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Hyers-Ulam Stability of Fifth Order Linear Differential Equations

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Abstract. In this paper, we study the Hyers-Ulam stability for the fifth-order linear differential equation. In particular, we treat ς as an arrangement of differential equation and in the form

$$\varsigma^{v}(x) + \eta_1 \varsigma^{iv}(x) + \eta_2 \varsigma^{'''}(x) + \eta_3 \varsigma^{''}(x) + \eta_4 \varsigma^{'}(x) + \eta_5 \varsigma(x) = \Omega(x)$$

where $\varsigma \in c^5[k,l]$, $\Omega \in [k,l]$. We demonstrate that $\varsigma^{v}(x) + \eta_1 \varsigma^{iv}(x) + \eta_2 \varsigma^{'''}(x) + \eta_3 \varsigma^{''}(x) + \eta_4 \varsigma^{'}(x) + \eta_5 \varsigma(x) = \Omega(x)$ has the Hyers-Ulam stability. Two illustrative examples are given to represent the effectiveness of the proposed method. Fifth-order linear differential equations find applications in a wide range of fields, from engineering and control theory to physics, biology, and beyond. These equations are powerful tools for modeling systems with complex dynamics that involve multiple interacting forces or rates of change. Understanding and analyzing their stability and behavior can lead to significant advancements in the design, control, and optimization of these systems.

2020 Mathematics Subject Classifications: 35B35

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1. Introduction

The Hyers-Ulam stability was presented by S.M. Ulam [23] to bring up the problem, suppose one has a function $\varsigma(t)$ which is near to solve an equation. Is there a exact solution $x(t)$ of the equation which is close to $\varsigma(t)$ (See [3, 5, 11]). In 1941, D.H. Hyers [6] response to the condition of Ulam for additive Cauchy equation in Banach space. A solution for Ulam's problems for linear mappings was demonstrated by Th. M. Rassias [21], thought about a mapping $g : \mathcal{E}_1 \to \mathcal{E}_2$ such that $t \to g(tx)$ is continuous in t for each fixed x. If that there exists $\theta \geq 0$ and $0 \leq p < 1$ such that

$$\|g(x+y) - g(x) - g(y)\| = \vartheta(\|x\|^p + \|\varsigma\|^p),\ \forall x, y \in \mathcal{E}_1.$$

After that, numerous mathematicians have investigate Ulam's problem in different ways (see [2, 9, 10, 18, 19, 24]). A Hyers-Ulam-Rassias problem is the differential equation $\varphi(g, \varsigma, \varsigma', \varsigma'', ..., \varsigma^n) = 0$ has the Hyers-Ulam-Rassias stability with respect to ϑ if there exist a constant $\mathcal{M} > 0$ to such an extent that for given a function ς such that $|\varphi(g, \varsigma, \varsigma', \varsigma'', ..., \varsigma^n)| \leq \vartheta(t)$. Then there exists a solution ς_c of the differential equation such that $|\varsigma(t) - \varsigma_c(t)| \leq \mathcal{M}\vartheta(t)$.

Meaning of Hyers-Ulam-Rassias stability importance implies that, assuming one is considering a Hyers-Ulam-Rassias stability system, one doesn't need to arrive at the exact solution. This is very useful for many applications for example statistical research, optimization, biology and financial aspects and so on.

In the past decades many of the researchers has been concentrated on the Hyers-Ulam stability of linear differential equations (see [7, 8, 25]). Likewise Jung has demonstrated the Hyers-Ulam stability of linear differential equations by using the Laplace transform method (see [22]) and authors in [20] studied Ulam stability of linear differential equations using fourier transform. Recently, authors in [4], researched Hyers-Ulam stability of an n-variable quartic functional equation. To the best of author's knowledge, Hyers-Ulam stability approaches to linear differential equation of order five has not been studied so far, which motivates the present study.

Linear differential equations have been studied extensively across various fields like physics, engineering, and applied mathematics. Early research primarily focused on first- and second-order differential equations, which are simpler to analyze both analytically and numerically. Fifth-order differential equations frequently arise in models related to advanced mechanical systems, fluid dynamics, or even quantum mechanics. They are often used in beam theory (e.g., the bending of beams), electromagnetic theory, and more complex vibrational systems. In control theory and signal processing, fifth-order differential equations can model systems with higher-order dynamics, particularly in cases involving feedback systems or circuits. Stability in the context of differential equations refers to the behavior of solutions as they respond to small changes in initial conditions or the forcing function. In simpler terms, stability determines whether a small perturbation will grow or decay over time.

The literature on stability analysis for differential equations has traditionally focused on lower-order systems (especially second-order). However, the theory has been extended to higher-order systems, including fifth-order equations, particularly in the study of physical systems with complex dynamics.The concept of Hyers-Ulam stability examines whether a differential equation exhibits stability when subjected to small perturbations in the functional form. Specifically, it asks whether the approximate solution remains close to the exact solution.Much of the early work on Hyers-Ulam stability focused on first- and second-order linear differential equations, but more recent studies have extended this analysis to higher-order equations, including fifth-order systems. These studies are particularly important because higher-order systems are more sensitive to perturbations, making the stability analysis more complex.

Recent works in the field have explored conditions under which fifth-order differential equations admit unique solutions. These results typically depend on the properties of the coefficients and boundary conditions. With increasing computational power, researchers have developed sophisticated numerical techniques to approximate solutions of fifth-order equations. Finite element methods and spectral methods have become popular in this regard. Current research extends classical stability theorems, such as Lyapunov stability, to fifth-order systems. This often involves formulating and solving Lyapunov functions for these complex systems to determine conditions under which solutions are stable. Several recent studies have focused on extending Hyers-Ulam stability to fifth-order differential equations. These works build on classical stability results but adapt them to the unique challenges posed by higher-order systems. The key focus of this research has been to identify conditions under which small deviations in the approximate solution lead to bounded deviations in the actual solution, thus extending Hyers-Ulam's classical framework to a more complex domain.

The study of fifth-order linear differential equations is motivated by both practical and theoretical needs. From a practical standpoint, these equations model complex real-world systems that involve higher-order dynamics and feedback loops, such as in engineering, physics, and control theory. From a theoretical perspective, studying fifth-order equations enhances understanding of stability, existence, uniqueness, and the development of more sophisticated numerical methods. As these equations play a crucial role in accurately describing advanced systems, their study not only fills important gaps in the literature but also leads to advancements in applied mathematics and various scientific fields.

Before diving into the mathematical derivations, offer a brief overview of the problem being solved and the key assumptions made. For example, if you're proving stability for a specific class of fifth-order linear differential equations, start by clearly stating. The general form of the fifth-order equation you're focusing on (e.g., constant coefficients or variable coefficients).The conditions under which the stability analysis is conducted, such as the smoothness of the solution or boundary conditions. We consider a fifth-order linear differential equation with constant coefficients, where we assume that the solutions are continuous and differentiable up to the fifth derivative. These assumptions are essential for ensuring that the equation can be analyzed within the framework of Hyers-Ulam stability.

In practical applications, stability often refers to the system's ability to resist or dampen perturbations. When you discuss an example showing Hyers-Ulam stability, interpret it by explaining how small deviations in the inputs (such as initial conditions or external forces) do not lead to exponential or uncontrolled growth in the system's response. In this example, the fifth-order differential equation models the motion of a mechanical system with multiple feedback loops. The stability results demonstrate that even with slight variations in the input forces or initial velocities, the motion remains bounded. This is particularly important in control systems, where small inaccuracies in the measurement or control signals can lead to significant errors in the system's behavior. By ensuring Hyers-Ulam stability, we know that such small deviations will not cause instability, ensuring the system operates predictably. "Many physical systems governed by fifth-order differential equations, such as beam vibrations or wave propagation, rely on stability for consistent performance. When interpreting examples, explain how the stability results ensure that the system remains predictable and controllable, even when exposed to small disturbances. In the case of wave propagation, the stability result indicates that small perturbations in the medium (such as density or pressure) will not cause the wave to grow uncontrollably in amplitude. This property is crucial for the design of wave guides or telecommunications systems, where small variations in the physical properties of the medium can lead to large-scale disruptions if the system is unstable."

One of the most natural future directions is extending the methods and stability results developed for linear fifth-order differential equations to nonlinear systems. In practical applications, many systems exhibit nonlinear behavior, especially when dealing with large deviations from equilibrium or complex feedback mechanisms. For instance, in engineering or physics, systems often become nonlinear when subjected to strong external forces or interactions between components. Boundary value problems for fifth-order systems often arise in the study of elastic structures and fluid dynamics. Investigating how the stability results extend to these problems would be essential for expanding the applicability of the current results. Boundary conditions can introduce additional constraints or complexities that affect the stability behavior. The stability of fifth-order linear differential equations opens several exciting future research directions. Potential extensions include generalizing the results to nonlinear and fractional differential equations, handling systems with variable coefficients, investigating boundary value problems, and exploring applications in control theory. Additionally, addressing the challenges of non-classical solutions and weak forms provides a pathway for further theoretical development. These future directions not only deepen the understanding of stability in complex systems but also offer practical tools for tackling real-world problems in various scientific and engineering fields.

The paper concludes by summarizing the key findings and contributions. We discuss the significance of extending Hyers-Ulam stability to fifth-order equations and suggest potential directions for future research, including the extension of these results to nonlinear equations and other higher-order systems.

Let $\mathcal{X}$ be a normed space over a scalar field $\mathcal{K}$ and let $\mathcal{I}$ be an open span. Expect that

$c_0, c_1, ..., c_n$ are fixed components of $\mathcal{K}$. We say that the differential equation

$$c_n(t)\varsigma^{(n)}(t) + c_{n-1}(t)\varsigma^{(n-1)}(t) + ... + c_1(t)\varsigma'(t) + c_0\varsigma(t) + h(t) = 0 \tag{1.1}$$

has the Hyers–Ulam stability, if for any function $g : \mathcal{I} \to \mathcal{X}$ satisfies the differential inequality

$$\|c_n(t)\varsigma^{(n)}(t) + c_{n-1}(t)\varsigma^{(n-1)}(t) + \ + c_1(t)\varsigma'(t) + c_0\varsigma(t) + h(t)\| \leq \epsilon,$$

for all $t \in I$ and for some $\epsilon \geq 0$. Then there exists a solution $f : \mathcal{I} \to \mathcal{X}$ of (1.1) such that $\|g(t) - f(t)\| \leq \mathcal{K}(\epsilon)$, for any $t \in I$, where $K(\epsilon)$ is an articulation for ϵ as it were.

