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Multi-step Residual Power Series Method for Solving Stiff Systems

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Abstract. In this paper, an efficient algorithm based on the residual power series method (RPSM) is presented to solve stiff systems of Caputo fractional order. We apply the RPSM on subintervals to get approximate solutions of these types of systems. The RPSM has advantages that it is suitable to solve linear and nonlinear systems and it gives high accurate results. Modifying this technique to multi-step RPSM considerably reduces the number of arithmetic operations and so reduces the time, especially when dealing with Stiff systems. Several numerical examples are given to show the efficiency, simplicity and the accuracy of the proposed method. Comparing classical RPSM with the new multi-step scheme shows that multi-step RPSM controls the convergence behaviour of the stiff systems. That is, the comparison reveals that MS-FRPSM reduces both absolute and residual errors. More iterations and a smaller step size lead to higher accuracy. Moreover, in MS-FRPSM, the intervals of convergence for the series solution will increase.

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Key Words and Phrases: Multi-step method, Residual power series method, Stiff system, Numerical solution, Caputo fractional derivative

1. Introduction

In various fields such as chemical kinetics, aerodynamics, ballistics, electrical circuit theory, and missile guidance, there are specific types of differential equations that pose significant challenges for solution using classical numerical methods [6]. As a result, many analytic and numerical methods have been developed to deal with such equations. Examples for these methods and the real-life applications of differential equations can be found in [14], [15],[23],[1], and [2].
Curtiss and Hirschfelder were the first to bring attention to these challenging differential equations, commonly referred to as stiff equations [8]. The mathematical stiffness of a problem arises from the disparate rates of various processes within the considered physical

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models. This phenomenon occurs when certain solution components decay significantly faster than others, characterized by terms such as $e^{-\lambda t}$ where $\lambda > 0$.
In the last few years, several analytical methods have been exploited to give approximate solutions for stiff systems of ordinary differential equations such as homotopy perturbation method [4], block method [3], second derivative multistep method [22], variational iteration method [5], and multi-step reproducing kernel Hilbert space method [12].
The residual power series method is one of the efficient numerical techniques employed for solving various types of differential equations. For a comprehensive understanding of this method and its applications in solving differential equations as well as integro-differential equations of distinct nature, readers are encouraged to refer to the relevant literature[16],[11], [10],[13], [21],[17], [20].

Unfortunately, when attempting to employ this method to tackle a stiff system, it encounters limitations. The resulting approximate solutions remain valid only within a narrow interval, displaying slow convergence or complete divergence when applied to a broader range. It gives numerical results with rapidly increasing error of approximation. Moreover, increasing the number of iterations results in a necessity of a large computer memory and large time to run. To overcome these drawbacks, we formulate an algorithm that depends on dividing any time interval into small subintervals and applying the RPSM on each subinterval. We call this technique a multi-step residual power series method (MS- RPSM). However, fractional differential equations (FDEs) become widely referred because of their capabilities to model several problems in engineering and science such as mechanical systems, dynamical systems, heat transfer, control theory, mixed convection flows, unification of diffusion, image processing, and wave propagation phenomenon [18],[7],[19],[25], and [24]. This is because the fascinating properties of distinct fractional operators, such as the ability to capture long-range dependencies and memory effects in various systems. As a result, we apply in this work the MS-RPSM as a treatment to get accurate solutions for stiff systems with the well-known Caputo fractional derivative.

2. Basics of Caputo operators and RPSM

Fractional residual power series method (FRPSM) is an analytical as well as numerical method based on the concept on the error function and the generalized Taylor series. In fact, it produces solutions in the form of rapidly convergent fractional power series (FPS) with computationally straightforward components. Basic definitions that are related to FPS and Caputo derivative are given below.

Definition 1. *[9] A power series representation of the form*

$$\sum_{k=0}^{\infty} b_k (z-z_0)^{k\beta} = b_0 + b_1 (z-z_0)^{\beta} + b_2 (z-z_0)^{2\beta} + \cdots,$$

where $0 \leq m-1 < \beta \leq m \in \mathbb{N}$, is called FPS about $z = z_0$ and b_k, $k = 0, 1, 2, \ldots$ are the coefficients for the series.
In order to present the definition of Caputo derivative, we need to define the space $C_{\gamma}^{n}[t_0, b]$

as follows: Let $q(t)$ be a function defined on $[t_0, b]$ and γ be a real number. We say that $q(t)$ is in the space $C_\gamma[t_0, b]$ if there is a continuous function $q_1(t)$ on $[t_0, b]$ such that $q(t) = t^\rho q_1(t)$ for a real number $\rho > C_\gamma$. Moreover, it is said to be in the space $C^n_\gamma[t_0, b]$ if its classical nth derivative is in the space $C_\gamma[t_0, b]$.

Definition 2. *[19] Assume that $n-1 < \beta \le n$, $n \in \mathbb{N}$, and $q(t) \in C^n_{-1}[t_0, b]$. Then the Caputo derivative of $q(t)$ of order β is given by*

$$ {}^C_{t_0}D^\beta_t q(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_{t_0}^t \frac{d^n q(\tau)}{d\tau^n}(t-\tau)^{n-\beta-1}\, d\tau, & t > t_0, \\ \frac{d^n q(t)}{dt^n}, & \beta = n. \end{cases} \tag{1} $$

Now, we summarize the FRPSM as described in [10] through the following algorithm.
Algorithm 2.1 To get the FPS solution of the fractional initial value problem (FIVP)

$$ {}^C_{t_0}D^\beta_t q(t) = B(t, q(t)), \quad t \ge t_0, \quad 0 < \beta \le 1, \tag{2} $$
$$ q(t_0) = q_0, $$

where B is a linear or nonlinear function of t and the unknown $q(t)$, and $q_0 \in \mathbb{R}$ is the initial condition, we follow the steps:
Step 1: Assume the solution of (2) is of the FPS form

$$ q(t) = \sum_{v=0}^{\infty} b_v (t-t_0)^{v\beta}. \tag{3} $$

Step 2: Use the initial condition $q(t_0) = q_0$ as the zeroth coefficient of the FPS. That is, $b_0 = q_0$.
Step 3: Define the R-th truncated FPS as

$$ q_R(t) = b_0 + \sum_{v=1}^{R} b_v (t-t_0)^{v\beta}. \tag{4} $$

Step 4: Define the residual function and the N-th residual function, respectively, as

$$ \text{Resid}_q(t) = {}^C_0D^\beta_t q(t) - B(t, q(t)), \tag{5} $$
$$ \text{Resid}_{q,N}(t) = {}^C_0D^\beta_t q_N(t) - B(t, q_N(t)). \tag{6} $$

Obviously,

$$ \text{Resid}_q(t) = \lim_{N\to\infty} \text{Resid}_{q,N}(t) = 0, $$

and

$$ {}^C_0D^\beta_t \text{Resid}_q(t) = 0, \quad \text{for} \quad t \ge 0. $$

Step 5: For $k = 0, 1, 2, \ldots, R-1$, compute

$$ ({}^C_0D^{(k-1)\beta}_t \text{Resid}_{q,N})(t_0) = 0. $$

Step 6: Solve the resulting equation in Step 5 to get the coefficient b_R.
Step 7: Repeat Steps 3-6 N-times to get the N-th truncated FPS where N can be chosen to achieve the required accuracy.

