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Sufficient Conditions For Sakaguchi Type Functions of Order β

S. P. Goyal[1,*], Pramila Vijaywargiya[1], Pranay Goswami[2]

[1] *Department of Mathematics, University of Rajasthan, Jaipur-302004, India*
[2] *Department of Mathematics, Amity University Rajasthan, Jaipur-302002, India*

Abstract. In this paper, we obtain some sufficient conditions for Sakaguchi type function of order β, defined on the open unit disk. Several interesting consequences of our results are also pointed out.

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1. Introduction

Let A_n be the class of all functions $f(z) = z + a_{n+1}z^{n+1} + \ldots$, which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} ; |z| < 1\}$ and let $A_1 = A$.

A function $f(z) \in A_n$ is said to be in class $S_n(\beta, t)$, if it satisfies

$$Re\left\{\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right\} > \beta, \quad (|t| \leq 1, t \neq 1) \tag{1}$$

for some $\beta (0 \leq \beta < 1)$ and for all $z \in \Delta$. For $n = 1$, this class is reduced to $S(\beta, t)$ (see, [5]). The class $S(0, -1)$ was introduced by Sakaguchi [7]. Therefore, a function $f(z) \in S(\beta, -1)$ is called Sakaguchi function of order β (see, [1]). Recently Owa et al. [5], Goyal and Goswami [2] have discussed some properties for functions $f(z) \in S(\beta, t)$.

In this paper, we obtain some sufficient conditions for functions $f(z) \in S_n(\beta, t)$. To prove our results, we need the following :

Lemma 1 ([4]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that Φ is a mapping from $\mathbb{C}^2 \times \Delta$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \Delta$, and for all real x, y such that $y \leq -n(1+x^2)/2$. If the function $p(z) = 1+c_n z^n+\ldots$ is analytic in Δ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \Delta$, then $Re(p(z)) > 0$.*

*Corresponding author.

Email addresses: somprg@gmail.com (S. Goyal), pramilavijay1979@gmail.com (P. Vijaywargiya), pranaygoswami83@gmail.com (P. Goswami)

2. Main Results

Theorem 1. *If* $f(z)\in A_n$ *satisfies*

$$Re\left[\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\left\{\frac{\alpha zf''(z)}{f'(z)}+\frac{\alpha tzf'(tz)}{f(z)-f(tz)}+1\right\}\right]>$$
$$\alpha\beta\left\{\beta+\frac{n}{2}(1-t)-(1-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(1-t)$$

for

$$(z\in\Delta,\ 0\le\alpha\le 1,\ 0\le\beta<1,\ |t|\le 1 \text{ and } t\neq 1),$$

then $f(z)\in S_n(\beta,t)$.

Proof. Define $p(z)$ by

$$\frac{(1-t)zf'(z)}{f(z)-f(tz)}=(1-\beta)p(z)+\beta.$$

Then $p(z)=1+c_nz^n+\ldots$ and is an analytic in Δ.
A computation shows that

$$\frac{zf''(z)}{f'(z)}+\frac{tzf'(tz)}{f(z)-f(tz)}=\frac{(1-t)(1-\beta)zp'(z)+[(1-\beta)p(z)+\beta]^2-(1-t)[(1-\beta)p(z)+\beta]}{(1-t)[(1-\beta)p(z)+\beta]}$$

and hence

$$\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\left[\frac{\alpha zf''(z)}{f'(z)}+\frac{\alpha tzf'(tz)}{f(z)-f(tz)}+1\right]$$
$$=\alpha(1-t)(1-\beta)zp'(z)+\alpha(1-\beta)^2p^2(z)+(1-\beta)[2\alpha\beta+(1-\alpha)(1-t)]p(z)+\beta[\alpha\beta+(1-\alpha)(1-t)]$$
$$=\Phi(p(z),zp'(z);z)\ \text{(say)},$$

where

$$\Phi(r,s;z)=\alpha(1-t)(1-\beta)s+\alpha(1-\beta)^2r^2+(1-\beta)[2\alpha\beta+(1-\alpha)(1-t)]r+\beta[\alpha\beta+(1-\alpha)(1-t)].$$