The stability of differential equation explored by [1] if $\epsilon > 0$, a differentiable function $g : \mathcal{I} \to \mathcal{R}$ fulfills the differential disparity $|\varsigma'(t) - \varsigma(t)| \leq \epsilon$, where $\mathcal{I}$ is an open subinterval of $\mathcal{R}$, at that point there exists a differentiable function $g_0 : \mathcal{I} \to \mathcal{R}$ fulfilling $g_0'(t) = g_0(t)$ such that $|g(t) - g_0(t)| \leq 3\epsilon$, for all $t \in \mathcal{I}$.

Li and Shen [12] have explored the Hyers–Ulam stability of the linear differential equations of the second order

$$\varsigma''(x) + \alpha\varsigma'(x) + \beta\varsigma(x) = g(x), \tag{1.2}$$

where $\varsigma \in \mathcal{C}^2[a, b], g \in \mathcal{C}[a, b]$ and $-\infty < a < b < \infty$. In fact they demonstrated that if the condition $\lambda^2 + \alpha\lambda + \beta = 0$ has two distinctive positive roots, at that point the condition $\varsigma''(x) + \alpha\varsigma'(x) + \beta\varsigma(x) = g(x)$ has the Hyers–Ulam stability. Recently, Luo [13–17, 26] investigated the Hyers-Ulam stability results of differential equations in fractional order.

In this paper we will examine the stability of differential equations of fifth order. In section 2, we will give a vital and adequate condition all together that the fifth order linear differential equation and established Hyers-Ulam stability constant under those conditions. In section 3, we will apply this results to fifth order differential equations by numerical examples.

Definition:1.1 [19] Let $\mathcal{I}$ be any interval and let $z : \mathcal{I} \to \mathcal{R}^n$, $\mathcal{A} : \mathcal{I} \to \mathcal{R}^{n\times n}$, $\mathcal{B} : \mathcal{I} \to \mathcal{R}^n$ at that point

$$z'(t) + \mathcal{A}(t)z(t) + \mathcal{B}(t) = 0 \tag{1.3}$$

is Hyers-Ulam-Rassias stable as for $\varphi : \mathcal{I} \to [0, \infty)$ with $\|z(t)\| = \sum_{i=1}^n |z_i(t)|$, if there exists a real consistent $\mathcal{K} > 0$ with the end goal that for every arrangement $s \in \mathcal{C}^1(\mathcal{I}, \mathcal{R}^n)$ of the inequality

$$\|z'(t) + \mathcal{A}(t)z(t) + \mathcal{B}(t)\| \leq \psi(t),$$

there exists an answer $z \in \mathcal{C}^1(\mathcal{I}, \mathcal{R}^n)$ of condition (1.3) with

$$\|s(t) - z(t)\| \leq \mathcal{K}\psi(t), \ \forall t \in \mathcal{I}.$$

Definition:1.2 [21] For a nonempty set $\mathcal{X}$, a function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a generalized metric on $\mathcal{X}$ if and only if d satisfies :

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$, for all $x, y \in \mathcal{X}$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in \mathcal{X}$.

Definition:1.3 [12] We denote (1.2) has the Hyers-Ulam stability if there exists a steady $v_1 > 0$ with the accompanying property, for every $\varepsilon > 0, \varsigma \in c^2[k, l]$, if

$$|\varsigma^{''} + a\varsigma^{'} + b\varsigma| \leq \varepsilon, \tag{1.4}$$

as such, $\exists\ u \in c^2[k, l]$ that satisfies:

$$|u^{''} + au^{'} + bu| = 0, \tag{1.5}$$

such that $|\varsigma(x) - u(x)| < v_1\epsilon$. We denote v_1 be a Hyers-Ulam stability constant for (1.2).
Definition:1.4 [12] We indicate that the augmentation of (1.2) has the Hyers-Ulam stability, if there exists a consistent $v_1 > 0$ with the going with property, for each $\varepsilon > 0$, $\varsigma \in c^3[k, l]$, if

$$|\varsigma^{'''} + a\varsigma^{''} + b\varsigma^{'} + c\varsigma| \leq \varepsilon, \tag{1.6}$$

as such, $\exists\ u \in c^3[k, l]$ that satisfies:

$$|u^{'''} + au^{''} + bu^{'} + cu| = 0, \tag{1.7}$$

such that $|\varsigma(x) - u(x)| < v_1\epsilon$, where v_1 is a Hyers-Ulam stability constant for (1.6).
Definition:1.5 [12] We mean that the augmentation of (1.6) has the Hyers-Ulam stability, if there exists a consistent $v_1 > 0$ with the going with property, for each $\varepsilon > 0$, $\varsigma \in c^4[k, l]$, if

$$|\varsigma^{iv} + \eta_1\varsigma^{'''} + \eta_2\varsigma^{''} + \eta_3\varsigma^{'} + \eta_4\varsigma| \leq \varepsilon, \tag{1.8}$$

as such, $\exists\ u \in c^4[k, l]$ that satisfies:

$$|u^{iv} + \eta_1 u^{'''} + \eta_2 u^{''} + \eta_3 u^{'} + \eta_4 u| = 0, \tag{1.9}$$

such that $|\varsigma(x) - u(x)| < v_1\epsilon$, where v_1 is a stability constant for (1.8).
Definition:1.6 We indicate that the augmentation of (1.8) has the Hyers-Ulam stability, if there exists a consistent $v_1 > 0$ with the going with property, for each $\varepsilon > 0$, $\varsigma \in c^5[k, l]$, if

$$|\varsigma^{v} + \eta_1\varsigma^{iv} + \eta_2\varsigma^{'''} + \eta_3\varsigma^{''} + \eta_4\varsigma^{'} + \eta_5\varsigma| \leq \varepsilon, \tag{1.10}$$

as such, $\exists\ u \in c^5[k, l]$ that satisfies:

$$|u^{v} + \eta_1 u^{iv} + \eta_2 u^{'''} + \eta_3 u^{''} + \eta_4 u^{'} + \eta_5 u| = 0, \tag{1.11}$$

such that $|\varsigma(x) - u(x)| < v_1\epsilon$, where v_1 is a constant for (1.10).

2. Main results

In this section, the authors discussed Hyers-Ulam stability of linear differential equation and also the critical consequences of this investigation are given in the accompanying hypothesis.

Lemma 2.1. The differential equation $\varsigma^{v}(x)+\eta_1\varsigma^{iv}(x)+\eta_2\varsigma^{'''}(x)+\eta_3\varsigma^{''}(x)+\eta_4\varsigma^{'}(x)+\eta_5\varsigma(x)=\Omega(x)$ has the Hyers - Ulam stability, where $\varsigma\in c^5[k,l]$ and $\Omega\in[a,b]$.

proof: Suppose that v_1,v_2,v_3,v_4,v_5 are the (real or complex) roots of $m^5+\eta_1 m^5+\eta_1 m^4+\eta_2 m^3+\eta_3 m^2+\eta_4 m+\eta_5=0$ with $p_1=\Re v_1, p_2=\Re v_2, p_3=\Re v_4, p_4=\Re v_3, p_5=\Re v_5$. Here $\Re$ denotes the real part. Let $\varepsilon>0$ and $\varsigma\in c^5[k,l]$ with

$$|\varsigma^{v}(x)+\eta_1\varsigma^{iv}(x)+\eta_2\varsigma^{'''}(x)+\eta_3\varsigma^{''}(x)+\eta_4\varsigma^{'}(x)+\eta_5\varsigma(x)-\Omega(x)|=0 \tag{2.1}$$

and let

$$\begin{aligned} g(x)=\varsigma^{iv}(x)+(v_1+\eta_1)\varsigma^{'''}(x)+(v_1^2+\eta_1 v_1+\eta_2)\varsigma^{''}(x)\\ +(v_1^3+\eta_1 v_1^2+\eta_2 v_1+\eta_3)\varsigma^{'}(x)+\\ (v_1^4+\eta_1 v_1^3+\eta_2 v_1^2+\eta_3 v_1+\eta_4)\varsigma(x). \end{aligned} \tag{2.2}$$

Then we obtain

$$\begin{aligned} g'(x)=\varsigma^{v}(x)+(v_1+\eta_1)\varsigma^{iv}(x)+(v_1^2+\eta_1 v_1+\eta_2)\varsigma^{'''}(x)\\ +(v_1^3+\eta_1 v_1^2+\eta_2 v_1+\eta_3)\varsigma^{''}(x)+(v_1^4+\eta_1 v_1^3+\eta_2 v_1^2+\eta_3 v_1+\eta_4)\varsigma^{'}(x)\\ +(v_1^5+\eta_1 v_1^4+\eta_2 v_1^3+\eta_3 v_1^2+\eta_4 v_1+\eta_5)\varsigma(x), \end{aligned} \tag{2.3}$$

