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Some Properties for Certain Subclasses of Spiral-Like and Robertson Analytic Functions

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Abstract. Making use of the definition of subordination, we introduce certain subclasses of spiral-like and Robertson functions in the open unit disk and study some important results such as convolution results, coefficients estimate, subordination properties and Fekete-Szego problems for these subclasses. Further, some known and new outcomes which follow as special cases of our outcomes are also mentioned.

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1. Introduction

Denote $\mathcal{A}$ the family of all analytic functions of the form:

$$\psi(\xi) = \xi + \sum_{j=2}^{\infty} \rho_j \xi^j \tag{1}$$

in $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$. Let Ω be the family of analytic functions $\omega(\xi)$ in $\mathbb{U}$ that satisfy the conditions $\omega(0) = 0$ and $|\omega(\xi)| < 1\,(\xi \in \mathbb{U})$. If $\psi(\xi)$ and $\phi(\xi)$ are analytic in $\mathbb{U}$, we say that $\psi(\xi)$ is subordinate to $\phi(\xi)$, written $\psi(\xi) \prec \phi(\xi)$ if there exists $\omega(\xi) \in \Omega$, such that $\psi(\xi) = \phi(\omega(\xi))\,(\xi \in \mathbb{U})$ (see [6] and [13]).

For functions $\psi(\xi)$ given by (1) and $\phi(\xi)$ given by

$$\phi(\xi) = \xi + \sum_{j=2}^{\infty} \sigma_j \xi^j, \tag{2}$$

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the convolution of the functions $\psi(\xi)$ and $\phi(\xi)$ is defined by

$$(\psi * \phi)(\xi) = \xi + \sum_{j=2}^{\infty} \rho_j \ \sigma_j \xi^j = (\phi * \psi)(\xi). \tag{3}$$

For $|\gamma| < \frac{\pi}{2}$ and $-1 \leq D < C \leq 1$, a function $\psi(\xi)$ of $\mathcal{A}$ is said to be in $\mathcal{S}^\gamma[C, D]$ if it it satisfies the following subordination condition:

$$\mathrm{e}^{i\gamma} \frac{\xi\psi'(\xi)}{\psi(\xi)} \prec \cos\gamma \ \left(\frac{1+C\xi}{1+D\xi}\right) + i \ \sin\gamma, \tag{4}$$

also, let $\mathcal{K}^\gamma[C, D]$ denote the subfamily of all functions $\psi(\xi)$ in $\mathcal{A}$ satisfying the condition that $\xi\psi'(\xi) \in \mathcal{S}^\gamma[C, D]$. $\mathcal{S}^\gamma[C, D]$ and $\mathcal{K}^\gamma[C, D]$ are the subfamilies of spirallike and Robertson functions respectively studied by several authors earlier ([15], [4, 5]). We note that $\mathcal{S}^0[C, D] = \mathcal{S}[C, D]$, $\mathcal{K}^0[C; D] = \mathcal{K}[C; D]$ with $-1 \leq D < C \leq 1$, where the subfamilies $\mathcal{S}[C, D]$ and $\mathcal{K}[C; D]$ of Janowski functions are introduced and studied by many authors (see [1], [2], [7], [9], [10], [21] and [20]). Also, we have $\mathcal{S}^0[1-2\lambda, -1] = \mathcal{S}(\lambda)$ and $\mathcal{K}^0[1-2\lambda, -1] = \mathcal{K}(\lambda)$ with $0 \leq \lambda < 1$, where $\mathcal{S}^*(\lambda)$ and $\mathcal{K}(\lambda)$ denote the subfamilies of $\mathcal{A}$ that consists, respectively, of starlike of order λ and convex of order λ in $\mathbb{U}$ (see [17] and [19]).

Making use of the subordination, we combine the subfamilies $\mathcal{S}^\gamma[C, D]$ and $\mathcal{K}^\gamma[C, D]$ into a new subfamily $\mathcal{SK}^\gamma[\alpha, \beta; C, D]$ of $\mathcal{A}$ as follows:

Definition 1. *A function $\psi(\xi) \in \mathcal{A}$ is said to be in the subfamily $\mathcal{SK}^\gamma[\alpha, \beta; C, D]$ if it satisfies the following condition:*

$$\mathrm{e}^{i\gamma} \left[\frac{(\alpha+\beta)\xi\psi'(\xi) + \beta\xi^2\psi''(\xi)}{\alpha\psi(\xi) + \beta\xi\psi'(\xi)}\right] \prec \cos\gamma \ \left(\frac{1+C\xi}{1+D\xi}\right) + i \ \sin\gamma \tag{5}$$

$$\left(\xi \in \mathbb{U}; \alpha, \beta \geq 0; |\gamma| < \frac{\pi}{2}; -1 \leq D < C \leq 1\right).$$

We note that

(i) $\mathcal{SK}^\gamma[\alpha, 0; C, D] = \mathcal{S}^\gamma[C, D]$ (see [15])

$$\mathcal{S}^\gamma[C, D] = \left\{\psi(\xi) \in \mathcal{A} : \mathrm{e}^{i\gamma}\left[\frac{\xi\psi'(\xi)}{\psi(\xi)}\right] \prec \cos\gamma \ \left(\frac{1+C\xi}{1+D\xi}\right) + i \ \sin\gamma\right\};$$

(ii) $\mathcal{SK}^\gamma[0, \beta; C, D] = \mathcal{K}^\gamma[C, D]$ (see [4, 5])

$$\mathcal{K}^\gamma[C, D] = \left\{\psi(\xi) \in \mathcal{A} : \mathrm{e}^{i\gamma}\left[1 + \frac{\xi\psi''(\xi)}{\psi'(\xi)}\right] \prec \cos\gamma \ \left(\frac{1+C\xi}{1+D\xi}\right) + i \ \sin\gamma\right\};$$

(iii) $\mathcal{SK}^\gamma[\alpha, \beta; 1-2\lambda, -1] = \mathcal{SK}^\gamma(\alpha, \beta; \lambda)$ $(0 \leq \lambda < 1)$

$$\mathcal{SK}^\gamma(\alpha, \beta; \lambda) = \left\{\psi(\xi) \in \mathcal{A} : \Re\left\{\mathrm{e}^{i\gamma}\left[\frac{(\alpha+\beta)\xi\psi'(\xi) + \beta\xi^2\psi''(\xi)}{\alpha\psi(\xi) + \beta\xi\psi'(\xi)}\right]\right\} > \lambda\cos\gamma\right\};$$

(iv) $\mathcal{SK}^{\gamma}[\alpha,0;1-2\lambda,-1]=\mathcal{S}^{\gamma}(\lambda)\,(0\leq\lambda<1)$ (see [12] and [11])

$$\mathcal{S}^{\gamma}(\lambda)=\left\{\psi(\xi)\in\mathcal{A}:\Re\left[e^{i\gamma}\frac{\xi\psi'(\xi)}{\psi(\xi)}\right]>\lambda\cos\gamma\right\};$$

