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ε-Łukasiewicz Fuzzy UP (BCC)-Ideals: a New Frontier in UP (BCC)-Algebras

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Abstract. This paper presents the development of ε-Łukasiewicz fuzzy sets using the Łukasiewicz t-norm derived from a given fuzzy set. These ε-Łukasiewicz fuzzy sets are subsequently applied to UP (BCC)-algebras. In addition, the paper introduces the concept of ε-Łukasiewicz fuzzy UP (BCC)-ideals and examines their various properties. Three specific subsets, termed the $\in$-set, q-set, and O-set, are constructed, with an exploration of the conditions under which these subsets qualify as UP (BCC)-ideals.

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1. Introduction

Zadeh [21] originally proposed the concept of fuzzy sets. Fuzzy set theory finds numerous real-world applications, and many researchers have extensively explored its principles. Following the introduction of fuzzy sets, various studies have focused on their generalizations. The intersection of fuzzy sets with other uncertainty models, such as soft and rough sets, has been explored in [1–3]. Modern technology enables sophisticated inferences and problem-solving capabilities, particularly in handling theme variations through programming. Łukasiewicz logic, governed by the Łukasiewicz t-norm, represents a non-classical, multi-valued logic initially formulated in the early 20th century with three truth values. One significant extension is the ε-Łukasiewicz fuzzy set, derived from the Łukasiewicz logic, a non-classical, many-valued logic. The ε-Łukasiewicz fuzzy set is based on the Łukasiewicz

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t-norm and t-conorm, which define fuzzy operations such as intersection, union, and complement. The parameter ε is introduced to provide additional flexibility and control over the set's fuzziness level. Fundamental concepts of the ε-Łukasiewicz fuzzy set can be found in [5, 6, 14].

ε-Łukasiewicz fuzzy sets are highly applicable across various fields by effectively modelling uncertainty and imprecision. In decision-making systems, they enable the incorporation of ambiguous information, allowing for more nuanced evaluations of alternatives. In medical diagnosis, these fuzzy sets help assess unclear symptoms and test results, leading to personalized treatment plans. In financial analysis, they enhance risk assessment by capturing market uncertainties, while in artificial intelligence, they improve knowledge representation and reasoning, enabling AI systems to make human-like decisions. Overall, ε-Łukasiewicz fuzzy sets offer valuable insights and practical solutions in uncertain environments across multiple disciplines.

Iampan [9] introduced UP-algebras as a novel algebraic structure. Somjanta et al. [20] and Guntasow et al. [7] applied fuzzy set theory within the framework of UP-algebras. Dokkhamdang et al. [4] introduced the concept of fuzzy UP-subalgebras with thresholds in UP-algebras. Poungsumpao et al. [16] studied fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras in terms of upper t-(strong) level subsets and lower t-(strong) level subsets. Senapati et al. [18] pioneered the concept of cubic sets within UP-subalgebras and UP-ideals in the framework of UP-algebras. Their research delved into the intricate relationships between cubic UP-subalgebras and cubic UP-ideals, revealing new insights into their structural connections. Senapati et al. [19] explored the concept of interval-valued intuitionistic fuzzy sets, applying it to both UP-subalgebras and UP-ideals in UP-algebras. Their work examined the homomorphic images and inverse images of these interval-valued intuitionistic fuzzy UP-subalgebras and UP-ideals, providing deeper insights into their structural behaviour. Jana et al. [11] introduced the concept of quasi-coincidence between an intuitionistic fuzzy point and an intuitionistic fuzzy set. They further developed and explored the notions of $(\in, \in \vee q)$-intuitionistic fuzzy BCI-subalgebras within the framework of BCI-algebras, offering new perspectives on their structure and properties. UP-algebras (see [9]) and BCC-algebras (see [15]) are identified as the same concept, as demonstrated by Jun et al. [13] in 2022. For consistency with Komori's initial characterization in 1984, our research team will adopt the term BCC rather than UP in subsequent investigations and publications.

In this paper, we utilize the Łukasiewicz t-norm to introduce the concept of ε-Łukasiewicz fuzzy sets derived from a given fuzzy set, applying this framework to BCC-algebras. We define ε-Łukasiewicz fuzzy BCC-ideals and explore their properties. Conditions are established for an ε-Łukasiewicz fuzzy set to qualify as an ε-Łukasiewicz fuzzy BCC-subalgebra, and we characterize these structures. Additionally, we introduce three specific subsets—referred to as $\in$-set, q-set, and O-set—and determine the conditions under which they can function as BCC-ideals.

2. Preliminaries

The concept of BCC-algebras (referenced in [15]) can be reformulated without the condition (2.6) as follows:

An algebra $X = (X, \circ, 0)$ of type $(2, 0)$ is called a *BCC-algebra* (see [8]) if it satisfies the following conditions:

$$(\forall x, y, z \in X)((y \circ z) \circ ((x \circ y) \circ (x \circ z)) = 0) \tag{2.1}$$
$$(\forall x \in X)(0 \circ x = x) \tag{2.2}$$
$$(\forall x \in X)(x \circ 0 = 0) \tag{2.3}$$
$$(\forall x, y \in X)(x \circ y = 0, y \circ x = 0 \Rightarrow x = y) \tag{2.4}$$

Following this, we will denote X as a BCC-algebra $(X, \circ, 0)$ unless stated otherwise.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \circ y = 0) \tag{2.5}$$

In X, the following assertions are valid (see [9]).

$$(\forall x \in X)(x \leq x) \tag{2.6}$$
$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{2.7}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow z \circ x \leq z \circ y) \tag{2.8}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow y \circ z \leq x \circ z) \tag{2.9}$$
$$(\forall x, y, z \in X)(x \leq y \circ x, \text{ in particular, } y \circ z \leq x \circ (y \circ z)) \tag{2.10}$$
$$(\forall x, y \in X)(y \circ x \leq x \Leftrightarrow x = y \circ x) \tag{2.11}$$
$$(\forall x, y \in X)(x \leq y \circ y) \tag{2.12}$$
$$(\forall a, x, y, z \in X)(x \circ (y \circ z) \leq x \circ ((a \circ y) \circ (a \circ z))) \tag{2.13}$$
$$(\forall a, x, y, z \in X)(((a \circ x) \circ (a \circ y)) \circ z \leq (x \circ y) \circ z) \tag{2.14}$$
$$(\forall x, y, z \in X)((x \circ y) \circ z \leq y \circ z) \tag{2.15}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z \circ y) \tag{2.16}$$
$$(\forall x, y, z \in X)((x \circ y) \circ z \leq x \circ (y \circ z)) \tag{2.17}$$
$$(\forall a, x, y, z \in X)((x \circ y) \circ z \leq y \circ (a \circ z)) \tag{2.18}$$

