{n=1}|\sigma_n|^p < +\infty\}$. *Then* $\delta(\sigma,\tau) = (\sum_{n=1}^{+\infty}|\sigma_n - y_n|^p)^{\frac{1}{p}}$ *is a complete b-MS on* X *with* $h = 2^{\frac{1}{p}}$, *consider IFS-valued maps* $S, \mathcal{T} : X \to IFS(X)$ *and* $S\sigma, \mathcal{T}\sigma$ *are IFSs such that* $\mu_{\mathcal{T}\sigma} : X \to [0,1]$, $\mu_{S\sigma} : X \to [0,1]$ *are membership functions of* $\mathcal{T}\sigma$ *and* $S\sigma$

respectively and $\nu_{\mathcal{T}\sigma} : X \to [0,1]$ *and* $\nu_{S\sigma} : X \to [0,1]$ *are non-membership functions of* $\mathcal{T}\sigma$ *and* $S\sigma$ *respectively with* $\mu_{\mathcal{T}\sigma}(\mathfrak{t})+\nu_{\mathcal{T}\sigma}(\mathfrak{t}) \leq 1$ *and* $\mu_{S\sigma}(\mathfrak{t})+\nu_{S\sigma}(\mathfrak{t}) \leq 1$ *for all* $\mathfrak{t} \in X$.

$$\mu_{T_\sigma}(\mathfrak{t}) = \begin{cases} \frac{1}{8}, & if\ \ t\ contains\ finite\ number\ of\ zero \\ \frac{9}{10}, & if\ \ t\ contains\ infinite\ number\ of\ zero \\ \frac{1}{100}, & if\ \ t\ contains\ no\ zero \end{cases}$$

$$\nu_{T_\sigma}(\mathfrak{t}) = \begin{cases} \frac{1}{5}, & if\ \ t\ contains\ finite\ number\ of\ zero \\ \frac{1}{120}, & if\ \ t\ contains\ infinite\ number\ of\ zero \\ \frac{1}{7}, & if\ \ t\ contains\ no\ zero \end{cases}$$

$$\mu_{S_\sigma}(\mathfrak{t}) = \begin{cases} \frac{1}{50}, & if\ \ t\ contains\ no\ zero \\ \frac{2}{5}, & if\ \ t\ contains\ finite\ number\ of\ zero \\ \frac{7}{10}, & if\ \ t\ contains\ infinite\ number\ of\ zero \end{cases}$$

$$\nu_{S_\sigma}(\mathfrak{t}) = \begin{cases} \frac{1}{9}, & if\ \ t\ contains\ no\ zero \\ \frac{3}{7}, & if\ \ t\ contains\ finite\ number\ of\ zero \\ \frac{2}{15}, & if\ \ t\ contains\ infinite\ number\ of\ zero \end{cases}$$

Let $\ltimes = \frac{1}{2}$ *and* $\beta = \frac{1}{6}$, *then*
$[T_\sigma]_{(\ltimes,\beta)} = \{t\ contains\ infinitely\ many\ zeros\}$

and

$[S_\sigma]_{(\ltimes,\beta)} = \{t\ contains\ infinitely\ many\ zeros\}$ *Clearly* $S\sigma, \mathcal{T}\sigma \in IFS(X)$ *for each* $\sigma \in X$. *Define the functions* $\gamma : X \times X \to \mathbb{R}_+$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *by*

$$\gamma(\sigma,\tau) = \begin{cases} 7, & if\ \ \sigma = y \\ 10, & elsewhere. \end{cases}$$

and $\varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{4}$ *for all* $t > 0$. *Let* $\wp(\mathfrak{a},\mathfrak{b}) = \frac{\mathfrak{b}-\mathfrak{a}}{10}$ *for all* $\mathfrak{a},\mathfrak{b} \in \mathbb{R}_+$. *Obviously* $\wp \in \mathcal{Z}$ *and* $\varphi \in \Lambda_b$
. *Also,*

$$\aleph([S_\sigma]_{(\ltimes,\beta)}, [T_\sigma]_{(\ltimes,\beta)} = \max(\sum_{i=1}^{\infty} |\sigma_i - y_i|^p)^{\frac{1}{p}}$$

Clearly,

$$\wp(\gamma(\sigma,\tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [S\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(\mathcal{T},S)}(\sigma,\tau))) \geq 0$$

As it can be seen that the pair (S,T) *is* γ*-admissible,* $\aleph$*-continuous and* $[S\sigma]_{(\ltimes,\beta)}$, $[\mathcal{T}\sigma]_{(\ltimes,\beta)}$ *are proximal for each* $\sigma \in X$. *Since the conditions in Theorem 2 are fulfilled, we see that* S *and* $\mathcal{T}$ *have many common IFFPs.*