(v) $\mathcal{SK}^{\gamma}[\alpha,0;1,-1]=\mathcal{S}(\gamma)$ (see [23])

$$\mathcal{S}(\gamma)=\left\{\psi(\xi)\in\mathcal{A}:\Re\left[e^{i\gamma}\frac{\xi\psi'(\xi)}{\psi(\xi)}\right]>0\right\};$$

(vi) $\mathcal{SK}^{\gamma}[0,\beta;1-2\lambda,-1]=\mathcal{K}^{\gamma}(\lambda)\,(0\leq\lambda<1)$ (see [12] and [11])

$$\mathcal{K}^{\gamma}(\lambda)=\left\{\psi(\xi)\in\mathcal{A}:\Re\left\{e^{i\gamma}\left[1+\frac{\xi\psi''(\xi)}{\psi'(\xi)}\right]\right\}>\lambda\cos\gamma\right\};$$

(vii) $\mathcal{SK}^{\gamma}[0,\beta;1,-1]=\mathcal{K}(\gamma)$ (see [23])

$$\mathcal{K}(\gamma)=\left\{\psi(\xi)\in\mathcal{A}:\Re\left\{e^{i\gamma}\left[1+\frac{\xi\psi''(\xi)}{\psi'(\xi)}\right]\right\}>0\right\};$$

(viii) $\mathcal{SK}^{0}[\alpha,0;C,D]=\mathcal{S}[C,D]$ (see [9] and [10])

$$\mathcal{S}[C,D]=\left\{\psi(\xi)\in\mathcal{A}:\frac{\xi\psi'(\xi)}{\psi(\xi)}\prec\frac{1+C\xi}{1+D\xi}\right\};$$

(ix) $\mathcal{SK}^{0}[0,\beta;C,D]=\mathcal{K}[C,D]$ (see [9], [10] and [2])

$$\mathcal{K}[C,D]=\left\{\psi(\xi)\in\mathcal{A}:1+\frac{\xi\psi''(\xi)}{\psi'(\xi)}\prec\frac{1+C\xi}{1+D\xi}\right\};$$

(x) $\mathcal{SK}^{0}[\alpha,\beta;C,D]=\mathcal{SK}[\alpha,\beta;C,D]$

$$\mathcal{SK}[\alpha,\beta;C,D]=\left\{\psi(\xi)\in\mathcal{A}:\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^{2}\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}\prec\frac{1+C\xi}{1+D\xi}\right\};$$

(xi) $\mathcal{SK}^{0}[\alpha,\beta;1-2\lambda,-1]=\mathcal{SK}(\alpha,\beta;\lambda)\,(0\leq\lambda<1)$

$$\mathcal{SK}(\alpha,\beta;\lambda)=\left\{\psi(\xi)\in\mathcal{A}:\Re\left(\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^{2}\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}\right)>\lambda\right\},$$

$\mathcal{SK}(\alpha,0;\lambda)=\mathcal{S}(\lambda)$ and $\mathcal{SK}(\alpha,\beta;\lambda)=\mathcal{K}(\lambda)$ (see [17]);

(xii) $\mathcal{SK}^{0}[\alpha,\beta;(1-2\lambda)\,\eta,-\eta]=\mathcal{SK}(\alpha,\beta;\lambda,\eta)\,(0\leq\lambda<1,0<\eta\leq1)$

$$\mathcal{SK}(\alpha,\beta;\lambda,\eta)=\left\{\psi(\xi)\in\mathcal{A}:\left|\frac{\frac{(\alpha+\beta)\xi\psi'(\xi)+\beta\xi^{2}\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}-1}{\frac{(\alpha+\beta)\xi\psi'(\xi)+\beta\xi^{2}\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}+1-2\lambda}\right|<\eta\right\},$$

$\mathcal{SK}(\alpha,0;\lambda,\eta)=\mathcal{S}(\lambda,\eta)$ and $\mathcal{SK}(\alpha,\beta;\lambda,\eta)=\mathcal{K}(\lambda,\eta)$ (see [8]).

The aim of the present investigation is to define a general subfamily $\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$ of spirallike and Robertson functions. We then investigate some convolution properties, membership characterizations, coefficient estimates and subordination result for this subfamily. Furthermore, Fekete-Szegö problems and several inequalities are studied. Various corollaries and consequences of most of our outcomes are connected with earlier outcomes related to the field of investigation here.

2. Convolution Properties

We suppose throughout this paper that $\alpha,\beta\geq 0$, $|\chi|=1$, $-1\leq D<C\leq 1$, $|\gamma|<\frac{\pi}{2}$, $\xi\in\mathbb{U}$ and $\psi(\xi)\in\mathcal{A}$ given by (1).

Theorem 1. *$\psi(\xi)\in\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$ if and only if*

$$\frac{1}{\xi}\left[\psi(\xi)*\frac{\xi-\left(\frac{\alpha-\beta}{\alpha+\beta}+\frac{\alpha+2\beta}{\alpha+\beta}\Lambda\right)\xi^{2}+\frac{\alpha}{\alpha+\beta}\Lambda\xi^{3}}{(1-\xi)^{3}}\right]\neq 0, \tag{6}$$

where Λ is given by

$$\Lambda=\Lambda(\chi,\gamma,C,D)=\frac{(1+D\chi)\,e^{i\gamma}+(C-D)\cos\gamma\chi}{(C-D)\cos\gamma\chi}. \tag{7}$$

Proof. If $\psi(\xi)\in\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$, then there is a function $\omega(\xi)\in\Omega$ such that

$$e^{i\gamma}\left[\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^{2}\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}\right]=\cos\gamma\left(\frac{1+C\omega(\xi)}{1+D\omega(\xi)}\right)+i\,\sin\gamma, \tag{8}$$

hence

$$e^{i\gamma}\left[\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^{2}\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}\right]\neq\cos\gamma\left(\frac{1+C\chi}{1+D\chi}\right)+i\,\sin\gamma\quad(|\chi|=1),$$

which is equivalent to

$$\frac{1}{\xi}\left\{(1+D\chi)\,e^{i\gamma}\left[(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^{2}\psi''(\xi)\right]\neq 0\right.$$
$$\left.-\left[e^{i\gamma}+(C\cos\gamma+i\,D\sin\gamma)\,\chi\right]\left[\alpha\psi(\xi)+\beta\xi\psi'(\xi)\right]\right\}\neq 0 \tag{9}$$

