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Numerical Treatment for the Caputo-Fabrizio Fractional Smoking Model using an Efficient Numerical Technique

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Abstract. Smoking is a prevalent social behavior widely practiced worldwide, especially in settings such as schools and some significant gatherings. Based on the World Health Organization (WHO), smoking is the third leading source of human mortality and the primary preventable cause of disease. This study introduces a highly efficient simulation technique to analyze and solve the Caputo-Fabrizio (CF) fractional smoking model. We numerically solve the fractional integral equations (FIEs) with Simpson's 1/3 rule, an effective numerical integration technique. We concentrate on clarifying the stability/convergence of the proposed strategy. We juxtapose the outcomes derived using the Runge-Kutta method (RK4) with those obtained through the implemented methodology. The findings indicate that the used technique provides a simple and efficient instrument for simulating the solution of these models. The principal advantage of the proposed method is its dependence on a limited number of straightforward steps, devoid of long-term consequences or reliance on a perturbation parameter.

2020 Mathematics Subject Classifications: 34A12, 41A30, 47H10, 65N20.

Key Words and Phrases: Smoking model, Caputo-Fabrizio fractional derivative, Numerical integration, RK4 method

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1. Introduction

According to a report by the WHO, many smokers pass away in their prime years. More than 5 million deaths globally occur each year as a result of smoking's effects on various body processes; by 2030, this number could increase to 8 million ([23], [27]). Smokers are 70% more likely to experience a heart attack compared to non-smokers. Lung

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cancer occurs 10% more frequently in smokers than in non-smokers. Smokers often live 10 to 13 years less than others do. Researchers work to extend people's lives to reduce smoking. To best explain cigarette smoking, many academics have investigated several efficient smoking models. They categorize individuals as potential, chain, or permanently abstinent smokers. Probably around 1997 [12], the smoking mathematical model came into existence. More recently, in 2007, a survey conducted in Korea by Ham [14] recorded the various stages and practices of smoking among students. Liu et al. [20] in 2008 refined Ham's model to incorporate a new category of temporarily abstaining smokers. In [14], the authors enhanced an integer-order model by incorporating a novel category of infrequent smokers and a dynamic interaction. The smoking models have also been provided by several authors in integer and fractional orders [10].

In the last few years, there are many novel techniques and topics have been presented in the field of fractional analysis, for example, a study of the susceptible-infected-recovered epidemiological model for pediatric diseases is performed by implementing the q-homotopy analysis transform technique (HATM) [25], as previously employed by the authors in [26] to numerically solve fractional Burger's equations. Linear, nonlinear, fractional, and fuzzy stochastic problems discussed in ([4], [7]). There are many real-life problems were discussed recently in many scientific papers, for example, the mathematical fractional-order model of a novel Coronavirus's Caputo growth was examined in [2]. Also, there are many theoretical studies that support the numerical techniques, for example, the existence, uniqueness, and synchronization of a discrete-time fractional tumor growth model, along with numerical discoveries and essential criteria for a modified-ABC fractional order smoking model solution, were delineated in ([3]-[5]).

Scientists are currently studying more effective fractional operators. In order to address the issue of singularity and achieve accurate and reliable modeling outcomes, an enhanced fractional-order Caputo-Fabrizio derivative has been formulated. This derivative incorporates a non-singular kernel, as proposed by Caputo and Fabrizio, leading to improved efficiency and robustness in recent years. The use of Laplace transformation to convert it to integer power is regarded as a constructive method. Consequently, in some scenarios, we can readily compute the precise solution. The citation for this information is [9]. The paper [21] explores the examination of fractional operators and presents novel characteristics associated with them. Unlike traditional integer derivatives, fractional-order derivatives have emerged as significant solutions to intricate situations in recent years. In the paper [19], a new and effective model for uncertain fractional currency is established. The solution to the fractional differential equations (FDEs) is calculated using the Caputo derivative, with the Mittag-Leffler function being employed. The study [28] looks at how to use the CF operator to find analytical solutions for fractional Volterra integral-differential equations. Furthermore, a work conducted in [11] has examined the intricate dynamics of the Omicron variant of COVID-19 by employing CF-fractional operators. In addition, a numerical method is created that includes an exponential law kernel to analyze and model the spread of the infection.

Many mathematicians found a challenging to create numerical and analytical solutions for the FDEs. Due to the lack of accurate solutions for many physical problems with fractional models, some academics have shown a strong interest in developing numerical solutions for the FDEs ([16],[17]). The Adams-Bashforth method, incorporating the CF operator, is formulated in [24]. This method involves three steps and can be used to solve both linear and nonlinear FDEs. Additionally, it possesses diverse uses in resolving chaotic systems with fractional orders. The authors [15] have developed a trapezoidal approach for the effective resolution of fractional differential equations (FDEs). This technique utilizes the CF operator and achieves a convergence order of two. Moreover, the convergence and stability of this scheme have been thoroughly investigated. Here in this research, we implement Simpson's 1/3 approach for addressing the FDEs via the CF-derivative. This method achieves a high level of accuracy, with an order of four, as detailed in our work. The proposed fractional Simpson's 1/3 approach offers superior accuracy compared to current methods and is straightforward to implement.