3. Description of the MS-RPSM

In this section, we provide a modification to Algorithm 2.1 to get the MS-FRPSM in order to solve stiff systems of Caputo fractional order, which may be of the specific form

$$ {}_t^C D_a^{\beta} G_i(t) = F_i(t, G_1, G_2, \ldots, G_m), \quad G_i(a) = c_i, \tag{7} $$

$a \geq 0$, $t \in [a, b], i = 1, 2, \ldots, m, \quad m \in \mathbb{N}, \quad 0 < \beta \leq 1.$
To clarify the MS-RPSM in which we will use to obtain the approximate solution of system (7), we split the interval [a,b] into subintervals $[t_{n-1}, t_n]$,$n = 1, 2, \ldots, M$, of step size $h = \frac{b-a}{M}$ and nodes $t_0 = a$, $t_n = a + nh$. Then we solve each FIVP on its corresponding subinterval:

$$ {}_t^C D_{t_{n-1}}^{\beta} \, {}^nG_i(t) = F_i\left(t, {}^nG_1, {}^nG_2, \ldots, {}^nG_m\right), \quad 0 < \beta \leq 1, \tag{8} $$

$$ {}^1G_i(t_0) = c_i, \quad {}^nG_i(t_{n-1}) = {}^{n-1}G_i(t_{n-1}), $$

$$ t \in [t_{n-1}, t_n], \quad i = 1, 2, \ldots, m, \quad m \in \mathbb{N}. $$

Now, we assume the solution of system (7) has the form:

$$ G_i(t) = \begin{cases} {}^1G_i(t), & \text{if } t \in [a, t_1], \\ {}^2G_i(t), & \text{if } t \in [t_1, t_2], \\ \vdots & \\ {}^MG_i(t), & \text{if } t \in [t_{M-1}, b], \end{cases} = \begin{cases} \sum_{v=0}^{\infty} \frac{a_{v1}(t-t_0)^{v\beta}}{\Gamma(v\beta+1)} & \text{if } t \in [a, t_1], \\ \sum_{v=0}^{\infty} \frac{a_{v2}(t-t_1)^{v\beta}}{\Gamma(v\beta+1)} & \text{if } t \in [t_1, t_2], \\ \vdots & \\ \sum_{v=0}^{\infty} \frac{a_{vM}(t-t_{M-1})^{v\beta}}{\Gamma(v\beta+1)} & \text{if } t \in [t_{M-1}, b], \end{cases} $$

and the N-th truncated solution has the form

$$ G_{i,N}(t) = \begin{cases} {}^1G_{i,N}(t), & \text{if } t \in [a, t_1], \\ {}^2G_{i,N}(t), & \text{if } t \in [t_1, t_2], \\ \vdots & \\ {}^MG_{i,N}(t), & \text{if } t \in [t_{M-1}, b], \end{cases} = \begin{cases} \sum_{v=0}^{N} \frac{a_{v1}(t-t_0)^{v\beta}}{\Gamma(v\beta+1)} & \text{if } t \in [a, t_1], \\ \sum_{v=0}^{N} \frac{a_{v2}(t-t_1)^{v\beta}}{\Gamma(v\beta+1)} & \text{if } t \in [t_1, t_2], \\ \vdots & \\ \sum_{v=0}^{N} \frac{a_{vM}(t-t_{M-1})^{v\beta}}{\Gamma(v\beta+1)} & \text{if } t \in [t_{M-1}, b], \end{cases} $$

for each $i = 1, 2, \ldots, m$. Hence the residual function can be written as follows:

$$ \text{Resid}_{G_i}(t) = \begin{cases} {}_t^C D_a^{\beta} \, {}^1G_i(t) - F_i(t, {}^1G_1, {}^1G_2, \ldots, {}^1G_m), & \text{if } t \in [a, t_1], \\ {}_t^C D_{t_1}^{\beta} \, {}^2G_i(t) - F_i(t, {}^2G_1, {}^2G_2, \ldots, {}^2G_m), & \text{if } t \in [t_1, t_2], \\ \vdots & \\ {}_t^C D_{t_{M-1}}^{\beta} \, {}^MG_i(t) - F_i(t, {}^MG_{1,N}, {}^MG_{2,N}, \ldots, {}^MG_{m,N}), & \text{if } t \in [t_{M-1}, b], \end{cases} $$

and the Nth residual function has the form:

$$\text{Resid}_{G_{i,N}}(t) = \begin{cases} {}_t^C D_a^\beta \, {}^1G_{i,N}(t) - F_i(t, {}^1G_{1,N}, {}^1G_{2,N}, \dots, {}^1G_{m,N}), & t \in [a, t_1] \\ {}_t^C D_{t_1}^\beta \, {}^2G_{i,N}(t) - F_i(t, {}^2G_{1,N}, {}^2G_{2,N}, \dots, {}^2G_{m,N}), & t \in [t_1, t_2] \\ \vdots & \vdots \\ {}_t^C D_{t_{M-1}}^\beta \, {}^MG_{i,N}(t) - F_i(t, {}^MG_{1,N}, {}^MG_{2,N}, \dots, {}^MG_{m,N}), & t \in [t_{M-1}, b] \end{cases}$$

for each $i = 1, 2, \dots, m$.

For more illustrations, we apply the FRPSM to the system with $n = 1$. That is, to the first subinterval on which the system has the form:

$${}_t^C D_a^\beta \, {}^1G_i(t) = F_i(t, {}^1G_1, {}^1G_2, \dots, {}^1G_m), \quad 0 < \beta \leq 1, \quad t \in [a, t_1], \quad {}^1G_i(a) = c_i, \tag{9}$$

to get the Nth truncated FPS of the form:

$${}^1G_{i,N}(t) = \sum_{v=0}^{N} \frac{a_{v1}(t-a)^{v\beta}}{\Gamma(v\beta + 1)}, \quad t \in [a, t_1], \quad i = 1, 2, \dots, m.$$

After that, we take the second system, i.e., when $n = 1$, which has the form

$${}_t^C D_{t_1}^\beta \, {}^2G_i(t) = F_i(t, {}^2G_1, {}^2G_2, \dots, {}^2G_m), \quad 0 < \beta \leq 1, \quad t \in [t_1, t_2], \quad {}^2G_i(t_1) = {}^1G_i(t_1). \tag{10}$$

Applying the FRPSM on system (10) gives us the Nth truncated FPS:

$${}^2G_{i,N}(t) = \sum_{v=0}^{N} \frac{a_{v2}(t-t_1)^{v\beta}}{\Gamma(v\beta + 1)}, \quad t \in [t_1, t_2], \quad i = 1, 2, \dots, m.$$

Repeating this process for the M systems, we get the approximate solution for the stiff system in (7) in the form of a piecewise-defined function in which its components are the Nth truncated FPS:

$$G_{i,N}(t) = \begin{cases} {}^1G_{i,N}(t), & t \in [a, t_1], \\ {}^2G_{i,N}(t), & t \in [t_1, t_2], \\ \vdots & \vdots \\ {}^MG_{i,N}(t), & t \in [t_{M-1}, b], \end{cases}$$

and the exact solution can be obtained from $\lim_{N\to\infty} G_{i,N}(t)$.

4. Numerical applications

In this section, we test the efficiency of the MS-FRPSM in solving stiff system through solving three numerical examples. We make a comparison between classical FRPS and MS-FRPSM in some examples to see the importance of minimizing the interval length to obtain better approximations. Since the exact solutions of fractional stiff systems are not available, we compute the residual errors to check our computations. In all of our examples, we used Mathematica 10 software.

Example 4.1 Consider the following linear fractional system:

$$
\begin{gathered}
{}_t^C D_0^{\beta} G(t) = -G(t) + 95W(t), \quad t \in [0,1], \quad 0 < \beta \leq 1, \\
{}_t^C D_0^{\beta} W(t) = -G(t) + 97W(t), \\
G(0) = 1, \quad W(0) = 1.
\end{gathered} \tag{11}
$$

The exact solution for the system (11) when $\beta = 1$ is

$$
G(t) = \frac{1}{47} e^{-96t}(-48 + 95e^{94t}), \quad W(t) = \frac{1}{47} e^{-96t}(-48 + e^{94t}).
$$

If we try to solve (11) using the classical FRPSM, we assume the solution has the FPS:

$$
G(t) = 1 + \sum_{v=1}^{\infty} \frac{a_v}{\Gamma(1+v\beta)} t^{v\beta}, \quad \text{and} \quad W(t) = 1 + \sum_{v=1}^{\infty} \frac{b_v}{\Gamma(1+v\beta)} t^{v\beta}.
$$

And now we define the N-th truncated series for the forms

$$
G_N(t) = 1 + \sum_{v=1}^{N} \frac{a_v}{\Gamma(1+v\beta)} t^{v\beta}, \quad \text{and} \quad W_N(t) = 1 + \sum_{v=1}^{N} \frac{b_v}{\Gamma(1+v\beta)} t^{v\beta}.
$$

The N-th truncated residual function are:

$$
Resid_{G,N} =_t^C D_0^{\beta} G_N(t) + G_N(t) - 95W_N(t), \quad t \in [0,1] Resid_{W,N} =_t^C D_0^{\beta} W_N(t) + G_N(t) - 97W_N(t).
$$