It is easy to verify that

$$\psi(\xi)*\frac{\xi}{1-\xi}=\psi(\xi), \tag{10}$$

$$\psi(\xi)*\frac{\xi}{(1-\xi)^{2}}=\xi\psi'(\xi), \tag{11}$$

and

$$\psi(\xi)*\frac{2\xi^{2}}{(1-\xi)^{3}}=\xi^{2}\psi''(\xi). \tag{12}$$

Using (10),(11) and (12) in (9), we obtain

$$\frac{1}{\xi}\left\{(1+D\chi)\,\mathrm{e}^{i\gamma}\left[\psi(\xi)*\frac{(\alpha+\beta)\,\xi}{(1-\xi)^2}+\psi(\xi)*\frac{2\beta\xi^2}{(1-\xi)^3}\right]\right.$$
$$\left.-\left[\mathrm{e}^{i\gamma}+(C\cos\gamma+i\,D\sin\gamma)\,\chi\right]\left[\psi(\xi)*\frac{\alpha\xi}{1-\xi}+\psi(\xi)*\frac{\beta\xi}{(1-\xi)^2}\right]\right\}$$

$$=\frac{(\alpha+\beta)(D-C)\cos\gamma\chi}{\xi}\left\{\psi(\xi)*\frac{\xi-\left(\frac{\alpha-\beta}{\alpha+\beta}+\frac{\alpha+2\beta}{\alpha+\beta}\left[\frac{(1+D\chi)\mathrm{e}^{i\gamma}+(C-D)\cos\gamma\chi}{(C-D)\cos\gamma\chi}\right]\right)\xi^2+\frac{\alpha}{\alpha+\beta}\left[\frac{(1+D\chi)\mathrm{e}^{i\gamma}+(C-D)\cos\gamma\chi}{(C-D)\cos\gamma\chi}\right]\xi^3}{(1-\xi)^3}\right\}$$

$$=\frac{(\alpha+\beta)(D-C)\cos\gamma\chi}{\xi}\left\{\psi(\xi)*\frac{\xi-\left(\frac{\alpha-\beta}{\alpha+\beta}+\frac{\alpha+2\beta}{\alpha+\beta}\Lambda\right)\xi^2+\frac{\alpha}{\alpha+\beta}\Lambda\xi^2}{(1-\xi)^3}\right\}\neq 0$$

which shows the necessary condition of Theorem 1.

Reversely, since, the assumption (9) is equivalent to (6), we get that

$$\mathrm{e}^{i\gamma}\left[\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^2\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}\right]\neq\cos\gamma\,\left(\frac{1+C\chi}{1+D\chi}\right)+i\,\sin\gamma, \tag{13}$$

if we denote

$$\varphi(\xi)=\mathrm{e}^{i\gamma}\left[\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^2\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}\right]$$

and

$$\psi(\xi)=\cos\gamma\,\left(\frac{1+C\xi}{1+D\xi}\right)+i\,\sin\gamma,$$

the relation (13) proves that $\varphi(\mathbb{U})\cap\psi(\partial\mathbb{U})=\emptyset$. Thus, the simply-connected domain $\varphi(\mathbb{U})$ is subset of a connected component of $\mathbb{C}\backslash\psi(\partial\mathbb{U})$. From here, using the fact that $\varphi(0)=\psi(0)=\mathrm{e}^{i\gamma}$ together with the univalence $\psi(\xi)$, it follows that $\varphi(\xi)$ subordinate to $\psi(\xi)$, which leads in fact the subordination (7), i.e. $\psi(\xi)\in\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$. This completes Theorem 1.

Putting $\gamma=0$ in Theorem 1, we get

Corollary 1. $\psi(\xi)\in\mathcal{SK}[\alpha,\beta;C,D]$ *if and only if*

$$\frac{1}{\xi}\left[\psi(\xi)*\frac{\xi-\left(\frac{\alpha-\beta}{\alpha+\beta}+\frac{\alpha+2\beta}{\alpha+\beta}\Lambda_1\right)\xi^2+\frac{\alpha}{\alpha+\beta}\Lambda_1\xi^3}{(1-\xi)^3}\right]\neq 0,$$

where Λ_1 *is given by*

$$\Lambda_1=\frac{1+C\chi}{(C-D)\,\chi}. \tag{14}$$

Putting $\beta = 0$ in Theorem 1, we get

Corollary 2. *[4] $\psi(\xi) \in \mathcal{S}^{\gamma}[C, D]$ if and only if*

$$\frac{1}{\xi}\left[\psi(\xi) * \frac{\xi - \Lambda\xi^2}{(1-\xi)^2}\right] \neq 0,$$

where Λ is given by (7).

Putting $\alpha = 0$ in Theorem 1, we get

Corollary 3. *[5, Lemma 3 with $n = 1$] $\psi(\xi) \in \mathcal{K}^{\gamma}[C, D]$ if and only if*

$$\frac{1}{\xi}\left[\psi(\xi) * \frac{\xi - (2\Lambda - 1)\xi^2}{(1-\xi)^3}\right] \neq 0,$$

where Λ is given by (7).

Taking $C = 1 - 2\lambda\,(0 \leq \lambda < 1)$ and $D = -1$ in Theorem 1, we get

Corollary 4. *$\psi(\xi) \in \mathcal{SK}^{\gamma}(\alpha, \beta; \lambda)$ if and only if*

$$\frac{1}{\xi}\left[\psi(\xi) * \frac{\xi - \left(\frac{\alpha-\beta}{\alpha+\beta} + \frac{\alpha+2\beta}{\alpha+\beta}\Lambda_2\right)\xi^2 + \frac{\alpha}{\alpha+\beta}\Lambda_2\xi^3}{(1-\xi)^3}\right] \neq 0,$$

where

$$\Lambda_2 = \frac{(1-\chi)\,\mathrm{e}^{i\gamma} + 2(1-\lambda)\cos\gamma\chi}{2(1-\lambda)\cos\gamma\chi}.$$

Theorem 2. *$\psi(\xi) \in \mathcal{SK}^{\gamma}[\alpha, \beta; C, D]$ if and only if*

$$1 - \sum_{j=2}^{\infty}\left(\frac{\alpha + \beta j}{\alpha + \beta}\right)\frac{(j-1)(1 + D\chi)\,\mathrm{e}^{i\gamma} - (C - D)\cos\gamma\chi}{(C - D)\cos\gamma\chi}\rho_j\xi^{j-1} \neq 0. \tag{15}$$

Proof. From Theorem 1, we have $\psi(\xi) \in \mathcal{SK}^{\gamma}[\alpha, \beta; C, D]$ if and only if

$$\frac{1}{\xi}\left[\psi(\xi) * \frac{\xi - \left(\frac{\alpha-\beta}{\alpha+\beta} + \frac{\alpha+2\beta}{\alpha+\beta}\Lambda\right)\xi^2 + \frac{\alpha}{\alpha+\beta}\Lambda\xi^3}{(1-\xi)^3}\right] \neq 0 \tag{16}$$

for all Λ given by (7). The left hand side of (16) can be written as

$$\frac{1}{\xi}\left[\psi(\xi) * \left(\frac{\alpha\Lambda}{\alpha+\beta}\frac{\xi}{1-\xi} + \frac{\alpha+\beta-\alpha\Lambda}{\alpha+\beta}\frac{\xi}{(1-\xi)^2} + \frac{\beta(1-\Lambda)}{\alpha+\beta}\frac{2\xi}{(1-\xi)^3}\right)\right]$$

$$= \frac{1}{\xi}\left[\frac{\alpha\Lambda}{\alpha+\beta}\psi(\xi) + \frac{\alpha+\beta-\alpha\Lambda}{\alpha+\beta}\xi\psi'(\xi) + \frac{\beta(1-\Lambda)}{\alpha+\beta}\xi^2\psi''(\xi)\right]$$

$$= 1-\sum_{j=2}^{\infty}\left(\frac{\beta(\Lambda-1)}{\alpha+\beta}j^{2}+\frac{(\alpha-\beta)\Lambda-\alpha}{\alpha+\beta}j-\frac{\alpha\Lambda}{\alpha+\beta}\right)\rho_{j}\xi^{j-1}$$

$$= 1-\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)\frac{(j-1)(1+D\chi)\,\mathrm{e}^{i\gamma}-(C-D)\cos\gamma\chi}{(C-D)\cos\gamma\chi}\rho_{j}\xi^{j-1}.$$

Hence, the proof is completed.