$\forall\ x\in[k,l]$ and

$$|g'(x)-v_1 g(x)-\Omega(x)|\le\varepsilon \tag{2.4}$$

$$\begin{aligned} |g'(x)-v_1 g(x)-\Omega(x)|=|\varsigma^{v}(x)+(v_1+\eta_1)\varsigma^{iv}(x)+(v_1^2+\eta_1 v_1+\eta_2)\varsigma^{'''}(x)\\ +(v_1^3+\eta_1 v_1^2+\eta_2 v_1+\eta_3)\varsigma^{''}(x)+(v_1^4+\eta_1 v_1^3+\eta_2 v_1^2+\eta_3 v_1+\eta_4)\varsigma^{'}(x)\\ +(v_1^5+\eta_1 v_1^4+\eta_2 v_1^3+\eta_3 v_1^2+\eta_4 v_1+\eta_5)\varsigma(x)-v_1[\varsigma^{iv}(x)+(v_1+\eta_1)\varsigma^{'''}(x)\\ +(v_1^2+\eta_1 v_1+\eta_2)\varsigma^{''}(x)+(v_1^3+\eta_1 v_1^2+\eta_2 v_1+\eta_4)\varsigma^{'}(x)\\ +(v_1^4+\eta_1 v_1^3+\eta_2 v_1^2+\eta_3 v_1+\eta_4)\varsigma(x)]-\Omega(x)| \end{aligned} \tag{2.5}$$

$$\begin{aligned} |g'(x)-v_1 g(x)-\omega(x)|=|\varsigma^{v}(x)+v_1\varsigma^{iv}(x)+\eta_1\varsigma^{iv}(x)+v_1^2\varsigma^{'''}(x)+\\ \eta_1 v_1\varsigma^{'''}(x)+\eta_2\varsigma^{'''}(x)+v_1^3\varsigma^{''}(x)+\eta_1 v_1^2\varsigma^{''}(x)+\\ \eta_2 v_1\varsigma^{''}(x)+\eta_3\varsigma^{''}(x)+v_1^4\varsigma^{'}(x)+\eta_1 v_1^3\varsigma^{'}(x)\\ +\eta_2 v_1^2\varsigma^{'}(x)+\eta_3 v_1\varsigma^{'}(x)+\eta_4\varsigma^{'}(x)+v_1^5\varsigma(x)+\\ \eta_1 v_1^4\varsigma(x)+\eta_2 v_1^3\varsigma(x)+\eta_3 v_1^2\varsigma(x)+\eta_4 v_1\varsigma(x)+\\ \eta_5\varsigma(x)-v_1[\varsigma^{iv}(x)+v_1\varsigma^{'''}(x)+\eta_1\varsigma^{'''}(x)\\ +v_1^2\varsigma^{''}(x)+\eta_1 v_1\varsigma^{''}(x)+\eta_2\varsigma^{''}(x)+v_1^3\varsigma^{'}(x)+\\ \eta_1 v_1^2\varsigma^{'}(x)+\eta_2 v_1\varsigma^{'}(x)+\eta_3\varsigma^{'}(x)+v_1^4\varsigma(x)+\\ \eta_1 v_1^3\varsigma(x)+\eta_2 v_1^2\varsigma(x)+\eta_3 v_1\varsigma(x)+\eta_4\varsigma(x)]-\Omega(x)| \end{aligned} \tag{2.6}$$

$$|g'(x)-v_1g(x)-\Omega(x)| = |\varsigma^{v}(x)+\eta_1\varsigma^{iv}(x)+\eta_2\varsigma^{'''}(x)+\eta_3\varsigma^{''}(x)+\eta_4\varsigma^{'}(x)+\eta_5\varsigma(x)-\Omega(x)| \leq \varepsilon \tag{2.7}$$

$$|g'(x)-v_1g(x)-\Omega(x)| \leq \varepsilon. \tag{2.8}$$

Equivalently 'g' satisfies

$$-\varepsilon \leq g^{'}(x)-v_1g(x)-\Omega(x) \leq \varepsilon. \tag{2.9}$$

Multiplying the equation by $e^{-v_1(x-k)}$, we get

$$-\varepsilon e^{-v_1(x-k)} \leq g^{'}(x)e^{-v_1(x-k)} - v_1g(x)e^{-v_1(x-k)} - \Omega(x)e^{-v_1(x-k)} \leq \varepsilon e^{-v_1(x-k)}. \tag{2.10}$$

Without loss of consensus, we may expect to be that $v_1 > 1$, thus

$$\begin{aligned}-v_1\varepsilon e^{-v_1(x-k)} \leq g^{'}(x)e^{-v_1(x-k)} - v_1g(x)e^{-V_1(x-k)} - \Omega(x)e^{-v_1(x-k)}\\ \leq v_1\varepsilon e^{-v_1(x-k)},\end{aligned} \tag{2.11}$$

for any $x \in [k,l]$. Integrating (2.11) from x to l, we get

$$\begin{aligned}-v_1\varepsilon\left(\frac{e^{-v_1(x-k)}}{-v_1}\right)_x^l \leq g(l)e^{-v_1(l-k)} - v_1\frac{g(x)e^{-v_1(x-k)}}{v_1} - \int_x^l \Omega(t)e^{-v_1(t-k)}dt\\ \leq v_1\varepsilon\left(\frac{e^{-v_1(x-k)}}{-v_1}\right)_x^l\end{aligned}$$

$$\begin{aligned}-\varepsilon\left(\frac{e^{-v_1(l-k)}-e^{-v_1(x-k)}}{-1}\right) \leq g(l)e^{-v_1(l-k)} - g(x)e^{-v_1(x-k)} - \int_x^l \Omega(t)e^{-v_1(t-k)}dt\\ \leq \varepsilon\left(\frac{e^{-v_1(l-k)}-e^{-v_1(x-k)}}{-1}\right)\end{aligned}$$

$$\begin{aligned}-\varepsilon\left(e^{-v_1(x-k)}-e^{-v_1(l-k)}\right) \leq g(l)e^{-v_1(l-k)} - g(x)e^{-v_1(x-k)} - \int_x^l \Omega(t)e^{-v_1(t-k)}dt\\ \leq \varepsilon\left(e^{-v_1(x-k)}-e^{-v_1(l-k)}\right)\end{aligned} \tag{2.12}$$

$$\begin{aligned}-\varepsilon e^{-v_1(x-k)} \leq g(l)e^{-v_1(l-k)} - \varepsilon e^{-v_1(l-k)} - g(x)e^{-v_1(x-k)} - \int_x^l \Omega(t)e^{-v_1(t-k)}dt\\ \leq \varepsilon\left(e^{-v_1(x-k)}-e^{-v_1(l-k)}\right)\end{aligned} \tag{2.13}$$

$$\begin{aligned}-\varepsilon e^{-v_1(x-k)} \leq g(l)e^{-v_1(l-k)} - \varepsilon e^{-v_1(l-k)} - g(x)e^{-v_1(x-k)} - \int_x^l \Omega(t)e^{-v_1(t-k)}dt\\ \leq \varepsilon e^{-v_1(x-k)}.\end{aligned}$$

Multiplying the equation by $e^{v_1(x-k)}$, we get

$$\begin{aligned} -\varepsilon e^{-v_1(x-k)} e^{v_1(x-k)} &\leq g(l) e^{-v_1(l-k)} e^{v_1(x-k)} - \varepsilon e^{-v_1(x-k)} e^{v_1(x-k)} \\ &\quad - g(x) e^{-v_1(x-k)} e^{v_1(x-k)} - \int_x^l \Omega(t) e^{-v_1(t-k)} dt e^{v_1(x-k)} \\ &\leq \varepsilon e^{-v_1(x-k)} e^{v_1(x-k)} \end{aligned} \tag{2.14}$$

$$-\varepsilon \leq (g(l)-\varepsilon) e^{-v_1 l + v_1 k + v_1 x - v_1 k} - g(x) - \int_x^l \Omega(t) e^{-v_1 t + v_1 k + v_1 x - v_1 k} dt \leq \varepsilon$$

$$-\varepsilon \leq (g(l)-\varepsilon) e^{v_1(x-l)} - g(x) - \int_x^l \Omega(t) e^{v_1(x-t)} dt \leq \varepsilon$$

$$-\varepsilon \leq g(l) e^{v_1(x-l)} - \varepsilon e^{v_1(x-l)} - g(x) - \int_x^l \Omega(t) e^{v_1(x-t)} dt \leq \varepsilon$$

$$-\varepsilon \leq g(l) e^{v_1(x-l)} - \varepsilon e^{v_1(x-l)} - g(x) - e^{v_1 x} \int_x^l \Omega(t) e^{-v_1 t} dt \leq \varepsilon. \tag{2.15}$$

Let $\chi(x) = g(l) e^{v_1(x-l)} - e^{v_1 x} \int_x^l \Omega(t) e^{-v_1 t} dt$. Then $\chi(x)$ satisfies $\chi^{'}(x) - v_1 \chi(x) - \Omega(x) = 0$ by $\chi^{'}(x) = v_1 \chi(x) + \Omega(x), x \in [k,l]$.

$$|\chi(x) - g(x)| = |e^{v_1(x-l)} g(l) - g(x) - e^{v_1 x} \int_x^l \Omega(t) e^{-v_1 t} dt|$$

$$|\chi(x) - g(x)| \leq e^{p_1 x} \int_x^l |e^{-v_1 t}| |g'(t) - v_1 g(t) - \Omega(t)| dt$$

$$|\chi(x) - g(x)| \leq \varepsilon e^{p_1 x} \int_x^l e^{-p_1 t} dt. \tag{2.16}$$

If $p_1 \neq 0$, then

$$\begin{aligned} |\chi(x) - g(x)| &\leq \varepsilon e^{p_1 x} \int_x^l e^{-p_1 t} dt \\ &\leq \varepsilon e^{p_1 x} \left[\frac{e^{-p_1 l}}{-p_1} - \frac{e^{-p_1 x}}{-p_1} \right] \\ &\leq \frac{\varepsilon}{-p_1} e^{p_1 x} \left[e^{-p_1 l} - e^{-p_1 x} \right] \\ &\leq \frac{\varepsilon}{-p_1} \left[e^{p_1 x} e^{-p_1 l} - e^{p_1 x} e^{-p_1 x} \right] \\ &\leq \frac{\varepsilon}{-p_1} \left[e^{p_1(x-l)} - 1 \right] \end{aligned}$$

$$|\chi(x) - g(x)| \leq \frac{\varepsilon}{p_1} \left[1 - e^{-p_1(l-x)} \right]; x \in [k,l]. \tag{2.17}$$

If $p_1 = 0$, then

$$|\chi(x) - g(x)| \le \varepsilon e^{p_1 x} \int_x^l e^{-p_1 t} dt$$

$$|\chi(x) - g(x)| \le \varepsilon \int_x^l dt$$

$$|\chi(x) - g(x)| \le \varepsilon(l - x); x \in [k, l]$$

$$|\chi(x) - g(x)| \le \varepsilon(l - k); x \in [k, l]. \tag{2.18}$$

Therefore,

$$|\chi(x) - g(x)| \le \begin{cases} \frac{[1-e^{-\chi(l-k)}]\varepsilon}{\chi}; & if \quad \chi \neq 0 \\ (l-k)\varepsilon; & if \quad \chi = 0, \end{cases} \tag{2.19}$$

$\forall\ x \in [k, l]$.