When $N = 1$, we solve $\text{Resid}_{W,1}(0) = 0$ and $\text{Resid}_{G,1}(0) = 0$ to get the values of a_1 and b_1 as:

$$
a_1 = 94, \quad b_1 = -98.
$$

When $N = 2$, we solve ${}_0^C D_t^{\beta} \text{Resid}_{W,2}(0) = 0$ and ${}_t^C D_0^{\beta} \text{Resid}_{G,2}(0) = 0$ to get the values of a_2 and b_2 as

$$
a_2 = -9404, \quad b_2 = 9412.
$$

When $N = 3$, we solve ${}_t^C D_0^{2\beta} \text{Resid}_{W,3}(0) = 0$ and ${}_t^C D_0^{2\beta} \text{Resid}_{G,3}(0) = 0$ to get the values of a_3 and b_3 as

$$
a_3 = 903544, \quad b_3 = -903560,
$$

and so on up to the Nth coefficients which can be obtained by solving ${}_t^C D_0^{(N-1)\beta} \text{Resid}_{W,N}(0) = 0$ and ${}_t^C D_0^{(N-1)\beta} \text{Resid}_{G,N}(0) = 0$. Some of the first coefficients are obtained as follows:

$$
a_4 = -86741744, \quad b_4 = 86741776,
$$

$$a_5 = 832710464, \quad b_5 = -832710528,$$

$$a_6 = -799412210624, \quad b_6 = 79912210752,$$

$$a_7 = 76743572232064, \quad b_7 = -76743572232320,$$

$$a_8 = -7367382934302464, \quad b_8 = 7367382934302976,$$

$$a_9 = 707268761693085184, \quad b_9 = -707268761693086208,$$

$$a_{10} = -67897801122536274944, \quad b_{10} = 67897801122536276992.$$

In fact, we use Mathematica software to get the 30th truncated FPS of $G(t)$ and $W(t)$, and compute some numerical and display some graphical results. Table 1 shows the values of the exact and approximate solutions when $\beta = 1$ in addition to the absolute error. Also, it presents numerical solutions for different values of β. The residual error function is computed in Table 2 for $\beta = 1, 0.99, 0.98, 0.97$ and the approximate solutions for the same values of β are shown in Figure 1.

Table 1 Numerical results for classical FRPSM for example 4.1

t_i	$G(t)$	$G_{30}(t)$, $\beta = 1$	$\lvert G(t) - G_{30}(t)\rvert$, $\beta = 1$	$G_{30}(t)$, $\beta = 0.99$	$G_{30}(t)$, $\beta = 0.98$	$G_{30}(t)$, $\beta = 0.97$
0.0	1.00000	1.00000	0.00000	1.00000	1.00000	1.00000
0.1	1.65481	1.65454	0.000269072	1.64314	1.62541	1.57125
0.2	1.35490	-468189	468191	-2.19151×10^{6}	-1.02179×10^{24}	-4.74525×10^{7}
0.3	1.10930	-1.13099×10^{11}	1.13099×10^{11}	-4.64964×10^{11}	-1.90422×10^{24}	-7.76854×10^{12}
0.4	0.908218	-7.27568×10^{14}	7.27568×10^{14}	-2.72922×10^{15}	-1.01996×10^{24}	-3.79751×10^{16}
0.5	0.74359	-6.45204×10^{17}	6.45204×10^{17}	-2.25504×10^{18}	-7.85291×10^{24}	-2.72471×10^{19}
0.6	0.608797	-1.63816×10^{20}	1.63816×10^{20}	-5.40562×10^{20}	-1.77741×10^{24}	-5.82343×10^{21}
0.7	0.498441	-1.75739×10^{22}	1.75739×10^{22}	-5.52509×10^{22}	-1.73098×10^{24}	-5.40409×10^{23}
0.8	0.408089	-1.00451×10^{24}	1.004514×10^{24}	-3.02892×10^{24}	-9.10183×10^{24}	-2.72564×10^{25}
0.9	0.334115	-3.55209×10^{25}	3.55209×10^{25}	-1.03246×10^{26}	-2.99082×10^{24}	-8.63426×10^{26}
1.0	0.27355	-8.60381×10^{26}	8.60381×10^{26}	-2.42025×10^{28}	-6.78535×10^{24}	-1.89592×10^{28}
t_i	$W(t)$	$W_{30}(t)$, $\beta = 1$	$\lvert W(t) - W_{30}(t)\rvert$, $\beta = 1$	$W_{30}(t)$, $\beta = 0.99$	$W_{30}(t)$, $\beta = 0.98$	$W_{30}(t)$, $\beta = 0.97$
0.0	1.00000	1.00000	0.00000	1.00000	1.00000	1.00000
0.1	0.01735	- 0.0170816	0.000269072	-0.0143377	-0.00539508	0.0398389
0.2	0.0142621	468191	468191	2.19151×10^{6}	1.02179×10^{7}	4.74525×10^{7}
0.3	0.0116768	1.13099×10^{11}	1.13099×10^{11}	4.64964×10^{11}	1.90422×10^{12}	7.76854×10^{12}
0.4	0.00956019	7.27568×10^{14}	7.27568×10^{14}	2.72922×10^{15}	1.01996×10^{16}	3.79751×10^{16}
0.5	0.00782722	6.45204×10^{17}	6.45204×10^{17}	2.25504×10^{18}	7.85291×10^{18}	2.72471×10^{19}
0.6	0.00640839	1.63816×10^{20}	1.63816×10^{20}	5.40562×10^{20}	1.77741×10^{21}	5.82343×10^{21}
0.7	0.00524674	1.75739×10^{22}	1.75739×10^{22}	5.52509×10^{22}	1.73098×10^{23}	5.40409×10^{23}
0.8	0.003517	1.00451×10^{24}	1.00451×10^{24}	3.02892×10^{24}	9.10183×10^{24}	-2.72564×10^{25}
0.9	0.003517	3.55209×10^{25}	3.55209×10^{25}	1.03246×10^{26}	2.99082×10^{26}	8.63426×10^{26}
1.0	0.00287947	8.60381×10^{26}	8.60381×10^{26}	2.42025×10^{28}	6.78535×10^{24}	1.89592×10^{28}

We note from Tables 1-2 and Figure 1 that the absolute and the residual errors are very large and still rapidly increase. Hence, we have to apply the MS-FRPSM so we may reduce the errors. We solve system (11) again with a step size of $h = 0.1$, so we divide the interval into 10 subintervals. Assuming the solution has the form:

Table 2 Some numerical values of the residual function for classical FRPSM, example 4.1

t_i	$\lvert Resid_{(G,30)}(t)\rvert$				$\lvert Resid_{(W,30)}(t)\rvert$			
	$\beta = 1$	$\beta = 0.99$	$\beta = 0.98$	$\beta = 0.97$	$\beta = 1$	$\beta = 0.99$	$\beta = 0.98$	$\beta = 0.97$
0.1	3.27×10^{-3}	3.00522	2.13794	2.21345	4.49×10^{-3}	2.10×10^{8}	9.80×10^{8}	4.55×10^{9}
0.2	4.49×10^{7}	2.10×10^{8}	9.80×10^{8}	4.55×10^{9}	1.08×10^{13}	4.46×10^{13}	1.82×10^{14}	7.45×10^{14}
0.3	1.08×10^{13}	4.46×10^{13}	1.82×10^{14}	7.45×10^{14}	6.98×10^{16}	2.62×10^{17}	9.79×10^{17}	3.64×10^{18}
0.4	6.98×10^{16}	2.62×10^{17}	9.79×10^{17}	3.64×10^{18}	6.19×10^{19}	2.16×10^{20}	7.53×10^{20}	2.61×10^{21}
0.5	6.19×10^{19}	2.16×10^{20}	7.53×10^{20}	2.61×10^{21}	1.57×10^{22}	5.18×10^{22}	1.70×10^{23}	5.59×10^{23}
0.6	1.57×10^{22}	5.18×10^{22}	1.70×10^{23}	5.59×10^{23}	1.68×10^{24}	5.30×10^{24}	1.66×10^{25}	5.18×10^{25}
0.7	1.68×10^{24}	5.30×10^{24}	1.66×10^{25}	5.18×10^{25}	9.64×10^{25}	2.90×10^{26}	8.73×10^{26}	2.61×10^{27}
0.8	9.64×10^{25}	2.90×10^{26}	8.73×10^{26}	2.61×10^{27}	3.41×10^{27}	9.91×10^{27}	2.87×10^{28}	8.28×10^{28}
0.9	3.41×10^{27}	9.91×10^{27}	2.87×10^{28}	8.28×10^{28}	8.25×10^{28}	2.32×10^{29}	6.51×10^{29}	1.82×10^{30}
1.0	8.25×10^{28}	2.32×10^{29}	6.51×10^{29}	1.82×10^{30}	8.25×10^{28}	2.32×10^{29}	6.51×10^{29}	1.82×10^{30}