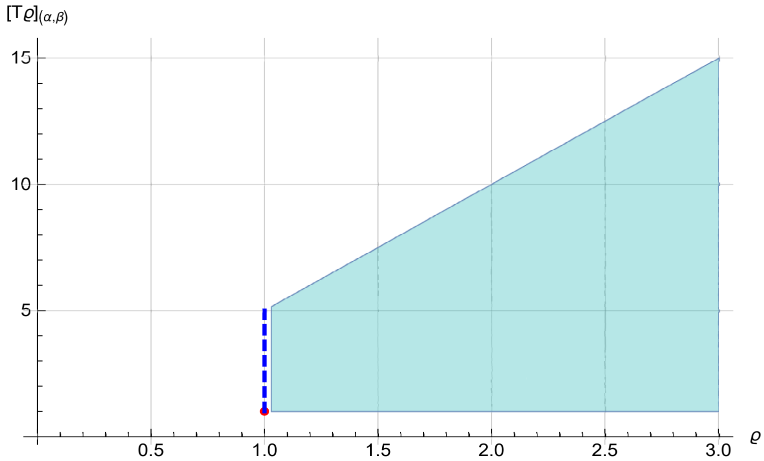

Figure 3: Graph representing common fixed points of the given mappings in example 25.

4. Consequences

In this section, we demonstrate how our primary theorem can be used to obtain a few intriguing fixed point conclusions, particularly when using different simulation function variations. All of the results reported here are also new as far as we can tell.

Corollary 1. *Let (X, δ, h) be a complete b-MS and $\mathcal{T}$ be an AHIF $\mathcal{Z}$-contraction regarding $\wp \in \mathcal{Z}$. Let's also consider that:*

(i) $\mathcal{T}$ *is a* γ*-admissible IFS-valued map;*

(ii) there are $\sigma_0 \in X$ *and* $\sigma_1 \in [\mathcal{T}\sigma_0]_{(\ltimes,\beta)}$ *such that* $\gamma(\sigma_0, \sigma_1) \geq 1$*, where* $(\ltimes, \beta) \in (0, 1] \times [0, 1)$*;*

(iii) $\mathcal{T}$ *is* $\aleph$*-continuous;*

(iv) $[\mathcal{T}\sigma]_{(\ltimes,\beta)}$ *is proximal for every* $\sigma \in X$*.*

Then, $\mathcal{T}$ *has at least one IFFP in* X*.*

Proof. If we take the mapping $\mathcal{T}=S$ in Theorem 2, then it can easily be seen that the AHIF $\mathcal{Z}$-contraction $\mathcal{T}$ has many IFFPs in X.

Corollary 2. *Let (X, δ, h) be a complete b-MS and $\mathcal{T}$ be an IFS-valued map satisfying:*

(i) $\mathcal{T}$ *is* γ*-admissible IFS-valued map;*

(ii) there exists $\sigma_0 \in X$ *and* $\sigma \in [\mathcal{T}\sigma_0]_{(\ltimes,\beta)}$ *such that* $\gamma(\sigma_0, \sigma_1) \geq 1$ *where* $(\ltimes, \beta) \in (0, 1] \times [0, 1)$*;*

(iii) $\mathcal{T}$ *is* $\aleph$*-continuous;*

(iv) $[\mathcal{T}\sigma]_{(\ltimes,\beta)}$ *is proximal for each* $\sigma \in X$*.*

Additionally, assume that there exists $\wp \in \mathcal{Z}$, $\varphi \in \Lambda_b$ and an operator $\gamma : X \times X \to \mathbb{R}_+$ in a way that for all $\sigma, \tau \in X$,

$$\wp(\gamma(\sigma,\tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}), \varphi([\mathcal{A}(\sigma,\tau)]^{\frac{1}{r}})) \geq 0 \tag{21}$$

Then, $\mathcal{T}$ has at least one IFFP in X.

Proof. Set $\rho(\tau,\kappa) = \phi(\kappa) - \tau$ for all $\tau, \kappa \in \mathbb{R}_+$ in Theorem 25. Then, 21 follows easily. Note that $\phi(\kappa) - \tau \in \mathcal{Z}$. Consequently Theorem 25 can be applied to find $u \in X$ such that $u \in [Tu]_{(\ltimes,\beta)}$.

Corollary 3. *Let (X, δ) be a complete b-MS and $\mathcal{T} : X \to IFS(X)$ be an IFS-valued map satisfying the condition:*

$$\gamma(\sigma,\tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) \leq \varphi(M^r_{\mathcal{T}}(\sigma,\tau)) \tag{22}$$

for all $\sigma, \tau \in X$, where $\varphi \in \Lambda_b$ and $\gamma : X \times X \to \mathbb{R}_+$ is a function. Furthermore, it can be assumed that:

(i) $\mathcal{T}$ is γ-admissible;

(ii) there are $\sigma_0 \in X$ and $\sigma_1 \in [\mathcal{T}\sigma_0]_{(\ltimes,\beta)}$ with $\gamma(\sigma_0, \sigma_1) \geq 1$

(iii) $\mathcal{T}$ is $\aleph$-continuous;

(iv) $[\mathcal{T}\sigma]_{(\ltimes,\beta)}$ is proximal for each $\sigma \in X$;

Then there is $\varsigma \in X$ such that $\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes,\beta)}$.