Definition 1. *[9] A nonempty subset S of X is called*

(1) a BCC-subalgebra of X if it satisfies the following property:

$$(\forall x, y \in S)(x \circ y \in S) \tag{2.19}$$

(2) a BCC-ideal of X if it satisfies the following properties:

$$0 \in S \tag{2.20}$$
$$(\forall x, y, z \in X)(x \circ (y \circ z), y \in S \Rightarrow x \circ z \in S) \tag{2.21}$$

A *fuzzy set* [21] in a nonempty set X is defined to be a function $\mu : X \to [0,1]$, where $[0,1]$ is the unit closed interval of real numbers.

Definition 2. *[20] A fuzzy set μ in X is said to be*

(1) a fuzzy BCC-subalgebra of X if it satisfies the following property:

$$(\forall x, y \in X)(\mu(x \circ y) \geq \min\{\mu(x), \mu(y)\}) \tag{2.22}$$

(2) a fuzzy BCC-ideal of X if it satisfies the following properties:

$$(\forall x \in X)(\mu(0) \geq \mu(x)) \tag{2.23}$$

$$(\forall x, y, z \in X)(\mu(x \circ z) \geq \min\{\mu(x \circ (y \circ z)), \mu(y)\}) \tag{2.24}$$

A fuzzy set μ in a set X of the form

$$\mu(x) = \begin{cases} t \in (0,1] & \text{if } x = a \\ 0 & \text{otherwise,} \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $[a/t]$.

For a fuzzy set μ in a set X, we say that a fuzzy point $[a/t]$ is

(1) contained in μ, denoted by $[a/t] \in \mu$, (see [17]) if $\mu(a) \geq t$,

(2) quasi-coincident with μ, denoted by $[a/t]q\mu$, (see [17]) if $\mu(a) + t > 1$.

Proposition 1. *If μ is a fuzzy set in a set X and $\varepsilon \in (0,1)$, then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} satisfies the following property:*

(1) $(\forall x, y \in X)(\mu(x) \geq \mu(y) \Rightarrow L^{\varepsilon}_{\mu}(x) \geq L^{\varepsilon}_{\mu}(y))$

(2) $(\forall x \in X)([x/\varepsilon]q\mu \Rightarrow L^{\varepsilon}_{\mu}(x) = \mu(x) + \varepsilon - 1)$

(3) $(\forall x \in X, \forall \delta \in (0,1))(\varepsilon \geq \delta \Rightarrow L^{\varepsilon}_{\mu}(x) \geq L^{\delta}_{\mu}(x))$

3. ε-Łukasiewicz fuzzy BCC-ideals of BCC-algebras

In this section, we revisit the concept of ε-Łukasiewicz fuzzy sets and introduce an innovative idea: ε-Łukasiewicz fuzzy BCC-ideals.

Definition 3. *Let μ be a fuzzy set in a set X and let $\varepsilon \in [0,1]$. A function $L^{\varepsilon}_{\mu} : X \to [0,1]$; $x \mapsto \max\{0, \mu(x) + \varepsilon - 1\}$ is called an ε-Łukasiewicz fuzzy set of μ in X.*

Definition 4. *[10] Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X is called an ε-Łukasiewicz fuzzy BCC-subalgebra of X if it satisfies the following property:*

$$(\forall x, y \in X, \forall t_a, t_b \in (0,1])([x/t_a] \in L^{\varepsilon}_{\mu}, [y/t_b] \in L^{\varepsilon}_{\mu} \Rightarrow [(x \circ y)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}) \tag{3.1}$$

Definition 5. *Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X is called an ε-Łukasiewicz fuzzy BCC-ideal of X if it satisfies the following properties:*

$$(\forall x \in X, \forall t_a \in (0,1])([x/t_a] \in L^{\varepsilon}_{\mu} \Rightarrow [0/t_a] \in L^{\varepsilon}_{\mu}) \tag{3.2}$$

$$(\forall x, y, z \in X, \forall t_a, t_b \in (0,1]) \left(\begin{array}{l} [(x \circ (y \circ z))/t_a] \in L^{\varepsilon}_{\mu}, [y/t_b] \in L^{\varepsilon}_{\mu} \\ \Rightarrow [(x \circ z)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu} \end{array} \right) \tag{3.3}$$

Example 1. *[7] Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:*

$\circ$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then X is a BCC-algebra. Define a fuzzy set μ as follows:

$$\mu : X \to [0,1]; x \mapsto \begin{cases} 0.6 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2 \\ 0.2 & \text{if } x = 3 \end{cases}$$

Given $\varepsilon = 0.85$, the ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X is given as follows:

$$L^{\varepsilon}_{\mu} : X \to [0,1]; x \mapsto \begin{cases} 0.45 & \text{if } x = 0 \\ 0.25 & \text{if } x = 1 \\ 0.15 & \text{if } x = 2 \\ 0.05 & \text{if } x = 3 \end{cases}$$

Then L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-ideal of X.