Letting $\gamma = 0$ in Theorem 2, we obtain

Corollary 5. *$\psi(\xi) \in \mathcal{SK}[\alpha, \beta; C, D]$ if and only if*

$$1-\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)\frac{(j-1)(1+D\chi)-(C-D)\chi}{(C-D)\chi}\rho_{j}\xi^{j-1}\neq 0.$$

Taking $\beta = 0$ in Theorem 2, we get

Corollary 6. *$\psi(\xi) \in \mathcal{S}^{\gamma}[C, D]$ if and only if*

$$1-\sum_{j=2}^{\infty}\frac{(j-1)(1+D\chi)\,\mathrm{e}^{i\gamma}-(C-D)\cos\gamma\chi}{(C-D)\cos\gamma\chi}\rho_{j}\xi^{j-1}\neq 0.$$

Taking $\alpha = 0$ in Theorem 2, we get

Corollary 7. *$\psi(\xi) \in \mathcal{K}^{\gamma}[C, D]$ if and only if*

$$1-\sum_{j=2}^{\infty}j\,\frac{(j-1)(1+D\chi)\,\mathrm{e}^{i\gamma}-(C-D)\cos\gamma\chi}{(C-D)\cos\gamma\chi}\rho_{j}\xi^{j-1}\neq 0.$$

Taking $C = 1-2\lambda\,(0 \leq \lambda < 1)$ and $D = -1$ in Theorem 2, we get

Corollary 8. *$\psi(\xi) \in \mathcal{SK}^{\gamma}(\alpha, \beta; \lambda)$ if and only if*

$$1-\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)\frac{(j-1)(1-\chi)\,\mathrm{e}^{i\gamma}-2(1-\lambda)\cos\gamma\chi}{2(1-\lambda)\cos\gamma\chi}\rho_{j}\xi^{j-1}\neq 0. \tag{17}$$

3. Membership characterizations

Now we obtain several sufficient conditions for the subfamily $\mathcal{SK}^{\gamma}[\alpha, \beta; C, D]$.

Theorem 3. *Let $\psi(\xi) \in \mathcal{A}$ and let μ be a real number with $0 \leq \mu < 1$. If*

$$\left|\frac{(\alpha+\beta)\xi\psi'(\xi)+\beta\xi^{2}\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}-1\right|\leq 1-\mu \quad (\xi\in\mathbb{U}), \tag{18}$$

then $\psi(\xi) \in \mathcal{SK}^{\gamma}[\alpha, \beta; C, D]$ provided that

$$|\gamma|\leq\cos^{-1}\left[\frac{(1-\mu)(1-D)}{C-D}\right]. \tag{19}$$

Proof. From (18) it follows that

$$\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^2\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}=1+(1-\mu)\,\omega(\xi)\,,$$

where $\omega(\xi)\in\Omega$. We have

$$\begin{aligned}\Re\left\{e^{i\gamma}\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^2\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}\right\} &= \Re\left\{e^{i\gamma}\right\}+(1-\mu)\,\Re\left\{e^{i\gamma}\omega(\xi)\right\}\\ &\geq \cos\gamma-(1-\mu)\left|e^{i\gamma}\omega(\xi)\right|\\ &> \cos\gamma-(1-\mu)\\ &\geq \left(\frac{1-C}{1-D}\right)\cos\gamma\end{aligned}$$

provided that $|\gamma|\leq\cos^{-1}\left[\frac{(1-\mu)(1-D)}{C-D}\right]$. Thus, the proof is completed.

Putting $\mu=1-\frac{(C-D)\cos\gamma}{(1-D)}$ in Theorem 3, we obtain

Corollary 9. *If* $\psi(\xi)\in\mathcal{A}$ *with*

$$\left|\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^2\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}-1\right|\leq\frac{(C-D)\cos\gamma}{(1-D)}\quad(\xi\in\mathbb{U})\,,\tag{20}$$

then $\psi(\xi)\in\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$.

Putting $\gamma=0$ in Corollary 9, we obtain

Corollary 10. *If* $\psi(\xi)\in\mathcal{A}$ *with*

$$\left|\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^2\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}-1\right|\leq\frac{C-D}{1-D}\quad(\xi\in\mathbb{U})\,,$$

then $\psi(\xi)\in\mathcal{SK}[\alpha,\beta;C,D]$.

Putting $\beta=0$ in Corollary 9, we obtain

Corollary 11. *If* $\psi(\xi)\in\mathcal{A}$ *with*

$$\left|\frac{\xi\psi'(\xi)}{\psi(\xi)}-1\right|\leq\frac{(C-D)\cos\gamma}{(1-D)}\quad(\xi\in\mathbb{U})\,,$$

then $\psi(\xi)\in\mathcal{S}^{\gamma}[C,D]$.

Putting $\alpha=0$ in Corollary 9, we obtain

Corollary 12. *If* $\psi(\xi) \in \mathcal{A}$ *with*

$$\left|\frac{\xi^2\psi''(\xi)}{\xi\psi'(\xi)}\right| \leq \frac{(C-D)\cos\gamma}{(1-D)} \quad (\xi \in \mathbb{U}),$$

then $\psi(\xi) \in \mathcal{K}^\gamma[C,D]$.

Taking $C = 1-2\lambda\,(0 \leq \lambda < 1)$ and $D = -1$ in Corollary 9, we obtain

Corollary 13. *If* $\psi(\xi) \in \mathcal{A}$ *with*

$$\left|\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^2\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}-1\right| \leq (1-\lambda)\cos\gamma \quad (\xi \in \mathbb{U}),$$

then $\psi(\xi) \in \mathcal{SK}^\gamma(\alpha,\beta;\lambda)$.

In the next theorem, we obtain a coefficients theorem for $\mathcal{SK}^\gamma[\alpha,\beta;C,D]$.

Theorem 4. $\psi(\xi) \in \mathcal{SK}^\gamma[\alpha,\beta;C,D]$ *if*

$$\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)[(1-D)(j-1)+(C-D)\cos\gamma]\,|\rho_j| \leq (C-D)\cos\gamma. \tag{21}$$

Proof. From Corollary 9, it suffices to prove that (20) is satisfied. We have

$$\begin{aligned}\left|\frac{(\alpha+\beta)\,\xi\psi'(\xi)+\beta\xi^2\psi''(\xi)}{\alpha\psi(\xi)+\beta\xi\psi'(\xi)}-1\right| &= \left|\frac{\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)(j-1)\,\rho_j\xi^{j-1}}{1+\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)\rho_j\xi^{j-1}}\right| \\ &< \frac{\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)(j-1)\,|\rho_j|}{1-\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)|\rho_j|}.\end{aligned}$$

The last expression is bounded above by $\frac{(C-D)\cos\gamma}{(1-D)}$, if

$$\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)(j-1)\,|\rho_j| \leq \frac{(C-D)\cos\gamma}{(1-D)}\left[1-\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)|\rho_j|\right]$$

which is equivalent to

$$\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)[(1-D)(j-1)+(C-D)\cos\gamma]\,|\rho_j| \leq (C-D)\cos\gamma.$$

This completes the Theorem 4.

