EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 3061-3078
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Atomic Solution of Third Order Fractional Abstract Cauchy Problem

Rami Alkhateeb[1,*], Ghaith Awwad[2]

[1] *Department of Basic Sciences, Al-Ahliyya Amman University, Amman, Jordan*
[2] *Department of Mathematics, University of Jordan, Amman, Jordan*

Abstract. In this paper, we find an atomic solution of the fractional abstract Cauchy problem of order three. The fractional derivative used is the conformable derivative. The main idea of the proofs are based on theory of tensor product of Banach space.

2020 Mathematics Subject Classifications: 34A55, 26A33, 34G10

Key Words and Phrases: Fractional derivatives, abstract Cauchy problem, atom function, conformable derivative, tensor product of Banach space

1. Introduction

Let X be a Banach space and $I = [0, 1]$. Let $C(I)$ be the Banach space of all real valued continuous function on I under the sup-norm, and $C(I, X)$ be the Banach space of all continuous functions defined on I with values on X.

In recent years, many researchers were devoted to the problem

$$\begin{aligned} Bu^{\prime} &= Au(t) + f(t)z \\ u(0) &= x_0, \end{aligned}$$

where $u \in C^{\prime}(I, X)$ and A, B are densely defined linear operators on the codomain u.

This is called the Abstract Cauchy Problem which is: If $f = 0$ or $z = 0$, then the equation is homogenous, otherwise it is called non-homogenous. Now in the non homogenous problem we have two cases: (i) The first case, u is unknown and f is given. In this case the problem is called a direct problem. (ii) The second case, u and f are unknown. In this case the problem is called an inverse problem, and some other conditions and informations should be given. If B is not invertible, then the equation is called non-degenerate.

There are many different techniques to solve Abstract Cauchy Problem in case (i). Tensor product is one of such techniques.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5433

Email addresses: r.alkhateeb@ammanu.edu.jo (R. Alkhateeb), ghaithmawwad@gmail.com (G. Awwad)

In [4], a new definition called $\alpha-$conformable fractional derivative was introduced, which says that: If $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty) \to \mathbb{R}$. For $x \in E$, let:

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}. \tag{1}$$

If the limit exists then it is called the $\alpha-$conformable fractional derivative of f at x. For $x = 0, D^{\alpha} f(0) = \lim_{x \to 0} D^{\alpha} f(0)$ if such limit exists.

The new definition satisfies:

(i) $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, for all $a, b \in \mathbb{R}$.

(ii) $D^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point t, with $g(t) \neq 0$. Then

(iii) $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$.

(iv) $D^{\alpha}\left(\frac{f}{g}\right) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$.

We list here the fractional derivatives of certain functions,

(v) $D^{\alpha}(t^p) = pt^{p-\alpha}$.

(vi) $D^{\alpha}_{\alpha}(\sin\left(\frac{1}{\alpha}t^{\alpha}\right)) = \cos\left(\frac{1}{\alpha}t^{\alpha}\right)$.

(vii) $D^{\alpha}_{\alpha}(\cos\left(\frac{1}{\alpha}t^{\alpha}\right)) = -\sin\left(\frac{1}{\alpha}t^{\alpha}\right)$.

(viii) $D^{\alpha}(e^{\frac{1}{\alpha}t^{\alpha}}) = e^{\frac{1}{\alpha}t^{\alpha}}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

One should notice that function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $D_{\frac{1}{2}}(f)(t) = 1$. Hence $D_{\frac{1}{2}}(f)(0) = 1$. But $D_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

For more on fractional calculus and its applications we refer to [1], [2], and [5].

2. Atomic solution

Let X and Y be two Banach space and X^* be the dual of X. Assume $x \in X$ and $y \in Y$. Define the map $x \otimes y$: $X^* \longrightarrow Y$, by $x \otimes y(x^*) = \langle x, x^* \rangle\, y$ for all $x^* \in X^*$.

It is well known that $x \otimes y$ is a bounded linear operator and $\|x \otimes y\| = \|x\| \, \|y\|$. The operator $x \otimes y$ is called an atom. The set $X \otimes Y = span \{x \otimes y : x \in X \text{ and } y \in Y\}$ is subspace of $L\left(X^*, Y\right)$.

If the sum of two atoms is an atom, then either the first components are dependent or the second are dependent.

An equation of the form

$$D^{2\alpha}u(t) + AD^{\alpha}u(t) + Bu(t) = f(t) \tag{2}$$

is called the fractional abstract Cauchy problem of order two, where v and f are nice functions from $(0, \infty)$ to the Banach space X, $A$and B are closed linear operator on X.

A solution of this equation of the form $v = u \otimes x$ .is called an atomic solution,where $v(t) = u(t)x$.In this paper, we are interested in finding an atomic solution of the third order vector valued fractional differential equations.

3. Main results

In this section, we prove some nice result containing certain solution of atomic problem. Consider the equation

$$v^{3\alpha}(t) + Av^{2\alpha}(t) + Bv^{\alpha}(t) = f(t), \tag{3}$$

where A and B are closed operators, $f(t)$ is given and u is the unknown equation (3) was discussed in [5] for the first order. Hence,we discuss third order.