The manuscript is structured as follows: Section 2 provides the first knowledge concerning fractional order integrals and derivatives. Section 3 delineates the smoking system and its CF-fractional representation. Section 4 delineates the derivation of Simpson's 1/3 rule for the CF-fractional integral. Section 5 gives the convergence analysis of the proposed method. Section 6 outlines the numerical implementation of the proposed strategy for addressing the smoking system. Section 7 presents the results and discussions of the numerical simulation, while Section 8 provides the conclusions.

2. Preliminaries

Fractional derivatives have been accurately defined by mathematical models employing a non-singular kernel have accurately defined fractional derivatives. This method improves the system's capacity to precisely depict and record memory effects. This definition is the most significant one employed in the development of fractional calculus theory [9]. This part provides a succinct overview of the fundamental definitions of fractional calculus involving a non-singular kernel.

Definition 1. *[9]*

For $\psi(t) \in \mathbb{H}^1(0,a), 0 < \gamma < 1$. Then the CF fractional derivative ${}^{CF}D^{\gamma}\psi(t)$ and its FI ${}^{CF}I^{\gamma}\psi(t)$, respectively are given as follows:

$$ {}^{CF}D^{\gamma}\psi(t) := \frac{1}{1-\gamma}\int_0^t \text{Exp}\Big[-\frac{\gamma}{1-\gamma}(t-\tau)\Big]\dot{\psi}(\tau)d\tau, $$

$$ {}^{CF}I^{\gamma}\psi(t) := (1-\gamma)\psi(t) + \gamma\int_0^t \psi(\tau)d\tau. \tag{1} $$

3. Description of the fractional smoking system

Now, the suggested approach to solving the fractional smoking model will be put into practice. Because mathematical modeling has been an important tool for pandemic grasp in recent decades [13], we can use it to stop the spread of tobacco usage. The current study incorporates the improved smoking model, which involves the CF-fractional derivative, to enhance the understanding of both qualitative and numerical analysis [6].

$$\begin{aligned}
&{}^{CF}D^{\varrho}\psi_1(t) = \sigma - \ell_1\,\psi_1(t)\psi_2(t) + \gamma\,\psi_4(t) - \beta\,\psi_1(t), \qquad \psi_1(0) = \psi_1^0,\\
&{}^{CF}D^{\varrho}\psi_2(t) = \ell_1\,\psi_1(t)\psi_2(t) - \ell_2\,\psi_2(t)\psi_3(t) - (\sigma_1 + \beta)\,\psi_2(t), \qquad \psi_2(0) = \psi_2^0,\\
&{}^{CF}D^{\varrho}\psi_3(t) = \ell_2\,\psi_2(t)\psi_3(t) - (\theta + \sigma_2 + \beta)\,\psi_3(t)), \qquad \psi_3(0) = \psi_3^0,\\
&{}^{CF}D^{\varrho}\psi_4(t) = \theta\,\psi_3(t) - (\alpha + \beta + \gamma)\,\psi_4(t), \qquad \psi_4(0) = \psi_4^0,\\
&{}^{CF}D^{\varrho}\psi_5(t) = \alpha\,\psi_4(t) - \beta\,\psi_5(t), \qquad \psi_5(0) = \psi_5^0.
\end{aligned} \tag{2}$$

This concept classifies the total population into 5 categories, denoted as ψ_1, ψ_2, ψ_3, ψ_4, and ψ_5, representing vulnerable smokers, individuals who ingest tobacco, occasional smokers, regular smokers, and individuals who have quit smoking at a specific time. The description of the constants in the system (2) are stated as follows:

(i) σ is the rate at which individuals are recruited through migration or birth;

(ii) ℓ_1 is the rate at which the vulnerable population transitions into the snuffing class;

(iii) ℓ_2 is the frequency of snuffing increases among occasional smokers;

(iv) θ is the conversion rate of occasional smokers transitioning to regular smoking;

(v) α, β, γ are the rates of departure, natural mortality, and recovery, respectively;

(vi) σ_1, σ_2 are the mortality rate among individuals who use snuff and the mortality rate caused by smoking, respectively.

4. Derivation Simpson's-1/3 rule for CF-fractional integral

This section presents the formulation of the fractional Simpson's-1/3 strategy (FSR) for solving CF-FDEs [8]. This aim will be achieved through the following steps:

(i) Considering the subsequent γ-order IVP:

$$ {}^{CF}D^{\gamma}u(t) = f(u(t)), \qquad u(0) = u_0. \tag{3}$$

(ii) Applying the CF-fractional integral operator on the IVP (3) and applying Proposition 3 in [1] and formula (1), we get:

$$u(t) = u_0 + {}^{CF}I^{\gamma} f\,(u(t)) = u_0 + (1-\gamma) f(u(t)) + \gamma \int_0^t f(u(s))ds. \tag{4}$$