Putting $\gamma = 0$ in Theorem 4, we obtain

Corollary 14. *$\psi(\xi) \in \mathcal{SK}[\alpha,\beta;C,D]$ if*

$$\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)\left[(1-D)(j-1)+C-D\right]|\rho_j| \leq C-D. \tag{22}$$

Putting $\beta = 0$ in Theorem 4, we obtain

Corollary 15. *$\psi(\xi) \in \mathcal{S}^{\gamma}[C,D]$ if*

$$\sum_{j=2}^{\infty}\left[(1-D)(j-1)+(C-D)\cos\gamma\right]|\rho_j| \leq (C-D)\cos\gamma. \tag{23}$$

Putting $\alpha = 0$ in Theorem 4, we obtain

Corollary 16. *$\psi(\xi) \in \mathcal{K}^{\gamma}[C,D]$ if*

$$\sum_{j=2}^{\infty} j\left[(1-D)(j-1)+(C-D)\cos\gamma\right]|\rho_j| \leq (C-D)\cos\gamma. \tag{24}$$

Taking $C = 1-2\lambda\,(0 \leq \lambda < 1)$ and $D = -1$ in Theorem 4, we obtain

Corollary 17. *$\psi(\xi) \in \mathcal{SK}^{\gamma}(\alpha,\beta;\lambda)$ if*

$$\sum_{j=2}^{\infty}\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)\left[j-1+(1-\lambda)\cos\gamma\right]|\rho_j| \leq (1-\lambda)\cos\gamma. \tag{25}$$

4. Subordination result

Before proving our subordination result for $\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$, we shall make use the following definitions and a lemma.

Definition 2. *[24] We say that a complex sequence $\{\sigma_j\}_{j=1}^{\infty}$ is a subordinating factor sequence (SFS) if, whenever $\psi(\xi) = \xi + \sum\limits_{j=2}^{\infty}\rho_j\xi^j$ is univalent (analytic) and convex in $\mathbb{U}$, we have*

$$\sum_{j=1}^{\infty}\rho_j\;\sigma_j\;\prec\psi(\xi) \quad (\rho_1 = 1;\xi\in\mathbb{U}). \tag{26}$$

Lemma 1. *[24] The complex sequence $\{\sigma_j\}_{j=1}^{\infty}$ is a subordinating factor sequence (SFS) if and only if*

$$\Re\left\{1+2\sum_{j=1}^{\infty}\sigma_j\;\xi^j\right\} > 0 \quad (\xi\in\mathbb{U}). \tag{27}$$

Theorem 5. *Let* $\psi(\xi) \in \mathcal{SK}^{\gamma}[\alpha, \beta; C, D]$ *satisfy the coefficient inequality (21) and let* $\phi(\xi) \in \mathcal{K}$*, then*

$$\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{2\left[(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]\right]}(\psi*\phi)(\xi) \prec \phi(\xi) \tag{28}$$

and

$$\Re\{\psi(\xi)\} > -\frac{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}. \tag{29}$$

The constant factor

$$\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{2\left[(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]\right]}$$

in (28) cannot be replaced by a larger number.

Proof. Let $\psi(\xi) \in \mathcal{SK}^{\gamma}[\alpha, \beta; C, D]$ satisfy the coefficient inequality (21) and suppose that

$$\phi(\xi) = \xi + \sum_{j=2}^{\infty} \sigma_j \xi^j \in \mathcal{K}.$$

Then, by Definition 2, the condition (28) will hold true if

$$\left\{\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{2\left[(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]\right]}\rho_j\right\}_{j=1}^{\infty}$$

is a subordinating factor sequence, with $\sigma_1 = 1$. From Lemma 1, it is equivalent to the inequality

$$\Re\left\{1+\sum_{j=1}^{\infty}\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}\rho_j\ \xi^j\right\} > 0 \quad (\xi \in \mathbb{U}). \tag{30}$$

By noting the fact that

$$\left(\frac{\alpha+\beta j}{\alpha+\beta}\right)\left[\frac{(1-D)(j-1)+(C-D)\cos\gamma}{(C-D)\cos\gamma}\right]$$

is an increasing for $j \geq 2$. In view of (21), when $|\xi| = r < 1$, we have

$$\Re\left\{1+\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}\sum_{j=1}^{\infty}\rho_j\ \xi^j\right\}$$

$$
\begin{aligned}
&= \Re\left\{1+\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}\xi\right. \\
&\quad \left.+\frac{\sum_{j=2}^{\infty}\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]\,\rho_j\ \xi^j}{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}\right\} \\
&\geq 1-\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}r \\
&\quad -\frac{\sum_{j=2}^{\infty}\left(\frac{\alpha+j\beta}{\alpha+\beta}\right)[(1-D)(j-1)+(C-D)\cos\gamma]\,|\rho_j|\ r^j}{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]} \\
&\geq 1-\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}r \\
&\quad -\frac{(C-D)\cos\gamma}{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}r \\
&= 1-r>0 \quad (|\xi|=r<1).
\end{aligned}
$$

This proves (30) and (28). The inequality (29) follows from (28) by letting

$$\phi(\xi)=\frac{\xi}{1-\xi}=\xi+\sum_{j=2}^{\infty}\xi^j\in\mathcal{K}.$$

The sharpness of the multiplying factor in (28) can be established by considering a function

$$\Psi(\xi)=\xi-\frac{(C-D)\cos\gamma}{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}\xi^2.$$

Clearly $\Psi\in\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$ satisfy (21). Using (28) we infer that

$$\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{2\left\{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]\right\}}\Psi(\xi)\prec\frac{\xi}{1-\xi},$$

and it follows that

$$\min_{|\xi|\leq r}\left\{\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{2\left\{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]\right\}}\Re\{\Psi(\xi)\}\right\}=-\frac{1}{2}.$$

This shows that the constant

$$\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]}{2\left\{(C-D)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1-D+(C-D)\cos\gamma]\right\}}$$

cannot be replaced by any larger one.

For $\gamma = 0$ in Theorem 5, we get

Corollary 18. *Let $\psi(\xi) \in \mathcal{SK}[\alpha,\beta;C,D]$ satisfy the coefficient inequality (22) and let $\phi(\xi) \in \mathcal{K}$, then*

$$\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)(1-2D+C)}{2\left[C-D+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)(1-2D+C)\right]}(\psi*\phi)(\xi) \prec \phi(\xi) \tag{31}$$

and

$$\Re\{\psi(\xi)\} > -\frac{C-D+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)(1-2D+C)}{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)(1-2D+C)}. \tag{32}$$

The constant factor

$$\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)(1-2D+C)}{2\left[C-D+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)(1-2D+C)\right]}$$

in (31) cannot be replaced by a larger number.