Theorem 1. *Let μ be a fuzzy set in X. Then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X is an ε-Łukasiewicz fuzzy BCC-ideal of X if and only if it satisfies the following properties:*

$$(\forall x \in X)(L^{\varepsilon}_{\mu}(0) \geq L^{\varepsilon}_{\mu}(x)) \tag{3.4}$$

$$(\forall x, y, z \in X)(L^{\varepsilon}_{\mu}(x \circ z) \geq \min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\}) \tag{3.5}$$

Proof. Assume that L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-ideal of X. Let $x \in X$. Since $[x/L^{\varepsilon}_{\mu}(x)] \in L^{\varepsilon}_{\mu}$, we have $[0/L^{\varepsilon}_{\mu}(x)] \in L^{\varepsilon}_{\mu}$ by (3.2), and so $L^{\varepsilon}_{\mu}(0) \geq L^{\varepsilon}_{\mu}(x)$. Note that $[(x\circ(y\circ z))/L^{\varepsilon}_{\mu}(x\circ(y\circ z))] \in L^{\varepsilon}_{\mu}, [y/L^{\varepsilon}_{\mu}(y)] \in L^{\varepsilon}_{\mu}$ for all $x, y, z \in X$. It follows from (3.3) that $[L^{\varepsilon}_{\mu}(x\circ z)/\min\{L^{\varepsilon}_{\mu}(x\circ(y\circ z)), L^{\varepsilon}_{\mu}(y)\}] \in L^{\varepsilon}_{\mu}$, that is, $L^{\varepsilon}_{\mu}(x\circ z) \geq \min\{L^{\varepsilon}_{\mu}(x\circ(y\circ z)), L^{\varepsilon}_{\mu}(y)\}$ for all $x, y, z \in X$.

Conversely, let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy set satisfying the conditions (3.4) and (3.5). If $[x/t] \in L^{\varepsilon}_{\mu}$ for all $x \in X$ and $t \in (0,1]$, then $L^{\varepsilon}_{\mu}(0) \geq L^{\varepsilon}_{\mu}(x) \geq t$ for all $x \in X$ by (3.4). Hence, $[0/t] \in L^{\varepsilon}_{\mu}$. Let $x, y, z \in X$ and $t_a, t_b \in (0,1]$ be such that $[(x\circ(y\circ z))/t_a] \in L^{\varepsilon}_{\mu}$ and $[y/t_b] \in L^{\varepsilon}_{\mu}$. Then $L^{\varepsilon}_{\mu}(x \circ (y \circ z)) \geq t_a$ and $L^{\varepsilon}_{\mu}(y) \geq t_b$. It follows from (3.5) that $L^{\varepsilon}_{\mu}(x \circ z) \geq \min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\} \geq \min\{t_a, t_b\}$. Hence, $[((x \circ z)/\min\{t_a, t_b\} \in L^{\varepsilon}_{\mu}$. Therefore, L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-ideal of X.

Proposition 2. *If L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-ideal of X, then*

$$(\forall x, y \in X)(y \leq x \Rightarrow L^{\varepsilon}_{\mu}(y) \leq L^{\varepsilon}_{\mu}(x)). \tag{3.6}$$

Proof. Let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy BCC-ideal of X. Let $x, y \in X$ be such that $y \leq x$. Then

$$\begin{aligned}
L^{\varepsilon}_{\mu}(x) &= L^{\varepsilon}_{\mu}(0 \circ x) \\
&= \max\{0, \mu(0 \circ x) + \varepsilon - 1\} \\
&\geq \max\{0, \min\{\mu(0 \circ (y \circ x)), \mu(y)\} + \varepsilon - 1\} \\
&= \max\{0, \min\{\mu(y \circ x) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}\} \\
&= \min\{\max\{0, \mu(0) + \varepsilon - 1\}, \max\{0, \mu(y) + \varepsilon - 1\}\} \\
&= \min\{L^{\varepsilon}_{\mu}(0), L^{\varepsilon}_{\mu}(y)\} \\
&= L^{\varepsilon}_{\mu}(y).
\end{aligned}$$

Proposition 3. *If L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-ideal of X, then*

$$(\forall w, x, y, z \in X)(x \leq w \circ (y \circ z) \Rightarrow L^{\varepsilon}_{\mu}(x \circ z) \geq \min\{L^{\varepsilon}_{\mu}(w), L^{\varepsilon}_{\mu}(y)\}). \tag{3.7}$$

Proof. Let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy BCC-ideal of X. Let $w, x, y, z \in X$ be such that $x \leq w \circ (y \circ z)$. Then

$$\begin{aligned}
L^{\varepsilon}_{\mu}(x \circ z) &= \max\{0, \mu(x \circ z) + \varepsilon - 1\} \\
&\geq \max\{0, \min\{\mu(x \circ (y \circ z)), \mu(y)\} + \varepsilon - 1\} \\
&\geq \max\{0, \min\{\min\{\mu(x \circ (w \circ (y \circ z))), \mu(w)\}, \mu(y)\} + \varepsilon - 1\} \\
&= \max\{0, \min\{\min\{\mu(0), \mu(w)\}, \mu(y)\} + \varepsilon - 1\} \\
&= \max\{0, \min\{\mu(w), \mu(y)\} + \varepsilon - 1\} \\
&= \max\{0, \min\{\mu(w) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}\} \\
&= \min\{\max\{0, \mu(w) + \varepsilon - 1\}, \max\{0, \mu(y) + \varepsilon - 1\}\} \\
&= \min\{L^{\varepsilon}_{\mu}(w), L^{\varepsilon}_{\mu}(y)\}.
\end{aligned}$$

Proposition 4. *If L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-ideal of X, then*

$$(\forall x, y, z \in X)(x \leq y \circ z \Rightarrow L^{\varepsilon}_{\mu}(x \circ z) \geq L^{\varepsilon}_{\mu}(y)). \tag{3.8}$$

Proof. Let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy BCC-ideal of X. Let $x, y, z \in X$ be such that $x \leq y \circ z$ in X. By Proposition 3, put $w = 0$. By (3.7), we have $x \leq 0 \circ (y \circ z)$. Hence, $L^{\varepsilon}_{\mu}(x \circ z) \geq \min\{L^{\varepsilon}_{\mu}(0), L^{\varepsilon}_{\mu}(y)\} = L^{\varepsilon}_{\mu}(y)$.

Proposition 5. *Every ε-Łukasiewicz fuzzy BCC-ideal of X is an ε-Łukasiewicz fuzzy BCC-subalgebra of X.*

Proof. Let L^{ε}_{μ} ba an ε-Łukasiewicz fuzzy BCC-ideal of X and let $x, y \in X$. By (2.10), we have $x \leq y \circ x$. It follows from (3.6) that $L^{\varepsilon}_{\mu}(y \circ x) \geq L^{\varepsilon}_{\mu}(x) \geq \min\{L^{\varepsilon}_{\mu}(y), L^{\varepsilon}_{\mu}(x)\}$.

The following example shows that the converse of Proposition 5 is not generally true.

