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Extending Approximations Spaces Using Nearly Open Sets, Subset Neighborhoods, and Ideals: A Medical Application

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Abstract. The close resemblance between rough sets and topology arises from the analogy between topological operators and rough approximations. This connection fosters combined studies between them. Therefore, this paper is created new approximations by leveraging topological concepts. Additionally, it is depicted that how a specific combination of ideals is utilized to approach rough from a topological perspective. As, ideals are valuable topological tools for reducing uncertainty. So, ideal structures are used to create new generalized approximation spaces that minimize vagueness. Initially, new topologies concepts are proposed relying on various types of the subset neighborhoods via ideals, and their relationships are analyzed. Thereafter, new approximations are derived from the proposed topological concepts. Moreover, all the present results are compared with earlier models to highlight the advantages and merits of the current technique. The present manners are more precise than previous approaches as they are particularly valuable for reducing vagueness. More importantly, three distinct perspectives are presented to elicit membership functions. To emphasize the importance of this paper, a numerical example related to Chikungunya disease is provided. This enables specialists to accurately assess the factors influencing Chikungunya disease. So, specialists and consultants can handle insufficient data regarding disease symptoms, resulting in easier and more accurate patient diagnoses. The study wraps up with a summary and proposals for future research.

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1. Introduction

The issues of imprecision in information systems used for data analysis have long been a concern. Many researchers, especially those specializing in artificial intelligence, had sought effective tools to address these challenges. One such tool that have been proposed

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is rough set theory. It is emerged as a non-statistical method for analyzing the data [37, 38]. This theory addresses complex real-life problems by partitioning a data set with inherent uncertainty into three distinct regions: the lower approximation, which encompasses the confirmed information; the upper approximation, which includes information whose membership in the set cannot be definitively determined; and the boundary area, represented by the gap between the upper and lower approximations. Vagueness is related to the boundary, where objects cannot be clearly classified. Consequently, the level of set's ambiguity, depends on the boundary contains element or not. A nonempty boundary indicates insufficient knowledge to precisely define the set. Therefore, a key goal is to minimize the boundary and enhance the set's accuracy.

In standard model, approximations were defined in terms of equivalence classes. However, the equivalence relations were overly constraining for applications. To broaden the aspects of this theory, researchers have suggested various extensions. For instance, equivalence classes have been replaced with other models in several studies (see: [34, 36]). Additionally, neighborhood systems extend rough set theory by substituting equivalence classes with neighborhoods in the definition of approximations (see: [3, 5–13, 16, 25, 42]).

Topology is a prevalent across nearly all branches of mathematics (see: [29, 40]). It has become a significant unifying idea in mathematics. The upper and lower approximations correspond to the closure and interior of a set. Therefore, different studies have explored the intersection of topological space and rough set theory (see: [22, 33, 39, 44]). Many mathematicians redirected their attention to the near (or nearly) open concept as an extension of open sets in topology [4, 20]. Hosny [22], Hosny and Al-Shami [26] presented $\wp$-nearly open and $\wp$-nearly approximations via ideals. These manners generalize $\wp$-nearly open and $\wp$-nearly approximations. Meanwhile, various techniques have been developed to construct topological spaces using neighborhood systems and their generalization. In this direction, initial right neighborhoods were presented in [18]. Thereafter, the remaining seven notions of initial neighborhoods were suggested under the name subset neighborhoods in [10]. More recently, Yildirim [43] formed a topology based on the subset neighborhoods. Moreover, she [43] presented near open sets relying on various types of subset neighborhoods, namely $\mathbb{S}_{\wp}$-near open sets. In [43], approximations were established and evaluate related to [1, 2, 30] under the restricted condition of similarity relations.

A nonempty collection $\mathfrak{D}$ of subsets of a set V is known an ideal on V, if $\mathfrak{D}$ is closed under finite unions and subsets [28]. Ideal in topological spaces was first studied in (see: [32, 41]). Thereafter, high-quality papers presented (see: [17, 19, 28]). The primary advantage of incorporating ideals into this theory was their ability to reduce vagueness by refining the boundaries of concepts. So, it increases the certain knowledge, thereby enhancing the reliability of decision making methods. Accordingly, using ideals is a robust technique for clarifying and precisely defining concepts. So, the exploration of rough set theory with ideals has become a prominent and engaging research area, attracting significant attention from researchers (see [20, 21, 23, 24, 27, 35]). Hence, ideals have been widely applied within this theoretical framework.