Theorem 2.2. The differential equation
$\varsigma^{v}(x)+\eta_1\varsigma^{iv}(x)+\eta_2\varsigma^{'''}(x)+\eta_3\varsigma^{''}(x)+\eta_4\varsigma^{'}(x)+\eta_5\varsigma(x) = \Omega(x)$ has the Hyers-Ulam stability, Where $\varsigma \in c^5[k, l]$ and $\Omega \in [k, l]$. Therefore

$$|\Gamma(x) - \phi(x)| \le \begin{cases} \frac{[1-e^{-p_2(l-k)}][1-e^{-p_1(l-k)}]}{p_2 p_1}; & if\ p_1, p_2 \neq 0 \\ \frac{[1-e^{-p_2(l-k)}][l-k]\in}{p_2}; & if\ p_2 \neq 0, p_1 = 0 \\ \frac{[1-e^{-p_1(l-k)}][l-k]\in}{p_2}; & if\ p_2 = 0, p_1 \neq 0 \\ (l-k)^2 \in & if p_2, p_1 = 0 \end{cases} \tag{2.20}$$

with respect to $x \in [k, l]$.

Proof: Similar to the proof of lemma 2.1.
Let consider $\phi(x) = \varsigma'(x) + (v_2 + \eta_1)\varsigma(x)$ by

$$\phi'(x) = \varsigma''(x) + (v_2 + \eta_1)\varsigma'(x) + (v_2^2 + \eta_1 v_2 + \eta_2)\varsigma(x).$$

Also $|\phi'(x) - v_2\phi(x) - \chi(x)| = |\chi(x) - g(x)|$.
Then,

$$\begin{aligned} &|\phi'(x) - v_2\phi(x) - \chi(x)| \le \epsilon \\ &= |\varsigma'(x) + (v_2 + \eta_1)\varsigma'(x) + (v_2^2 + \eta_1 v_2 + \eta_2)\varsigma(x) - v_2(\varsigma'(x) + (v_2 + \eta_1)\varsigma(x) - \chi(x)| \\ &= |\varsigma'(x) + v_2\varsigma'(x) + \eta_1\varsigma'(x) + v_2^2\varsigma(x) + \eta_1 v_2\varsigma(x) + \eta_2\varsigma(x) - v_2\varsigma'(x) - v_2^2\varsigma(x) - \eta_1 v_2\varsigma(x) - \chi(x)| \\ &= |\varsigma'(x) + \eta_1\varsigma'(x) + \eta_2\varsigma(x) - \chi(x)| \end{aligned} \tag{2.21}$$

$$|\phi'(x) - v_2\phi(x) - \chi(x)| = |\varsigma''(x) - v_2\varsigma'(x) - v_2(\varsigma'(x) - v_2\varsigma(x)) - \chi(x)|$$

$$|\phi'(x) - v_2\phi(x) - \chi(x)| = |\varsigma''(x) + \eta_1\varsigma'(x) + \eta_2\varsigma(x) - \chi(x)| \le \in$$

$$|\phi'(x) - v_2\phi(x) - \chi(x)| \le \in . \tag{2.22}$$

Proportionally $'\phi'$ satisfies,

$$-\in\leq \phi'(x) - v_2\phi(x) - \chi(x) \leq\in . \tag{2.23}$$

Multiplying the equation by $e^{-v_2(x-k)}$, we get

$$-\in e^{-v_2(x-k)} \leq \phi'(x)e^{-v_2(x-k)} - v_2\phi(x)e^{-v_2(x-k)} - \chi(x)e^{-v_2(x-k)} \leq\in e^{-v_2(x-k)}. \tag{2.24}$$

Without loss of consensus, we may expect to be that $v_2 > 1$, thus

$$-v_2 \in e^{-v_2(x-k)} \leq \phi'(x)e^{-v_2(x-k)} - v_2\phi(x)e^{-v_2(x-k)} - \chi(x)e^{-v_2(x-k)} \leq\in v_2e^{-v_2(x-k)}, \tag{2.25}$$

$\forall\ x \in [k, l]$. Integrating (2.24) from x to l, we get

$$-\in \left(e^{-v_2(x-k)} - e^{-v_2(l-k)}\right) \leq \phi(l)e^{-v_2(l-k)} - \phi(x)e^{-v_2(x-k)} - \int_x^l \chi(t)e^{-v_2(t-k)}dt \leq\in \left(e^{-v_2(x-k)} - e^{-v_2(l-k)}\right)$$

$$-\in e^{-v_2(x-k)} \leq \phi(l)e^{-v_2(l-k)}-\in e^{-v_2(l-k)} - \phi(x)e^{-v_2(x-k)} - \int_x^l \chi(t)e^{-v_2(t-k)}dt \leq\in \left(e^{-v_2(x-k)}\right). \tag{2.26}$$

Multiplying the equation by $e^{v_2(x-k)}$, we get

$$-\in e^{-v_2(x-k)}e^{v_2(x-k)} \leq \phi(l)e^{-v_2(l-k)}e^{v_2(x-k)}-\in e^{-v_2(l-k)}e^{v_2(x-k)} - \phi(x)e^{-v_2(x-k)}e^{v_2(x-k)} - \int_x^l \chi(t)e^{-v_2(t-k)}e^{v_2(x-k)}dt \leq\in \left(e^{-v_2(x-k)}e^{v_2(x-k)}\right)$$

$$-\in\leq \phi(n)e^{-v_2(l-x)}-\in e^{-v_2(l-x)} - \phi(x) - e^{v_2x}\int_x^l \chi(t)e^{-v_2t}dt \leq\in . \tag{2.27}$$

Let $\Gamma(x) = \phi(l)e^{v_2(x-l)} - e^{v_2x}\int_x^l \chi(t)e^{-v_2t}dt$, for all $x \in [k, l]$.
Then $\Gamma'(x) - v_2\Gamma(x) - \chi(x) = 0$ is defined by $\Gamma'(x) = v_2\Gamma(x) + \chi(x), x \in [k, l]$ and

$$|\Gamma(x) - \phi(x)| = \left|e^{v_2(x-l)}\phi(l) - \phi(x) - e^{v_2x}\int_x^l \chi(t)e^{-v_2t}dt\right|$$

$$|\Gamma(x) - \phi(x)| = e^{p_2x}\left|\int_x^l [e^{-v_2t}][\phi'(t) - v_2\phi(t) - \chi(t)]dt\right|$$

$$|\Gamma(x) - \phi(x)| \leq\in e^{p_2x}\int_x^b e^{-p_2t}dt. \tag{2.28}$$

If $p_2 \neq 0$, then

$$|\Gamma(x)-\phi(x)| \leq \in e^{p_2 x}\int_x^l e^{-p_2 t}dt$$
$$|\Gamma(x)-\phi(x)| \leq \frac{\in}{-p_2}[e^{-p_2(l-x)}-1]$$
$$|\Gamma(x)-\phi(x)| \leq \frac{\in}{p_2}[1-e^{-p_2(l-x)}]; x\in[k,l]$$
$$|\Gamma(x)-\phi(x)| \leq \frac{\in}{p_2}[1-e^{-p_2(l-k)}]; x\in[k,l]. \tag{2.29}$$

If $p_2 = 0$, then

$$|\Gamma(x)-\phi(x)| \leq \in e^{p_2 x}\int_x^l e^{-p_2 t}dt$$
$$|\Gamma(x)-\phi(x)| \leq \in e^{0x}\int_x^l e^{-0t}dt$$
$$|\Gamma(x)-\phi(x)| \leq \in \int_x^l dt$$
$$|\Gamma(x)-\phi(x)| \leq \in (l-x); x\in[k,l]$$
$$|\Gamma(x)-\phi(x)| \leq \in (l-k); x\in[k,l]. \tag{2.30}$$

It follows from (2.19), we conclude our result (2.20).
Theorem.2.3. The differential equation
$\varsigma^{v}(x)+\eta_1\varsigma^{iv}(x)+\eta_2\varsigma^{'''}(x)+\eta_3\varsigma^{''}(x)+\eta_4\varsigma^{'}(x)+\eta_5\varsigma(x)=\Omega(x)$ has the Hyers-Ulam stability, Where $\varsigma\in c^5[k,l]$ and $\Omega\in[k,l]$. Therefore

$$|u(x)-\varsigma(x)| \leq \begin{cases} \frac{[1-e^{-p_3(l-k)}][1-e^{-p_1(l-k)}][1-e^{-p_2(l-k)}]\varepsilon}{p_3p_1p_2}, & \text{if } p_1,p_3,p_2\neq 0\\ \frac{[1-e^{-p_1(l-k)}][1-e^{-p_3(l-k)}](l-k)\varepsilon}{p_1p_3}, & \text{if } p_1,p_3\neq 0, p_2=0\\ \frac{[1-e^{-p_1(l-k)}][1-e^{-p_2(l-k)}](l-k)\varepsilon}{p_1p_2}, & \text{if } p_1,p_2\neq 0, p_3=0\\ \frac{[1-e^{-p_2(l-k)}][1-e^{-p_3(l-k)}](l-k)\varepsilon}{p_2p_3}, & \text{if } p_1=0, p_3,p_2\neq 0\\ \frac{[1-e^{-p_1(l-k)}](l-k)^2\varepsilon}{p_1}, & \text{if } p_2,p_3=0, p_1\neq 0\\ \frac{[1-e^{-p_2(l-k)}](l-k)^2\varepsilon}{p_2}, & \text{if } p_1,p_3=0, p_2\neq 0\\ \frac{[1-e^{-p_3(l-k)}](l-k)^2\varepsilon}{p_3}, & \text{if } p_1,p_2=0, p_3\neq 0\\ (l-k)^3\varepsilon, & \text{if } p,p_2,p_3=0 \end{cases} \tag{2.31}$$