(iii) Initially, we will employ a quadratic polynomial P_2 to estimate the integral function f in Equation (4). The function will be assessed at t_0, t_1, and t_2, where t_0 is less than

t_1 and t_1 is less than t_2. The interval is partitioned into two subintervals, denoted as $t_1 - t_0 = t_2 - t_1 = h$, resulting in a combined width of $2h$. The integration of the quadratic polynomial P_2 can be computed as follows:

$$I_2 f(u(t)) = \int_a^b f(u(t))dt \approx \int_{t_0}^{t_2} P_2(u(t))dt = \int_{t_0}^{t_2} \left[\sum_{j=0}^{2} L_j(t)\, f\left(u\left(t_j\right)\right) \right] dt,$$

where the second-order Lagrange polynomials $L_0(t)$, $L_1(t)$, and $L_2(t)$ are defined as follows:

$$L_j(t) = \prod_{i=0,\, i\neq j}^{2} \frac{(t-t_i)}{(t_j-t_i)}.$$

(iv) Integrating the first interpolant function $L_0(t)$, by taking $h = \frac{t_2 - t_0}{2}$ and substituting "$t = s + t_0$", gives us:

$$\int_{t_0}^{t_2} L_0(t)dt = \frac{1}{2h^2} \int_{t_0}^{t_0+2h} \left(t-t_1\right)\left(t-t_2\right) dt = \frac{1}{2h^2} \int_0^{2h} \left(s+t_0-t_2\right)\left(s+t_0-t_1\right) ds$$
$$= \frac{1}{2h^2} \int_0^{2h} (s-2h)(s-h)ds = \frac{h}{3}.$$

After making some simplifications to the rest of the terms, we have the following:

$$I_2(f) = \frac{h}{3}\left[f\left(u\left(t_0\right)\right) + 4f\left(u\left(t_1\right)\right) + f\left(u\left(t_2\right)\right)\right].$$

(v) By substituting in the equation (4), we get:

$$u\left(t_n\right) = u_0 + (1-\gamma) f\left(u\left(t_n\right)\right) + \frac{h}{3}\gamma\left[f\left(u\left(t_0\right)\right) + 4f\left(u\left(t_1\right)\right) + f\left(u\left(t_2\right)\right)\right], \quad n = 0, 1, 2.$$

(vi) To improve the accuracy of numerical integration, we partition $[a, b]$ to n subintervals as follows:
For any even number $n \geq 2$, we establish the following definitions:

$$h = \frac{b-a}{n} = t_{k+1} - t_k, \quad k = 0, 1, 2, \ldots, n.$$

Now, by using the quadrature rule for each pair of subintervals and implementing the Simpson's-1/3 rule to each $\left[t_{2k}, t_{2(k+1)}\right]$, $\quad k = 0, 1, 2, \ldots, \frac{n-2}{2}$, we can appoint the following formula:

$$I_n(f) = \sum_{k=0}^{\frac{n-2}{2}} \int_{t_{2k}}^{t_{2k+2}} f(u(t))dt = \sum_{k=0}^{\frac{n-2}{2}} \left(\frac{h}{3}\left[f\left(u\left(t_{2k}\right)\right) + 4f\left(u\left(t_{2k+1}\right)\right) + f\left(u\left(t_{2k+2}\right)\right)\right] \right).$$

(vii) By referring u_j as the approximate solution of $u(t_j)$ and using Eq.(4), we can write the FSR for the CF-FDE (3), for $j = 0(1)(n-1)$:

$$u_{j+1} = u_0 + (1-\gamma) f(u_{j+1}) + \gamma \frac{h}{3}\Big[f(u(t_0)) + 4\sum_{i=2,4,6}^{j} f(u(t_i)) + 2\sum_{j=1,3,5}^{j-1} f(u(t_j)) + f(u(t_{j+1}))\Big].$$

This formula can be rewritten in a compact form as follows:

$$u_{j+1} = u_0 + (1-\gamma) f(u_{j+1}) + \gamma\, h \sum_{r=0}^{j+1} \xi_r\, f(u_r), \qquad j = 0, 1, 2, \ldots, n-1, \tag{5}$$

where ξ_r are the weights of the FSR and are defined as:

$$\xi_r = \begin{cases} 1/3, & r = 0,\ n+1, \\ 2/3, & r = 1, 3, 5, \ldots, \\ 4/3, & r = 2, 4, 6, \ldots. \end{cases}$$

5. Convergence analysis

In this section, we are going to collect some theorems concerning the stability and error analysis of the given IVP (3), and the regulated numerical scheme which investigated and proved in ([8], [22]).

Theorem 1. *[22]*

Let us assume a continuous function f: $[0, T] \times \mathbb{R} \to \mathbb{R}$ with $\gamma \in (0, 1)$ that satisfies the Lipschitz condition:

$$|f(u(t_1)) - f(u(t_2))| \leq \epsilon\, |u(t_1) - u(t_2)|, \qquad \epsilon > 0. \tag{6}$$

Then the IVP (3) has a unique solution on $C[0, T]$ under the condition:

$$\frac{(2(1-\gamma) + 2\gamma T)\epsilon}{(2-\gamma)} < 1.$$

Lemma 1.

Suppose that $f(u(t)) \in C^4([a, b])$, then the error of numerical scheme (5) is estimated by:

$$\left| \int_{t_0}^{t_{n+1}} f(u(s))ds - \gamma h \sum_{i=0}^{n+1} \xi_i f(u(t_i)) \right| \leq\ h^4,$$

where $\hat{c} = \frac{(b-a)f^{(4)}(\zeta)}{180}$, for some constant $a < \zeta < b$, $h = \frac{b-a}{n}$, and $t_k = a + hk$, $k = 0, 1, \ldots, n+1$.