Example 2. *[7] Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:*

$\circ$	0	1	2	3
0	0	1	2	3
1	0	0	1	2
2	0	0	0	1
3	0	0	0	0

Then X is a BCC-algebra. Define a fuzzy set μ as follows:

$$\mu : X \to [0, 1]; x \mapsto \begin{cases} 1 & \text{if } x = 0 \\ 0.6 & \text{if } x = 1 \\ 0.4 & \text{if } x = 2 \\ 0.1 & \text{if } x = 3 \end{cases}$$

Given $\varepsilon = 0.9$, the ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X is given as follows:

$$L^{\varepsilon}_{\mu} : X \to [0, 1]; x \mapsto \begin{cases} 0.9 & \text{if } x = 0 \\ 0.5 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2 \\ 0 & \text{if } x = 3 \end{cases}$$

Then L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-subalgebra of X but not an ε-Łukasiewicz fuzzy BCC-ideal of X because $L^{\varepsilon}_{\mu}(0 \circ 3) = L^{\varepsilon}_{\mu}(3) = 0 \ngeq 0.3 = \min\{0.3, 0.5\} = \min\{L^{\varepsilon}_{\mu}(0 \circ (1 \circ 3)), L^{\varepsilon}_{\mu}(1)\}$ by Theorem 1.

Theorem 2. *If μ is a fuzzy BCC-ideal of X, then its ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X is an ε-Łukasiewicz fuzzy BCC-ideal of X.*

Proof. Assume that μ is a fuzzy BCC-ideal of X. Let $x \in X$ and $t_a \in (0, 1]$ be such that $[x/t_a] \in L^{\varepsilon}_{\mu}$. Then $L^{\varepsilon}_{\mu}(x) \geq t_a$. Thus

$$\begin{aligned} L^{\varepsilon}_{\mu}(0) &= \max\{0, \mu(0) + \varepsilon - 1\} \\ &\geq \max\{0, \mu(x) + \varepsilon - 1\} \\ &= L^{\varepsilon}_{\mu}(x) \\ &\geq t_a. \end{aligned}$$

Then $[0/t_a] \in L^{\varepsilon}_{\mu}$. Let $x, y, z \in X$ and $t_a, t_b \in (0, 1]$ be such that $[x \circ (y \circ z)/t_a] \in L^{\varepsilon}_{\mu}$ and $[y/t_b] \in L^{\varepsilon}_{\mu}$. Then $L^{\varepsilon}_{\mu}(x \circ (y \circ z)) \geq t_a$ and $L^{\varepsilon}_{\mu}(y) \geq t_b$. Thus

$$\begin{aligned} L^{\varepsilon}_{\mu}(x \circ z) &= \max\{0, \mu(x \circ z) + \varepsilon - 1\} \\ &\geq \max\{0, \min\{\mu(x \circ (y \circ z)), \mu(y)\} + \varepsilon - 1\} \\ &= \max\{0, \min\{\mu(x \circ (y \circ z)) + \varepsilon - 1, \mu(y) + \varepsilon - 1\}\} \\ &= \min\{\max\{\mu(x \circ (y \circ z)) + \varepsilon - 1\}, \max\{\mu(y) + \varepsilon - 1\}\} \\ &= \min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\} \\ &\geq \min\{t_a, t_b\}. \end{aligned}$$

Then $[(x \circ z)/\min\{t_a, t_b\}] \in L^\varepsilon_\mu$. Hence, L^ε_μ is an ε-Łukasiewicz fuzzy BCC-ideal of X.

Let μ be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set L^ε_μ of μ in X and $t \in (0,1]$, consider the sets

$$(L^\varepsilon_\mu, t)_\in = \{x \in X : [x/t] \in L^\varepsilon_\mu\},$$
$$(L^\varepsilon_\mu, t)_q = \{x \in X : [x/t] q L^\varepsilon_\mu\},$$

which are called the $\in$-set and q-set, respectively, of L^ε_μ (with value t).

We explore the conditions under which the $\in$-set and q-set of ε-Łukasiewicz fuzzy sets can be a BCC-ideal.

Theorem 3. *Let L^ε_μ be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. Then the $\in$-set $(L^\varepsilon_\mu, t)_\in$ of L^ε_μ with value $t \in (0.5, 1]$ is a BCC-ideal of X if and only if the following assertions are valid:*

$$(\forall x \in X)(\max\{L^\varepsilon_\mu(0), 0.5\} \geq L^\varepsilon_\mu(x)) \tag{3.9}$$
$$(\forall x, y, z \in X)(\max\{L^\varepsilon_\mu(x \circ z), 0.5\} \geq \min\{L^\varepsilon_\mu(x \circ (y \circ z)), L^\varepsilon_\mu(y)\}) \tag{3.10}$$

Proof. Assume that the $\in$-set $(L^\varepsilon_\mu, t)_\in$ of L^ε_μ with value $t \in (0.5, 1]$ is a BCC-ideal of X. If the condition (3.9) is not valid, then there exists $a \in X$ such that $\max\{L^\varepsilon_\mu(0), 0.5\} < L^\varepsilon_\mu(a)$. Thus $L^\varepsilon_\mu(a) \in (0.5, 1]$ and $L^\varepsilon_\mu(a) > L^\varepsilon_\mu(0)$. If we take $t = L^\varepsilon_\mu(a)$, then $[a/t] \in L^\varepsilon_\mu$, that is, $a \in (L^\varepsilon_\mu, s)_\in$ and $0 \notin (L^\varepsilon_\mu, t)_\in$. This is a contradiction and so $L^\varepsilon_\mu(x) \leq \max\{L^\varepsilon_\mu(0), 0.5\}$ for all $x \in X$. Now, if the condition (3.10) is not valid, then there exist $a, b, c \in X$ such that $\max\{L^\varepsilon_\mu(a \circ c), 0.5\} < \min\{L^\varepsilon_\mu(a \circ (b \circ c)), L^\varepsilon_\mu(b)\}$. If we take $s = \min\{L^\varepsilon_\mu(a \circ (b \circ c)), L^\varepsilon_\mu(b)\}$, then $s \in (0.5, 1]$ and $[a \circ (b \circ c)/s], [b/s] \in L^\varepsilon_\mu$, that is, $a \circ (b \circ c), b \in (L^\varepsilon_\mu, s)_\in$. Since $(L^\varepsilon_\mu, s)_\in$ is a BCC-ideal of X, we have $a \circ c \in (L^\varepsilon_\mu, s)_\in$. But $[(a \circ c)/s] \in L^\varepsilon_\mu$ implies $a \circ c \notin (L^\varepsilon_\mu, s)_\in$, a contradiction. Thus, $\max\{L^\varepsilon_\mu(x \circ z), 0.5\} \geq \min\{L^\varepsilon_\mu(x \circ (y \circ z)), L^\varepsilon_\mu(y)\}$ for all $x, y, z \in X$.