Taking $\beta = 0$ in Theorem 5, we get

Corollary 19. *Let $\psi(\xi) \in \mathcal{S}^{\gamma}[C,D]$ satisfy the coefficient inequality (23) and let $\phi(\xi) \in \mathcal{K}$, then*

$$\frac{1-D+(C-D)\cos\gamma}{2[2(C-D)\cos\gamma+1-D]}(\psi*\phi)(\xi) \prec \phi(\xi) \tag{33}$$

and

$$\Re\{\psi(\xi)\} > -\frac{2(C-D)\cos\gamma+1-D}{1-D+(C-D)\cos\gamma}. \tag{34}$$

The constant factor

$$\frac{1-D+(C-D)\cos\gamma}{2[2(C-D)\cos\gamma+1-D]}$$

in (33) cannot be replaced by a larger number.

Taking $\alpha = 0$ in Theorem 5, we get

Corollary 20. *Let $\psi(\xi) \in \mathcal{K}^{\gamma}[C, D]$ satisfy the coefficient inequality (24) and let $\phi(\xi) \in \mathcal{K}$, then*

$$\frac{1-D+(C-D)\cos\gamma}{3(C-D)\cos\gamma+2(1-D)}(\psi * \phi)(\xi) \prec \phi(\xi) \tag{35}$$

and

$$\Re\{\psi(\xi)\} > -\frac{3(C-D)\cos\gamma+2(1-D)}{2[1-D+(C-D)\cos\gamma]}. \tag{36}$$

The constant factor

$$\frac{1-D+(C-D)\cos\gamma}{3(C-D)\cos\gamma+2(1-D)}$$

in (35) cannot be replaced by a larger number.

Taking $C = 1-2\lambda\,(0 \leq \lambda < 1)$ and $D = -1$ in Theorem 5, we get

Corollary 21. *Let $\psi(\xi) \in \mathcal{SK}^{\gamma}(\alpha, \beta; \lambda)$ satisfy the coefficient inequality (25) and let $\phi(\xi) \in \mathcal{K}$, then*

$$\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1+(1-\lambda)\cos\gamma]}{2\left\{(1-\lambda)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1+(1-\lambda)\cos\gamma]\right\}}(\psi * \phi)(\xi) \prec \phi(\xi) \tag{37}$$

and

$$\Re\{\psi(\xi)\} > -\frac{(1-\lambda)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1+(1-\lambda)\cos\gamma]}{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1+(1-\lambda)\cos\gamma]}. \tag{38}$$

The constant factor

$$\frac{\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1+(1-\lambda)\cos\gamma]}{2\left\{(1-\lambda)\cos\gamma+\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)[1+(1-\lambda)\cos\gamma]\right\}}$$

in (37) cannot be replaced by a larger number.

5. Fekete-Szegö problems

The Fekete-Szegö problem consists in finding upper-bounds for $\left|\rho_3 - \mu\rho_2^2\right|$ for various subfamilies of analytic functions (see [3], [16], [18] and [22]). In order to get upper-bounds for $\left|\rho_3 - \mu\rho_2^2\right|$ for the subfamily $\mathcal{SK}^{\gamma}[\alpha, \beta; C, D]$ the next lemma is required.

Lemma 2. *[14, p.108]Let $\omega \in \Omega$ be given by*

$$\omega(\xi) = \sum_{j=1}^{\infty} \omega_j\ \xi^j \quad (\xi \in \mathbb{U}).$$

Then

$$|\omega_1| \leq 1, \quad |\omega_2| \leq 1 - |\omega_1|^2, \tag{39}$$

and

$$\left|\omega_2 - \nu\, \omega_1^2\right| \leq \max\{1, |\nu|\}, \tag{40}$$

for any complex number $\nu \in \mathbb{C}$. *The functions* $\omega(\xi) = \xi$ *and* $\omega(\xi) = \xi^2$ *or one of their rotations show that both inequalities (39) and (40) are sharp.*

First we obtain upper-bounds for $\left|\rho_3 - \mu\rho_2^2\right|$ with $\mu \in \mathbb{R}$.

Theorem 6. *Let* $\psi(\xi) \in \mathcal{SK}^\gamma[\alpha, \beta; C, D]$ *and let* $\mu \in \mathbb{R}$. *Then*

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \begin{cases} \frac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\left[-D + (C-D)\left(1 - \frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right)\right] & (\mu \leq \vartheta_1) \\ \frac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)} & (\vartheta_1 \leq \mu \leq \vartheta_2) \\ \frac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\left[D - (C-D)\left(1 - \frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right)\right] & (\mu \geq \vartheta_2) \end{cases} \tag{41}$$

where

$$\vartheta_1 = \frac{(\alpha+2\beta)^2(C-2D-1)}{2(\alpha+\beta)(\alpha+3\beta)(C-D)} \tag{42}$$

$$\vartheta_2 = \frac{(\alpha+2\beta)^2(C-2D+1)}{2(\alpha+\beta)(\alpha+3\beta)(C-D)}. \tag{43}$$

Proof. Suppose that $\psi(\xi)$ is in $\mathcal{SK}^\gamma[\alpha, \beta; C, D]$. Then, from the definition of the subclass $\mathcal{SK}^\gamma[\alpha, \beta; C, D]$, there exists

$$\omega(\xi) = \omega_1\xi + \omega_2\xi^2 + \omega_3\xi^3 + ... \in \Omega$$

such that

$$e^{i\gamma}\left[\frac{(\alpha+\beta)\xi\psi'(\xi) + \beta\xi^2\psi''(\xi)}{\alpha\psi(\xi) + \beta\xi\psi'(\xi)}\right] = \cos\gamma\left(\frac{1 + C\omega(\xi)}{1 + D\omega(\xi)}\right) + i\,\sin\gamma \quad (\xi \in \mathbb{U}). \tag{44}$$

We have

$$e^{i\gamma}\left[\frac{(\alpha+\beta)\xi\psi'(\xi) + \beta\xi^2\psi''(\xi)}{\alpha\psi(\xi) + \beta\xi\psi'(\xi)}\right] = e^{i\gamma} + e^{i\gamma}\left(\frac{\alpha+2\beta}{\alpha+\beta}\right)\rho_2\xi$$
$$+e^{i\gamma}\left[\frac{2(\alpha+3\beta)}{\alpha+\beta}\rho_3 - \frac{(\alpha+2\beta)^2}{(\alpha+\beta)^2}\rho_2^2\right]\xi^2 + ... \tag{45}$$

and

$$\cos\gamma\left(\frac{1+C\omega(\xi)}{1+D\omega(\xi)}\right) + i\sin\gamma = e^{i\gamma} + (C-D)\cos\gamma\omega_1\xi + (C-D)\cos\gamma\left(\omega_2 - D\omega_1^2\right)\xi^2 + ... \tag{46}$$

By using (45) and (46), equating the coefficients of ξ and ξ^2 on both sides of (44), we have

$$\rho_2 = \frac{(\alpha+\beta)(C-D)e^{-i\gamma}\cos\gamma}{\alpha+2\beta}\,\omega_1 \tag{47}$$

and

$$\rho_3 = \frac{(\alpha+\beta)(C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma}{2(\alpha+3\beta)}\left[\omega_2 + \left(-D + (C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma\right)\omega_1^2\right]. \tag{48}$$