The stability and error analysis of the regulated numerical scheme are investigated and proved meanwhile the following theorems [8].

Theorem 2.

The newly designed fractional numerical technique (5) exhibits conditional stability.

Theorem 3.

The recently developed fractional numerical method exhibits conditional convergence of order four, as stated in the equation (5):

$$\|u(t_{n+1}) - u_{n+1}\| \leq \mathfrak{C} h^4,$$

where $\mathfrak{C} = \gamma \hat{c}\, c_h$.

6. Numerical implementation

Most of the existing numerical methods converge slowly for this kind of problem and this results in inaccurate approximations. In this work, the discretized system of CF-fractional differential equations is numerically integrated using Simpson's 1/3 technique. We will develop a numerical method to illustrate the numerical behavior of the CF fractional smoking system. To this end, let us revisit the model presented in (2):

$$^{CF}D^{\varrho}\, \bar{\Psi}(t) = \mathbb{F}(\bar{\Psi}(t), t), \tag{7}$$

where

$$\bar{\Psi}(t) = [\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t), \psi_5(t)]^T, \qquad \bar{\Psi}(0) = [\psi_1(0), \psi_2(0), \psi_3(0), \psi_4(0), \psi_5(0)]^T,$$

$$\mathbb{F}(\bar{\Psi}(t), t) = \begin{pmatrix} \mathbf{f}_1(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, t) \\ \mathbf{f}_2(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, t) \\ \mathbf{f}_3(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, t) \\ \mathbf{f}_4(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, t) \\ \mathbf{f}_5(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, t) \end{pmatrix} = \begin{pmatrix} \sigma - \ell_1\, \psi_1(t)\psi_2(t) + \gamma\, \psi_4(t) - \beta\, \psi_1(t) \\ \ell_1\, \psi_1(t)\psi_2(t) - \ell_2\, \psi_2(t)\psi_3(t) - (\sigma_1 + \beta)\, \psi_2(t) \\ \ell_2\, \psi_2(t)\psi_3(t) - (\psi + \sigma_2 + \beta)\, \psi_3(t) \\ \theta\, \psi_3(t) - (\alpha + \beta + \gamma)\, \psi_4(t) \\ \alpha\, \psi_4(t) - \beta\, \psi_5(t) \end{pmatrix}. \tag{8}$$

Applying the CF fractional integral operator on Eq.(7) and using Proposition 3 in [1] and formula (1), we get:

$$\bar{\Psi}(t) = \bar{\Psi}(0) + {}^{CF}I^{\varrho}\, \mathbb{F}(\bar{\Psi}(t), t) = \bar{\Psi}(0) + (1-\varrho)\mathbb{F}(\bar{\Psi}(t), t) + \varrho \int_0^t \mathbb{F}(\bar{\Psi}(s), s) ds. \tag{9}$$

Applying the derived Simpson's-1/3 rule for the integration on the RHS of (9), to get the following numerical scheme as constructed in the formula (5):

$$\bar{\Psi}_{j+1} = \bar{\Psi}(0) + (1-\varrho)\mathbb{F}(\bar{\Psi}_{j+1}, t_{j+1}) + \varrho\, h \sum_{r=0}^{j+1} \xi_r\, \mathbb{F}(\bar{\Psi}_r, t_r), \qquad j = 0, 1, 2, \ldots, n-1, \tag{10}$$

where the weights ξ_r, $r = 0, 1, ..., j+1$ of the fractional Simpson's-1/3 rule are defined in (7).

So, the system given in (2) transforms into a system of algebraic equations as follows (for $j = 0, 1, 2, \ldots, n-1$):

$$\begin{aligned} \psi_{k,j+1} = \psi_{k,0} &+ (1-\varrho)\mathbf{f}_k(\psi_{1,j+1}, \psi_{2,j+1}, \psi_{3,j+1}, \psi_{4,j+1}, \psi_{5,j+1}, t_{j+1}) \\ &+ \varrho\, h \sum_{r=0}^{j+1} \xi_r\, \mathbf{f}_k(\psi_{1,r}, \psi_{2,r}, \psi_{3,r}, \psi_{4,r}, \psi_{5,r}, t_r), \qquad k = 1(1)5, \end{aligned} \tag{11}$$

where the functions $\mathbf{f}_k$ are defined in (8).

7. Numerical simulation

This section aims to evaluate the efficacy of the suggested approach by simulating a solution throughout the interval $[0, 50]$, whereby we examine the system (2) with varying values of ϱ, β, and distinct initial solutions (I.Cs) [18]. In all figures, we utilize identical quantities for the following constants:

$$\sigma = 0.1, \quad \theta = \alpha = 0.05, \quad \ell_1 = \gamma = \sigma_1 = \sigma_2 = 0.003, \quad \ell_2 = \beta = 0.002.$$

We examine the subsequent two scenarios about the I.Cs:

(i) **Small values:** $\psi_1^0 = 40, \;\; \psi_2^0 = 30, \;\; \psi_3^0 = 20, \;\; \psi_4^0 = 10, \;\; \psi_5^0 = 5;$

(ii) **Large values:** $\psi_1^0 = 75, \;\; \psi_2^0 = 60, \;\; \psi_3^0 = 45, \;\; \psi_4^0 = 30, \;\; \psi_5^0 = 15.$

Furthermore, we provide a comparison of the findings derived from the suggested technique with those given by using the RK4 method to assess the quality and accuracy of the given scheme. The numerical results obtained for the analyzed model using the introduced numerical scheme are presented in Figures 1-3.