$\forall\ x\in[k,l]$.
proof: Let us consider $\varsigma(x)=u''(x)+(v_2+\eta_1)u'(x)+(v_2^2+\eta_1v_2+\eta_2)u(x)$.
Then we obtain

$$\varsigma'(x)=u^{'''}(x)+(v_2+\eta_1)u^{''}(x)+(v_2^2+\eta_1v_2+\eta_2)u'(x)+(v_2^3+\eta_1v_2^2+\eta_2v_2+\eta_3)u(x)$$

and

$$|\varsigma'(x) - v_2\varsigma(x) - \Gamma(x)| = |u'''(x) + (v_2+\eta_1)u''(x) + (v_2^2+\eta_1 v_2+\eta_2)u'(x)$$
$$+(v_2^3+\eta_1 v_2^2+\eta_2 v_2+\eta_3)u(x) - v_2[u''(x) + (v_2+\eta_1)u'(x)$$
$$+(v_2^2+\eta_1 v_2+\eta_2)u(x)] - \Gamma(x)|$$
$$|\varsigma'(x) - v_2\varsigma(x) - \Gamma(x)| = |u'''(x) + v_2 u''(x) + \eta_1 u''(x) + v_2^2 u'(x)$$
$$+\eta_1 v_2 u'(x) + \eta_2 u'(x) + v_2^3 u(x) + \eta_1 v_2^2 u(x) + \eta_2 v_2 u(x) + \eta_3 u(x)$$
$$-v_2[u''(x) + v_2 u'(x) + \eta_1 u'(x) + v_2^2 u(x) + \eta_1 v_2 u(x) + \eta_2 u(x)] - \Gamma(x)|$$
$$|\varsigma'(x) - v_2\varsigma(x) - \Gamma(x)| = |u'''(x) + \eta_1 u''(x) + \eta_2 u'(x) + \eta_3 u(x) - \Gamma(x)| \leq \in$$
$$|\varsigma'(x) - v_2\varsigma(x) - \Gamma(x)| \leq \in . \tag{2.32}$$

Proportionally ς satisfies,

$$-\varepsilon \leq \varsigma'(x) - v_2\varsigma(x) - \Gamma(x) \leq \varepsilon. \tag{2.33}$$

Multiplying the equation by $e^{-v_2(x-k)}$, we get

$$-\varepsilon e^{-v_2(x-k)} \leq e^{-v_2(x-k)}\varsigma'(x) - v_2\varsigma(x)e^{-v_2(x-k)} - \Gamma(x)e^{-v_2(x-k)} \leq .\varepsilon e^{-v_2(x-k)} \tag{2.34}$$

Without loss of consensus, we may expect to be that $v_2 > 1$, thus

$$-v_2\varepsilon e^{-v_2(x-k)} \leq e^{-v_2(x-k)}\varsigma'(x) - v_2\varsigma(x)e^{-v_2(x-k)} - \Gamma(x)e^{-v_2(x-k)} \leq v_2\varepsilon e^{-v_2(x-k)}, \tag{2.35}$$

for any $x \in [k,l]$. Integrating (2.34) from x to l, we obtain

$$-\varepsilon e^{-v_2(x-k)} \leq \varsigma(n)e^{-v_2(l-k)} - \varepsilon e^{-v_2(l-k)} - \varsigma(x)e^{-v_2(x-k)} - \int_x^l \Gamma(t)e^{-v_2(t-a)}dt \leq \varepsilon e^{-v_2(x-k)}. \tag{2.36}$$

Multiplying the equation by $e^{v_2(x-k)}$, we get

$$-\varepsilon e^{-v_2(x-k)}e^{v_2(x-k)} \leq \varsigma(l)e^{-v_2(l-k)}e^{v_2(x-k)} - \varepsilon e^{-v_2(l-k)}e^{v_2(x-k)} - \varsigma(x)e^{-v_2(x-k)}e^{v_2(x-k)}$$
$$-\int_x^l \Gamma(t)e^{-v_2(t-k)}dt e^{v_2(x-k)}$$
$$\leq \varepsilon e^{-v_2(x-k)}e^{v_2(x-k)}$$

$$-\varepsilon \leq \varsigma(b)e^{v_2(x-l)} - \varepsilon e^{v_2(x-l)} - \varsigma(x) - e^{v_2 x}\int_x^l \Gamma(t)e^{-v_2 t}dt \leq \varepsilon. \tag{2.37}$$

Let $u(x) = \varsigma(l)e^{v_2(x-l)} - e^{v_2 x}\int_x^l \Gamma(t)e^{-v_2 t}dt$ then $u'(x) - v_2 u(x) - \Gamma(x) = 0$ by $u'(x) = v_2 u(x) + \Gamma(x); x \in [k,l]$

$$|u(x) - \varsigma(x)| = \left|\varsigma(l)e^{v_2(x-l)} - \varsigma(x) - e^{v_2(x)}\int_x^l \Gamma(t)e^{-v_2 t}dt\right|$$
$$|u(x) - \varsigma(x)| = |e^{v_2 x}|\left|\varsigma(l)e^{-v_2 n} - e^{-v_2 x}\varsigma(x)\int_x^l \Gamma(t)e^{-v_2 t}dt\right|$$

$$|u(x)-\varsigma(x)| \leq e^{p_2 x}\int_x^l e^{-p_2 t}dt. \tag{2.38}$$

If $p_2 \neq 0$, then

$$|u(x)-\varsigma(x)| \leq \varepsilon e^{p_2 x}\int_x^l e^{-p_2 t}dt$$
$$|u(x)-\varsigma(x)| \leq \frac{\varepsilon}{-p_2}\left[e^{-p_2(l-x)}-1\right]; x\in[k,l]$$
$$|u(x)-\varsigma(x)| \leq \frac{\varepsilon}{p_2}\left[1-e^{-p_2(l-x)}\right]; x\in[k,l]$$
$$|u(x)-\varsigma(x)| \leq \frac{\varepsilon}{p_2}\left[1-e^{-p_2(l-k)}\right]; x\in[k,l]. \tag{2.39}$$

If $p_2 = 0$, then

$$|u(x)-\varsigma(x)| \leq \varepsilon e^{p_2 x}\int_x^l e^{-p_2 t}dt$$
$$|u(x)-\varsigma(x)| \leq \varepsilon\int_x^l dt$$
$$|u(x)-\varsigma(x)| \leq \varepsilon(l-x); x\in[k,l]$$
$$|u(x)-\varsigma(x)| \leq \varepsilon(l-k); x\in[k,l]. \tag{2.40}$$

If follows from (2.20), we conclude our result (2.31),
$\forall\ x \in [k,l]$.

Theorem.2.4. The differential equation
$\varsigma^{v}(x)+\eta_1\varsigma^{iv}(x)+\eta_2\varsigma^{'''}(x)+\eta_3\varsigma^{''}(x)+\eta_4\varsigma^{'}(x)+\eta_5\varsigma(x)=\Omega(x)$ has the Hyers-Ulam stability, where $\varsigma\in c^5[k,l]$ and $\Omega\in[k,l]$. Then

$$|T(x)-\chi(x)| \leq \begin{cases} \frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right]\varepsilon}{p_1p_2p_3p_4}, & \text{if } p_1,p_2,p_3,p_4 \neq 0 \\ \frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right](l-k)\varepsilon}{p_1p_2p_3}, & \text{if } p_1,p_2,p_3 \neq 0, p_4=0 \\ \frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)\varepsilon}{p_1p_2p_4}, & \text{if } p_1,p_2,p_4 \neq 0, p_3=0 \\ \frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)\varepsilon}{p_1p_3p_4}, & \text{if } p_1,p_3,p_4 \neq 0, p_2=0 \\ \frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)\varepsilon}{p_2p_3p_4}, & \text{if } p_2,p_3,p_4 \neq 0, p_1=0 \\ \frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right](l-k)^2\varepsilon}{p_1p_2}, & \text{if } p_1,p_2 \neq 0, p_3,p_4=0 \\ \frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_3(l-k)}\right](l-k)^2\varepsilon}{p_1p_3}, & \text{if } p_1,p_3 \neq 0, p_2,p_4=0 \\ \frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)^2\varepsilon}{p_1p_4}, & \text{if } p_1,p_4 \neq 0, p_2,p_3=0 \\ \frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right](l-k)^2\varepsilon}{p_2p_3}, & \text{if } p_2,p_3 \neq 0, p_1,p_4=0 \\ \frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)^2\varepsilon}{p_2p_4}, & \text{if } p_2,p_4 \neq 0, p_1,p_3=0 \\ \frac{\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)^2\varepsilon}{p_3p_4}, & \text{if } p_3,p_4 \neq 0, p_1,p_2=0 \\ \frac{\left[1-e^{-p_1(l-k)}\right](l-k)^3\varepsilon}{p_1}, & \text{if } p_1 \neq 0, p_2,p_3,p_4=0 \\ \frac{\left[1-e^{-p_2(l-k)}\right](l-k)^3\varepsilon}{p_2}, & \text{if } p_2 \neq 0, p_1,p_3,p_4=0 \\ \frac{\left[1-e^{-p_3(l-k)}\right](l-k)^3\varepsilon}{p_3}, & \text{if } p_3 \neq 0, p_1,p_2,p_4=0 \\ \frac{\left[1-e^{-p_4(l-k)}\right](l-k)^3\varepsilon}{p_4}, & \text{if } p_4 \neq 0, p_1,p_2,p_3=0 \\ (l-k)^4\varepsilon, & \text{if } p_1,p_2,p_3,p_4=0 \end{cases} \tag{2.41}$$

with respect to $x \in [k,l]$

proof: Similar to the proof of theorem (2.3), let $\in > 0$ and $\Omega \in [k,l]$.