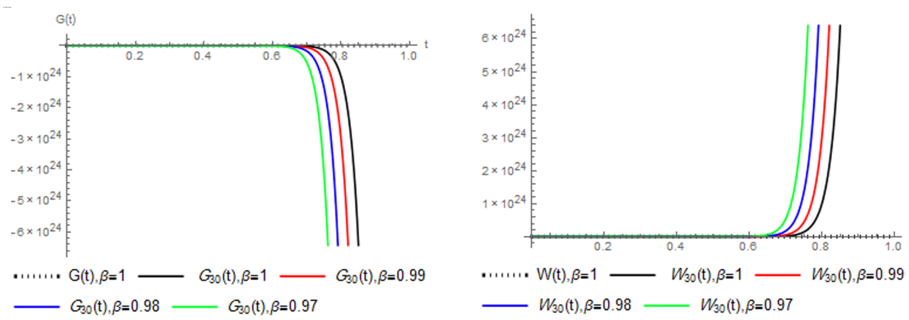

Figure 1 : Approximate solution curves using classical FRPSM, example 4.1

$$G(t) = \begin{cases} {}^{1}G(t) & \text{for } t \in [0, 0.1], \\ {}^{2}G(t) & \text{for } t \in [0.1, 0.2], \\ {}^{3}G(t) & \text{for } t \in [0.2, 0.3], \\ {}^{4}G(t) & \text{for } t \in [0.3, 0.4], \\ {}^{5}G(t) & \text{for } t \in [0.4, 0.5], \\ {}^{6}G(t) & \text{for } t \in [0.5, 0.6], \\ {}^{7}G(t) & \text{for } t \in [0.6, 0.7], \\ {}^{8}G(t) & \text{for } t \in [0.7, 0.8], \\ {}^{9}G(t) & \text{for } t \in [0.8, 0.9], \\ {}^{10}G(t) & \text{for } t \in [0.9, 1], \end{cases} \qquad W(t) = \begin{cases} {}^{1}W(t) & \text{for } t \in [0, 0.1], \\ {}^{2}W(t) & \text{for } t \in [0.1, 0.2], \\ {}^{3}W(t) & \text{for } t \in [0.2, 0.3], \\ {}^{4}W(t) & \text{for } t \in [0.3, 0.4], \\ {}^{5}W(t) & \text{for } t \in [0.4, 0.5], \\ {}^{6}W(t) & \text{for } t \in [0.5, 0.6], \\ {}^{7}W(t) & \text{for } t \in [0.6, 0.7], \\ {}^{8}W(t) & \text{for } t \in [0.7, 0.8], \\ {}^{9}W(t) & \text{for } t \in [0.8, 0.9], \\ {}^{10}W(t) & \text{for } t \in [0.9, 1], \end{cases}$$

and the Nth truncated FPS solution has the form:

$$G(t)=\begin{cases} {}^{1}G_N(t) & \text{for } t\in[0,0.1],\\ {}^{2}G_N(t) & \text{for } t\in[0.1,0.2],\\ {}^{3}G_N(t) & \text{for } t\in[0.2,0.3],\\ {}^{4}G_N(t) & \text{for } t\in[0.3,0.4],\\ {}^{5}G_N(t) & \text{for } t\in[0.4,0.5],\\ {}^{6}G_N(t) & \text{for } t\in[0.5,0.6],\\ {}^{7}G_N(t) & \text{for } t\in[0.6,0.7],\\ {}^{8}G_N(t) & \text{for } t\in[0.7,0.8],\\ {}^{9}G_N(t) & \text{for } t\in[0.8,0.9],\\ {}^{10}G_N(t) & \text{for } t\in[0.9,1], \end{cases} \qquad W(t)=\begin{cases} {}^{1}W_N(t) & \text{for } t\in[0,0.1],\\ {}^{2}W_N(t) & \text{for } t\in[0.1,0.2],\\ {}^{3}W_N(t) & \text{for } t\in[0.2,0.3],\\ {}^{4}W_N(t) & \text{for } t\in[0.3,0.4],\\ {}^{5}W_N(t) & \text{for } t\in[0.4,0.5],\\ {}^{6}W_N(t) & \text{for } t\in[0.5,0.6],\\ {}^{7}W_N(t) & \text{for } t\in[0.6,0.7],\\ {}^{8}W(t) & \text{for } t\in[0.7,0.8],\\ {}^{9}W_N(t) & \text{for } t\in[0.8,0.9],\\ {}^{10}W_N(t) & \text{for } t\in[0.9,1], \end{cases}$$

${}^{n}G_N(t)=\sum_{v=0}^{N}\frac{a_{vn}(t-t_{n-1})^{v\beta}}{\Gamma(v\beta+1)}$ and ${}^{n}W_N(t)=\sum_{v=0}^{N}\frac{b_{vn}(t-t_{n-1})^{v\beta}}{\Gamma(v\beta+1)}$.

For $N=3$, the results are as follows:

$$ {}^{1}G_3(t)=1-\frac{9404t^{2\beta}}{\Gamma(2\beta+1)}+\frac{903544t^{3\beta}}{\Gamma(3\beta+1)}+\frac{94t^{\beta}}{\Gamma(\beta+1)}, $$

$$ \begin{aligned} {}^{2}G_3(t)=1&+\frac{94(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{9404(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{903544(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{3.277291397699514(t-0.1)^{\beta}}{\Gamma(\beta+1)}\\ &+\frac{3.5022880095564233\ (t-0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{286.01572342177724(t-0.1)^{3\beta}}{\Gamma(3\beta+1)}, \end{aligned} $$

$$ \begin{aligned} {}^{3}G_3(t)=&1+\frac{90.72270860230049(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{9400.497711990443(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\ &+\frac{903830.0157234218(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{2.709793687108408(t-0.2)^{\beta}}{\Gamma(\beta+1)}\\ &+\frac{5.418576468893888(t-0.2)^{2\beta}}{\Gamma(2\beta+1)}-\frac{10.74010602678672(t-0.2)^{3\beta}}{\Gamma(3\beta+1)}, \end{aligned} $$

$$\begin{aligned}
{}^{4}G_{3}(t) =& 1-\frac{2.709793687108408(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{90.72270860230049(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&+\frac{5.418576468893888(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{9400.497711990443(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{10.74010602678672(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{903830.0157234218\ (0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&-\frac{2.218600227456377(t-0.3)^{\beta}}{\Gamma(\beta+1)}+\frac{4.437200120106484(t-0.3)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{8.874368098810919(t-0.3)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}$$

$$\begin{aligned}
{}^{5}G_{3}(t) =& 1-\frac{2.709793687108408(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{90.72270860230049(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{2.218600227456377(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{5.418576468893888(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{9400.497711990443(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{4.437200120106484(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{10.74010602678672(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{903830.0157234218(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&-\frac{8.874368098810919(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{1.816436237919282(t-0.4)^{\beta}}{\Gamma(\beta+1)}\\
&+\frac{3.632872475726769(t-0.4)^{2\beta}}{\Gamma(2\beta+1)}-\frac{7.265744940721219(t-0.4)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}$$

$$\begin{aligned}
{}^{6}G_{3}(t) =& 1-\frac{4.52622992502769(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{90.72270860230049(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{2.218600227456377(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{9.051448944620656(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{9400.497711990443(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{4.437200120106484(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{18.00585096750794(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{903830.0157234218(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&-\frac{8.874368098810919(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{1.4871722089907775(t-0.5)^{\beta}}{\Gamma(\beta+1)}\\
&+\frac{2.9743444179828282(t-0.5)^{2\beta}}{\Gamma(2\beta+1)}-\frac{5.948688836087882(t-0.5)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}$$