Conversely, suppose that L^ε_μ satisfies the conditions (3.9) and (3.10). For every $t \in (0.5, 1]$, we have $0.5 < t \leq L^\varepsilon_\mu(x) \leq \max\{L^\varepsilon_\mu(0), 0.5\}$ for all $x \in (L^\varepsilon_\mu, t)_\in$ by (3.9). Then $0 \in (L^\varepsilon_\mu, t)_\in$. Let $t \in (0.5, 1]$ and $x, y, z \in X$ be such that $x \circ (y \circ z) \in (L^\varepsilon_\mu, t)_\in$ and $y \in (L^\varepsilon_\mu, t)_\in$. Then $L^\varepsilon_\mu(x \circ (y \circ z)) \geq t$ and $L^\varepsilon_\mu(y) \geq t$, which imply from (3.10) that $0.5 < t \leq \min\{L^\varepsilon_\mu(x \circ (y \circ z)), L^\varepsilon_\mu(y)\} \leq \max\{L^\varepsilon_\mu(x \circ z), 0.5\}$. Thus, $[(x \circ z)/t] \in L^\varepsilon_\mu$, that is, $x \circ z \in (L^\varepsilon_\mu, t)_\in$. Hence, $(L^\varepsilon_\mu, t)_\in$ is a BCC-ideal of X for $t \in (0.5, 1]$.

In Theorem 3, if $t \notin (0.5, 1]$, that is, there exists at least one $t \leq 0.5$, then Theorem 3 is incorrect, as shown in the following example.

Example 3. *Let $X = \{0, 1, 2, 3\}$ with the following Cayley table:*

$\circ$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	0
3	0	1	2	0

Then X is a BCC-algebra. Define a fuzzy set μ as follows:

$$\mu : X \to [0,1]; x \mapsto \begin{cases} 0.72 & \text{if } x = 0 \\ 0.68 & \text{if } x = 1 \\ 0.61 & \text{if } x = 2 \\ 0.57 & \text{if } x = 3 \end{cases}$$

Given $\varepsilon = 0.42$, the ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X is given as follows:

$$L^{\varepsilon}_{\mu} : X \to [0,1]; x \mapsto \begin{cases} 0.14 & \text{if } x = 0 \\ 0.10 & \text{if } x = 1 \\ 0.03 & \text{if } x = 2 \\ 0 & \text{if } x = 3 \end{cases}$$

Then $(L^{\varepsilon}_{\mu}, 0.25)_{\in} = \emptyset$ is not a BCC-ideal of X.

Theorem 4. *Let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. If μ is a fuzzy BCC-ideal of X, then the q-set $(L^{\varepsilon}_{\mu}, t)_q$ of L^{ε}_{μ} with value $t \in (0,1]$ is a BCC-ideal of X.*

Proof. Assume that μ is a fuzzy BCC-ideal of X and let $t \in (0,1]$. If $0 \notin (L^{\varepsilon}_{\mu}, t)_q$, then $[0/t]\overline{q}L^{\varepsilon}_{\mu}$, that is, $L^{\varepsilon}_{\mu}(0) + t \leq 1$. Since $L^{\varepsilon}_{\mu}(0) \geq L^{\varepsilon}_{\mu}(x)$ for $x \in (L^{\varepsilon}_{\mu}, t)_q$, it follows that $L^{\varepsilon}_{\mu}(x) \leq L^{\varepsilon}_{\mu}(0) \leq 1 - t$. Hence, $[x/t]\overline{q}L^{\varepsilon}_{\mu}$, and so $x \notin (L^{\varepsilon}_{\mu}, t)_q$. This contradiction is thus $0 \in (L^{\varepsilon}_{\mu}, t)_q$. Let $t \in (0,1]$ and $x, y, z \in (L^{\varepsilon}_{\mu}, t)_q$ be such that $x \circ (y \circ z) \in (L^{\varepsilon}_{\mu}, t)_q$ and $y \in (L^{\varepsilon}_{\mu}, t)_q$. Then $[x \circ (y \circ z)/t]_q L^{\varepsilon}_{\mu}$ and $[y/t]_q L^{\varepsilon}_{\mu}$, that is, $L^{\varepsilon}_{\mu}(x \circ (y \circ z)) + t > 1$ and $L^{\varepsilon}_{\mu}(y) + t > 1$. It follows from Theorem 1 and Theorem 2 that $L^{\varepsilon}_{\mu}(x \circ z) + t \geq \min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\} + t = \min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)) + t, L^{\varepsilon}_{\mu}(y) + t\} > 1$. Thus, $[(x \circ z)/t]qL^{\varepsilon}_{\mu}$. So $x \circ z \in (L^{\varepsilon}_{\mu}, t)_q$. Hence, $(L^{\varepsilon}_{\mu}, t)_q$ is a BCC-ideal of X.

Theorem 5. *Let μ be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X, if the q-set $(L^{\varepsilon}_{\mu}, t)_q$ is a BCC-ideal of X, then L^{ε}_{μ} satisfies the following properties:*

$$(\forall t_a \in (0, 0.5])(0 \in (L^{\varepsilon}_{\mu}, t_a)_{\in}) \tag{3.11}$$

$$(\forall x, y, z \in X, \forall t_a, t_b \in (0, 0.5]) \left(\begin{array}{l} [x \circ (y \circ z)/t_a]qL^{\varepsilon}_{\mu}, [y/t_b]qL^{\varepsilon}_{\mu} \\ \Rightarrow (x \circ z) \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in} \end{array} \right) \tag{3.12}$$