Proof. Take $\wp := \wp(\mathfrak{a}, \mathfrak{b}) = \varphi(\mathfrak{b}) - \mathfrak{a}$ for each $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ in corollary (1). Then (22) follows easily. Notice that $\varphi(\mathfrak{b}) - \mathfrak{a} \in \mathcal{Z}$. Accordingly, by utilizing corollary (1), $\varsigma \in X$ can be determined such that $\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes,\beta)}$.

The ensuing generalization and fuzzification of Rhoades' outcome [23] are as follows:

Corollary 4. *Let (X, δ, h) be a complete b-MS and $\mathcal{T} : X \to IFS(X)$ be an IFS-valued map satisfying the following:*

$$\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) \leq \varphi(M^r_{\mathcal{T}}(\sigma,\tau)) - \varphi^2(M^r_{\mathcal{T}}(\sigma,\tau)) \tag{23}$$

for all $\sigma, \tau \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is lower semicontinuous and $\varphi^{-1}(0) = \{0\}$. Further let us suppose that:

(i) $\mathcal{T}$ is $\aleph$-continuous;

(ii) $[\mathcal{T}\sigma]_{(\ltimes,\beta)}$ is proximal for each $\sigma \in X$.

Then, there exists $\varsigma \in X$ such that $\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes,\beta)}$.

Proof. In corollary (1), take $\wp := \wp(\mathfrak{a},\mathfrak{b}) = \mathfrak{b} - \varphi(\mathfrak{b}) - \mathfrak{a}$ for all $\mathfrak{a},\mathfrak{b} \in \mathbb{R}_+$ and $\gamma(\sigma,\tau) = 1$ for all $\sigma,\tau \in X$. Then (23) is attained. Observe that $\mathfrak{b} - \varphi(\mathfrak{b}) - \mathfrak{a} \in \mathcal{Z}$. Thus, by corollary (1), $\mathcal{T}$ has an IFFP in X.

Corollary 5. *Nadler's type [16]) Let (X,δ,h) be a complete b-MS and $\mathcal{T} : X \to IFS(X)$ be an IFS-valued map satisfying:*

$$\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) \leq \lambda\delta(\sigma,\tau) \tag{24}$$

for all $\sigma,\tau \in X$, where $\lambda \in (0,1)$. Assume also that:

(i) $\mathcal{T}$ is $\aleph$-continuous;

(ii) $[\mathcal{T}\sigma]_{(\ltimes,\beta)}$ is proximal for each $\sigma \in X$.

Then $\mathcal{T}$ has an IFFP in X.

Proof. Consider $\gamma(\sigma,\tau) = 1$, $\wp := \wp(\mathfrak{a},\mathfrak{b}) = \lambda\mathfrak{b} - \mathfrak{a}$ for all $\mathfrak{a},\mathfrak{b} \in \mathbb{R}_+$ and insert $\varphi(\mathfrak{a}) = \lambda\mathfrak{a}$ for all $\mathfrak{a} \geq 0$, with $\lambda \in (0,1)$ in corollary (1). Then (24) is achievable. It can be obsereved that $\lambda\mathfrak{b} - \mathfrak{a} \in \mathcal{Z}$. Corollary (1) yields that there is $\varsigma \in X$ such that $\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes,\beta)}$.

The subsequent corollary builds upon Heilpern's initial metric FP theorem [15]:

Corollary 6. *Let (X,δ,h) ba e complete b-MS and $\mathcal{T} : X \to W(X)$ be an IFS-valued map satisfying:*

$$\delta_+\infty(\mathcal{T}\sigma,\mathcal{T}\tau) \leq \lambda\delta(\sigma,\tau) \tag{25}$$

for all $\sigma,\tau \in X$, where $\lambda \in (0,1)$. In addition, suppose that:

(i) $\mathcal{T}$ is $\aleph$-continuous;

(ii) $\mathcal{T}\sigma$ is proximal for each $\sigma \in X$.

Then, there is $\varsigma \in X$ such that $\{\varsigma\} \in X$ such as $\{\varsigma\} \subset \mathcal{T}\varsigma$.

Proof. Since $\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) \leq \delta_{+\infty}(\mathcal{T}\sigma,\mathcal{T}\tau)$ for all $\sigma,\tau \in X$, by employing Corollary 5, we can identify $\varsigma \in X$ so that $\{\varsigma\} \subset \mathcal{T}\varsigma = [\mathcal{T}\varsigma]_{(\ltimes,\beta)} \in W(X)$.

The definition of single-valued γ-admissible map proposed by Samet in [25] is provided here:

Definition 28. *Let $O : X \to X$ and $\gamma : X \times X \to \mathbb{R}_+$ be mappings. Then, O is named as γ-admissible if for all $x,\tau \in X$,*

$$\gamma(\sigma,\tau) \geq 1 \;\Rightarrow\; \gamma(O\sigma, O\tau) \geq 1. \tag{26}$$

Chifu and Karapinar came up with this corollary without relying on triangular γ-orbital admissibility of O:

Corollary 7. *Let (X, δ, h) be a complete b-MS and $O : X \to X$ be a γ-admissible single-valued mapping fulfilling:*

$$\wp(\gamma(\sigma,\tau)\aleph(O\sigma, O\tau), \varphi(M_O^r(\sigma,\tau)) \geq 0 \tag{27}$$

for all $\sigma, \tau \in X$, where $\varphi \in \Lambda_b$,

$$M_O^r(\sigma,\tau) = \begin{cases} [W(\sigma,\tau)]^{\frac{1}{r}}, & \text{for } r > 0,\ \sigma,\tau \in X \\ D(\sigma,\tau), & \text{for } r = 0,\ \sigma,\tau \in X, \end{cases}$$

$$\begin{aligned} W(\sigma,\tau) &= k_1(\delta(\sigma,\tau))^r + k_2(\delta(\sigma,O\sigma))^r + k_3(\delta(\tau,O\tau))^r \\ &+k_4\left(\frac{\delta(\tau,O\tau)(1+\delta(\sigma,O\sigma))}{1+\delta(\sigma,\tau)}\right)^r + k_5\left(\frac{\delta(\tau,O\sigma)(1+\delta(\sigma,O\tau))}{1+\delta(\sigma,\tau)}\right)^r \end{aligned}$$

and

$$\begin{aligned} D(\sigma,\tau) &= (\delta(\sigma,\tau))^{k_1} \times (\delta(\sigma,O\sigma))^{k_2} \times (\delta(\tau,O\tau))^{k_3} \\ &\times\left(\frac{\delta(\tau,O\tau)(1+\delta(\sigma,O\sigma))}{1+\delta(\sigma,\tau)}\right)^{k_4} \times \left(\frac{\delta(\sigma,O\tau)+\delta(\tau,O\sigma)}{2h}\right)^{k_5} \end{aligned}$$

with $r \geq 0$ and $k_i \geq 0$ $(i = \overline{1,5})$ such that $\sum_{i=1}^{5} k_i = 1$. Then, there exists $\varsigma \in X$ such that $O\varsigma = \varsigma$.

Proof. Let $(\ltimes, \beta) \in (0,1] \times [0,1)$ and for each $\sigma \in X$, consider an IFS-valued map $\mathcal{T}$ and for $\sigma \in X$, $\mathcal{T}\sigma$ is an IFS such that $\mu_{\mathcal{T}\sigma} : X \to [0,1]$ is membership function and $\nu_{\mathcal{T}\sigma} : X \to [0,1]$ is nonmembership function. We define these maps as

$$\mu_{\mathcal{T}\sigma}(\mathfrak{a}) = \begin{cases} \ltimes, & \text{if } \mathfrak{a} = O\sigma \\ 0, & \text{if } \mathfrak{a} \neq O\sigma, \end{cases}$$

and

$$\nu_{\mathcal{T}\sigma}(\mathfrak{a}) = \begin{cases} 0, & \text{if } \mathfrak{a} = O\sigma \\ \beta, & \text{if } \mathfrak{a} \neq O\sigma, \end{cases}$$

Then,

$$[\mathcal{T}x]_{(\ltimes,\beta)} = \{\mathfrak{a} \in X : \mu_{\mathcal{T}\sigma}(\mathfrak{a}) \geq \ltimes \text{ and } \nu_{\mathcal{T}\sigma}(\mathfrak{a}) \leq \beta\}.$$

Obviously, $\{O\sigma\} \in \mathcal{Q}(X)$. Notice that in this case, $\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) = \delta(O\sigma, O\tau)$. Therefore, corollary (1) can be applied to obtain $\varsigma \in X$ such that $\varsigma \in [\mathcal{T}\varsigma]_{(\ltimes,\beta)} = \{O\varsigma\}$, which further implies that $\varsigma = O\varsigma$.

5. Application

One of the most helpful subfields of fixed theory, FP theory in partially ordered sets has many applications, including solving matrix equations and solving boundary value issues. See [3, 20, 21] for a few articles that go in this direction. Here, we will utilize our principal discovery to derive its counterpart within the framework of ordered b-MS. Importantly, a b-MS can be equipped with a partial ordering. To elaborate, if $(X, \preceq)$ is a partially ordered set, then $(X, \delta, h, \preceq)$ is recognized as an ordered b-MS. As a result,we define, $\sigma, \tau \in X$ are comparable if either $\sigma \preceq \tau$ or $\tau \preceq \sigma$ is fulfilled. Consider $\pounds, \Re \subseteq X$, then $\pounds \preceq \Re$ if for all $\mathfrak{l} \in \pounds$, there is $\mathfrak{r} \in \Re$ with $\mathfrak{l} \preceq \mathfrak{r}$.