$$
\begin{aligned}
{}^{7}G_3(t) =& 1-\frac{6.013402134018468(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{90.72270860230049(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{2.218600227456377(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{12.025793362603483(0.1)^{2.0\beta}}{\Gamma(2.0\beta+1)}\\
&-\frac{9400.497711990443(0.1)^{2.0\beta}}{\Gamma(2.0\beta+1)}+\frac{4.437200120106484(0.1)^{2.0\beta}}{\Gamma(2.0\beta+1)}\\
&-\frac{23.95453980359582(0.1)^{3.0\beta}}{\Gamma(3.0\beta+1.0)}+\frac{903830.0157234218(0.1)^{3.0\beta}}{\Gamma(3.0\beta+1)}\\
&-\frac{8.874368098810919(0.1)^{3.0\beta}}{\Gamma(3.0\beta+1)}-\frac{1.2175936226236566(t-0.6)^{\beta}}{\Gamma(\beta+1.0)}\\
&+\frac{2.4351872452475702(t-0.6)^{2.0\beta}}{\Gamma(2.0\beta+1)}-\frac{4.870374490519822\ (t-0.6)^{3.0\beta}}{\Gamma(3.0\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
{}^{8}G_3(t) =& 1-\frac{7.230995756642124(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{90.72270860230049(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{2.218600227456377(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{14.460980607851054(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{9400.497711990443(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{4.437200120106484(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{28.824914294115644(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{903830.0157234218\cdot(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&-\frac{8.874368098810919(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{0.9968813435936204(t-0.7)^{\beta}}{\Gamma(\beta+1)}\\
&+\frac{1.9937626871880143(t-0.7)^{2\beta}}{\Gamma(2\beta+1)}-\frac{3.9875253744502994(t-0.7)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
{}^{9}G_3(t) =& 1-\frac{7.230995756642124\ (0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{90.72270860230049(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{2.218600227456377(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{0.9968813435936204(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{14.460980607851054(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{9400.497711990443(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{4.437200120106484(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{1.9937626871880143(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{28.824914294115644(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&+\frac{903830.0157234218(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{8.874368098810919(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{3.9875253744502994(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&-\frac{0.8161774131697912(t-0.8)^{\beta}}{\Gamma(\beta+1)}+\frac{1.6323548263398586(t-0.8)^{2\beta}}{\Gamma(2\beta+1)}-\frac{3.2647096527062445(t-0.8)^{3\beta}}{\Gamma(3\beta+1)}.
\end{aligned}
$$

$$
\begin{aligned}
{}^{10}G_3(t) =& 1-\frac{8.047173169811915\cdot(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{90.72270860230049(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{2.218600227456377(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{0.9968813435936204(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{16.093335434190912(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{9400.497711990443(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{4.437200120106484(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{1.9937626871880143(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{32.08962394682189(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&+\frac{903830.0157234218(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{8.874368098810919(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{3.9875253744502994(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&-\frac{0.66822954813(t-0.9)^{\beta}}{\Gamma(\beta+1)}+\frac{1.33645909626(t-0.9)^{2\beta}}{\Gamma(2\beta+1)}-\frac{2.67291819253(t-0.9)^{3\beta}}{\Gamma(3\beta+1)}.
\end{aligned}
$$

$$
{}^{1}W_3(t)=1+\frac{9412\,t^{2\beta}}{\Gamma(2\beta+1)}-\frac{903560\,t^{3\beta}}{\Gamma(3\beta+1)}-\frac{98\,t^{\beta}}{\Gamma(\beta+1)},
$$

$$
\begin{aligned}
{}^{2}W_3(t) =& 1-\frac{98(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{9412(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{903560(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{0.0023683853879674643(t-0.1)^{\beta}}{\Gamma(\beta+1)}\\
&+\frac{3.04755801506667(t-0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{299.1154154710234(t-0.1)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
{}^{3}W_3(t) =& 1-\frac{97.99763161461203(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{9415.047558015067(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&-\frac{903859.115415471(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{0.02851350296616295(t-0.2)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{0.056016100609398656(t-0.2)^{2\beta}}{\Gamma(2\beta+1)}+\frac{0.0149852902177372(t-0.2)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
{}^{4}W_3(t) =& 1+\frac{0.02851350296616295(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{97.99763161461203(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{0.056016100609398656(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{9415.047558015067(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{0.0149852902177372(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{903859.115415471(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&+\frac{0.023353683080527432(t-0.3)^{\beta}}{\Gamma(\beta+1)}-\frac{0.04670703135478371(t-0.3)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{0.0933819213075534(t-0.3)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
{}^{5}W_3(t) =& 1+\frac{0.02851350296616295(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{97.99763161461203(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&+\frac{0.02335368308052743 2(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{0.056016100609398656(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{9415.047558015067(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{0.04670703135478371(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{0.0149852902177372(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{903859.115415471(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&+\frac{0.0933819213075534(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{0.01912038145060513(t-0.4)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{0.03824076278941546(t-0.4)^{2\beta}}{\Gamma(2\beta+1)}+\frac{0.07648151484653454(t-0.4)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
{}^{6}W_3(t) =& 1+\frac{0.04763388441676808(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{97.99763161461203(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{0.023353683080527432(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{0.09425686339881412(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{9415.047558015067(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{0.04670703135478371(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{0.09146680506427174(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{903859.115415471(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{0.0933819213075534(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&+\frac{0.015654444305179482(t-0.5)^{\beta}}{\Gamma(\beta+1)}-\frac{0.03130888861163217(t-0.5)^{2\beta}}{\Gamma(2\beta+1)}+\frac{0.06261777734550833(t-0.5)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
{}^{7}W_3(t) =& 1+\frac{0.06328832872194756(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{97.99763161461203(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{0.023353683080527432(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{0.12556575201044629(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{9415.047558015067(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{0.04670703135478371(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{0.15408458240978007(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{903859.115415471(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{0.0933819213075534(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&+\frac{0.012816774974988565(t-0.6)^{\beta}}{\Gamma(\beta+1)}-\frac{0.025633549950234258(t-0.6)^{2\beta}}{\Gamma(2\beta+1)}+\frac{0.05126709992515507(t-0.6)^{3\beta}}{\Gamma(3\beta+1)}
\end{aligned}
$$

$$
\begin{aligned}
{}^{8}W_3(t) =& 1+\frac{0.07610510369693613(0.1)^{\beta}}{\Gamma(\beta+1)}-\frac{97.99763161461203(0.1)^{\beta}}{\Gamma(\beta+1)}+\frac{0.023353683080527432(0.1)^{\beta}}{\Gamma(\beta+1)}\\
&-\frac{0.15119930196068054(0.1)^{2\beta}}{\Gamma(2\beta+1)}+\frac{9415.047558015067(0.1)^{2\beta}}{\Gamma(2\beta+1)}-\frac{0.04670703135478371(0.1)^{2\beta}}{\Gamma(2\beta+1)}\\
&+\frac{0.20535168233493514(0.1)^{3\beta}}{\Gamma(3\beta+1)}-\frac{903859.115415471(0.1)^{3\beta}}{\Gamma(3\beta+1)}+\frac{0.0933819213075534(0.1)^{3\beta}}{\Gamma(3\beta+1)}\\
&+\frac{0.010493487827309411(t-0.7)^{\beta}}{\Gamma(\beta+1)}-\frac{0.020986975655392648(t-0.7)^{2\beta}}{\Gamma(2\beta+1)}+\frac{0.041973951385060104(t-0.7)^{3\beta}}{\Gamma(3\beta+1)},
\end{aligned}
$$

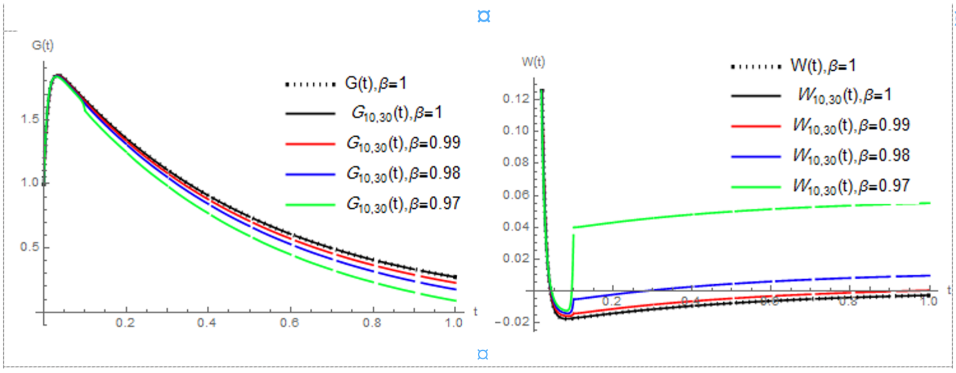

Figure 2 :Approximate solution curves using MS-FRPSM, example 4.1

$$\begin{aligned}
{}^{9}W_3(t) =& 1 + \frac{0.07610510369693613(0.1)^{\beta}}{\Gamma(\beta+1)} - \frac{97.99763161461203(0.1)^{\beta}}{\Gamma(\beta+1)} + \frac{0.02335368308052743 2(0.1)^{\beta}}{\Gamma(\beta+1)} + \frac{0.010493487827309411(0.1)^{\beta}}{\Gamma(\beta+1)} \\
&- \frac{0.15119930196068054(0.1)^{2\beta}}{\Gamma(2\beta+1)} + \frac{9415.047558015067(0.1)^{2\beta}}{\Gamma(2\beta+1)} - \frac{0.04670703135478371(0.1)^{2\beta}}{\Gamma(2\beta+1)} - \frac{0.020986975655392648(0.1)^{2\beta}}{\Gamma(2\beta+1)} \\
&+ \frac{0.20535168233493514(0.1)^{3\beta}}{\Gamma(3\beta+1)} - \frac{903859.115415471(0.1)^{3\beta}}{\Gamma(3\beta+1)} + \frac{0.0933819213075534(0.1)^{3\beta}}{\Gamma(3\beta+1)} + \frac{0.041973951385060104(0.1)^{3\beta}}{\Gamma(3\beta+1)} \\
&+ \frac{0.008591341191263868(t-0.8)^{\beta}}{\Gamma(\beta+1)} - \frac{0.017182682382804 07(t-0.8)^{2\beta}}{\Gamma(2\beta+1)} + \frac{0.03436536479212293(t-0.8)^{3\beta}}{\Gamma(3\beta+1)} + 1,
\end{aligned}$$