Theorem 1. *The equation $v^{3\alpha}(t) + Av^{2\alpha}(t) + Bv^{\alpha}(t) = f(t)$ with the initial conditions $v(0) = 2x_0, v^{\alpha}(0) = x_0$, and $v^{2\alpha}(0) = x_0$ has an atomic solution .where A and B are closed operators on X,and f is a given atomic function, $f : [0, \infty] \to X$.*

Now, we are looking for atomic solution of (3). So put $v(t) = u(t)x, u(t) : [0, \infty] \to R, x$ is an element in the Banach space X, and consider the case when $u(0) = 1$, then the initial conditions given in Theorem 1 will be as follows:

$$\begin{pmatrix} v(0) = u(0)x = 2x_0 \text{ which implies that } x = 2x_0 \\ v^{\alpha}(0) = x_0, \text{ and } v^{2\alpha}(0) = x_0. \end{pmatrix} \tag{4}$$

Further assume f to be an atom: $f = h \otimes z$ where $h : [0, \infty] \to R$,and $z \in x$. So, (3) becomes:

$$u^{3\alpha} \otimes x + u^{2\alpha} \otimes Ax + u^{\alpha} \otimes Bx = f(t).$$

This can be written as:

$$u^{3\alpha} \otimes x + u^{2\alpha} \otimes Ax + u^{\alpha} \otimes Bx = h \otimes z. \tag{5}$$

There are four cases that must be discussed when solving the equation (5) as follows:

Case one: $u^{3\alpha} \otimes x + u^{2\alpha} \otimes Ax$ is an atom.

In this case we have two situations:

(i) $u^{3\alpha} = u^{2\alpha}$.

(ii) $x = Ax$.

Let us take situation (1).So equation (5) becomes:

$$u^{3\alpha} \otimes (x + Ax) + u^{\alpha} \otimes Bx = h \otimes z, \tag{6}$$

where h and z are given. So we have two cases:

(a) $u^{3\alpha} = u^{\alpha} = h = u^{2\alpha}$.

(b) $x + Ax = Bx = z$.

In case (a), we have three cases:

(i) $u^{3\alpha} - u^{\alpha} = 0$.

This case can be solved as in [3]:

$$r^3 - r = r(r-1)(r+1) = 0,$$

which gives $r_1 = 0, r_2 = 1$, and $r_3 = -1$.

Consequently,

$$u(t) = c_1 + c_2 e^{\frac{t^\alpha}{\alpha}} + c_3 e^{-\frac{t^\alpha}{\alpha}}.$$

So by (4), we have

$$c_1 + c_2 + c_3 = 2x_0,\ c_2 - c_3 = x_0,\ \text{and } c_2 + c_3 = x_0,$$

which implies that $c_1 = x_0, c_2 = x_0$, and $c_3 = 0$. Hence, we have

$$u(t) = x_0 + x_0 e^{\frac{t^\alpha}{\alpha}}. \tag{7}$$

(ii) $u^{\alpha} = h$.

Using (7), we have

$$h = x_0 e^{\frac{t^\alpha}{\alpha}}.$$

Hence for an atomic solution to exist, h must $= x_0 e^{\frac{t^\alpha}{\alpha}}$.

(iii) $u^{3\alpha} = h$.

By using (7), we have

$$h = x_0 e^{\frac{t^\alpha}{\alpha}}.$$

Since $u^{3\alpha} = u^{\alpha} = h = u^{2\alpha} = x_0 e^{\frac{t^\alpha}{\alpha}}$, then(6) becomes

$$x + Ax + Bx = z.$$

So

$$(I + A + b)\, x = z,$$

This means z will be in the intersection of the ranges $(I + A + B)$.

Consequently, there is an atomic solution in this situation.

In case (b), equation (6) becomes

$$u^{3\alpha} \otimes (x + Ax) + u^{\alpha} \otimes Bx = h \otimes z.$$

So,

$$u^{3\alpha} + u^{\alpha} = h.$$

This is third order homogenous linear fractional differential equation. To solve it, we follow the variation of parameters method.

The homogenous part can solved as in, [3].

$$r^3 + r = r(r^2 + 1) = 0,$$

which implies that $r_1 = 0, r_2 = i$, and $r_3 = -i$.

Hence

$$u_h(t) = c_1 + c_2 \cos\left(\frac{t^{\alpha}}{\alpha}\right) + c_3 \sin\left(\frac{t^{\alpha}}{\alpha}\right).$$

By the assumption (4), we get $c_1 = 3x_0, c_2 = -x_0$, and $c_3 = x_0$.