For all real x and y satisfying $y\le -n(1+x^2)/2$, we have

$$\begin{aligned}Re[\Phi(ix,y;z)]&=\alpha(1-t)(1-\beta)y-\alpha(1-\beta)^2x^2+\beta[\alpha\beta+(1-\alpha)(1-t)]\\&\le\alpha(1-t)(1-\beta)\left\{-n(1+x^2)/2\right\}-\alpha(1-\beta)^2x^2+\beta[\alpha\beta+(1-\alpha)(1-t)]\\&=\frac{-\alpha n}{2}(1-t)(1-\beta)-\left\{\frac{\alpha n}{2}(1-t)(1-\beta)+\alpha(1-\beta)^2\right\}x^2+\beta[\alpha\beta+(1-\alpha)(1-t)]\\&\le\frac{-\alpha n}{2}(1-t)(1-\beta)+\beta[\alpha\beta+(1-\alpha)(1-t)]\\&=\alpha\beta\left\{\beta+\frac{n}{2}(1-t)-(1-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(1-t).\end{aligned}$$

Let $\Omega = \left\{w;\ Re(w) > \alpha\beta\left\{\beta + \frac{n}{2}(1-t) - (1-t)\right\} + \left\{\beta - \frac{n\alpha}{2}\right\}(1-t)\right\}$.
Then $\Phi(p(z), zp'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \le -n(1+x^2)/2$, $z \in \Delta$. By an application of Lemma 1, the result fellows.

On taking $t = -1$, in the Theorem 1, we have following

Corollary 1. *If $f(z) \in A_n$ satisfies*

$$Re\left[\frac{zf'(z)}{f(z)-f(-z)}\left\{\frac{\alpha z f''(z)}{f'(z)} - \frac{\alpha z f'(-z)}{f(z)-f(-z)} + 1\right\}\right] > \frac{\alpha\beta}{4}\{\beta + n - 2\} + \left\{\frac{2\beta - n\alpha}{4}\right\}$$

for

$$(z \in \Delta,\ 0 \le \alpha \le 1,\ 0 \le \beta < 1),$$

then $f(z) \in S_n(\beta, -1)$.

By taking $\beta = 0$ in Corollary 1, we have

Corollary 2. *If $f(z) \in A_n$ satisfies*

$$Re\left[\frac{zf'(z)}{f(z)-f(-z)}\left\{\frac{\alpha z f''(z)}{f'(z)} - \frac{\alpha z f'(-z)}{f(z)-f(-z)} + 1\right\}\right] > \frac{-n\alpha}{4}$$

where

$$(z \in \Delta,\ 0 \le \alpha \le 1),$$

then $f(z) \in S_n(0, -1)$.

If we take $t = 0$ in the Theorem 1, we have the following

Corollary 3 ([6])**.** *If $f(z) \in A_n$ satisfies*

$$Re\left[\frac{zf'(z)}{f(z)}\left\{\frac{\alpha z f''(z)}{f'(z)} + 1\right\}\right] > \alpha\beta\left\{\beta + \frac{n}{2} - 1\right\} + \left\{\beta - \frac{n\alpha}{2}\right\}$$

for

$$(z \in \Delta,\ 0 \le \alpha \le 1,\ 0 \le \beta < 1),$$

then $f(z) \in S_n(\beta, 0) = S_n^*(\beta)$.

If we take $\beta = 0$ and $n = 1$ in Corollary 3, we have

Corollary 4 ([3])**.** *If $f(z) \in A$ satisfies*

$$Re\left[\frac{zf'(z)}{f(z)}\left\{\frac{\alpha z f''(z)}{f'(z)} + 1\right\}\right] > \frac{-\alpha}{2}\ \ (z \in \Delta),$$

for some $\alpha(\alpha \ge 0)$, then $f(z) \in S_1(0,0) = S^*$.

If we take $\beta = \frac{\alpha}{2}$ and $n = 1$, in Corollary 3, we get the following

Corollary 5 ([3]). *If $f(z) \in A$ satisfies*

$$Re\left[\frac{zf'(z)}{f(z)}\left\{\frac{\alpha zf''(z)}{f'(z)}+1\right\}\right] > \frac{-\alpha^2}{4}(1-\alpha) \quad (z \in \Delta),$$

for some $\alpha(0 \le \alpha < 2)$, then $f(z) \in S_1(\frac{\alpha}{2}, 0) = S^*(\frac{\alpha}{2})$.