Let us consider $\chi(x) = \varsigma^{'''}(x) + (v_3+\eta_1)\varsigma^{''}(x) + (v_3^2+\eta_1 u+\eta_2)\varsigma'(x) + (v_3^3+\eta_1 u^2+\eta_2 v_3+\eta_3)\varsigma(x)$,

we obtain

$$\chi'(x) = \varsigma^{iv}(x) + (v_3+\eta_1)\varsigma^{'''}(x) + (v_3^2+\eta_1 v_3+\eta_2)\varsigma^{''}(x)$$
$$+(v_3^3+\eta_1 v_3^2+\eta_2 v_3+\eta_3)\varsigma'(x) + (v_3^4+\eta_1 v_3^3+\eta_2 v_3^2+\eta_3 v_3+\eta_4)\varsigma(x),$$

for all $x \in [k,l]$. Then

$$\left|\chi'(x) - v_3(\chi(x)) - u(x)\right| \leq \varepsilon \tag{2.42}$$

$$\begin{aligned} \left|\chi'(x) - v_3\chi(x) - u(x)\right| &= |\varsigma^{iv}(x) + (v_3+\eta_1)\varsigma^{'''}(x) \\ &+(v_3^2+\eta_1 v_3+\eta_2)\varsigma^{''}(x) \\ &+(v_3^3+\eta_1 v_3^2+\eta_2 v_3+\eta_3)\varsigma'(x) \\ &+(v_3^4+\eta_1 v_3^3+\eta_2 v_3^2+\eta_3 v_3+\eta_4)\varsigma(x) \\ &+v_3[y^{'''}(x) + (v_3+\eta_1)\varsigma^{''}(x) \\ &+(v_3^2+\eta_1 v_3+\eta_2)\varsigma'(x) + v_3^3+\eta_1 v_3^2+\eta_2 v_3+\eta_3)\varsigma(x)] - u(x)| \end{aligned}$$

$$\begin{aligned}\left|\chi'(x)-v_3\chi(x)-u(x)\right| &= |\varsigma^{iv}(x)+v_3\varsigma'''(x)+\eta_1\varsigma'''(x)\\ &+v_3^2\varsigma''(x)+\eta_1v_3\varsigma''+\eta_2\varsigma''(x)\\ &+v_3^3\varsigma'(x)+\eta_1v_3^2\varsigma'(x)+\eta_2v_3\varsigma'(x)+\eta_3\varsigma'(x)+v_3^4\varsigma(x)+\eta_1v_3^3\varsigma(x)\\ &+\eta_2v_3^2\varsigma(x)+\eta_3v_3\varsigma(x)+\eta_4\varsigma(x)-v_3\varsigma'''(x)-v_3^2\varsigma''-\eta_1v_3\varsigma''(x)\\ &-v_3^3\varsigma'(x)-\eta_1v_3^2\varsigma'(x)-\eta_2v_3\varsigma'(x)-v_3^4\varsigma(x)-\eta_1v_3^3\varsigma(x)-\eta_2v_3^2\varsigma(x)\\ &-\eta_3v_3\varsigma(x)-u(x)|\\ \left|\chi'(x)-v_3\chi(x)-u(x)\right| &= \left|\varsigma^{iv}(x)+\eta_1\varsigma'''(x)+\eta_2\varsigma''(x)+\eta_3\varsigma'(x)+\eta_4\varsigma(x)-u(x)\right|\\ \left|\chi'(x)-v_3\chi(x)-u(x)\right| &\le \varepsilon. \end{aligned} \tag{2.43}$$

Proportionally χ satisfies,

$$-\varepsilon \le \chi'(x)-v_3\chi(x)-u(x)\le \varepsilon. \tag{2.44}$$

Multiplying the equation by $e^{-v_3(x-k)}$, we get

$$-\varepsilon e^{-v_3(x-k)} \le \chi'(x)e^{-v_3(x-k)}-v_3\chi(x)e^{-v_3(x-k)}-u(x)e^{-v_3(x-k)}\le \varepsilon e^{-v_3(x-k)} \tag{2.45}$$

Without loss of consensus, we may expect to be that $v_3>1$, thus

$$-\varepsilon v_3 e^{-v_3(x-k)} \le \chi'(x)e^{-v_3(x-k)}-v_3\chi(x)e^{-v_3(x-k)}-u(x)e^{-v_3(x-k)}\le \varepsilon e^{-v_3(x-k)}, \tag{2.46}$$

for all $x\in[k,l]$. Integrating (2.45) from x to l, we get

$$\begin{aligned}-\varepsilon(e^{-v_3(x-k)}-e^{-v_3(l-k)}) &\le \chi(l)e^{-v_3(x-k)}-\chi(x)e^{-v_3(x-k)}-\int_x^l u(t)e^{-v_3(t-k)}dt\\ &\le \varepsilon(e^{-v_3(x-k)}-e^{-v_3(l-k)})\end{aligned}$$

$$\begin{aligned}-\varepsilon e^{-v_3(x-k)} \le \chi(l)e^{-v_3(l-k)}-\varepsilon e^{-v_3(l-k)}-\chi(x)e^{-v_3(x-k)}-\int_x^l u(t)e^{-v_3(t-k)}dt\\ \le \varepsilon e^{-v_3(x-k)}.\end{aligned} \tag{2.47}$$

Multiplying the equation by $e^{v_3(x-k)}$, we get

$$\begin{aligned}-\varepsilon e^{-v_3(x-k)}e^{v_3(x-k)} &\le \chi(l)e^{-v_3(l-k)}e^{v_3(x-k)}-\varepsilon e^{-v_3(l-k)}e^{v_3(x-k)}-\chi(x)e^{-v_3(x-k)}e^{v_3(x-k)}\\ &-\int_x^l u(t)e^{-v_3(t-k)}dte^{v_3(x-k)}\le \varepsilon e^{-v_3(x-k)}e^{v_3(x-k)}\end{aligned}$$

$$-\varepsilon \le \chi(l)e^{v_3(x-k)}-\varepsilon e^{v_3(x-k)}-\chi(x)-e^{v_3x}\int_x^l u(t)e^{-v_3(t)}dt\le\varepsilon. \tag{2.48}$$

Let $T(x)=\chi(l)e^{v_3(x-l)}-e^{v_3x}\int_x^l\phi(t)e^{-v_3t}dt$ and $T(x)$ satisfies $T'(x)-v_3T(x)-u(x)=0$ by $T'(x)=v_3T(x)+u(x)$. Then

$$|T(x)-\chi(x)|=\left|\chi(l)e^{v_3(x-l)}-\chi(x)-e^{v_3x}\int_x^l u(t)e^{-v_3t}dt\right|$$

$$|T(x) - \chi(x)| \leq \varepsilon e^{p_4 x} \int_x^l e^{-p_4 t} dt. \tag{2.49}$$

If $p_4 \neq 0$, then

$$\begin{aligned}
|T(x) - \chi(x)| &\leq \varepsilon e^{p_4 x} \int_x^l e^{-p_4 t} dt \\
|T(x) - \chi(x)| &\leq \frac{\varepsilon}{-p_4} \left[e^{-p_4(l-x)} - 1 \right]; x \in [k, l] \\
|T(x) - \chi(x)| &\leq \frac{\varepsilon}{p_4} \left[1 - e^{-p_4(l-x)} \right]; x \in [k, l] \\
|T(x) - \chi(x)| &\leq \frac{\varepsilon}{p_4} \left[1 - e^{-p_4(l-k)} \right]; x \in [k, l].
\end{aligned} \tag{2.50}$$

If $p_4 = 0$, then

$$\begin{aligned}
|T(x) - \chi(x)| &\leq \varepsilon e^{p_4 x} \int_x^l e^{-p_4 t} dt \\
|T(x) - \chi(x)| &\leq \varepsilon \int_x^l dt \\
|T(x) - \chi(x)| &\leq \varepsilon (l - x); x \in [k, l] \\
|T(x) - \chi(x)| &\leq \varepsilon (l - k); x \in [k, l].
\end{aligned} \tag{2.51}$$

It follows from (2.31), we conclude our result (2.41), for all $x \in [k, l]$.
Let $\in > 0$ and $\varsigma \in c^5[k, l]$.
Theorem.2.5 The differential equation $\varsigma^{v}(x) + \eta_1 \varsigma^{iv}(x) + \eta_2 \varsigma^{'''}(x) + \eta_3 \varsigma^{''}(x) + \eta_4 \varsigma^{'}(x) +$

$\eta_5\varsigma(x) = \Omega(x)$ has the Hyers-Ulam stability, where $\varsigma \in c^5[k,l]$ and $\Omega \in [k,l]$, therefore,

$$|K(x)-L(x)| \leq \begin{cases}
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right]\left[1-e^{-p_5(l-k)}\right]\varepsilon}{p_1p_2p_3p_4p_5} \; if \;\; p_1,p_2,p_3,p_4,p_5 \neq 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)\varepsilon}{p_1p_2p_3p_4} \; if \;\; p_1,p_2,p_3,p_4 \neq 0 \; p_5 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)\varepsilon}{p_1p_2p_3p_5} \; if \;\; p_1,p_2,p_3,p_5 \neq 0 \; p_4 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_4(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)\varepsilon}{p_1p_2p_4p_5} \; if \;\; p_1,p_2,p_4,p_5 \neq 0 \; p_3 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)\varepsilon}{p_1p_3p_4p_5} \; if \;\; p_1,p_3,p_4,p_5 \neq 0 \; p_2 = 0 \\
\frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)\varepsilon}{p_2p_3p_4p_5} \; if \;\; p_2,p_3,p_4,p_5 \neq 0 \; p_1 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right](l-k)^2\varepsilon}{p_1p_2p_3} \; if \;\; p_1,p_2,p_3 \neq 0 \; p_4,p_5 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)^2\varepsilon}{p_1p_2p_4} \; if \;\; p_1,p_2,p_4 \neq 0 \; p_3,p_5 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)^2\varepsilon}{p_1p_3p_4} \; if \;\; p_1p_3p_4 \neq 0 \; p_2,p_5 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^2\varepsilon}{p_1p_2p_5} \; if \;\; p_1,p_2,p_5 \neq 0 \; p_3,p_4 = 0 \\
\frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_4(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^2\varepsilon}{p_2p_4p_5} \; if \;\; p_2,p_4,p_5 \neq 0 \; p_1,p_3 = 0 \\
\frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^2\varepsilon}{p_2p_3p_5} \; if \;\; p_2,p_3,p_5 \neq 0 \; p_1,p_4 = 0 \\
\frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^2\varepsilon}{p_2p_4p_5} \; if \;\; p_2,p_4,p_5 \neq 0 \; p_1,p_4 = 0 \\
\frac{\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^2\varepsilon}{p_3p_4p_5} \; if \;\; p_3,p_4,p_5 \neq 0 \; p_1,p_2 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^2\varepsilon}{p_1p_3p_5} \; if \;\; p_1,p_3,p_5 \neq 0 \; p_2,p_4 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_4(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^2\varepsilon}{p_1p_4p_5} \; if \;\; p_1,p_4,p_5 \neq 0 \; p_2,p_3 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_2(l-k)}\right](l-k)^3\varepsilon}{pq} if \;\; p_1,p_2 \neq 0 \; p_3,p_4,p_5 = 0 \\
\frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_3(l-k)}\right](l-k)^3\varepsilon}{qr} if \;\; p_2,p_3 \neq 0 \; p_1,p_4,p_5 = 0 \\
\frac{\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)^3\varepsilon}{rs} if \;\; p_3,p_4 \neq 0 \; p_1,p_2,p_5 = 0 \\
\frac{\left[1-e^{-p_4(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^3\varepsilon}{sn} if \;\; p_4,p_5 \neq 0 \; p_1,p_2,p_3 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_3(l-k)}\right](l-k)^3\varepsilon}{pr} if \;\; p_1,p_3 \neq 0 \; p_2,p_4,p_5 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)^3\varepsilon}{ps} if \;\; p_1,p_4 \neq 0 \; p_2,p_3,p_5 = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^3\varepsilon}{pn} if \;\; p_1,p_5 \neq 0 \; p_2,p_3,p_4 = 0 \\
\frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_4(l-k)}\right](l-k)^3\varepsilon}{qs} if \;\; p_2,p_4 \neq 0 \; p_1,p_3,p_5 = 0 \\
\frac{\left[1-e^{-p_2(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^3\varepsilon}{qn} if \;\; p_2,p_5 \neq 0 \; p_1,p_3,p_4 = 0 \\
\frac{\left[1-e^{-p_3(l-k)}\right]\left[1-e^{-p_5(l-k)}\right](l-k)^3\varepsilon}{rn} if \;\; p_3,p_5 \neq 0 \; p_1,p_2, = 0 \\
\frac{\left[1-e^{-p_1(l-k)}\right](l-k)^4\varepsilon}{p_1} if \;\; p_1 \neq 0 \; p_2,p_3,p_4,p_5 = 0 \\
\frac{\left[1-e^{-p_2(l-k)}\right](l-k)^4\varepsilon}{p_2} if \;\; p_2 \neq 0 \; p_1,p_3,p_4,p_5 = 0 \\
\frac{\left[1-e^{-p_3(l-k)}\right](l-k)^4\varepsilon}{p_3} if \;\; p_3 \neq 0 \; p_1,p_2,p_4,p_5 = 0 \\
\frac{\left[1-e^{-p_4(l-k)}\right](l-k)^4\varepsilon}{p_4} if \;\; p_4 \neq 0 \; p_1,p_2,p_3,p_5 = 0 \\
\frac{\left[1-e^{-p_5(l-k)}\right](l-k)^4\varepsilon}{p_5} if \;\; p_5 \neq 0 \; p_1,p_2,p_3,p_4 = 0 \\
(b-a)^5\varepsilon if p_1,p_2,p_3,p_4,p_5 = 0.
\end{cases} \tag{2.52}$$