Proof. Let $x, y, z \in X$ and $t_a, t_b \in (0, 0.5]$. If $0 \notin (L^{\varepsilon}_{\mu}, t_a)_{\in}$, then $[0/t_a]\overline{\in}L^{\varepsilon}_{\mu}$ and so $L^{\varepsilon}_{\mu}(0) < t_a \leq 1 - t_a$ since $t_a \leq 0.5$. Hence, $[0/t_a]\overline{q}L^{\varepsilon}_{\mu}$. This contradiction is hence $0 \in (L^{\varepsilon}_{\mu}, t_a)_{\in}$. Let $x, y, z \in X$ be such that $x \circ (y \circ z) \in (L^{\varepsilon}_{\mu}, t)_q$ and $y \in (L^{\varepsilon}_{\mu}, t)_q$. Then $[x \circ (y \circ z)/t]qL^{\varepsilon}_{\mu}$ and $[y/t]qL^{\varepsilon}_{\mu}$, that is, $L^{\varepsilon}_{\mu}(x \circ (y \circ z)) > 1 - t$ and $L^{\varepsilon}_{\mu}(y) > 1 - t$. It follows that $L^{\varepsilon}_{\mu}(x \circ z) \geq \min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\} > 1 - t$. Thus, $[x \circ z/t]qL^{\varepsilon}_{\mu}$ and so $x \circ z \in (L^{\varepsilon}_{\mu}, t)_q$. Hence, $(L^{\varepsilon}_{\mu}, t)_q$ is a BCC-ideal of X.

Theorem 6. *If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X satisfies the following properties:*

$$(\forall x \in X, \forall t \in (0.5, 1])([x/t]qL^{\varepsilon}_{\mu} \Rightarrow [0/t] \in (L^{\varepsilon}_{\mu})) \tag{3.13}$$

$$(\forall x, y, z \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x \circ (y \circ z)/t_a]qL^{\varepsilon}_{\mu}, [y/t_b]qL^{\varepsilon}_{\mu} \\ \Rightarrow [(x \circ z)/\max\{t_a, t_b\}] \in L^{\varepsilon}_{\mu} \end{array} \right) \tag{3.14}$$

then the nonempty $\in$*-set* $(L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ *of* L^{ε}_{μ} *is a BCC-ideal of* X *for all* $t_a, t_b \in (0.5, 1]$.

Proof. Let $t_a, t_b \in (0.5, 1]$ and assume that the $\in$-set $(L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ of L^{ε}_{μ} is nonempty. Then there exists $x \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$, and so $L^{\varepsilon}_{\mu}(x) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, that is, $[x/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. Hence, $[0/\max\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$ by (3.13), and thus $0 \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$. Let $x, y, z \in X$ be such that $x \circ (y \circ z) \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ and $y \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$. Then $L^{\varepsilon}_{\mu}(x \circ (y \circ z)) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$ and $L^{\varepsilon}_{\mu}(y) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, that is, $[(x \circ (y \circ z))/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$ and $[y/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. It follows from (3.14) that $[(x \circ z)/\max\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. Hence, $x \circ z \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$, and therefore $(L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ is a BCC-ideal of X for all $t_a, t_b \in (0.5, 1]$.

Theorem 7. *If an* ε*-Lukasiewicz fuzzy set* L^{ε}_{μ} *in* X *satisfies the conditions* (3.13) *and*

$$(\forall x, y, z \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x \circ (y \circ z)/t_a]qL^{\varepsilon}_{\mu}, [y/t_b]qL^{\varepsilon}_{\mu} \\ \Rightarrow [(x \circ z)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu} \end{array} \right), \tag{3.15}$$

then the nonempty $\in$*-set* $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_{\in}$ *of* L^{ε}_{μ} *is a BCC-ideal of* X *for all* $t_a, t_b \in (0.5, 1]$.

Proof. Let $t_a, t_b \in (0.5, 1]$ and assume that the $\in$-set $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_{\in}$ of L^{ε}_{μ} is nonempty. Then there exists $x \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_{\in}$, and so $L^{\varepsilon}_{\mu}(x) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$, that is, $[x/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. Hence, $[0/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$ by (3.13), and thus $0 \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_{\in}$. Let $x, y, z \in X$ be such that $x \circ (y \circ z) \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_{\in}$ and $y \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_{\in}$. Then $L^{\varepsilon}_{\mu}(x \circ (y \circ z)) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$ and $L^{\varepsilon}_{\mu}(y) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$, that is, $[(x \circ (y \circ z))/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$ and $[y/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. It follows from (3.15) that $[(x \circ z)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. Hence, $x \circ z \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_{\in}$, and therefore $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_{\in}$ is a BCC-ideal of X for all $t_a, t_b \in (0.5, 1]$.

Theorem 8. *If an* ε*-Lukasiewicz fuzzy set* L^{ε}_{μ} *in* X *satisfies the conditions* (3.13) *and*

$$(\forall x \in X, \forall t \in (0.5, 1])([x/t]qL^{\varepsilon}_{\mu} \Rightarrow [0/t] \in (L^{\varepsilon}_{\mu})) \tag{3.16}$$

$$(\forall x, y, z \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x \circ (y \circ z)/t_a]qL^{\varepsilon}_{\mu}, [y/t_b]qL^{\varepsilon}_{\mu} \\ \Rightarrow [(x \circ z)/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu} \end{array} \right) \tag{3.17}$$

then the nonempty $\in$*-set* $(L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ *of* L^{ε}_{μ} *is a BCC-ideal of* X *for all* $t_a, t_b \in (0.5, 1]$.

Proof. Let $y \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ for $t_a, t_b \in (0.5, 1]$. Then $L^{\varepsilon}_{\mu}(y) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, and so $[y/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. Hence, $[(x \circ y)/\max\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$ for all $x \in X$ by (3.16), which implies that $x \circ y \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ for all $x \in X$. Let $x, y \in$

$(L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ for $t_a, t_b \in (0.5, 1]$. Then $L^{\varepsilon}_{\mu}(x) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$ and $L^{\varepsilon}_{\mu}(y) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, that is, $[x/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$ and $[y/\max\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. It follows from (3.17) that $[(x \circ (y \circ z)/\max\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$ for all $z \in X$. Hence, $x \circ (y \circ z) \in (L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ for all $z \in X$. Therefore, $(L^{\varepsilon}_{\mu}, \max\{t_a, t_b\})_{\in}$ of L^{ε}_{μ} is a BCC-ideal of X for all $t_a, t_b \in (0.5, 1]$.