$$\begin{aligned}
{}^{10}W_3(t) =& 1 + \frac{0.0846964448882(0.1)^{\beta}}{\Gamma(\beta+1)} - \frac{97.99763161461203(0.1)^{\beta}}{\Gamma(\beta+1)} + \frac{0.023353683080527432(0.1)^{\beta}}{\Gamma(\beta+1)} + \frac{0.010493487827309411(0.1)^{\beta}}{\Gamma(\beta+1)} \\
&- \frac{0.16838198434348461(0.1)^{2\beta}}{\Gamma(2\beta+1)} + \frac{9415.047558015067(0.1)^{2\beta}}{\Gamma(2\beta+1)} - \frac{0.04670703135478371(0.1)^{2\beta}}{\Gamma(2\beta+1)} - \frac{0.020986975655392648(0.1)^{2\beta}}{\Gamma(2\beta+1)} \\
&+ \frac{0.23971704712705807(0.1)^{3\beta}}{\Gamma(3\beta+1)} - \frac{903859.115415471(0.1)^{3\beta}}{\Gamma(3\beta+1)} + \frac{0.0933819213075534(0.1)^{3\beta}}{\Gamma(3\beta+1)} + \frac{0.041973951385060104(0.1)^{3\beta}}{\Gamma(3\beta+1)} \\
&+ \frac{0.007033995243471 98(t-0.9)^{\beta}}{\Gamma(\beta+1)} - \frac{0.014067990487040993(t-0.9)^{2\beta}}{\Gamma(2\beta+1)} + \frac{0.02813598098340719(t-0.9)^{3\beta}}{\Gamma(3\beta+1)}.
\end{aligned}$$

The MS-FRPS is carried out for $N = 30$ and some values of the approximate numerical solutions, together with the absolute errors for $\beta = 1$, are presented in Table 3. The approximate solution curves and values of the residual errors for different values of β are given in Figure 2 and Table 4, respectively.

Obviously, the MS-FRPSM gives more accurate results than those found with classical FRPSM in Tables 1 and 2, and Figure 1. However, the residual error is still large, which can be decreased by performing more iterations.

Example 4.2: Consider the following nonhomogeneous fractional stiff system:

$$\begin{aligned}
{}_{t}^{C}D_{0}^{\beta}G(t) &= -G(t) - 15W(t) - 15e^{-t}, \quad t \in [0,1], \quad 0 < \beta \leq 1, \\
{}_{t}^{C}D_{0}^{\beta}W(t) &= 15G(t) - W(t) - 15e^{-t}, \\
G(0) &= 1, \quad W(0) = 1.
\end{aligned} \tag{12}$$

The exact solution for the system (12) when $\beta = 1$ is:

$$G(t) = e^{-t}, \quad W(t) = e^{-t}.$$

Table 3 Numerical results for MS-FRPSM for example 4.1

t_i	$G(t)$	$G_{30}(t)$, $\beta = 1$	$\lvert G(t) - G_{30}(t)\rvert$, $\beta = 1$	$G_{30}(t)$, $\beta = 0.99$	$G_{30}(t)$, $\beta = 0.98$	$G_{30}(t)$, $\beta = 0.97$
0.0	1.00000	1.00000	0.00000	1.00000	1.00000	1.00000
0.1	1.65481	1.65454	0.000269072	1.64314	1.62541	1.57125
0.2	1.35490	1.35490	1.0734×10^{-7}	1.33634	1.31133	1.24976
0.3	1.10930	1.10930	3.65752×10^{-11}	1.08487	1.05391	0.986293
0.4	0.90822	0.90822	1.72085×10^{-13}	0.87899	0.84315	0.77058
0.5	0.74359	0.74359	1.94289×10^{-13}	0.71043	0.67059	0.59397
0.6	0.60880	0.60880	5.3435×10^{-14}	0.57242	0.52931	0.44938
0.7	0.49844	0.49844	3.57769×10^{-13}	0.45943	0.41365	0.33099
0.8	0.40809	0.40809	1.32228×10^{-13}	0.36692	0.31895	0.23407
0.9	0.33411	0.33411	2.32314×10^{-14}	0.29118	0.24141	0.15471
1.0	0.27355	0.27355	2.37588×10^{-14}	0.22917	0.17793	0.08974
t_i	$W(t)$	$W_{30}(t)$, $\beta = 1$	$\lvert W(t) - W_{30}(t)\rvert$, $\beta = 1$	$W_{30}(t)$, $\beta = 0.99$	$W_{30}(t)$, $\beta = 0.98$	$W_{30}(t)$, $\beta = 0.97$
0.0	1.00000	1.00000	0.00000	1.00000	1.00000	1.00000
0.1	-0.0173506	-0.0170816	0.000269072	-0.0143377	-0.00539508	0.0398389
0.2	-0.0142621	-0.0142620	1.07339×10^{-7}	-0.0114419	-0.00241976	0.0429069
0.3	-0.0116768	-0.0116768	3.64694×10^{-11}	-0.00879496	0.00028984	0.0456802
0.4	-0.00956019	-0.00956019	5.91012×10^{-13}	-0.00662777	0.00250836	0.0479508
0.5	-0.00782722	-0.00782722	3.56415×10^{-13}	-0.00485343	0.00432473	0.0498099
0.6	-0.00640839	-0.00640839	6.70992×10^{-13}	-0.00340072	0.00581185	0.0513319
0.7	-0.00524674	-0.00524674	5.19546×10^{-13}	-0.00221135	0.0070294	0.0525781
0.8	-0.00429567	-0.00429567	2.34957×10^{-13}	-0.00123757	0.00802624	0.0535984
0.9	-0.00351700	-0.00351700	1.33380×10^{-13}	-0.000440303	0.00884239	0.0544337
1.0	-0.00287947	-0.00287947	1.06159×10^{-13}	0.000212441	0.0095106	0.0551176

Table 4 Some numerical values of the residual function for MS-FRPSM, Example 4.1

t_i	$\lvert \text{Resid}_{G,30}(t)\rvert$			$\lvert \text{Resid}_{W,30}(t)\rvert$		
	$\beta = 1$	$\beta = 0.99$	$\beta = 0.98$	$\beta = 1$	$\beta = 0.99$	$\beta = 0.98$
0.1	3.27729	3.00522	2.13794	0.00236839	0.25239	1.10209
0.2	2.70979	2.42332	1.5412	0.0285135	0.22648	1.07661
0.3	2.2186	1.9204	1.02637	0.0233537	0.231763	1.08202
0.4	1.81644	1.50863	0.604851	0.0191204	0.236098	1.08646
0.5	1.48717	1.17151	0.259741	0.0156544	0.239646	1.09009
0.6	1.21759	0.895491	0.0228115	0.0128168	0.242552	1.09306
0.7	0.996881	0.669509	0.254146	0.0104935	0.24493	1.0955
0.8	0.816177	0.484491	0.443547	0.00859134	0.246878	1.09749
0.9	0.66823	0.333011	0.598615	0.007034	0.248472	1.09912
1.0	0.5471	0.208989	0.725574	0.00575895	0.249778	1.10046

To apply the MS-FRPSM, we divide the interval [0,1] into 10 subintervals, so we have step size h= 0.1. On each subinterval, we carry out 10 iterations of the FRPSM. Accurate results for stiff system in (12) are clear from the absolute errors listed in Table 5 and the residual errors in Table 6. Moreover, the approximate solution curves are displayed in Figure 3, while the behavior of the 10th residual functions is shown in Figure 4.