Theorem 29. *Let $(X, \delta, h, \preceq)$ be a complete ordered b-MS and $S, \mathcal{T}$ are IFS-valued maps. Consider that there is $\wp \in \mathcal{Z}$, $\varphi \in \Lambda_b$ and an operator $\gamma : X \times X \to \mathbb{R}_+$ such that*

$$\wp(\gamma(\sigma,\tau))\aleph([S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}, [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}), \varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau)) \geq 0 \tag{28}$$

for all $\sigma, \tau \in X$ with $[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))} \preceq [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}$ and

$$\wp(\gamma(\sigma,\tau))\aleph([\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}, [S\tau]_{(\ltimes(\tau),\beta(\tau))}), \varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau)) \geq 0 \tag{29}$$

for all $\sigma, \tau \in X$ with $[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))} \preceq [S\tau]_{(\ltimes(\tau),\beta(\tau))}$. We further require that the following conditions be satisfied:

(i) there is $\sigma_0 \in X$ and
a) $\sigma_1 \in [S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}$ such that $[S\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))} \preceq [\mathcal{T}\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$;
b) $\sigma_1 \in [\mathcal{T}\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))}$ such that $[\mathcal{T}\sigma_0]_{(\ltimes(\sigma_0),\beta(\sigma_0))} \preceq [S\sigma_1]_{(\ltimes(\sigma_1),\beta(\sigma_1))}$

(ii) for each $\sigma \in X$ and
a) $\tau \in [S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$ with $[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))} \preceq [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}$, we have $[\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))} \preceq [S\varpi]_{(\ltimes(\varpi),\beta(\varpi))}$ for all $\varpi \in [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}$;
b) $\tau \in [\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$ with $[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))} \preceq [S\tau]_{(\ltimes(\tau),\beta(\tau))}$, we have $[S\tau]_{(\ltimes(\tau),\beta(\tau))} \preceq [\mathcal{T}\varpi]_{(\ltimes(\varpi),\beta(\varpi))}$ for all $\varpi \in [S\tau]_{(\ltimes(\tau),\beta(\tau))}$

(iii) The pair $(S, \mathcal{T})$ is $\aleph$-continuous;

(iv) The sets $[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$ and $[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$ are proximal for each $\sigma \in X$.

Then, S and $\mathcal{T}$ have at least one common IFFP.

Proof. Let the function $\gamma : X \times X \to \mathbb{R}_+$ be defined by

$$\gamma(\sigma,\tau) = \begin{cases} 1, & \text{if } [S\sigma]_{(\ltimes(\sigma),\beta(\sigma))} \preceq [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))} \textit{ or } [\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))} \preceq [S\tau]_{(\ltimes(\tau),\beta(\tau))}, \\ 0, & \text{otherwise.} \end{cases}$$

To show that the pair $(S, \mathcal{T})$ is γ-admissible, take

(i) $\sigma \in X$ and $\tau \in [S\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$ with $\gamma(\sigma,\tau) \geq 1$ then, $[S\sigma]_{(\ltimes(\sigma),\beta(\sigma))} \preceq [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}$ and by hypothesis $(ii)(\mathfrak{a})$,we have $[\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))} \preceq [S\varpi]_{(\ltimes(\varpi),\beta(\varpi))}$ for all $\varpi \in [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}$. It follows that $\gamma(\tau,\varpi) \geq 1$ for all $\varpi \in [\mathcal{T}\tau]_{(\ltimes(\tau),\beta(\tau))}$.

(ii) $\sigma \in X$ and $\tau \in [\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))}$ with $\gamma(\sigma,\tau) \geq 1$ then, $[\mathcal{T}\sigma]_{(\ltimes(\sigma),\beta(\sigma))} \preceq [S\tau]_{(\ltimes(\tau),\beta(\tau))}$ and by hypothesis $(ii)(\mathfrak{b})$,we have $[S\tau]_{(\ltimes(\tau),\beta(\tau))} \preceq [\mathcal{T}\varpi]_{(\ltimes(\varpi),\beta(\varpi))}$ for all $\varpi \in [S\tau]_{(\ltimes(\tau),\beta(\tau))}$. It follows that $\gamma(\tau,\varpi) \geq 1$ for all $\varpi \in [S\tau]_{(\ltimes(\tau),\beta(\tau))}$.

Moreover, by inequalities (28) and (29), we find that the pair $(S,\mathcal{T})$ is an AHIF $\mathcal{Z}$-contraction regarding $\wp \in \mathcal{Z}$. As a result, it can be seen that all the axioms of the Theorem 2 are met. Thus S and $\mathcal{T}$ have at least one common IFFP in X.

Example 30. *Let $X = \mathbb{N}$ be a partially ordered set such that $\mathfrak{a} \preceq \mathfrak{b}$ if and only if $\mathfrak{b}|\mathfrak{a}$ for all $\mathfrak{a},\mathfrak{b} \in X$. Define $\delta(\sigma,\tau) = |\sigma-\tau|^3$ for all $\sigma,\tau \in X$. Then $(X,\delta,h=4)$ is a complete b-MS but not a MS. For each $\sigma \in X$ consider two IFS-valued maps $S,\mathcal{T} : X \to IFS(X)$ and for $\sigma \in X$, $\mathcal{T}\sigma, S\sigma$ are IFSs such that $\mu_{S\sigma}, \mu_{\mathcal{T}\sigma} : X \to [0,1]$ are membership functions and $\nu_{S\sigma}, \nu_{\mathcal{T}\sigma} : X \to [0,1]$ are non-membership functions of S and $\mathcal{T}$ respectively with $\mu_{S\sigma}(\mathfrak{a}) + \nu_{S\sigma}(\mathfrak{a}) \leq 1$ and $\mu_{\mathcal{T}\sigma}(\mathfrak{a}) + \nu_{\mathcal{T}\sigma}(\mathfrak{a}) \leq 1$ for all $\mathfrak{a} \in X$. We define these maps as:*
***Case 1**:*
If σ is even;