Hence

$$u_h(t) = 3x_0 - x_0 \cos\left(\frac{t^{\alpha}}{\alpha}\right) + x_0 \sin\left(\frac{t^{\alpha}}{\alpha}\right).$$

For the particular part, the ,we use variation of parameters introduced in [2]. Thus we have

$$u_p(t) = \sum_{m=1}^{3} u_m \int_b^t \frac{hW_m^{\alpha}(\tau)}{W^{\alpha}(\tau)\tau^{1-\alpha}} d\tau. \tag{8}$$

Where b is an arbitrary positive constant, and

$$W^{\alpha}[u_1(t), u_2(t), u_3(t)] = \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1^{\alpha}(t) & u_2^{\alpha}(t) & u_3^{\alpha}(t) \\ u_1^{2\alpha}(t) & u_2^{2\alpha}(t) & u_3^{2\alpha}(t) \end{vmatrix}, W_1^{\alpha} = \begin{vmatrix} 0 & u_2(t) & u_3(t) \\ 0 & u_2^{\alpha}(t) & u_3^{\alpha}(t) \\ 1 & u_2^{2\alpha}(t) & u_3^{2\alpha}(t) \end{vmatrix},$$

$$W_2^{\alpha} = \begin{vmatrix} u_1(t) & 0 & u_3(t) \\ u_1^{\alpha}(t) & 0 & u_3^{\alpha}(t) \\ u_1^{2\alpha}(t) & 1 & u_3^{2\alpha}(t) \end{vmatrix}, \text{ and } W_3^{\alpha} = \begin{vmatrix} u_1(t) & u_2(t) & 0 \\ u_1^{\alpha}(t) & u_2^{\alpha}(t) & 0 \\ u_1^{2\alpha}(t) & u_2^{2\alpha}(t) & h \end{vmatrix}.$$

Hence,

$$W^{\alpha}=\begin{vmatrix} 3x_0 & -x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) & x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) \\ 0 & x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) & x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) \\ 0 & x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) & -x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) \end{vmatrix}=-3x_0.$$

Which means that:

$$W_1^{\alpha}=\begin{vmatrix} 0 & -x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) & x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) \\ 0 & x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) & x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) \\ 1 & x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) & -x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) \end{vmatrix}=-x_0,$$

$$W_2^{\alpha}=\begin{vmatrix} 3x_0 & 0 & x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) \\ 0 & 0 & x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) \\ 0 & 1 & -x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) \end{vmatrix}=-3x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right),$$

and

$$W_3^{\alpha}=\begin{vmatrix} 3 & -x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) & 0 \\ 0 & x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right) & 0 \\ 0 & x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right) & 1 \end{vmatrix}=3x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right).$$

So, we have

$$u_1^{\alpha}(t)=\frac{W_1^{\alpha}}{W^{\alpha}}=\frac{1}{3},$$

$$u_2^{\alpha}(t)=\frac{W_2^{\alpha}}{W^{\alpha}}=\frac{-3x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right)}{-3x_0}=\cos\left(\frac{t^{\alpha}}{\alpha}\right),$$

and

$$u_3^{\alpha}(t)=\frac{W_3^{\alpha}}{W^{\alpha}}=\frac{3x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right)}{-3x_0}=-\sin\left(\frac{t^{\alpha}}{\alpha}\right).$$

Consequently,

$$u_p(t)=3x_0\int_b^t\frac{h}{3}\frac{d\tau^{\alpha}}{\tau^{\alpha-1}}-x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right)\int_b^t h\cos\left(\frac{\tau^{\alpha}}{\alpha}\right)\frac{d\tau^{\alpha}}{\tau^{\alpha-1}}-x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right)\int_b^t h\sin\left(\frac{\tau^{\alpha}}{\alpha}\right)\frac{d\tau^{\alpha}}{\tau^{\alpha-1}}.$$

So,

$$\begin{aligned} u(t) &= u_h(t)+u_p(t), \\ u(t) &= 3x_0-x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right)+x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right)+3x_0\int_b^t h\frac{d\tau^{\alpha}}{\tau^{\alpha-1}}-x_0\cos\left(\frac{t^{\alpha}}{\alpha}\right)\int_b^t h\cos\left(\frac{\tau^{\alpha}}{\alpha}\right)\frac{d\tau^{\alpha}}{\tau^{\alpha-1}} \\ &\quad -x_0\sin\left(\frac{t^{\alpha}}{\alpha}\right)\int_b^t h\sin\left(\frac{\tau^{\alpha}}{\alpha}\right)\frac{d\tau^{\alpha}}{\tau^{\alpha-1}}. \end{aligned}$$

This completes situation (1).

Considering situation (2), $x = Ax$, equation (5) becomes:

$$\left(u^{3\alpha} + u^{2\alpha}\right) \otimes x + u^{\alpha} \otimes Bx = h \otimes z.$$

This situation has two cases:

(a) $u^{3\alpha} + u^{2\alpha} = u^{\alpha} = h$.

(b) $x = Bx = z$.

In case (a), for the existence of an atomic solution, we have three situations:

(i) $u^{3\alpha} + u^{2\alpha} - u^{\alpha} = 0$.