Theorem 2. *Let $0 \le \beta < 1$, $|t| \le 1$, $t \ne 1$ with $-1 \le t + \beta < 1$,*

$$\lambda = (1-\beta)^2\left\{1-\beta+(1-t)\frac{n}{2}\right\}^2, \ \mu = \left\{(1-\beta)(1-t)\frac{n}{2} - (\beta^2-(1-t)\beta)\right\}^2,$$
$$\nu = \left\{(1-\beta)^2 + (\beta^2-(1-t)\beta)\right\}^2 \ \text{and} \ \sigma = (1-\beta)^2(2\beta-1+t)^2 \tag{2}$$

satisfy $(\lambda+\mu-\nu+\sigma)\beta^2 < (1-2\beta)\mu$.
Also suppose that u_0 be the positive real root of the equation

$$2\lambda(1-\beta)^2u^3 + \left\{(1-\beta)^2(2\lambda+\mu-\nu+\sigma)+3\lambda\beta^2\right\}u^2 + 2\beta^2(2\lambda+\mu-\nu+\sigma)u$$
$$+(\lambda+2\mu-\nu+\sigma)\beta^2 - (1-\beta)^2\mu = 0 \tag{3}$$

and

$$\rho^2 = \frac{(1-\beta)^2(1+u_0)}{(1-t)^2\left\{(1-\beta)^2u_o+\beta^2\right\}}[\lambda u_0^2 + (\lambda+\mu-\nu+\sigma)u_0 + \mu]. \tag{4}$$

Now if $f(z) \in A_n$ satisfies

$$\left|\left(\frac{(1-t)zf'(z)}{f(z)-f(tz)} - 1\right)\left(\frac{zf''(z)}{f'(z)} + \frac{tzf'(tz)}{f(z)-f(tz)}\right)\right| \le \rho \quad (z \in \Delta),$$

then $f(z) \in S_n(\beta, t)$.

Proof. Define $p(z)$ by

$$\frac{(1-t)zf'(z)}{f(z)-f(tz)} = (1-\beta)p(z) + \beta.$$

Then $p(z) = 1 + c_nz^n + \dots$ and is an analytic in Δ.
A computation shows that

$$\frac{zf''(z)}{f'(z)} + \frac{tzf'(tz)}{f(z)-f(tz)} = \frac{(1-t)(1-\beta)zp'(z) + [(1-\beta)p(z)+\beta]^2 - (1-t)[(1-\beta)p(z)+\beta]}{(1-t)[(1-\beta)p(z)+\beta]}$$

and hence

$$\left(\frac{(1-t)zf'(z)}{f(z)-f(tz)} - 1\right)\left(\frac{zf''(z)}{f'(z)} + \frac{tzf'(tz)}{f(z)-f(tz)}\right)$$
$$= \frac{(1-\beta)(p(z)-1)}{(1-t)[(1-\beta)p(z)+\beta]}\left\{(1-t)(1-\beta)zp'(z) + [(1-\beta)p(z)+\beta]^2 - (1-t)[(1-\beta)p(z)+\beta]\right\}$$

$$= \Phi(p(z), zp'(z); z).$$

Then, for all real x and y satisfying $y \leq -n(1+x^2)/2$, we have

$$\begin{aligned}|\Phi(ix, y; z)|^2 =& \frac{(1-\beta)^2(1+x^2)}{(1-t)^2[(1-\beta)^2x^2+\beta^2]} \\ &\times \left[\left\{(1-t)(1-\beta)y - \beta(1-t-\beta) - (1-\beta)^2x^2\right\}^2 + (1-\beta)^2(2\beta-1+t)^2x^2\right] \\ =& \frac{(1-\beta)^2(1+u)}{(1-t)^2[(1-\beta)^2u+\beta^2]} \\ &\times \left[\left\{(1-t)(1-\beta)y - \beta(1-t-\beta) - (1-\beta)^2u\right\}^2 + (1-\beta)^2(2\beta-1+t)^2u\right] \\ =& g(u, y)\end{aligned}$$

where $u = x^2 > 0$ and $y \leq -n(1+x^2)/2$.
Since

$$\frac{\partial g}{\partial y} = \frac{2(1-\beta)^3(1+u)}{(1-t)[(1-\beta)^2u+\beta^2]}\left\{(1-t)(1-\beta)y - \beta(1-t-\beta) - (1-\beta)^2u\right\} < 0,$$