$\forall x \in [k,l]$.

Proof: Let us consider,

$$L(x) = \varsigma^{iv}(x) + (v_5+\eta_1)\varsigma^{'''}(x) + (v_5^2+\eta_1 v_5+\eta_2)y^{''}(x) + (v_5^3 \\ +\eta_1 v_5^2+\eta_2 v_5+\eta_3)\varsigma'(x) + (v_5^4+\eta_1 v_5^3+\eta_2 v_5^2+\eta_3 v_5+\eta_4)\varsigma(x),$$

we obtain

$$L'(x) = \varsigma^{v}(x) + (v_5+\eta_1)\varsigma^{iv}(x) + (v_5^2+\eta_1 v_5+\eta_2)\varsigma^{'''}(x) \\ +(v_5^3+\eta_1 v_5^2+\eta_2 v_5+\eta_3)\varsigma^{''}(x) + (v_5^4+\eta_1 v_5^3+\eta_2 v_5^2+\eta_3 v_5+\eta_4)\varsigma'(x) \\ +(v_5^5+\eta_1 v_5^4+\eta_2 v_5^3+\eta_3 v_5^2+\eta_4 v_5+\eta_5)\varsigma(x),$$

$\forall\ x \in [k,l]$. Then

$$\left|L'(x) - v_5 L(x) - T(x)\right| \leq \varepsilon. \tag{2.53}$$

$$\begin{aligned}\left|L'(x) - v_5 L(x) - T(x)\right| = |&\varsigma^{v}(x) + (v_5+\eta_1)\varsigma^{iv}(x) + (v_5^2+\eta_1 v_5+\eta_2)\varsigma^{'''}(x) \\ &+(v_5^3+\eta_1 v_5^2+\eta_2 v_5+\eta_3)\varsigma^{''}(x) \\ &+(v_5^4+\eta_1 v_5^3+\eta_2 v_5^2+\eta_3 v_5+\eta_4)\varsigma'(x) \\ &+(v_5^5+\eta_1 v_5^4+\eta_2 v_5^3+\eta_3 v_5^2+\eta_4 v_5+\eta_5)\varsigma(x) \\ &-v_5[\varsigma^{iv}(x) + (v_5+\eta_1)y^{'''}(x) + (v_5^2+\eta_1 v_5+\eta_2)\varsigma^{''}(x) \\ &+(v_5^3+\eta_1 v_5^2+\eta_2 v_5+\eta_3)\varsigma'(x) + (v_5^4+\eta_1 v_5^3+\eta_2 v_5^2+\eta_3 v_5+\eta_4)\varsigma(x)] - T(x)|\end{aligned}$$

$$\begin{aligned}\left|L'(x) - v_5 L(x) - T(x)\right| = |&\varsigma^(x) + v_5\varsigma^{iv}(x) + \eta_1\varsigma^{iv}(x) + v_5^2\varsigma^{'''}(x) + \eta_1 v_5\varsigma^{'''}(x) + \eta_2\varsigma^{'''}(x) \\ &+v_5^3\varsigma^{''}(x) + \eta_1 v_5^2\varsigma^{''}(x) + \eta_2 v_5\varsigma^{''}(x) + \eta_3\varsigma^{''}(x) \\ &+v_5^5\varsigma(x) + \eta_1 v_5^4\varsigma(x) + \eta_2 v_5^3\varsigma(x) + \eta_3 v_5^2\varsigma(x) \\ &+\eta_4 v_5\varsigma(x) + \eta_5\varsigma(x) - v_5\varsigma^{iv}(x) - v_5^2\varsigma^{'''}(x) - v_5\eta_1\varsigma^{'''}(x) \\ &-v_5^3\varsigma^{''}(x) - \eta_1 v_5^2\varsigma^{''}(x) - \eta_2 v_5\varsigma^{''}(x) - v_5^4\varsigma'(x) \\ &-\eta_1 v_5^3\varsigma'(x) - \eta_2 v_5^2\varsigma'(x) - \eta_3 v_5\varsigma'(x) \\ &-(v_5^5\varsigma(x) - \eta_1 v_5^4\varsigma(x) - \eta_2 v_5^3\varsigma(x) - \eta_3 v_5^2\varsigma(x) \\ &-\eta_4 v_5\varsigma(x) - T(x)|\end{aligned}$$

$$\left|L'(x) - v_5 L(x) - T(x)\right| = \left|\varsigma^{v} + \eta_1\varsigma^{iv}(x) + \eta_2\varsigma^{'''}(x) + \eta_3\varsigma^{''}(x) + \eta_4\varsigma^{'}(x) + \eta_5\varsigma(x) - T(x)\right| \leq \varepsilon$$

$$\left|L'(x) - wL(x) - T(x)\right| \leq \varepsilon. \tag{2.54}$$

Proportionally L satisfies,

$$-\varepsilon \leq L'(x) - v_5 L(x) - T(x) \leq \varepsilon. \tag{2.55}$$

Multiplying the equation by $e^{-v_5(x-k)}$, we get

$$-\varepsilon e^{-v_5(x-k)} \leq L'(x)e^{-v_5(x-k)} - v_5 L(x)e^{-v_5(x-k)} - T(x)e^{-v_5(x-k)} \leq \varepsilon e^{-v_5(x-k)}. \tag{2.56}$$

Without loss of consensus, we may expect to be that $v_5 > 1$, thus

$$\begin{aligned} -\varepsilon e^{-v_5(x-k)} \leq L'(x)e^{-v_5(x-k)} - v_5 L(x)e^{-v_5(x-k)} \\ -T(x)e^{-v_5(x-k)} \leq \varepsilon e^{-v_5(x-k)}, \end{aligned} \tag{2.57}$$

$\forall\ x \in [k, l]$. Integrating (2.56) from x to l, we obtain

$$\begin{aligned} -\varepsilon\left(e^{-v_5(x-k)} - e^{-v_5(l-k)}\right) \leq L(l)e^{-v_5(l-k)} - L(x)e^{-v_5(x-k)} - \int_x^l T(x)e^{-v_5(t-k)}dt \\ \leq \varepsilon\left(e^{-v_5(x-k)} - e^{-v_5(l-k)}\right) \end{aligned}$$

$$\begin{aligned} -\varepsilon e^{-v_5(x-k)} \leq L(l)e^{-v_5(l-k)} - \varepsilon e^{-v_5(l-k)} - L(x)e^{-v_5(x-l)} - \int_x^l T(x)e^{-v_5(t-k)}dt \\ \leq \varepsilon e^{-v_5(x-k)}. \end{aligned} \tag{2.58}$$

Multiplying the equation by $e^{v_5(x-a)}$, we get

$$\begin{aligned} -\varepsilon e^{-v_5(x-k)}e^{v_5(x-k)} \leq L(l)e^{-v_5(l-k)}e^{v_5(x-k)} - \varepsilon e^{-v_5(l-k)}e^{v_5(x-k)} \\ -L(x)e^{-v_5(x-k)}e^{v_5(x-k)} - \int_x^l T(x)e^{-v_5(t-k)}dt e^{v_5(x-k)} \leq \varepsilon e^{-v_5(x-k)}e^{v_5(x-k)} \end{aligned}$$

$$-\varepsilon \leq L(l)e^{v_5(x-l)} - \varepsilon e^{v_5(x-l)} - L(x) - e^{v_5 x}\int_x^l T(t)e^{-v_5 t}dt \leq \varepsilon. \tag{2.59}$$