(i) Figure 1 illustrates the numerical solution for diverse values of $\varrho = 1.0$, 0.95, 0.85, 0.75, with $h = 0.1$, under the small values of the I.Cs.

(ii) Figure 2 illustrates a comparison of the results derived from the proposed technique against those obtained using the RK4 method at ($\varrho = 1$) with $h = 0.1$ under the small values of the I.Cs.

(iii) Figure 3 examines the impact of the natural death rate on the system, utilizing various values of $\beta = 0.002$, 0.004, 0.006, 0.008, at $\varrho = 0.93$, $h = 0.1$, under the large values of the I.Cs.

By looking closely at these three figures, we can assert and confirm that the numerical solution is based on the different values of ϱ, h, and β; and it is a clear indication that the presented numerical scheme has been well-implemented for solving the suggested system in its fractional form and with this type of derivatives with small and large values of the fractional order.

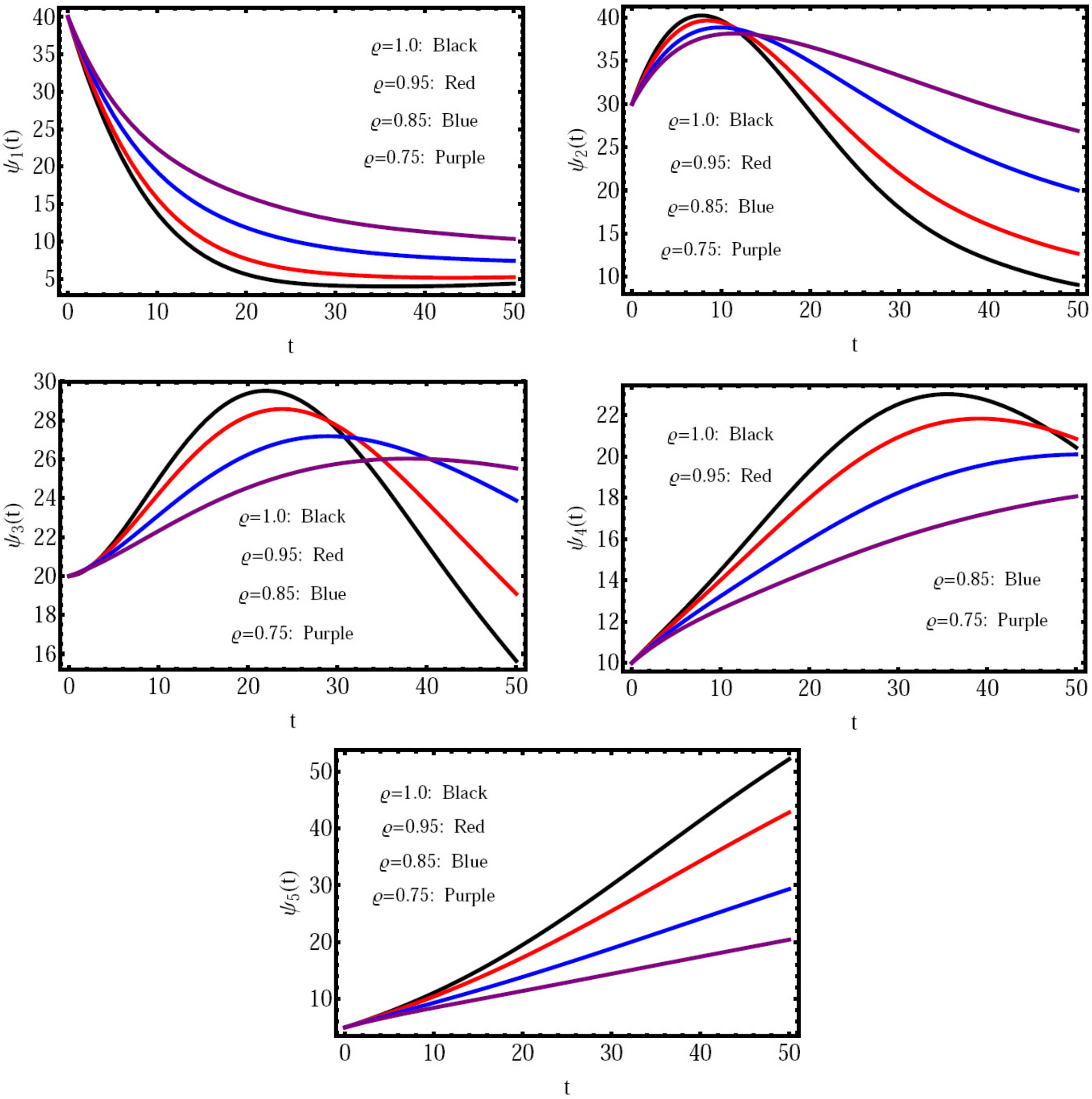

Fig. 1: The approximate solution $\psi_i(t)$, $i = 1, 2, 3, 4, 5$ against distinct values of ϱ with small initial values.