therefore we have

$$h(u) \;=\; g[u, -n(1+u)/2] \leq g(u, y),$$

where

$$h(u) = \frac{(1-\beta)^2(1+u)}{(1-t)^2\left\{(1-\beta)^2u+\beta^2\right\}}[\lambda u^2 + (\lambda+\mu-\nu+\sigma)u + \mu], \tag{5}$$

where λ, μ, ν *and* σ are given in (2).
Now differentaiting (5) and using $h'(u) = 0$, we get

$$\begin{gathered}2\lambda(1-\beta)^2u^3 + \left\{(1-\beta)^2(2\lambda+\mu-\nu+\sigma) + 3\lambda\beta^2\right\}u^2 + \\ 2\beta^2(2\lambda+\mu-\nu+\sigma)u + (\lambda+2\mu-\nu+\sigma)\beta^2 - (1-\beta)^2\mu = 0\end{gathered}$$

which is a cubic equation in u. Since u_0 is the positive real root of this equation we have $h(u) \geq h(u_0)$ and hence

$$|\Phi(ix, y; z)|^2 \geq h(u_0) = \rho^2.$$

Define $\Omega = \{w;\ |w| < \rho\}$, then $\Phi(p(z), zp'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \leq -n(1+x^2)/2$, $z \in \Delta$. Therefore by an application of Lemma 1. the result follows.

By taking $t = -1$, in Theorem 2, we have the following

Corollary 6. *Let* $0 \leq \beta < 1$, $\lambda_1 = (1-\beta)^2\left\{1-\beta+n\right\}^2$, $\mu_1 = \left\{(1-\beta)n - (\beta^2-2\beta)\right\}^2$, $\nu_1 = \left\{(1-\beta)^2 + (\beta^2-2\beta)\right\}^2$ *and* $\sigma_1 = 4(1-\beta)^4$, *satisfy* $(\lambda_1+\mu_1-\nu_1+\sigma_1)\beta^2 < (1-2\beta)\mu_1$. *Also suppose that* u_1 *be the positive real root of the equation*

$$2\lambda_1(1-\beta)^2u^3 + \left\{(1-\beta)^2(2\lambda_1+\mu_1-\nu_1+\sigma_1) + 3\lambda_1\beta^2\right\}u^2$$

$$+2\beta^2(2\lambda_1+\mu_1-\nu_1+\sigma_1)u+(\lambda_1+2\mu_1-\nu_1+\sigma_1)\beta^2-(1-\beta)^2\mu_1=0 \tag{6}$$

and

$$\rho_1^2=\frac{(1-\beta)^2(1+u_1)}{4\{(1-\beta)^2u_1+\beta^2\}}[\lambda_1 u_1^2+(\lambda_1+\mu_1-\nu_1+\sigma_1)u_1+\mu_1]. \tag{7}$$

Now if $f(z)\in A_n$ satisfies

$$\left|\left(\frac{2zf'(z)}{f(z)-f(-z)}-1\right)\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(-z)}{f(z)-f(-z)}\right)\right|\le\rho_1 \quad (z\in\Delta),$$

then $f(z)\in S_n(\beta,-1)$.

By taking $\beta=0$ in Corollary 6, we get the following

Corollary 7. *Let u_2 be the positive real root of the equation*

$$2(n+1)^2u^3+(3n^2+4n+5)u^2-n^2=0 \tag{8}$$

and

$$\rho_2^2=\frac{(1+u_2)}{4u_2}[(n+1)^2u_2^2+2(n^2+n+2)u_2+n^2]. \tag{9}$$

Now if $f(z)\in A_n$ satisfies

$$\left|\left(\frac{2zf'(z)}{f(z)-f(-z)}-1\right)\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(-z)}{f(z)-f(-z)}\right)\right|\le\rho_2 \quad (z\in\Delta),$$

then $f(z)\in S_n(0,-1)$.

By taking $n=1$ in Corollary 7, we have $u_3=0.266048\ldots$, thus we have the following result

Corollary 8. *If $f(z)\in A$ satisfies*

$$\left|\left(\frac{2zf'(z)}{f(z)-f(-z)}-1\right)\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(-z)}{f(z)-f(-z)}\right)\right|\le\rho_3 \quad (z\in\Delta),$$

where $\rho_3=2.0145979\ldots$, then $f(z)\in S(0,-1)$.

By taking $t=0$ in Theorem 2, we get a known result due to Ravichandran et al. [6, Thm. 2.5]. For $n=1$, $\beta=0=t$, our Theorem 2 reduces to another known result due to Li and Owa [3].

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