Let $K(x) = L(l)e^{v_5(x-l)} - e^{v_5 x}\int_x^l T(t)e^{-v_5 t}dt$. Then

$$\begin{aligned} |K(x) - L(x)| &= \left|L(l)e^{v_5(x-l)} - L(x) - e^{v_5 x}\int_x^l T(t)e^{-v_5 t}dt\right| \\ |K(x) - L(x)| &\leq e^{p_5 x}\int_x^l \left|e^{-v_5 t}\right| \left|L'(t) - v_5 L(t) - T(t)\right| dt \\ |K(x) - L(x)| &\leq e^{p_5 x}\int_x^l e^{-p_5 t} \left|L'(t) - v_5 L(t) - T(t)\right| dt \end{aligned}$$

$$|K(x) - L(x)| \leq \varepsilon e^{p_5 x}\int_x^l e^{-p_5 t}dt. \tag{2.60}$$

If $p_5 \neq 0$, then

$$\begin{aligned} |K(x) - L(x)| &\leq \varepsilon e^{p_5 x}\int_x^l e^{-p_5 t}dt \\ |K(x) - L(x)| &\leq \frac{\varepsilon}{-p_5}\left[e^{-p_5(l-x)} - 1\right] \\ |K(x) - L(x)| &\leq \frac{\varepsilon}{p_5}\left[1 - e^{-p_5(l-x)}\right]; x \in [k, l] \end{aligned}$$

$$|K(x) - L(x)| \leq \frac{\varepsilon}{p_5}\left[1 - e^{-p_5(l-k)}\right]; x \in [k,l]. \tag{2.61}$$

If $p_5 = 0$, then

$$|K(x) - L(x)| \leq \varepsilon e^{p_5 x}\int_x^l e^{-p_5 t}dt$$

$$|K(x) - L(x)| \leq \varepsilon \int_x^l dt$$

$$|K(x) - L(x)| \leq (l - x); x \in [k,l]$$

$$|K(x) - L(x)| \leq (l - k); x \in [k,l]. \tag{2.62}$$

It follow from (2.41), we desired our result (2.52), for all $x \in [k,l]$
Thus, the proof is completed.

3. Illustrative Examples

In this section, the following numerical examples are discussed to prove the usefulness of the theortical in this paper

Example 3.1 Consider the following differential equation

$$\tau^{v}(\mathcal{X}) + 2\tau^{iv}(\mathcal{X}) + \tau'''(\mathcal{X}) + \tau''(\mathcal{X}) + 2\tau'(\mathcal{X}) + 3\tau(\mathcal{X}) = \Omega(\mathcal{X}); \mathcal{X} \in [2,3]. \tag{3.1}$$

Suppose $\in > 0$, as such

$$|\tau^{v}(\mathcal{X}) + 2\tau^{iv}(\mathcal{X}) + \tau'''(\mathcal{X}) + \tau''(\mathcal{X}) + 2\tau'(\mathcal{X}) + 3\tau(\mathcal{X}) - \Omega(\mathcal{X})| \leq \in .\ \mathcal{X} \in [2,3]. \tag{3.2}$$

Suppose $v_1 = 1$, then

$$g(\mathcal{X}) = \tau^{iv}(\mathcal{X}) + 3\tau'''(\mathcal{X}) + 4\tau''(\mathcal{X}) + 5\tau'(\mathcal{X}) + 7\tau(\mathcal{X}) \quad and$$
$$g'(\mathcal{X}) = \tau^{v}(\mathcal{X}) + 3\tau^{iv}(\mathcal{X}) + 4\tau'''(\mathcal{X}) + 5\tau''(\mathcal{X}) + 7\tau'(\mathcal{X}) + 7\tau(\mathcal{X})\ \mathcal{X} \in [2,3].$$

The conditions (2.16), (2.29), (2.42) and (2.53) of theorem 2.4 are fulfilled. Consequently, there is a capacity $\mathcal{X} \in c^5[2,3]$, which is a gentle arrangement of

$$u^{v}(\mathcal{X}) + 2u^{iv}(\mathcal{X}) + u'''(\mathcal{X}) + u''(\mathcal{X}) + 2u'(\mathcal{X}) + 3u(\mathcal{X}) = \Omega(\mathcal{X})$$

that is satisfied by equation (3.2).

Example 3.2 Consider the following differential equation

$$\tau^{v}(\mathcal{X}) + 4\tau^{iv}(\mathcal{X}) + \tau'''(\mathcal{X}) + 2\tau''(\mathcal{X}) + \tau'(\mathcal{X}) + 0\tau(\mathcal{X}) = \Omega(\mathcal{X}); \mathcal{X} \in [3,2]. \tag{3.3}$$

Suppose $\in > 0$ and $\Omega \in [3,2]$, such that

$$|\tau^{v}(\mathcal{X}) + 4\tau^{iv}(\mathcal{X}) + \tau'''(\mathcal{X}) + 2\tau''(\mathcal{X}) + \tau'(\mathcal{X}) + 0\tau(\mathcal{X}) - \Omega(\mathcal{X})| \leq \in .\ \mathcal{X} \in [3,2]. \tag{3.4}$$

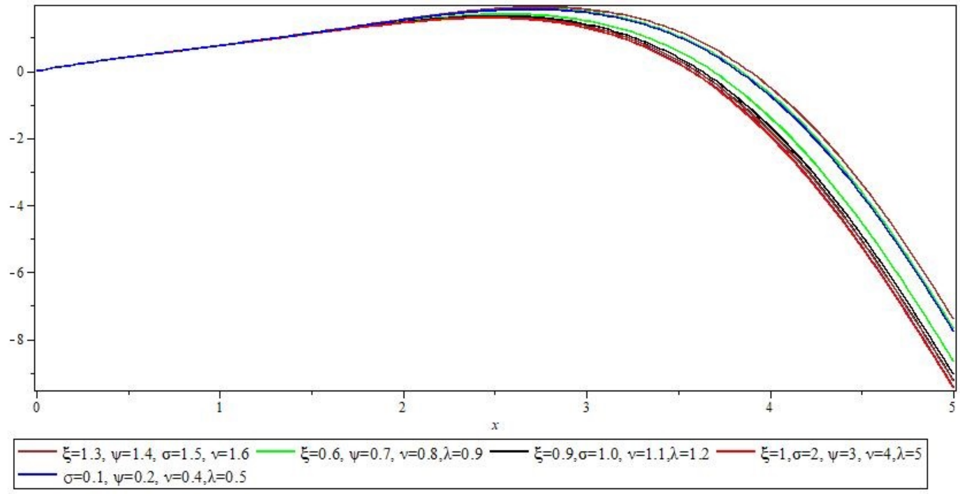

Figure 1: The solution of $\tau(\mathcal{X})$ by equation (3.2)

We take

$$g(\mathcal{X}) = \tau^{iv}(\mathcal{X}) + 5\tau'''(\mathcal{X}) + 6\tau''(\mathcal{X}) + 8\tau'(\mathcal{X}) + 7\tau(\mathcal{X}),\ \mathcal{X} \in [3,2].$$

Then,

$$g'(\mathcal{X}) = \tau^{v}(\mathcal{X}) + 5\tau^{iv}(\mathcal{X}) + 6\tau'''(\mathcal{X}) + 8\tau''(\mathcal{X}) + 7\tau'(\mathcal{X}) + 7\tau(\mathcal{X}),\ \mathcal{X} \in [3,2].$$

Such that,

$$|g'(\mathcal{X}) - g(\mathcal{X}) - \Omega(\mathcal{X})| = |\tau^{v}(\mathcal{X}) + 4\tau^{iv}(\mathcal{X}) + \tau'''(\mathcal{X}) + 2\tau''(\mathcal{X}) + \tau'(\mathcal{X}) + 0\tau(\mathcal{X}) - \Omega(\mathcal{X})| \leq \in,$$

for all $\mathcal{X} \in [k,l]$.
The conditions (2.16), (2.29), (2.42) and (2.53) of theorem 2.4 are fulfilled. Consequently, there is a capacity $\mathcal{X} \in c^5[2,3]$, which is a gentle arrangement of

$$u^{iv}(\mathcal{X}) + u'''(\mathcal{X}) + u''(\mathcal{X}) = \Psi(\mathcal{X})$$

that is satisfied by equation (3.4).

4. Conclusions

We have researched the Hyers-Ulam stability as for the linear differential equation of fifth-order in this investigation. The adequacy of the proposed technique has been shown in the numerical examples. In future work, the proposed scheme will be taken into account for time delay with disturbances.

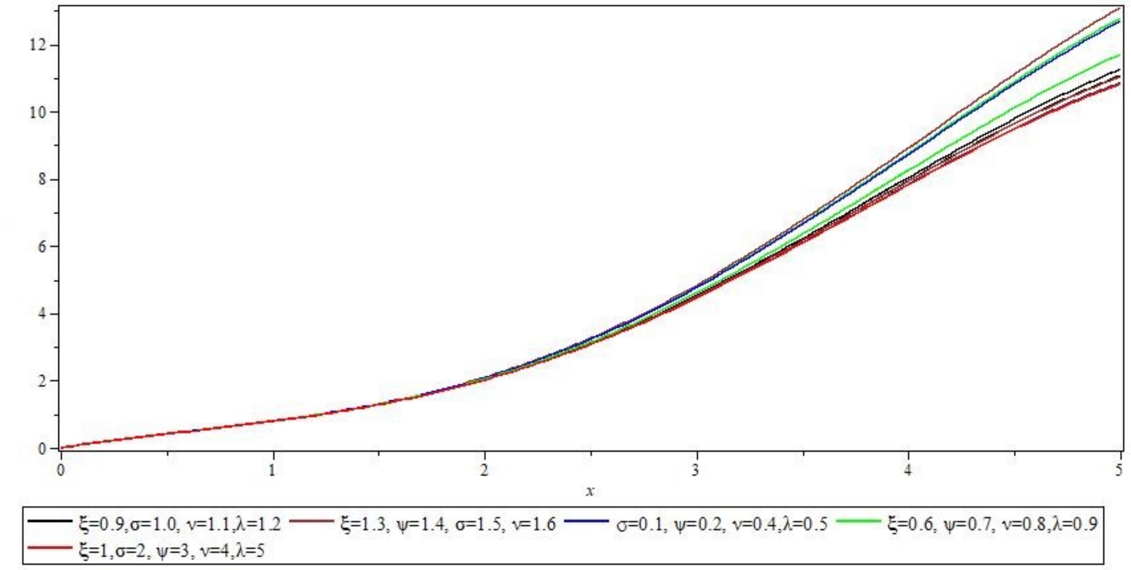

Figure 2: The solution of $\tau(\mathcal{X})$ by equation (3.4)

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