This can be solved as in [3],

$$r^3 + r^2 - r = r(r^2 + r - 1) = 0.$$

Which gives

$$r_1 = 0, r_2 = \frac{-1+\sqrt{5}}{2}, \text{ and } r_3 = \frac{-1-\sqrt{5}}{2}. \tag{9}$$

Then

$$u(t) = c_1 + c_2 e^{r_2\left(\frac{t^{\alpha}}{\alpha}\right)} + c_3 e^{r_3\left(\frac{t^{\alpha}}{\alpha}\right)}. \tag{10}$$

Now, by assumptions (4), we have

$$c_1 + c_2 + c_3 = 2x_0, r_2c_2 + r_3c_3 = x_0, \text{ and } r_2^2c_2 + r_3^2c_3 = x_0.$$

So,

$$c_1 = 2x_0 - c_2 - c_3, c_2 = \frac{r_3 - x_0}{r_2r_3 - r_2^2}, \text{ and } c_3 = \frac{r_2 - x_0}{r_2r_3 - r_3^2}. \tag{11}$$

From (9) and (11), we have

$$c_1 = 0, c_2 = \frac{3+\sqrt{5}}{5-\sqrt{5}}x_0 \text{ , and } c_3 = \frac{3-\sqrt{5}}{\sqrt{5}+5}x_0.$$

So, the equation (10) will be

$$u(t) = \frac{3+\sqrt{5}}{5-\sqrt{5}}x_0 e^{\frac{-1+\sqrt{5}}{2}\left(\frac{t^{\alpha}}{\alpha}\right)} + \frac{3-\sqrt{5}}{\sqrt{5}+5}x_0 e^{\frac{-1-\sqrt{5}}{2}\left(\frac{t^{\alpha}}{\alpha}\right)}. \tag{12}$$

(ii) $u^{\alpha} = h$.

For an atomic solution to exist h must equal to u^{α}.

Hence, from (12), we have

$$h = \frac{1+\sqrt{5}}{5-\sqrt{5}}x_0 e^{\frac{-1+\sqrt{5}}{2}\left(\frac{t^{\alpha}}{\alpha}\right)} + \frac{1-\sqrt{5}}{5+\sqrt{5}}x_0 e^{\frac{-1-\sqrt{5}}{2}\left(\frac{t^{\alpha}}{\alpha}\right)}.$$

(iii) $u^{3\alpha} + u^{2\alpha} = h$.

So,

$$\begin{aligned} h &= c_2(r_2^3 + r_2^2)e^{r_2(\frac{t^\alpha}{\alpha})} + c_3(r_3^3 + r_3^2)e^{r_3(\frac{t^\alpha}{\alpha})} \\ h &= \frac{1+\sqrt{5}}{5-\sqrt{5}}x_0 e^{\frac{-1+\sqrt{5}}{2}(\frac{t^\alpha}{\alpha})} + \frac{1-\sqrt{5}}{5+\sqrt{5}}x_0 e^{\frac{-1-\sqrt{5}}{2}(\frac{t^\alpha}{\alpha})}. \end{aligned} \tag{13}$$

Hence, the equations (12) and (13) are equal to h.

So, there is an atomic solution in this case.

In case (b), equation (5) becomes

$$\left(u^{3\alpha} + u^{2\alpha}\right) \otimes x + u^\alpha \otimes Bx = h \otimes z.$$

So,

$$u^{3\alpha} + u^{2\alpha} + u^\alpha = h. \tag{14}$$

This is third order homogenous linear fractional differential equation, to solve it we follow the variation of parameters method.

The homogenous part can solved as in [3].

$$r^3 + r^2 + r = r(r^2 + r + 1) = 0,$$

which gives $r_1 = 0, r_2 = \frac{-1+i\sqrt{3}}{2}$, and $r_3 = \frac{-1-i\sqrt{3}}{2}$.

Hence,

$$u_h(t) = c_1 + e^{-\frac{t^\alpha}{2\alpha}}\left(c_2 \cos\left(\frac{\sqrt{3}t^\alpha}{2\alpha}\right) + c_3 \sin\left(\frac{\sqrt{3}t^\alpha}{2\alpha}\right)\right).$$

By assumption (4), we have

$$c_1 = 4x_0, c_2 = -2x_0, \text{ and } c_3 = 0.$$

So,

$$u_h(t) = 4x_0 - 2x_0 e^{-\frac{t^\alpha}{2\alpha}} \cos\left(\frac{\sqrt{3}t^\alpha}{2\alpha}\right).$$

Since $c_3 = 0$, there is no particular part, and for an atomic solution to exist, h must equal *zero*. This completes situation (2), and hence, **Case one** is completed.

Case two: $(u^{3\alpha} \otimes x + u^\alpha \otimes Bx)$ is an atom.

In this case we have two situations:

(i) $u^{3\alpha} = u^{\alpha}$.

(ii) $x = Bx$.

Considering situation (1), equation (5) becomes:

$$u^{3\alpha} \otimes (x + Bx) + u^{2\alpha} \otimes Ax = h \otimes z. \tag{15}$$

So we have two cases:

(a) $u^{3\alpha}(t) = u^{2\alpha}(t) = h = u^{\alpha}$.

(b) $x + Bx = Ax = z$.

In case (a) we have three cases:

(i) $u^{3\alpha} - u^{2\alpha} = 0$.

This case can be solved as in [3]:

$$r^3 - r^2 = r^2(r-1) = 0,$$

which gives $r_1 = 0, r_2 = 0$, and $r_3 = 1$.

Consequently,

$$u(t) = c_1 + c_2\frac{t^{\alpha}}{\alpha} + c_3 e^{\frac{t^{\alpha}}{\alpha}}.$$

Hence, by the assumption (4), we have

$$c_1 = x_0\ , c_2 = 0, c_3 = x_0.$$

Hence,

$$u(t) = x_0 + x_0 e^{\frac{t^{\alpha}}{\alpha}}. \tag{16}$$

(ii) $u^{2\alpha} = h$.