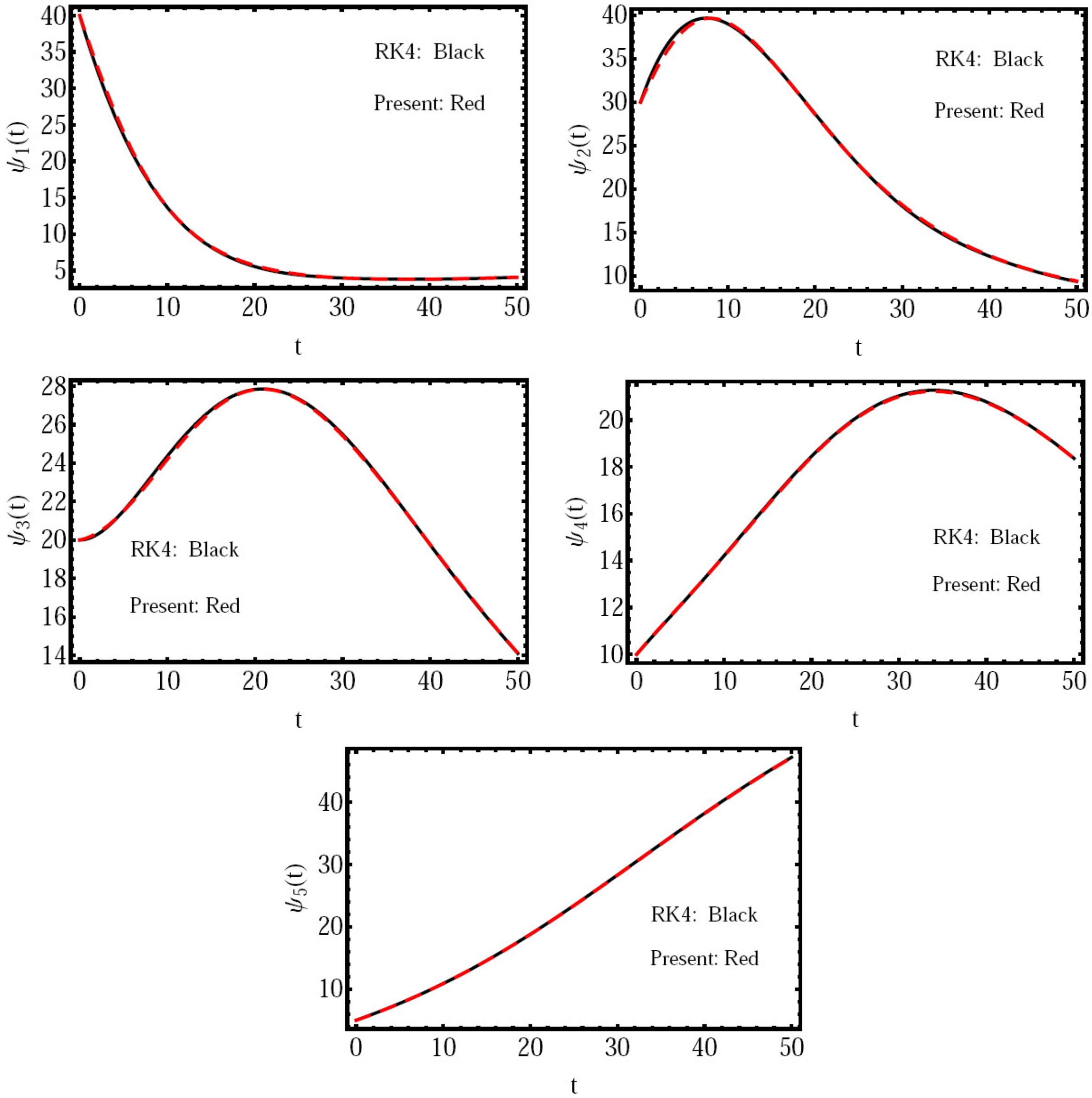

Fig. 2: The solution $\psi_i(t)$, $i = 1, 2, 3, 4, 5$ by the present and RK4 methods $\varrho = 1$.