Lemma 1. *Every ε-Łukasiewicz fuzzy ideal L^{ε}_{μ} of X satisfies the following property:*

$$(\forall x, y, z \in X)(L^{\varepsilon}_{\mu}(x \circ z) \geq \max\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\})$$

Proof. Note that $[(x \circ (y \circ z))/L^{\varepsilon}_{\mu}(x \circ (y \circ z))] \in L^{\varepsilon}_{\mu}$ and $[y/L^{\varepsilon}_{\mu}(y)] \in L^{\varepsilon}_{\mu}$ for all $x, y, z \in X$. It follows that $[(x \circ z)/\min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\}] \in L^{\varepsilon}_{\mu}$, that is, $L^{\varepsilon}_{\mu}(x \circ z) \geq \min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\}$ for all $x, y, z \in X$.

Theorem 9. *If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X satisfies the following properties:*

$$(\forall x \in X, \forall t \in (0.5, 1])([x/t]qL^{\varepsilon}_{\mu} \Rightarrow [0/t] \in (L^{\varepsilon}_{\mu})) \tag{3.18}$$

$$(\forall x, y, z \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x \circ (y \circ z)/t_a] \in L^{\varepsilon}_{\mu}, [y/t_b] \in L^{\varepsilon}_{\mu} \\ \Rightarrow [(x \circ z)/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu} \end{array} \right) \tag{3.19}$$

then the nonempty q-set $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$ of L^{ε}_{μ} is a BCC-ideal of X for all $t_a, t_b \in (0, 0.5]$.

Proof. Let $t_a, t_b \in (0, 0.5]$. If $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$ is nonempty, then there exists $x \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$. Hence, $L^{\varepsilon}_{\mu}(x) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$, which shows that $[x/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. It follows from (3.18) that $[0/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$. Thus, $0 \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$. Let $x, y, z \in X$ be such that $x \circ (y \circ z) \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$ and $y \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$. Then $L^{\varepsilon}_{\mu}(x \circ (y \circ z)) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$ and $L^{\varepsilon}_{\mu}(y) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$. Thus, $[(x \circ (y \circ z))/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$ and $[y/\min\{t_a, t_b\}] \in L^{\varepsilon}_{\mu}$. It follows from (3.19) that $[(x \circ z)/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu}$, that is, $x \circ z \in (L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$. Therefore, $(L^{\varepsilon}_{\mu}, \min\{t_a, t_b\})_q$ is a BCC-ideal of X.

Theorem 10. *If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} in X satisfies the conditions* (3.11) *and* (3.12)*, then the q-set $(L^{\varepsilon}_{\mu}, t)_q$ of L^{ε}_{μ} is a BCC-ideal of X for all $t \in (0.5, 1]$.*

Proof. Assume that L^{ε}_{μ} satisfies the conditions (3.11) and (3.12). The condition (3.11) induces $L^{\varepsilon}_{\mu}(0) + t \geq 2t > 1$, that is, $[0/t]qL^{\varepsilon}_{\mu}$. Hence, $0 \in (L^{\varepsilon}_{\mu}, t)_q$. Let $x, y, z \in X$ be such that $x \circ (y \circ z) \in (L^{\varepsilon}_{\mu}, t)_q$ and $y \in (L^{\varepsilon}_{\mu}, t)_q$. Then $[(x \circ (y \circ z))/t]qL^{\varepsilon}_{\mu}$ and $[y/t]qL^{\varepsilon}_{\mu}$. It follows from (3.12) that $x \circ z \in (L^{\varepsilon}_{\mu}, \min\{t, t\})_{\in} = (L^{\varepsilon}_{\mu}, t)_{\in}$. Hence, $L^{\varepsilon}_{\mu}(x \circ z) \geq t > 1 - t$, that is, $x \circ z \in (L^{\varepsilon}_{\mu}, t)_q$. Therefore, $(L^{\varepsilon}_{\mu}, t)_q$ is a BCC-ideal of X for all $t \in (0.5, 1]$.

Let μ be a fuzzy set in X. Consider an ε-Łukasiewicz fuzzy set L^{ε}_{μ} associated with μ in X. Define the set $O(L^{\varepsilon}_{\mu}) = \{x \in X : L^{\varepsilon}_{\mu}(x) > 0\}$, known as the O-set of L^{ε}_{μ}. It is noted that $O(L^{\varepsilon}_{\mu})$ can be expressed as $O(L^{\varepsilon}_{\mu}) = \{x \in X : \mu(x) + \varepsilon - 1 > 0\}$.

Theorem 11. *Let L^{ε}_{μ} be an ε-Łukasiewicz fuzzy set of a fuzzy set μ in X. If μ is a fuzzy BCC-ideal of X, then the O-set $O(L^{\varepsilon}_{\mu})$ of L^{ε}_{μ} is a BCC-ideal of X.*

Proof. Assume that μ is a fuzzy BCC-ideal of X. Then L^{ε}_{μ} is an ε-Łukasiewicz fuzzy BCC-ideal of X by Theorem 2. It is clear that $0 \in L^{\varepsilon}_{\mu}$. Let $x, y, z \in O(L^{\varepsilon}_{\mu})$ be such that $\mu(x \circ (y \circ z)) + \varepsilon - 1 > 0$ and $\mu(y) + \varepsilon - 1 > 0$. It follows from (3.5) that $L^{\varepsilon}_{\mu}(x \circ z) \geq \min\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\} = \min\{\mu(x \circ (y \circ z)) + \varepsilon - 1, \mu(y) + \varepsilon - 1\} > 0$. Thus, $x \circ z \in O(L^{\varepsilon}_{\mu})$. Hence, $O(L^{\varepsilon}_{\mu})$ is a BCC-ideal of X.

Theorem 12. *Let μ be a fuzzy set in X. If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X satisfies the following properties:*

$$(\forall x \in X, \forall t \in (0,1])([x/t]qL^{\varepsilon}_{\mu} \Rightarrow [0/t]qL^{\varepsilon}_{\mu}) \tag{3.20}$$

$$(\forall x, y, z \in X, \forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} [x \circ (y \circ z)/t_a] \in L^{\varepsilon}_{\mu}, [y/t_b] \in L^{\varepsilon}_{\mu} \\ \Rightarrow [(x \circ z)/\min\{t_a, t_b\}]qL^{\varepsilon}_{\mu} \end{array} \right) \tag{3.21}$$

then the O-set $O(L^{\varepsilon}_{\mu})$ of L^{ε}_{μ} is a BCC-ideal of X.