From (16), we get

$$h = x_0 e^{\frac{t^{\alpha}}{\alpha}}.$$

So for an atomic solution to exist h must $= x_0 e^{\frac{t^{\alpha}}{\alpha}}$.

(iii) $u^{3\alpha} = h$.

From (16), we get

$$h = x_0 e^{\frac{t^\alpha}{\alpha}}.$$

Since $u^{3\alpha} = u^{2\alpha} = u^{\alpha} = \ h = x_0 e^{\frac{t^\alpha}{\alpha}}$ in (ii) and (iii).

Consequently,

$$x + Bx + Ax = z.$$

So $(I+B+A)x = z$ which mean z will be in the intersection of the ranges of $(I + B + A)$
.

Hence, there is an atomic solution in situation (1).

Now, in case (b) $x + Bx = Ax = z$. Hence, $x + Bx = Ax, x + Bx = z$, and $Ax = z$. So, equation (5) becomes

$$u^{3\alpha} \otimes (x + Bx) + u^{2\alpha} \otimes Ax = h \otimes z. \tag{17}$$

Substitute the equation (17) in the equation (15), we get

$$\left(u^{3\alpha} + u^{2\alpha}\right) \otimes Ax = h \otimes z.$$

Hence,

$$u^{3\alpha} + u^{2\alpha} = h. \tag{18}$$

This is third order homogenous linear fractional differential equation, to solve it we follow the variation of parameters method.

The homogenous part can solved as in [3] as follows:

$$r^3 + r^2 = 0,$$

which gives $r_1 = 0, r_2 = 0$, and $r_3 = -1$.

Hence,

$$u_h(t) = c_1 + c_2 \frac{t^\alpha}{\alpha} + c_3 e^{-(\frac{t^\alpha}{\alpha})}. \tag{19}$$

So by the assumption (4), we have

$$c_1 = x_0, c_2 = 2x_0, c_3 = x_0.$$

Hence, (19) becomes

$$u_h(t) = x_0 + 2x_0 \frac{t^\alpha}{\alpha} + x_0 e^{-(\frac{t^\alpha}{\alpha})}.$$

For the particular part we use variation of parameters introduced in [2]. Thus, by using (8), the Wronskian will given by:

$$W^{\alpha} = \begin{vmatrix} x_0 & 2x_0\frac{t^{\alpha}}{\alpha} & x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \\ 0 & 2 & -x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \\ 0 & 0 & x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \end{vmatrix} = 2x_0 e^{-(\frac{t^{\alpha}}{\alpha})}.$$

So, we have

$$W_1^{\alpha} = \begin{vmatrix} 0 & 2x_0\frac{t^{\alpha}}{\alpha} & x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \\ 0 & 2x_0 & -x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \\ 1 & 0 & x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \end{vmatrix} = -2x_0 e^{-(\frac{t^{\alpha}}{\alpha})}\left(\frac{t^{\alpha}}{\alpha}+1\right),$$

$$W_2^{\alpha} = \begin{vmatrix} x_0 & 0 & x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \\ 0 & 0 & -x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \\ 0 & 1 & x_0 e^{-(\frac{t^{\alpha}}{\alpha})} \end{vmatrix} = x_0 e^{-(\frac{t^{\alpha}}{\alpha})},$$

and

$$W_3^{\alpha} = \begin{vmatrix} x_0 & 2x_0\frac{xt^{\alpha}}{\alpha} & 0 \\ 0 & 2x_0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2.$$

So,

$$u_1^{\alpha}(t) = \frac{W_1^{\alpha}}{W^{\alpha}} = \frac{-2x_0 e^{-(\frac{t^{\alpha}}{\alpha})}\left(\frac{t^{\alpha}}{\alpha}+1\right)}{2x_0 e^{-(\frac{t^{\alpha}}{\alpha})}} = -\left(\frac{t^{\alpha}}{\alpha}+1\right),$$

$$u_2^{\alpha}(t) = \frac{W_2^{\alpha}}{W^{\alpha}} = \frac{e^{-(\frac{t^{\alpha}}{\alpha})}}{2e^{-(\frac{t^{\alpha}}{\alpha})}} = \frac{1}{2},$$

and

$$u_3^{\alpha}(t) = \frac{W_3^{\alpha}}{W^{\alpha}} = \frac{2x_0}{2x_0 e^{-(\frac{t^{\alpha}}{\alpha})}} = e^{(\frac{t^{\alpha}}{\alpha})}.$$

Consequently,

$$u_p = -x_0\int_b^t h(\frac{t^{\alpha}}{\alpha}+1)\frac{dt^{\alpha}}{t^{\alpha-1}} + 2x_0\frac{t^{\alpha}}{\alpha}\int_b^t \frac{h}{2}\frac{dt^{\alpha}}{t^{\alpha-1}} + x_0 e^{-(\frac{t^{\alpha}}{\alpha})}\int_b^t h e^{\frac{t^{\alpha}}{\alpha}}\frac{dt^{\alpha}}{t^{\alpha-1}}.$$