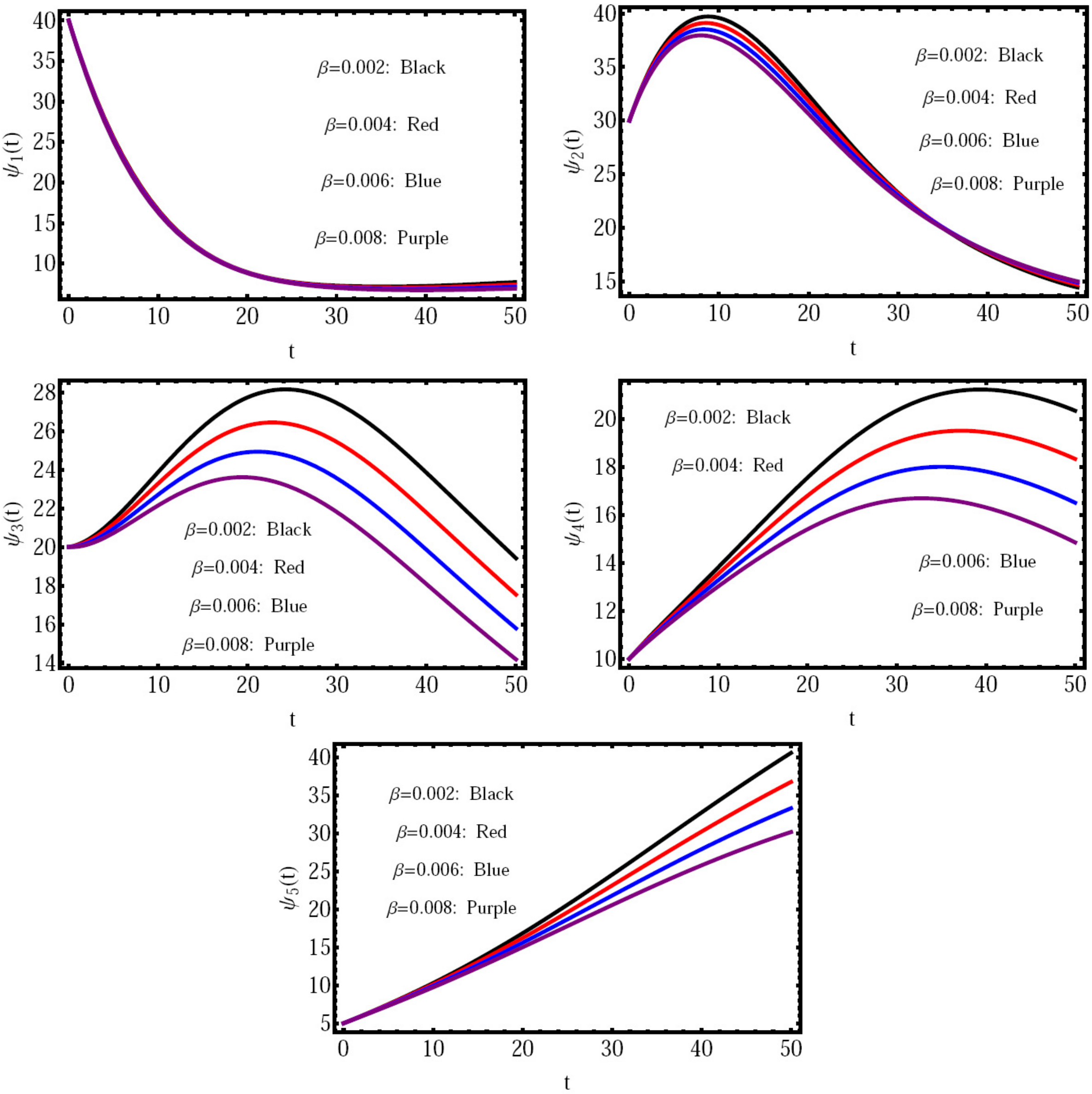

Fig. 3: The effect of β on the approximate solution $\psi_i(t)$, $i = 1, 2, 3, 4, 5$.

8. Conclusions and Discussions

This study seeks to apply fractional calculus tools and techniques to derive numerical solutions for the smoking mathematical system, utilizing the CF-fractional derivative operator. The numerical evaluation of the conversion system for fractional integral equations is conducted utilizing Simpson's 1/3 formula, attaining fourth-order accuracy. This study employed various values of the fractional order ϱ and a step size of $h = 0.1$ to obtain solutions for the mathematical model under investigation. Furthermore, we have ascertained that the suggested methodology is exceptionally proficient in evaluating this mathematical model. Furthermore, reducing the value of h allows us to control the accuracy of the numerical solutions. In conclusion, we have found that using the CF-fractional derivative operator makes the mathematical model under study in this research more suitable for numerical simulations. The RK4 technique produces similar graphical findings and results. Furthermore, our research reveals the precise and efficient nature of the suggested approach. In future studies, we want to utilize the same model as a generalization of the current work but with varying types of fractional derivatives.

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References

[1] T. Abdeljawad and D. Baleanu. On fractional derivatives with exponential kernel and their discrete versions. *Reports on Mathematical Physics*, 80:11–27, 2017.

[2] K. Abuasbeh, R. Shafqat, A. Alsinai, and M. Awadalla. Analysis of the mathematical modeling of COVID-19 by using a mild solution with delay caputo operator. *Symmetry*, 15:286, 2023.

[3] W. Al-sadi, Z. Wei, T. Q. S. Abdullah, A. Alkhazzan, and J. F. Gomez-Aguilar. Dynamical and numerical analysis of the hepatitis B virus treatment model through fractal-fractional derivative. *Mathematical Methods in the Applied Sciences*, 5:1–19, 2024.

[4] M. Alqhtani, J. F. Gomez Aguilar, K. M. Saad, Z. Sabir, and E. P. Careta. A scale conjugate neural network learning process for the nonlinear malaria disease model. *AIMS Mathematics*, 89:21106–2122, 2023.

[5] J. Alzabut, R. Dhineshbabu, A. G. M. Selvam, J. F. Gomez-Aguilar, and H. Khan. Existence, uniqueness and synchronization of a fractional tumor growth model in discrete time with numerical results. *Results in Physics*, 54:107030, 2023.