$$\mu_{\mathcal{T}\sigma}(\mathfrak{a}) = \begin{cases} \ltimes, & \textit{if } \mathfrak{a} = \frac{\sigma}{2} \\ \frac{\ltimes}{3}, & \textit{if } \mathfrak{a} = \sigma \\ 0, & \textit{elsewhere} \end{cases}$$

$$\nu_{\mathcal{T}\sigma}(\mathfrak{a}) = \begin{cases} \frac{\beta}{5}, & \textit{if } \mathfrak{a} = \frac{\sigma}{2} \\ \frac{\beta}{2}, & \textit{if } \mathfrak{a} = \sigma \\ \beta, & \textit{elsewhere} \end{cases}$$

$$\mu_{S\sigma}(\mathfrak{a}) = \begin{cases} \ltimes, & \textit{if } \mathfrak{a} = \frac{\sigma}{2} \\ 1-\frac{\ltimes}{3}, & \textit{if } \mathfrak{a} = \sigma \\ 0, & \textit{elsewhere} \end{cases}$$

$$\nu_{S\sigma}(\mathfrak{a}) = \begin{cases} \frac{\beta}{6}, & \textit{if } \mathfrak{a} = \frac{\sigma}{2} \\ \frac{\beta}{3}, & \textit{if } \mathfrak{a} = \sigma \\ \beta, & \textit{elsewhere.} \end{cases}$$

***Case 2**:*
If σ is odd;

$$\mu_{\mathcal{T}\sigma}(\mathfrak{a}) = \begin{cases} 1-\ltimes, & \textit{if } \mathfrak{a} = \sigma+1 \\ \ltimes, & \textit{if } \mathfrak{a} = \sigma \\ \ltimes^3, & \textit{elsewhere} \end{cases}$$

$$\nu_{\mathcal{T}\sigma}(\mathfrak{a}) = \begin{cases} 0, & \textit{if } \mathfrak{a} = \sigma+1 \\ \beta^2, & \textit{if } \mathfrak{a} = \sigma \\ \beta^4, & \textit{elsewhere} \end{cases}$$

$$\mu_{S\sigma}(\mathfrak{a}) = \begin{cases} \frac{\ltimes}{2}, & \text{if } \mathfrak{a} = \sigma + 1 \\ \ltimes, & \text{if } \mathfrak{a} = \sigma \\ 0, & \text{elsewhere} \end{cases}$$

$$\nu_{S\sigma}(\mathfrak{a}) = \begin{cases} \beta, & \text{if } \mathfrak{a} = \sigma + 1 \\ \beta^2, & \text{if } \mathfrak{a} = \sigma \\ \beta^3, & \text{elsewhere.} \end{cases}$$

Let $\ltimes = \frac{3}{5}$ *and* $\beta = \frac{1}{5}$. *Then*

$$[\mathcal{T}\sigma]_{(\ltimes,\beta)} = \begin{cases} \{\frac{\sigma}{2}\}, & \text{if } \sigma \text{ is even} \\ \{\sigma\}, & \text{if } \sigma \text{ is odd} \end{cases}$$

and

$$[S\sigma]_{(\ltimes,\beta)} = \begin{cases} \{\frac{\sigma}{2}\}, & \text{if } \sigma \text{ is even} \\ \{\sigma\}, & \text{if } \sigma \text{ is odd.} \end{cases}$$

Clearly $S\sigma, \mathcal{T}\sigma \in IFS(X)$ *for each* $\sigma \in X$. *Define the functions* $\gamma : X \times X \to \mathbb{R}_+$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *by*

$$\gamma(\sigma,\tau) = \begin{cases} 6, & \text{if } \sigma = \tau = 1 \\ \frac{1}{150}, & \text{if } \sigma,\tau \in \{2,3\} \text{ such that } \sigma \neq \tau \\ 0, & \text{otherwise.} \end{cases}$$

and $\varphi(\mathfrak{t}) = \frac{t}{4}$ *for all* $t > 0$. *Let* $\wp(\mathfrak{a},\mathfrak{b}) = \frac{1}{2}\mathfrak{b} - \mathfrak{a}$ *for all* $\mathfrak{a},\mathfrak{b} \in \mathbb{R}_+$. *Obviously* $\wp \in \mathcal{Z}$ *and* $\varphi \in \Lambda_b$

Now we verify conditions

$$\wp(\gamma(\sigma,\tau)\aleph([S\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau))) \geq 0$$

and

$$\wp(\gamma(\sigma,\tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [S\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(\mathcal{T},S)}(\sigma,\tau))) \geq 0$$

for $r > 0$ *under the following cases;*

Case 1:

If $\sigma = \tau = 1$, *then*

$[S\sigma]_{(\ltimes,\beta)} = [\mathcal{T}\tau]_{(\ltimes,\beta)} = \{1\}$ *this implies that*

$$\aleph([S\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) = 0$$

so,

$$\wp(6(0), \varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau))) \geq 0$$