It follows

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \tfrac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\left\{|\omega_2| + \left|-D + (C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma\left[1 - \tfrac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right]\right|\ |\omega_1|^2\right\}$$

Making use of Lemma 2 we have

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \tfrac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\left\{1 + \left(\left|-D + (C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma\left[1 - \tfrac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right]\right| - 1\right)\ |\omega_1|^2\right\}$$

or

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \tfrac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\left[1 + \left(\sqrt{D^2 + \Pi(\Pi - 2D)\cos^2\gamma} - 1\right)\ |\omega_1|^2\right], \tag{49}$$

where

$$\Pi = (C-D)\left[1 - \frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right]. \tag{50}$$

Denote by

$$H(x,y) = 1 + \left(\sqrt{D^2 + \Pi(\Pi - 2D)x^2} - 1\right)\ y^2$$

where $x = \cos\gamma$, $y = |\omega_1|$ and $(x,y) : [0,1] \times [0,1]$. Simple calculation shows that $H(x,y)$ does not have a local maximum at any interior point of the rectangle $(0,1) \times (0,1)$. Thus, the maximum must be attained at a boundary point. Since $H(x,0) = 1$, $H(0,y) = 1 + (|D| - 1)y^2 \leq 1$ and $H(1,1) = |\Pi - D|$, it follows that the maximal value of $H(x,y)$ may be $H(0,0) = 1$ or $H(1,1) = |\Pi - D|$. Hence, from (49) we obtain

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \tfrac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\max\{1, |\Pi - D|\}, \tag{51}$$

where Π is given by (50). Consider first the case $|\Pi - D| \geq 1$. If $\mu \leq \vartheta_1$, where ϑ_1 is given by (42), then $\Pi \geq 1 + D$ and from (51) we obtain

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \tfrac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\left[-D + (C-D)\left(1 - \tfrac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right)\right]$$

which is the first part of the inequality (41). If $\mu \geq \vartheta_2$, where ϑ_2 is given by (43), then $\Pi \leq D - 1$ and it follows from (51) that

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \tfrac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\left[D - (C-D)\left(1 - \tfrac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right)\right]$$

and this is the third part of (41).

Next, suppose $\vartheta_1 \leq \mu \leq \vartheta_2$. Then, $|\Pi - D| \leq 1$ and thus, from (51) we obtain

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \tfrac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}$$

which is the second part of the inequality (41).

For $\gamma = 0$ in Theorem 6, we obtain

Corollary 22. *Let* $\psi(\xi) \in \mathcal{SK}[\alpha, \beta; C, D]$ *and let* $\mu \in \mathbb{R}$. *Then*

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \begin{cases} \frac{(\alpha+\beta)(C-D)}{2(\alpha+3\beta)}\left[-D + (C-D)\left(1 - \frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right)\right] & (\mu \leq \vartheta_1) \\ \frac{(\alpha+\beta)(C-D)}{2(\alpha+3\beta)} & (\vartheta_1 \leq \mu \leq \vartheta_2) \\ \frac{(\alpha+\beta)(C-D)}{2(\alpha+3\beta)}\left[D - (C-D)\left(1 - \frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right)\right] & (\mu \geq \vartheta_2) \end{cases}$$

where ϑ_1 *and* ϑ_2 *are given by (42) and (43).*

Taking $\beta = 0$ in Theorem 6, we obtain

Corollary 23. *Let* $\psi(\xi) \in \mathcal{S}^\gamma[C, D]$ *and let* $\mu \in \mathbb{R}$. *Then*

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \begin{cases} \frac{(C-D)\cos\gamma}{2}\left[-D + (C-D)(1-2\mu)\right] & (\mu \leq \vartheta_1) \\ \frac{(C-D)\cos\gamma}{2} & (\vartheta_1 \leq \mu \leq \vartheta_2) \\ \frac{(C-D)\cos\gamma}{2}\left[D - (C-D)(1-2\mu)\right] & (\mu \geq \vartheta_2) \end{cases}$$

where

$$\vartheta_3 = \frac{C - 2D - 1}{2(C-D)}, \vartheta_4 = \frac{C - 2D + 1}{2(C-D)}.$$

Taking $\alpha = 0$ in Theorem 6, we obtain

Corollary 24. *Let* $\psi(\xi) \in \mathcal{K}^\gamma[C, D]$ *and let* $\mu \in \mathbb{R}$. *Then*

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \begin{cases} \frac{(C-D)\cos\gamma}{6}\left[-D + (C-D)\left(1 - \frac{3}{2}\mu\right)\right] & (\mu \leq \vartheta_5) \\ \frac{(C-D)\cos\gamma}{6} & (\vartheta_5 \leq \mu \leq \vartheta_6) \\ \frac{(C-D)\cos\gamma}{6}\left[D - (C-D)\left(1 - \frac{3}{2}\mu\right)\right] & (\mu \geq \vartheta_6) \end{cases}$$

where

$$\vartheta_5 = \frac{2(C - 2D - 1)}{3(C-D)}, \vartheta_6 = \frac{2(C - 2D + 1)}{3(C-D)}.$$

Taking $C = 1 - 2\lambda\,(0 \leq \lambda < 1)$ and $D = -1$ in Theorem 6, we obtain

Corollary 25. *Let* $\psi(\xi) \in \mathcal{SK}^\gamma(\alpha, \beta; \lambda)$ *and let* $\mu \in \mathbb{R}$. *Then*

$$\left|\rho_3 - \mu\rho_2^2\right| \leq \begin{cases} \frac{(\alpha+\beta)(1-\lambda)\cos\gamma}{(\alpha+3\beta)}\left[1 + 2(1-\lambda)\left(1 - \frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right)\right] & (\mu \leq \vartheta_7) \\ \frac{(\alpha+\beta)(1-\lambda)\cos\gamma}{(\alpha+3\beta)} & (\vartheta_7 \leq \mu \leq \vartheta_8) \\ \frac{(\alpha+\beta)(1-\lambda)\cos\gamma}{(\alpha+3\beta)}\left[-1 - 2(1-\lambda)\left(1 - \frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}\right)\right] & (\mu \geq \vartheta_8) \end{cases}$$

where

$$\vartheta_7 = \frac{(\alpha + 2\beta)^2(1-\lambda)}{2(\alpha+\beta)(\alpha+3\beta)(1-\lambda)}$$

$$\vartheta_8 = \frac{(\alpha + 2\beta)^2(2-\lambda)}{2(\alpha+\beta)(\alpha+3\beta)(1-\lambda)}.$$

We consider the Fekete-Szegö problem for the subclass $\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$ with complex parameter $\mu\in\mathbb{C}$.