Table 5 Numerical results for MS-FRPSM for example 4.2

t_i	$G(t)$	$G_{10}(t)$, $\beta = 1$	$\|G(t) - G_{10}(t)\|$, $\beta = 1$	$G_{10}(t)$, $\beta = 0.95$	$G_{10}(t)$, $\beta = 0.9$	$G_{10}(t)$, $\beta = 0.85$
0.0	1	1	0	1	1	1
0.1	0.904837	0.904837	1.11022 $\times 10^{-16}$	0.892109	0.878096	0.862774
0.2	0.818731	0.818731	2.22045 $\times 10^{-16}$	0.794485	0.767793	0.738607
0.3	0.740818	0.740818	2.22045 $\times 10^{-16}$	0.706152	0.667986	0.626256
0.4	0.67032	0.67032	2.22045 $\times 10^{-16}$	0.626224	0.577678	0.524597
0.5	0.606531	0.606531	1.11022 $\times 10^{-16}$	0.553903	0.495963	0.432612
0.6	0.548812	0.548812	1.11022 $\times 10^{-16}$	0.488463	0.422025	0.349380
0.7	0.496585	0.496585	5.55112 $\times 10^{-17}$	0.429252	0.355123	0.274069
0.8	0.449329	0.449329	0	0.375675	0.294587	0.205925
0.9	0.40657	0.40657	1.11022 $\times 10^{-16}$	0.327196	0.239812	0.144265
1.0	0.367879	0.367879	0	0.283331	0.19025	0.0884732
t_i	$W(t)$	$W_{30}(t)$, $\beta = 1$	$\|W(t) - W_{10}(t)\|$, $\beta = 1$	$W_{10}(t)$, $\beta = 0.95$	$W_{10}(t)$, $\beta = 0.9$	$W_{10}(t)$, $\beta = 0.85$
0.0	1.000000	1.000000	0.000000	1.000000	1.000000	1.000000
0.1	0.904837	0.904837	1.11022×10^{-16}	0.892109	0.878096	0.862774
0.2	0.818731	0.818731	1.11022×10^{-16}	0.794485	0.767793	0.738607
0.3	0.740818	0.740818	1.11022×10^{-16}	0.706152	0.667986	0.626256
0.4	0.670320	0.670320	1.11022×10^{-16}	0.626224	0.577678	0.524597
0.5	0.606531	0.606531	1.11022×10^{-16}	0.553903	0.495963	0.432612
0.6	0.548812	0.548812	0.000000	0.488463	0.422025	0.349380
0.7	0.496585	0.496585	1.66533×10^{-16}	0.429252	0.355123	0.274069
0.8	0.449329	0.449329	1.66533×10^{-16}	0.375675	0.294587	0.205925
0.9	0.406570	0.406570	5.55112×10^{-17}	0.327196	0.239812	0.144265
1.0	0.367879	0.367879	0.000000	0.283331	0.190250	0.0884732

Table 6 Some numerical values of the residual function for MS-FRPSM, example 4.2

t_i	$\|Resid_{G,10}(t)\|$			$\|Resid_{W,10}(t)\|$		
	$\beta = 0.95$	$\beta = 0.90$	$\beta = 0.85$	$\beta = 0.95$	$\beta = 0.90$	$\beta = 0.85$
0.1	0.00018	0.00018	0.00017	0.00011	0.00013	0.00015
0.2	0.00015	0.00015	0.00014	0.00012	0.00015	0.00020
0.3	0.00012	0.00011	0.00010	0.00012	0.00018	0.00023
0.4	0.00010	0.00009	0.00008	0.00013	0.00020	0.00027
0.5	0.00008	0.00007	0.00006	0.00013	0.00022	0.00030
0.6	0.00006	0.00005	0.00004	0.00014	0.00023	0.00033
0.7	0.00005	0.00004	0.00003	0.00014	0.00025	0.00036
0.8	0.00004	0.00003	0.00002	0.00015	0.00026	0.00039
0.9	0.00003	0.00002	0.00001	0.00016	0.00027	0.00041
1.0	0.00002	0.00001	7.2 $\times 10^{-6}$	0.00016	0.00029	0.00043

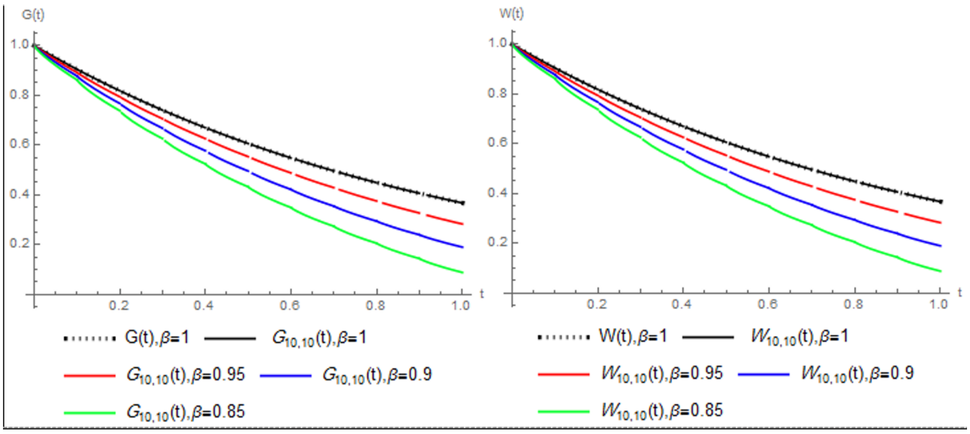

Figure 3 : Approximate solution curves using MS-FRPSM, example 4.2

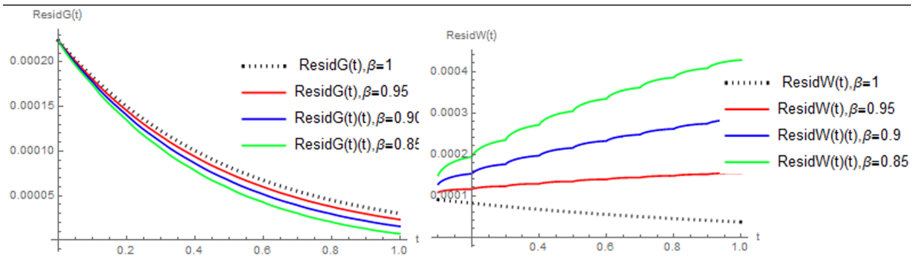

Figure 4 : The 10th residual function using MS-FRPSM, example 4.2

Example 4.3: Consider the following fractional stiff system:

$$ {}_{t}^{C}D_{0}^{\beta}G(t) = -2G(t) + W(t) + 2\sin(t), \quad t \in [0,5], \quad 0 < \beta \le 1, \tag{13} $$

$$ {}_{t}^{C}D_{0}^{\beta}W(t) = -3G(t) + 2\left(W(t) + \sin(t) - \cos(t)\right), $$

with initial conditions

$$ G(0) = 2, \quad W(0) = 3. \tag{14} $$

The exact solution of (13) when $\beta = 1$ is $G(t) = 2e^{-t} + \sin t$, $W(t) = 2e^{-t} + \cos t$. We solve the fractional stiff system in (13) using classical FRPSM and the MS-FRPSM with step size $h = 1$. In both methods, we compute the 10th truncated FPS solution, which is denoted by $G_{10,10}(t)$ and $W_{10,10}(t)$ for MS-FRPSM, and by $G_{10}(t)$ and $W_{10}(t)$ for classical FRPSM. A comparison between these two methods is carried out as follows: Table 7 compares the absolute errors for $\beta = 1$, while Table 8 and Table 9 represent comparisons between the residual errors for $\beta = 0.9$, $\beta = 0.8$, and $\beta = 0.7$. It is noticeable from these tables that multistep schemes contribute to reducing the error of approximations even though it is still high. The graphs of the 10th residual functions are shown in Figure 5 and Figure 6. On the other hand, a comparison between the curves of the 10th FPS using both methods for $\beta = 0.9$, $\beta = 0.8$, and $\beta = 0.7$ are presented in Figure 7 and Figure 8.