Hence,

$$\begin{aligned} u(t) &= u_h + u_p, \\ u(t) &= x_0 + 2x_0\frac{t^{\alpha}}{\alpha} + x_0 e^{-(\frac{t^{\alpha}}{\alpha})} - \int_b^t h(\frac{t^{\alpha}}{\alpha}+1)\frac{dt^{\alpha}}{t^{\alpha-1}} + \frac{t^{\alpha}}{\alpha}\int_b^t h\frac{dt^{\alpha}}{t^{\alpha-1}} + e^{-(\frac{t^{\alpha}}{\alpha})}\int_b^t h e^{\frac{t^{\alpha}}{\alpha}}\frac{dt^{\alpha}}{t^{\alpha-1}}. \end{aligned}$$

Now, we will take situation (2), so equation (5) becomes:

$$\big(u^{3\alpha}(t)+u^{\alpha}(t)\big)\otimes x+u^{2\alpha}(t)\otimes Ax=h\otimes z.$$

For this case we have two cases:

(a) $u^{3\alpha}+u^{\alpha}=u^{2\alpha}=h=u^{\alpha}$.

(b) $x=Ax=z$.

In case (a), for the existence of an atomic solution, we have five situations.

(i) $u^{3\alpha}-u^{2\alpha}+u^{\alpha}=0$.
So we have from [3]

$$r^3-r^2+r=r(r^2-r+1)=0,$$

which gives $r_1=0, r_2=\frac{1+i\sqrt{3}}{2}$, and $r_3=\frac{1-i\sqrt{3}}{2}$.
Then,

$$u(t)=c_1+e^{\frac{1}{2}\left(\frac{t^{\alpha}}{\alpha}\right)}\left(c_2\cos\left(\frac{\sqrt{3}t^{\alpha}}{2\alpha}\right)+c_3\sin\left(\frac{\sqrt{3}t^{\alpha}}{2\alpha}\right)\right).$$

By the assumption (4), we have

$$c_1=2x_0, c_2=0, \text{ and } c_3=\frac{2}{\sqrt{3}}x_0.$$

Hence,

$$u(t)=2x_0+\frac{2}{\sqrt{3}}x_0e^{\frac{1}{2}\left(\frac{t^{\alpha}}{\alpha}\right)}\sin\left(\frac{\sqrt{3}t^{\alpha}}{2\alpha}\right). \tag{20}$$

(ii) $u^{3\alpha}+u^{\alpha}=u^{\alpha}$.
$u^{3\alpha}=0$, from [3],we have

$$r^3=0.$$

Consequently,

$$r_1=r_2=r3=0.$$

So,

$$u(t)=c_1+c_2\left(\frac{t^{\alpha}}{\alpha}\right)+c_3\left(\frac{t^{\alpha}}{\alpha}\right)^2.$$

So, by assumption (10), we have

$$c_1 = 2x_0, c_2 = x_0, c_3 = \frac{x_0}{2}.$$

Hence,

$$u(t) = 2x_0 + x_0\left(\frac{t^\alpha}{\alpha}\right) + \frac{x_0}{2}\left(\frac{t^\alpha}{\alpha}\right)^2.$$

(iii) $u^{2\alpha} = h$.
So,from (20) we have,

$$h = x_0 e^{\frac{1}{2}\left(\frac{t^\alpha}{\alpha}\right)}\left(\cos\left(\frac{\sqrt{3}t^\alpha}{2\alpha}\right) - \frac{x_0}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t^\alpha}{2\alpha}\right)\right).$$

(iv) $u^\alpha = h$.
So, from (20) we have,

$$h = x_0 e^{\frac{1}{2}\left(\frac{t^\alpha}{\alpha}\right)}\left(\cos\left(\frac{\sqrt{3}t^\alpha}{2\alpha}\right) + \frac{x_0}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t^\alpha}{2\alpha}\right)\right).$$

(v) $u^{3\alpha} + u^\alpha = h$.
So, from (20), we have

$$h = x_0 e^{\frac{1}{2}\left(\frac{t^\alpha}{\alpha}\right)}\cos\left(\frac{\sqrt{3}t^\alpha}{2\alpha}\right).$$

Since we don't have same solution from (i), (ii), (iii), (iv), and (v), there is no an atomic solution in this case.

This completes situation (2), and hence, **Case two** is completed.

Case three: $(u^{2\alpha} \otimes Ax + u^\alpha \otimes Bx)$ is an atom. This has two situations:

(1) $u^{2\alpha} = u^\alpha$.

(2) $A\ x = Bx$.

Let us take situation (1), so equation (5) becomes:

$$u^{3\alpha} \otimes x + u^{2\alpha} \otimes (Ax + Bx) = h \otimes z.$$

So, we have two cases:

(a) $u^{3\alpha}(t) = u^{2\alpha}(t) = h = u^\alpha$.

(b) $x = Ax + Bx = z$.

In case (a), we have four situations:

(i) $u^{3\alpha} - u^{2\alpha} = 0$.