Similarly

$$\wp(\gamma(\sigma,\tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [S\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(\mathcal{T},S)}(\sigma,\tau))) \geq 0$$

Case 2:
If $\sigma, \tau \in \{2,3\}$ such that $\sigma \neq \tau$, Let $\sigma = 2$ and $\tau = 3$. Then $[S\sigma]_{(\ltimes,\beta)} = \{1\}$ and $[\mathcal{T}\tau]_{(\ltimes,\beta)} = \{3\}$ which implies that

$$\begin{aligned}\aleph([S\sigma]_{(\ltimes,\beta)}, [\mathcal{T}\tau]_{(\ltimes,\beta)}) &= \aleph(\{1\},\{3\}) \\ &= \delta(1,3) = 8\end{aligned}$$

$$\begin{aligned}M^r_{(S,\mathcal{T})}(2,3) &= [\mathcal{A}(2,3)]^{\frac{1}{r}} \\ &= \Bigg[k_1(\delta(2,3))^r + k_2(\delta(2,[S2]_{(\ltimes,\beta)}))^r + k_3(\delta(3,[\mathcal{T}3]_{(\ltimes,\beta)}))^r \\ &\quad + k_4\left(\frac{\delta(3,[\mathcal{T}3]_{(\ltimes,\beta)})(1+\delta(2,[S2]_{(\ltimes,\beta)}))}{1+\delta(2,3)}\right)^r \\ &\quad + k_5\left(\frac{\delta(3,[S2]_{(\ltimes,\beta)})(1+\delta(2,[\mathcal{T}3]_{(\ltimes,\beta)}))}{1+\delta(2,3)}\right)^r\Bigg]^{\frac{1}{r}}\end{aligned}$$

taking $k_1 = k_2 = \frac{1}{2}$ and $k_3 = k_4 = k_5 = 0$,

$$\begin{aligned}M^r_{(S,\mathcal{T})}(2,3) &= [\frac{1}{2}(1)^r + \frac{1}{2}(1)^r]^{\frac{1}{r}} \\ &= \{1\}^{\frac{1}{r}} = 1\end{aligned}$$

Therefore,

$$\wp\left(\frac{1}{150}(8), \varphi(1)\right) \geq 0$$

Similarly,

$$\wp(\gamma(\sigma,\tau)\aleph([\mathcal{T}\sigma]_{(\ltimes,\beta)}, [S\tau]_{(\ltimes,\beta)}), \varphi(M^r_{(\mathcal{T},S)}(\sigma,\tau))) \geq 0$$

Case 3
If $\sigma, \tau \in X - \{1,2,3\}$, then $\gamma(\sigma,\tau) = 0$ which implies that

$$\wp(0, \varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau)) = \frac{1}{2}\varphi(M^r_{(S,\mathcal{T})}(\sigma,\tau))) \geq 0$$

Similarly,

$$\wp(0, \varphi(M^r_{(\mathcal{T},S)}(\sigma,\tau))) \geq 0.$$

Moreover, it is clear that the pair $(S,\mathcal{T})$ is γ-admissible, $\aleph$-continuous and the sets $[S\sigma]_{(\ltimes,\beta)}$, $[\mathcal{T}\sigma]_{(\ltimes,\beta)}$ are proximal for each $\sigma \in X$. Hence S and $\mathcal{T}$ have at least one common IFFP in X.

6. Conclusion

In this paper, we introduce the concept of Admissible Hybrid Intuitionistic Fuzzy (AHIF) Z-contractions within the framework of complete b-metric spaces (b-MSs), a significant innovation that aims to extend and generalize existing results in intuitionistic fuzzy fixed point (IFFP) theory. By developing new definitions and constructing a theorem that ensures the existence of at least one common IFFP, our work advances the understanding of fixed points in the context of b-metric spaces, a structure that has gained considerable attention due to its flexibility in handling generalized distance measures.

The importance of our work lies in its ability to unify and expand upon prior theorems, which were limited in scope to more restrictive spaces and less generalized contractions. Through the introduction of AHIF Z-contractions, we establish a broader framework for studying common fixed points in intuitionistic fuzzy settings. Our results are not only theoretical but also practical, as demonstrated by the example that showcases the application of these findings in ordered b-metric spaces. This example illustrates the relevance of our work for real-world problems where common IFFPs play a crucial role, further highlighting the utility of our contributions.

Moreover, our work builds upon foundational fixed point theorems and intuitionistic fuzzy set theory by providing new pathways for exploration within the framework of b-metric spaces. The construction of AHIF Z-contractions and the corresponding fixed point results represent a significant extension of existing literature. These contributions refine and broaden the applicability of IFFP results, offering new perspectives for researchers interested in the interplay between b-metric spaces and intuitionistic fuzzy systems.

The introduction of the AHIF Z-contraction, along with its practical implications, signals a meaningful advancement in fixed point theory, particularly within the context of intuitionistic fuzzy systems in b-metric spaces. By demonstrating the existence of common fixed points in an ordered b-metric space, we not only validate our theoretical results but also provide a robust example that may inspire further applications across various domains. This research contributes to the growing body of work on generalized metric spaces and intuitionistic fuzzy systems, and we anticipate that it will foster additional exploration and innovation in these areas, leading to new developments in both theory and application.