Theorem 7. *Let $\psi(\xi)\in\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$ and let $\mu\in\mathbb{C}$. Then,*

$$\left|\rho_3-\mu\rho_2^2\right|\leq\frac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\max\left\{1,\left|D+(C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma\left[\frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}-1\right]\right|\right\}. \tag{52}$$

Proof. Assume that $\psi(\xi)\in\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$. Making use of (47) and (48) we obtain

$$\left|\rho_3-\mu\rho_2^2\right|\leq\frac{(\alpha+\beta)(C-D)\cos\gamma}{2(\alpha+3\beta)}\left|\omega_2-\left(D+(C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma\left[\frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}-1\right]\right)\,\omega_1^2\,\right|$$

The inequality (52) follows as an application of Lemma 2 with

$$\nu=D+(C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma\left[\frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}-1\right].$$

For $\gamma=0$ in Theorem 7, we obtain

Corollary 26. *Let $\psi(\xi)\in\mathcal{SK}[\alpha,\beta;C,D]$ and let $\mu\in\mathbb{C}$. Then,*

$$\left|\rho_3-\mu\rho_2^2\right|\leq\frac{(\alpha+\beta)(C-D)}{2(\alpha+3\beta)}\max\left\{1,\left|D+(C-D)\left[\frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}-1\right]\right|\right\}.$$

Taking $\beta=0$ in Theorem 7, we obtain

Corollary 27. *Let $\psi(\xi)\in\mathcal{S}^{\gamma}[C,D]$ and let $\mu\in\mathbb{C}$. Then,*

$$\left|\rho_3-\mu\rho_2^2\right|\leq\frac{(C-D)\cos\gamma}{2}\max\left\{1,\left|D+(C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma\,(2\mu-1)\right|\right\}.$$

Taking $\alpha=0$ in Theorem 7, we obtain

Corollary 28. *Let $\psi(\xi)\in\mathcal{K}^{\gamma}[C,D]$ and let $\mu\in\mathbb{C}$. Then,*

$$\left|\rho_3-\mu\rho_2^2\right|\leq\frac{(C-D)\cos\gamma}{6}\max\left\{1,\left|D+(C-D)\,\mathrm{e}^{-i\gamma}\cos\gamma\left(\tfrac{3}{2}\mu-1\right)\right|\right\}.$$

Taking $C=1-2\lambda\,(0\leq\lambda<1)$ and $D=-1$ in Theorem 6, we obtain

Corollary 29. *Let $\psi(\xi)\in\mathcal{SK}^{\gamma}(\alpha,\beta;\lambda)$ and let $\mu\in\mathbb{C}$. Then,*

$$\left|\rho_3-\mu\rho_2^2\right|\leq\frac{(\alpha+\beta)(1-\lambda)\cos\gamma}{(\alpha+3\beta)}\max\left\{1,\left|-1+2\,(1-\lambda)\,\mathrm{e}^{-i\gamma}\cos\gamma\left[\frac{2\mu(\alpha+\beta)(\alpha+3\beta)}{(\alpha+2\beta)^2}-1\right]\right|\right\}.$$

6. Conclusions

In our present investigation, we have defined a general subclass $\mathcal{SK}^{\gamma}[\alpha,\beta;C,D]$ of spirallike and Robertson analytic functions. For functions belonging to this subclass, we have derived some interesting results such as convolution properties, membership characterizations, coefficient estimates, subordination result and the Fekete-Szegö estimates. Furthermore, interesting corollaries and particular cases are shown for each of those results for particular choices of parameters found in the definition of this subclass. Our results are connected with those in several earlier works, which are related to the Geometric Function Theory. Moreover, these results can be extended to multivalent functions and meromophic functions.

References

[1] O P Ahuja. Families of analytic functions related to ruscheweyh derivatives and subordinate to convex functions. *Yokohama Math.*, 41:39–50, 1993.

[2] M K Aouf and T M Seoudy. Classes of analytic functions related to the dziok-srivastava operator. *Integral Transforms Spec. Funct.*, 22(6):423–430, 2011.

[3] M K Aouf and T M Seoudy. Fekete–szegö problem for certain subclass of analytic functions with complex order defined by q–analogue of ruscheweyh operator. *Constructive Math. Anal.*, 3(1):36–44, 2020.

[4] S S Bhoosnurmath and M V Devadas. Subclasses of spirallike functions defined by subordination. *J. of Analysis, Madras*, 4:173–183, 1996.

[5] S S Bhoosnurmath and M V Devadas. Subclasses of spirallike functions defined by ruschweyh derivatives. *Tamkang J. Math.*, 28:59–65, 1997.

[6] T Bulboacă. *Differential Subordinations and Superordinations, Recent Results.* House of Scientific Book Publ., Cluj-Napoca, 2005.

[7] R M Goel and B S Mehrok. On the coefficients of a subclass of starlike functions. *Indian J. Pure Appl. Math.*, 12:634–647, 1981.

[8] V P Gupta and P K Jain. Certain classes of univalent functions with negative coefficients ii. *Bull. Austral. Math. Soc.*, 15(3):467–473, 1976.

[9] W Janowski. Some extremal problems for certain families of analytic functions. *Bull. Polish Acad. Sci.*, 21:17–25, 1973.

[10] W Janowski. Some extremal problems for certain families of analytic functions. *Ann. Polon. Math.*, 28:297–326, 1973.

[11] F R Keogh and E P Merkes. A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.*, 20:8–12, 1969.

[12] R J Libera. Univalent a-spiral functions. *Canad. J. Math.*, 19:449–456, 1967.

[13] S S Miller and P T Mocanu. *Differential Subordination: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225.* Marcel Dekker Inc., New York and Basel, 2000.

[14] Z Nehari. *Conformal mapping.* McGraw-Hill, New-York, 1952.

[15] S V Nikitin. A class of regular functions, current problems in function theory (russian). *Rostov-Gos. Univ. Rostov-on-Don.*, 188:143–147, 1987.

[16] H Orhan, D Raducanu, M Caglar, and M Rayram. Coefficient estimates and other properties for a class of spirallike functions associated with a differential operator. *Abstr. Anal. Appl.*, Art. ID 415319:1–7, 2013.

[17] M S Robertson. On the theory of univalent functions. *Ann. Math.*, 37:374–408, 1936.

[18] T M Seoudy. Convolution results and fekete-szegö inequalities for certain classes of symmetric q-starlike and symmetric q-convex functions. *J. Math.*, Art. ID 8203921:1–11, 2022.

[19] H. Silverman. Univalent functions with negative coefficients. *Proc. Amer. Math. Soc.*, 51:109–116, 1975.

[20] H Silverman and E M Silvia. Subclasses of starlike functions subordinate to convex functions. *Canad. J. Math.*, 1:48–61, 1985.

[21] H Silverman, E M Silvia, and D Telage. Convolution conditions for convexity, starlikeness and spiral-likeness. *Math. Z.*, 162:125–130, 1978.

[22] H M Srivastava, T M Seoudy, and M K Aouf. A generalized conic domain and its applications to certain subclasses of multivalent functions associated with the basic (or q-) calculus. *AIMS Math.*, 6:6580–6602, 2021.

[23] L Špaček. Contribution à la theorie des fonctions univalents. *Cas. Mat. Fys.*, 62:12–19, 1932.

[24] H S Wilf. Subordinating factor sequence for convex maps of the unit circle. *Proc. Amer. Math. Soc.*, 12:689–693, 1961.