So, we can solve it as in [3]

$$r^3 - r^2 = r^2(r-1) = 0,$$

which gives $r_1 = 0, r_2 = 0$, and $r_3 = 1$. So,

$$u(t) = c_1 + c_2 \frac{t^\alpha}{\alpha} + c_3 e^{(\frac{t^\alpha}{\alpha})}. \tag{21}$$

By assumption (4), we have

$$c_1 = x_0, c_2 = 0, \text{ and } c_3 = x_0.$$

So, (21) becomes

$$u(t) = x_0 + x_0 e^{(\frac{t^\alpha}{\alpha})}. \tag{22}$$

(ii) $u^{2\alpha} = h$.

Using (21), we get

$$h = x_0 e^{(\frac{t^\alpha}{\alpha})}.$$

(iii) $u^{3\alpha} = u^\alpha$.

$$r^3 = r.$$

So, we have $r_1 = 0, r_2 = 1$, and $r_3 = -1$.

Hence,

$$u(t) = c_1 + c_2 e^{(\frac{t^\alpha}{\alpha})} + c_3 e^{-(\frac{t^\alpha}{\alpha})}.$$

By assumption (4), we have

$$c_1 = x_0.c_2 = x_0, \text{ and } c_3 = 0.$$

Consequently,

$$u(t) = x_0 + x_0 e^{(\frac{t^\alpha}{\alpha})}. \tag{23}$$

(iv) $u^{3\alpha} = h$.

Using (21), we get

$$h = x_0 e^{(\frac{t^\alpha}{\alpha})}.$$

Since $u^{3\alpha} = u^{2\alpha} = h = u^\alpha = x_0 e^{\frac{t^\alpha}{\alpha}}$, (6) becomes

$$x = Ax + Bx = z.$$

So,

$$(A + B)\, x = z,$$

or

$$(I)x = z.$$

Which means that z will be at the range of intersection of $(A + B)$ and I.

Consequently, there is atomic solution in this case.

In case (b), equation (5) becomes

$$u^{3\alpha} \otimes x + u^{2\alpha} \otimes (A + B)x = h \otimes z.$$

So,

$$u^{3\alpha} + u^{2\alpha} = h.$$

This is third order homogenous linear fractional differential equation, to solve it we follow the variation of parameters method. The homogenous and particular parts can be found similarly as (18) in the case (b) in situation (1) in **Case two, and**

$$u(t) = x_0 + 2x_0 \frac{t^\alpha}{\alpha} + x_0 e^{-(\frac{t^\alpha}{\alpha})} - \int_b^t h(\frac{t^\alpha}{\alpha} + 1)\frac{dt^\alpha}{t^{\alpha-1}} + \frac{t^\alpha}{\alpha}\int_b^t h\frac{dt^\alpha}{t^{\alpha-1}} + e^{-(\frac{t^\alpha}{\alpha})}\int_b^t he^{\frac{t^\alpha}{\alpha}}\frac{dt^\alpha}{t^{\alpha-1}}.$$

Now, we will take situation (2), so equation (5) becomes:

$$u^{3\alpha} \otimes x + (u^{2\alpha} + u^\alpha) \otimes Ax = h \otimes z.$$

So, we have two cases:

(a) $u^{3\alpha}(t) = u^{2\alpha}(t) + u^\alpha(t) = h.$

(b) $x = Ax = z.$

In case (a), for an atomic solution to exist we must have three situations

(i) $u^{3\alpha}(t) - u^{2\alpha}(t) - u^\alpha(t) = 0.$

So, we can solve it as in [3]

$$r^3 - r^2 - r = r(r^2 - r - 1) = 0,$$

which gives $r_1 = 0, r_2 = \frac{1+\sqrt{5}}{2}$, and $r_3 = \frac{1-\sqrt{5}}{2}$. So,

$$u(t) = c_1 + c_2 e^{r_2(\frac{t^\alpha}{\alpha})} + c_3 e^{r_3(\frac{t^\alpha}{\alpha})}.$$

By assumption (4), we have

$$c_1 = 2x_0, c_2 = \frac{x_0}{\sqrt{5}}, \text{ and } c_3 = \frac{-x_0}{\sqrt{5}}.$$

Hence,

$$u(t) = 2x_0 + \frac{x_0}{\sqrt{5}} e^{\frac{-1+\sqrt{5}}{2}(\frac{t^\alpha}{\alpha})} - \frac{x_0}{\sqrt{5}} e^{\frac{-1-\sqrt{5}}{2}(\frac{t^\alpha}{\alpha})}. \tag{24}$$

(ii) $u^{2\alpha}(t) + u^\alpha(t) = h.$

So from (24), we have

$$h = \left(1 + \frac{2}{\sqrt{5}}\right) x_0 e^{\frac{-1+\sqrt{5}}{2}(\frac{t^\alpha}{\alpha})} + \left(1 - \frac{2}{\sqrt{5}}\right) x_0 e^{\frac{-1-\sqrt{5}}{2}(\frac{t^\alpha}{\alpha})}.$$

So, for an atomic solution to exist h must equal $\left(1 + \frac{2}{\sqrt{5}}\right) x_0 e^{\frac{-1+\sqrt{5}}{2}(\frac{t^\alpha}{\alpha})} + \left(1 - \frac{2}{\sqrt{5}}\right) x_0 e^{\frac{-1-\sqrt{5}}{2}(\frac{t^\alpha}{\alpha})}$.