Open Problem: Discuss the limitations and potential challenges that may arise when extending the proven results to Suzuki-type fuzzy weak ϕ-contraction, and propose future research directions to address these issues.

Author Contributions: Conceptualization, M.R., N.S. and Q.M.; Formal analysis, N.S., M.A. and M.R.; Investigation, N.S., Q.M. and M.A.; Writing original draft preparation, Q.M., M.A. and I.R.; Writing,review and editing, N.S., I.R. and Q.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.
Data Availability Statement: The data used to support the findings of this study are available from the corresponding author upon request.
Conflict of Interest: The authors declare no conflict of interest.

References

[1] S. Aleksic et al. Picard's sequences in b-metric spaces. *Fixed Point Theory*, 21(1):35–46, 2020.

[2] H. H. Alsulami, E. Karapinar, F. Khojasteh, and A. F. Roldán-López de Hierro. A proposal to the study of contractions in quasi-metric spaces. *Discrete Dynamics in Nature and Society*, 2014.

[3] M. Arshad, A. Shoaib, M. Abbas, and A. Azam. Fixed points of a pair of kannan type mappings on a closed ball in ordered partial metric spaces. *Miskolc Mathematical Notes*, 14(3):769–784, 2013.

[4] K. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96, 1986.

[5] I. A. Bakhtin. The contraction principle in quasi metric spaces. *Funct. Anal. Unianowsk Gos. Ped. Inst*, 30:26–37, 1989.

[6] M. Boriceanu. Fixed point theory for multivalued generalized contraction on a set with two b-metrics. *Studia Universitatis Babes-Bolyai, Mathematica*, 2009.

[7] N. Bourbaki. *Topologie Generale*. Herman, Paris, France, 1974.

[8] S. Czerwik. Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, 1(1):5–11, 1993.

[9] S. Czerwik. Nonlinear set-valued contraction mappings in b-metric spaces. *Atti Sem. Mat. Fis. Univ. Modena*, 46:263–276, 1998.

[10] S. Czerwik, K. Dlutek, and S. Singh. Round-off stability of iteration procedures for operators in b-metric spaces. *J. Nat. Phys. Sci.*, 11, 1997.

[11] A. F. Roldán-López de Hierro, E. Karapinar, C. Roldán-López de Hierro, and J. Martínez-Moreno. Coincidence point theorems on metric spaces via simulation functions. *Journal of Computational and Applied Mathematics*, 275:345–355, 2015.

[12] R. George and B. Fisher. Some generalized results of fixed points in cone b-metric spaces. *Mathematica Moravica*, 17(2):39–50, 2013.

[13] M. Gulzar, D. Alghazzawi, M. H. Mateen, and N. Kausar. A certain class of t-intuitionistic fuzzy subgroups. *IEEE Access*, 8:163260–163268, 2020.

[14] M. Gulzar, M. H. Mateen, D. Alghazzawi, and N. Kausar. A novel applications of complex intuitionistic fuzzy sets in group theory. *IEEE Access*, 8:196075–196085, 2020.

[15] S. Heilpern. Fuzzy mappings and fixed point theorem. *Journal of Mathematical Analysis and Applications*, 83(2):566–569, 1981.

[16] S. B. Nadler Jr. Multi-valued contraction mappings. 1969.

[17] F. Khojasteh et al. A new approach to the study of fixed point theory for simulation functions. *Filomat*, 29(6):1189–1194, 2015.

[18] N. Konwar. Extension of fixed point results in intuitionistic fuzzy b metric space. *Journal of Intelligent and Fuzzy Systems*, 39(5):7831–7841, 2020.

[19] S. S. Mohammed and I. A. Fulatan. Fuzzy fixed point results via simulation functions. *Mathematical Sciences*, 16(2):137–148, 2022.

[20] J. J. Nieto and R. Rodríguez-López. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. *Acta Mathematica Sinica, English Series*, 23(12):2205–2212, 2007.

[21] A. Ran and M. Reurings. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proceedings of the American Mathematical Society*, 132(5):1435–1443, 2004.

[22] M. Rashid, A. Azam, F. Dar, F. Ali, and M. A. Al-Kadhi. A comprehensive study on advancement in hybrid contraction and graphical analysis of -fuzzy fixed points with application. *Mathematics*, 11(21):4489, 2023.

[23] B. Rhoades. Some theorems on weakly contractive maps. *Nonlinear Analysis: Theory, Methods and Applications*, 47(4):2683–2693, 2001.

[24] I. A. Rus. Generalized contractions and applications. 2001.

[25] B. Samet, C. Vetro, and P. Vetro. Fixed point theorems for a-contractive type mappings. *Nonlinear Analysis: Theory, Methods and Applications*, 75(4):2154–2165, 2012.

[26] Y. H. Shen, F. X. Wang, and W. Chen. A note on intuitionistic fuzzy mappings. 2012.

[27] S. L. Singh and B. Prasad. Some coincidence theorems and stability of iterative procedures. *Computers and Mathematics with Applications*, 55(11):2512–2520, 2008.