[6] Y. N. Anjam, R. Shafqat, I. E. Sarris, M. Rahma, S. Touseef, and M. Arshad. A fractional order investigation of the smoking model using the Caputo-Fabrizio differential operator. *Fractal Fract.*, 623:1–18, 2022.

[7] H. M. Arshad and R. Shafqat. Approximating solutions of fuzzy stochastic fractional integro-evolution equations with the averaging principle: Theory and applications. *International Journal of Advancements in Mathematics*, 22:150–163, 2022.

[8] S. Arshad, I. Saleem, A. Akgül, J. Huang, Y. Tang, and S. M. Eldin. A novel numerical method for solving the Caputo-Fabrizio fractional differential equation. *AIMS Mathematics*, 84:9535–9556, 2022.

[9] M. Caputo and M. Fabrizio. A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.*, 1:73–85, 2015.

[10] V. S. Erturk, G. Zaman, and S. Momani. A numeric analytic method for approximating a giving up smoking model containing fractional derivatives. *Comput. Math. Appl.*, 64:3068–3074, 2012.

[11] M. Farman, H. Besbes, K. S. Nisar, and M. Omri. Analysis and dynamical transmission of the COVID-19 model by using Caputo-Fabrizio derivative. *Alexandria Engineering Journal*, 66:597–606, 2023.

[12] C. C. Garsow, G. J. Salivia, and A. R. Herrera. Mathematical models for dynamics of tobacco use, recovery and relapse. Technical report, Cornell University, London, UK, 2000.

[13] I. Goldenberg, M. Jonas, A. Tenenbaum, V. Boyko, S. Matetzky, A. Shotan, S. Behar, and H. R. Reiss. Bezafibrate infarction prevention study group: Current smoking, smoking cessation, and the risk of sudden cardiac death in patients with coronary artery disease. *Arch. Int. Med.*, 163:2301–2305, 2003.

[14] O. K. Ham. Stages and processes of smoking cessation among adolescents. *West. J. Nurs. Res.*, 29:301–315, 2007.

[15] A. Jajarmi, S. Arshad, and D. Baleanu. A new fractional modeling and control strategy for the outbreak of dengue fever. *Physica A*, 535:1–10, 2019.

[16] M. M. Khader and K. M. Abualnaja. Galerkin-FEM for obtaining the numerical solution of the linear fractional klein-gordon equation. *Journal of Applied Analysis and Computation*, 9:261–270, 2019.

[17] M. M. Khader and M. Adel. Modeling and numerical simulation for covering the fractional COVID-19 model using spectral collocation-optimization algorithms. *Fractal and Fractional*, 6363:1–19, 2022.

[18] M. M. Khader and A. H. Tedjani. Numerical simulation for the fractional-order smoking model using a spectral collocation method based on the Gegenbauer wavelet polynomials. *Journal of Applied Analysis and Computation*, 14:847–863, 2024.

[19] Q. Liu, T. Jin, M. Zhu, C. Tian, F. Li, and D. Jiang. Uncertain currency option pricing based on the fractional differential equation in the Caputo sense. *Fractal Fract.*, 6:407–412, 2022.

[20] T. Liu, T. J. Deiss, M. W. Lippi, A. Jauregui, K. Vessel, S. Ke, A. Belzer, H. Zhuo, K. N. Kangelaris, and A. D. Gomez et al. Alternative tobacco product use in critically ill patients. *Int. J. Environ. Res. Public Health*, 17:1–15, 2020.

[21] J. Rong Loh, A. Isah, C. Phang, and Y. T. Toh. On the new properties of Caputo-Fabrizio operator and its application in deriving shifted legendre operational matrix. *Appl. Numer. Math.*, 132:138–153, 2018.

[22] J. Losada and J. J. Nieto. Properties of a new fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.*, 1:87–92, 2015.

[23] World Health Organisation. Monitoring tobacco use and prevention policies. executive summary. *World Health Organisation*, 2017.

[24] K. M. Owolabi and A. Atangana. Analysis and application of new fractional Adams-Bashforth scheme with Caputo-Fabrizio derivative. *Chaos Solitons Fract.*, 105:111–119, 2017.

[25] P. Veeresha, E. Ilhan, D. G. Prakasha, and H. M. Baskonus. A new numerical investigation of fractional order susceptible-infected-recovered epidemic model of childhood disease. *Alexandria Engineering Journal*, 61:1747–1756, 2022.

[26] P. Veeresha and D. G. Prakasha. A novel technique for (2n+1)-dimensional time-fractional coupled burgers equations. *Alexandria Engineering Journal*, 166:324–345, 2019.

[27] A. Yildirum and Y. Cherruault. Analytical approximate solution of a SIR epidemic model with constant vaccination strategy by homotopy perturbation method. *Emerald*, 38:1566–1581, 2009.

[28] F. Youbi, S. Momani, S. Hasan, and M. Al-Smadi. Effective numerical technique for nonlinear Caputo-Fabrizio systems of fractional volterra integro-differential equations in hilbert space. *Alexandria Engineering Journal*, 61:1778–1786, 2022.