(iii) $u^{3\alpha}(t) = h.$

So from (24), we have

$$h = c_2 r_2^3 e^{\frac{r_2 t^\alpha}{\alpha}} + c_3 r_3^3 e^{\frac{r_3 t^\alpha}{\alpha}}.$$

Hence,

$$h = (\frac{2}{\sqrt{5}} + 1) x_0 e^{\frac{1+\sqrt{5}}{2}\frac{t^\alpha}{\alpha}} + (1 - \frac{2}{\sqrt{5}}) x_0 e^{\frac{1-\sqrt{5}}{2}\frac{t^\alpha}{\alpha}}.$$

Since $u^{2\alpha}(t) + u^\alpha(t) = u^{3\alpha}(t) = h$ in (ii) and (iii), there is an atomic solution .
This completes situation (2), and hence, **Case three** is completed.

Case four: $(u^{3\alpha} \otimes x + u^{2\alpha} \otimes Ax + u^\alpha \otimes Bx)$ is an atom.
This has two situations:

(1) $u^{3\alpha} = u^{2\alpha} = u^\alpha = h.$

(2) $x = Ax = Bx = z.$

Considering situation (1), equation (5) becomes:

$$u^{3\alpha} \otimes (x + Ax + Bx) = h \otimes z.$$

So, we have seven cases:

(a) $u^{3\alpha} = u^{2\alpha}$.

We can solve it as in [3]

$$r^3 - r^2 = r^2(r-1) = 0.$$

Hence, $r_1 = 0, r_2 = 0$, and $r_3 = 1$.

Consequently,

$$u(t) = c_1 + c_2(\frac{t^\alpha}{\alpha}) + c_3 e^{\frac{t^\alpha}{\alpha}}.$$

By assumption (4), we have

$$c_1 = x_0, c_2 = 0, \text{ and } c_3 = x_0.$$

Hence,

$$u(t) = x_0 + x_0 e^{\frac{t^\alpha}{\alpha}}.$$

(b) $u^{3\alpha} = u^{\alpha}$.

We can solve it as in [3]

$$r^3 - r = r(r^2-1) = r(r-1)(r+1) = 0.$$

Hence, $r_1 = 0, r_2 = 1$, and $r_3 = -1$.

Consequently,

$$u(t) = c_1 + c_2 e^{\frac{t^\alpha}{\alpha}} + c_3 e^{-\frac{t^\alpha}{\alpha}}.$$

By assumption (4), we have

$$c_1 = x_0, c_2 = x_0, \text{ and } c_3 = 0.$$

Hence,

$$u(t) = x_0 + x_0 e^{\frac{t^\alpha}{\alpha}}.$$

(c) $u^{2\alpha} = u^{\alpha}$.

We can solve it as in [3]

$$(r^2 - r) = r(r-1) = 0.$$

Hence, $r_1 = 0$ and $r_2 = 1$.

Consequently,

$$u(t) = c_1 + c_2 e^{\frac{t^\alpha}{\alpha}}$$

By assumption (4), we have

$$c_1 = x_0 \text{ and } c_2 = x_0.$$

Hence,

$$u(t) = x_0 + x_0 e^{\frac{t^\alpha}{\alpha}}.$$

(d) $u^{3\alpha} = h$.

Consequently,

$$h = x_0 e^{\frac{t^\alpha}{\alpha}}.$$

So, for an atomic solution to exist h must equal $x_0 e^{\frac{t^\alpha}{\alpha}}$.

(e) $u^{2\alpha} = h = x_0 e^{\frac{t^\alpha}{\alpha}}$.

(f) $u^{\alpha} = h = x_0 e^{\frac{t^\alpha}{\alpha}}$.

Hence, (e) and (f) give the same result, there is an atomic solution in this case and $h = x_0 e^{\frac{t^\alpha}{\alpha}}$.

In situation (2), equation (5) will be

$$e^{\frac{t^\alpha}{\alpha}} \otimes x + e^{\frac{t^\alpha}{\alpha}} \otimes Ax + e^{\frac{t^\alpha}{\alpha}} Bx = e^{\frac{t^\alpha}{\alpha}} \otimes z.$$

So,

$$(I + A + B)x = z.$$

Hence, z is the image of x under $(I + A + B)$.
This completes situation (2), and hence, **Case four** is completed.

Acknowledgements

The authors are grateful to the reviewers for their careful reading and valuable suggestions.

References

[1] T Abdeljawad. On conformable fractional calculus. *J. Comput. Appl. Math.*, 279:57–66, 2015.

[2] M Alhorani, M Abuhammad, and R Khalil. Variation of parameters for local fractional nonhomogeneous linear-differential equation. *J. Math. Computer Sci.*, 16:140–146, 2016.

[3] M Alhorani, R Khalil, and I Aldarawi. Fractional cauchy euler differential equation. *J. Comput. Anal. Appl.*, 28:226–233, 2020.

[4] R Khalil, M Alhorani, A Yousef, and M Sababheh. A new definition of fractional derivative. *J. Comput. Appl. Math.*, 264:65–70, 2014.

[5] L Rabhi, M Alhorani, and R Khalil. Inhomogeneous conformable abstract cauchy problem. *Open Mathematics*, 19:690–705, 2021.