Proof. If $y \in O(L^{\varepsilon}_{\mu})$, then $\mu(y) > 1-\varepsilon$, that is, $[y/(1-\varepsilon)] \in \mu$. Hence, $[0/(1-\varepsilon)]qL^{\varepsilon}_{\mu}$ by (3.20), and thus $L^{\varepsilon}_{\mu}(0)+1-\varepsilon > 1$. Thus, $L^{\varepsilon}_{\mu}(0) > \varepsilon > 0$, which shows that $0 \in O(L^{\varepsilon}_{\mu})$. Let $x, y, z \in X$ be such that $x \circ (y \circ z) \in O(L^{\varepsilon}_{\mu})$ and $y \in O(L^{\varepsilon}_{\mu})$. Then $\mu(x \circ (y \circ z)) + \varepsilon - 1 > 0$ and $\mu(y) + \varepsilon - 1 > 0$. Since $[x \circ (y \circ z)/L^{\varepsilon}_{\mu}(x \circ (y \circ z))] \in L^{\varepsilon}_{\mu}$ and $[y/L^{\varepsilon}_{\mu}(y)] \in L^{\varepsilon}_{\mu}$. It follows from (3.21) that

$$[(x \circ z)/\max\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\}]qL^{\varepsilon}_{\mu}. \tag{3.22}$$

If $x \circ z \notin O(L^{\varepsilon}_{\mu})$, then $L^{\varepsilon}_{\mu}(x \circ z) = 0$. Thus,

$$\begin{aligned} & L^{\varepsilon}_{\mu}(x \circ z) + \max\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\} \\ & = \max\{L^{\varepsilon}_{\mu}(x \circ (y \circ z)), L^{\varepsilon}_{\mu}(y)\} \\ & = \max\{\max\{0, \mu(x \circ (y \circ z)) + \varepsilon - 1\}, \max\{0, \mu(y) + \varepsilon - 1\}\} \\ & = \max\{\mu(x \circ (y \circ z)) + \varepsilon - 1, \mu(y) + \varepsilon - 1\} \\ & = \max\{\mu(x \circ (y \circ z)), \mu(y)\} + \varepsilon - 1 \\ & \leq 1 + \varepsilon - 1 \\ & = \varepsilon \\ & \leq 1, \end{aligned}$$

which shows that (3.22) is not valid. This is a contradiction. So $x \circ z \in O(L^{\varepsilon}_{\mu})$. Hence, $O(L^{\varepsilon}_{\mu})$ is a BCC-ideal of X.

Theorem 13. *Let μ be a fuzzy set in X. If an ε-Łukasiewicz fuzzy set L^{ε}_{μ} of μ in X satisfies $[0/\varepsilon]q\mu$ and the following property:*

$$(\forall x, y, z \in X) \left(\begin{array}{l} [x \circ (y \circ z)/\varepsilon]q\mu, [y/\varepsilon]q\mu \\ \Rightarrow [(x \circ z)/\varepsilon]qL^{\varepsilon}_{\mu} \end{array} \right), \tag{3.23}$$

then the O-set $O(L^{\varepsilon}_{\mu})$ of L^{ε}_{μ} is a BCC-ideal of X.

Proof. If $[0/\varepsilon]q\mu$, then $\mu(0)+\varepsilon > 1$ and so $L^{\varepsilon}_{\mu}(0) = \max\{0, \mu(0)+\varepsilon-1\} = \mu(0)+\varepsilon-1 > 0$. Hence, $0 \in O(L^{\varepsilon}_{\mu})$. Let $x, y, z \in X$ be such that $x \circ (y \circ z) \in O(L^{\varepsilon}_{\mu})$ and $y \in O(L^{\varepsilon}_{\mu})$. Then $\mu(x \circ (y \circ z)) + \varepsilon - 1 > 0$ and $\mu(y) + \varepsilon - 1 > 0$. Hence, $L^{\varepsilon}_{\mu}(x \circ (y \circ z)) + 1 = \max\{0, \mu(x \circ (y \circ z)) + \varepsilon - 1\} + 1 = \mu(x \circ (y \circ z)) + \varepsilon - 1 + 1 = \mu(x \circ (y \circ z)) + \varepsilon > 1$ and $L^{\varepsilon}_{\mu}(y) + 1 = \max\{0, \mu(y) + \varepsilon - 1\} + 1 = \mu(y) + \varepsilon - 1 + 1 = \mu(y) + \varepsilon > 1$, that is, $[x \circ (y \circ z)/\varepsilon]qL^{\varepsilon}_{\mu}$ and $[y/\varepsilon]qL^{\varepsilon}_{\mu}$. It follows from (3.23) that $[(x \circ z)/\varepsilon] = [(x \circ z)/\varepsilon] \in L^{\varepsilon}_{\mu}$, which shows that $L^{\varepsilon}_{\mu}(x \circ z) \geq \varepsilon > 0$. Hence, $x \circ z \in O(L^{\varepsilon}_{\mu})$. Therefore, $O(L^{\varepsilon}_{\mu})$ is a BCC-ideal of X.

4. Conclusions

The concept of ε-Łukasiewicz fuzzy sets utilizing the Łukasiewicz t-norm was introduced by Jun [12]. This paper applies ε-Łukasiewicz fuzzy sets to BCC-ideals within BCC-algebras, introducing the concept of ε-Łukasiewicz fuzzy BCC-ideals and exploring their properties. The characterization of ε-Łukasiewicz fuzzy BCC-ideals is discussed, along with the relationship between fuzzy BCC-ideals and ε-Łukasiewicz fuzzy BCC-ideals. Conditions are provided under which ε-Łukasiewicz fuzzy sets qualify as ε-Łukasiewicz fuzzy BCC-ideals. Furthermore, conditions under which three subsets—$\in$-set, q-set, and O-set—can be BCC-ideals are explored.

This study's insights and findings are anticipated to be applied in future research concerning relevant algebraic systems. This includes exploring their utility as mathematical tools applicable to decision theory, medical diagnosis systems, automation systems, and other fields.

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