Table 7 A comparison between absolute errors using FRPSM and MS-FRPSM when $\beta = 1$, Example 4.3

	MS-FRPSM	FRPSM	MS-FRPSM	FRPSM
t_i	$\lvert G(t) - G_{10,10}(t)\rvert$	$\lvert G(t) - G_{10}(t)\rvert$	$\lvert W(t) - W_{10,10}(t)\rvert$	$\lvert W(t) - W_{10}(t)\rvert$
0.0	0.000000	0.000000	0.000000	0.000000
0.5	3.56963×10^{-11}	3.56963×10^{-11}	2.29745×10^{-11}	2.29745×10^{-11}
1.0	7.11208×10^{-8}	7.11208×10^{-8}	4.41523×10^{-8}	4.41523×10^{-8}
1.5	2.90986×10^{-8}	5.98460×10^{-6}	1.53817×10^{-8}	3.58103×10^{-6}
2.0	2.31792×10^{-8}	0.000137827	8.38990×10^{-8}	0.000079445
2.5	4.17414×10^{-8}	0.00156031	2.18289×10^{-7}	0.00086600
3.0	1.23302×10^{-7}	0.01127010	4.23894×10^{-7}	0.00602104
3.5	2.31435×10^{-7}	0.05969050	7.27017×10^{-7}	0.03069310
4.0	4.21251×10^{-7}	0.25188200	1.21487×10^{-6}	0.12466700
4.5	6.69059×10^{-7}	0.89344500	1.97749×10^{-6}	0.42573800
5.0	1.10148×10^{-6}	2.76316000	3.22386×10^{-6}	1.26819000

Table 8 A comparison between residual errors $\lvert Resid_G\rvert$ using MS-FRPSM and FRPSM, Example 4.3

t_i	MS-FRPSM			FRPSM		
	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$
0.5	0.413857	0.494220	0.566087	0.413857	0.494224	0.566087
1.0	0.231610	0.226850	0.181151	0.231610	0.226850	0.181151
1.5	0.425311	0.412688	0.331603	0.246721	0.084573	0.080431
2.0	0.608996	0.469273	0.283837	0.388434	0.132785	0.034489
2.5	0.788556	0.530022	0.210149	0.626232	0.442167	0.472802
3.0	0.864512	0.599768	0.327769	0.953735	1.09011	1.65211
3.5	0.638268	0.241984	0.148857	1.441670	2.29235	4.00186
4.0	0.424538	0.165570	0.053341	2.406320	4.70177	8.68621
4.5	0.082856	0.369207	0.585395	4.77180	9.92627	18.0669
5.0	0.431016	0.556547	0.628991	10.74120	21.3298	36.3969

Table 9 A comparison between residual errors $|Resid_W|$ using FRPSM and MS-FRPSM, Example 4.3

t_i	MS-FRPSM			FRPSM		
	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 0.8$	$\beta = 0.7$
0.5	7.61053	7.43003	7.23856	7.61053	7.43003	7.23856
1.0	4.35363	4.25856	4.15649	4.35363	4.25856	4.15649
1.5	2.27806	2.08497	1.83811	2.28767	2.16498	2.08242
2.0	1.68566	1.46167	1.19799	1.51504	1.26703	1.15864
2.5	1.85531	1.47128	0.98933	1.56928	1.26046	1.27452
3.0	1.86966	1.45236	1.00075	1.78177	1.79020	2.43064
3.5	0.92400	0.297301	0.34279	1.93607	3.02567	5.35909
4.0	0.22727	0.595633	0.89744	2.86553	6.44237	12.6162
4.5	1.17682	1.429970	1.54596	7.21321	16.2054	30.3349
5.0	0.812702	0.773654	0.63014	20.9921	41.3743	70.2115

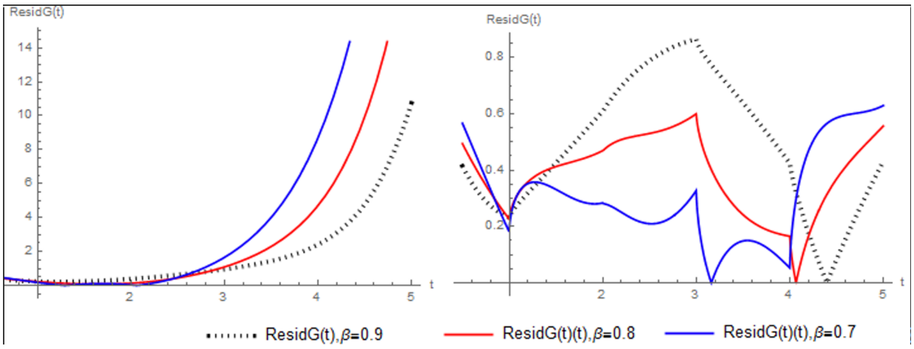

Figure 5 : Curves of the 10th residual functions for G(t) of system (13) using MS-FRPSM (right) and FRPSM (left)

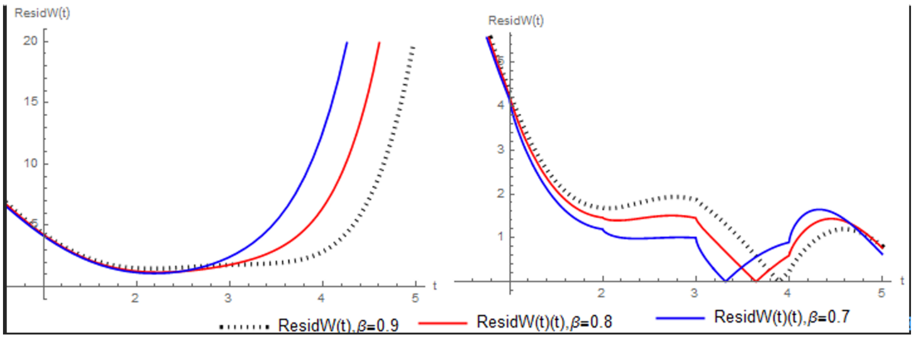

Figure 6 :Curves of the 10th residual functions for W(t) of system (13) using MS-FRPSM (right) and FRPSM (left)

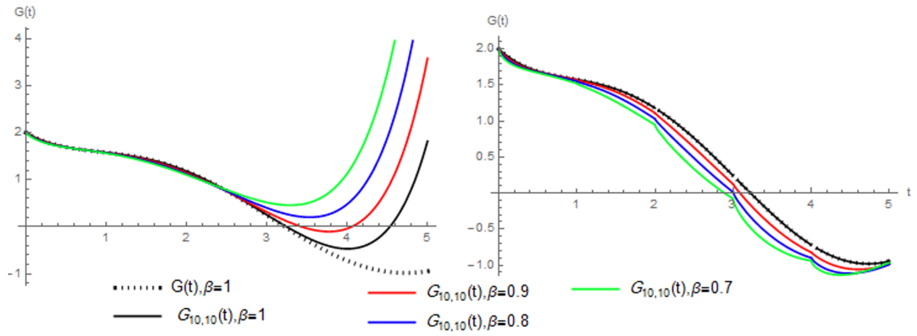

Figure 7 : The 10th FPS of G(t) for system (13) using MS-FRPSM (right) and FRPSM (left)

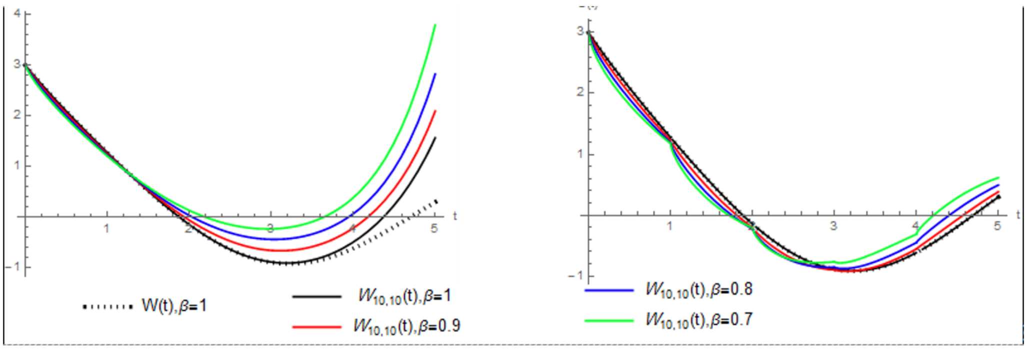

Figure 8 : The 10th FPS of W(t) for system (13) using MS-FRPSM (right) and FRPSM (left)

5. Conclusions

In this paper, we introduced a modified algorithm, namely, the MS-FRPSM, in an attempt to obtain approximate solutions of fractional stiff systems. Numerical examples were carried out to assess the applicability of the improved technique for solving these systems. We compared the exact solution for integer order stiff systems to the N-th truncated FPS, in addition to comparing the residual errors for fractional orders obtained using classical FRPSM and MS-FRPSM.

The comparison reveals that MS-FRPSM reduced both absolute and residual errors. More iterations and a smaller step size lead to higher accuracy. Moreover, MS-FRPSM increased the length of the convergence interval. Additionally, it does not require large computer memory and provides accurate numerical results in less time.

For future works, it is recommended to apply the MS-FRPSM for more types of equations with different fractional operators. It may be combined with some decomposition methods such as Adomian decomposition method to solve nonlinear FDEs.

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