The aim of this manuscript is to advance research in these directions. It highlights that ideals are crucial in this study. This work comprises seven sections. The fundamental definitions and properties are given in Section 2. Afterwards, Section 3 introduces and examines new $\mathbb{S}_\wp$-nearly open sets related to an ideal $\mathfrak{D}$, denoted by $\mathfrak{D}$-$\alpha_{\mathbb{S}_\wp}$-open, $\mathfrak{D}$-$\theta\beta_{\mathbb{S}_\wp}$-open, $\mathfrak{D}$-$P_{\mathbb{S}_\wp}$-open, $\mathfrak{D}$-$S_{\mathbb{S}_\wp}$-open, $\mathfrak{D}$-$\beta_{\mathbb{S}_\wp}$-open and $\mathfrak{D}$-$\theta\beta_{\mathbb{S}_\wp}$-open sets. After substituting $\mathfrak{D} = \{\emptyset\}$ into the current definitions, we find that the resulting definitions are equivalent to those defined by Yildirim [43]. This equivalence shows that the specific case of the current framework are coincided with Yildirim's definitions. So, the last definitions [43] are considered as a special case of the current ones. Moreover, the essential comparisons of these manners with the prior ones [1, 2, 22, 26, 30] are stated. Furthermore, it is demonstrated that $\mathfrak{D}$-$\alpha_{\mathbb{S}_\wp}O(V)$ and $\mathfrak{D}$-$P_{\mathbb{S}_\wp}O(V)$ are distinct (see Example 3.1), even though every element in $\alpha_{\mathbb{S}_\wp}$ is in $P_{\mathbb{S}_\wp}O(V)$ as noted in [43]. Section 4 seeks to exhibit approximations derived from $\mathbb{S}_\wp$-nearly open sets in the context of ideals. The relationships between these approximations and those proposed in [1, 2, 22, 26, 30, 43] are presented in Theorems 4.1, 4.2, 4.3 and Corollaries 4.1, 4.3, 4.5. Section 5 focuses on defining three types of membership functions. Thereafter, the core features and relationships of these functions are derived and compared to the earlier ones in [1, 22, 26]. In Section 6, the paper presents a medical application to show the practical applicability and effectiveness of the suggested models and exhibits how ideals play a crucial role in decision-making. Accordingly, the current manners allow specialists to classify people with Chikungunya disease easily and with high accuracy. Eventually, a summary of the work's contributions and recommendations directions are given.

2. Preliminaries

Definition 2.1. *[1, 14, 15, 31] Let Υ be an arbitrary binary relation on a finite set $V \neq \emptyset$ and $t \in V$. Then, the $\wp$-neighborhood of $t \in V$ (in brief, $\aleph_\wp(t)$), $\forall \wp \in \{R, L, I, U, \langle R\rangle, \langle L\rangle, \langle I\rangle, \langle U\rangle\}$ is introduced by:*

(i) *R-neighborhood:* $\aleph_R(t) = \{s \in V : (t, s) \in \Upsilon\}$.

(ii) *L-left neighborhood:* $\aleph_L(t) = \{a \in V : (s, t) \in \Upsilon\}$.

(iii) $\aleph_I(t) = \aleph_R(t) \cap \aleph_L(t)$.

(iv) $\aleph_U(t) = \aleph_R(t) \cup \aleph_L(t)$.

(v) $\aleph_{\langle R\rangle}(t) = \bigcap_{t\in\aleph_R(s)} \aleph_R(s)$.

(vi) $\aleph_{\langle L\rangle}(t) = \bigcap_{t\in\aleph_L(s)} \aleph_L(s)$.

(vii) $\aleph_{\langle I\rangle}(t) = \aleph_{\langle R\rangle}(t) \cap \aleph_{\langle L\rangle}(t)$.

(viii) $\aleph_{\langle U\rangle}(t) = \aleph_{\langle R\rangle}(t) \cup \aleph_{\langle L\rangle}(t)$.

(ix) *the triple* $(V, \Upsilon, \Pi_{\wp)}$ *is known as a* $\wp$*-neighborhood space (or* $\wp$*-NS for short), with* $\Pi_\wp$ *being a mapping from* V *to* $P(V)$ *that assigns each* $t \in V$ *with a* $\wp$*-neighborhood.*

Theorem 2.1. *[1, 2, 30] The topology on* V *derived from an approximation space* (V, Υ) *given by* $\tau_\wp = \{M \subseteq V : \aleph_\wp(t) \subseteq M, \forall t \in M\}, \forall \wp$*. Sets in* $\tau_\wp$ *are termed* $\wp$*-open, their complements are* $\wp$*-closed set and all* $\wp$*-closed set is denoted by* $\daleth_\wp$*.*

Definition 2.2. *[1, 2, 30] The* $\wp$*-lower and* $\wp$*-upper approximations, boundary region and accuracy of a set* M *are*

$$\underline{\mathfrak{N}}_\wp(M) = \cup\{N \in \tau_\wp : N \subseteq M\} = int_\wp(M) \text{ (represents the topological } \wp\text{-interior operator)},$$

$$\overline{\mathfrak{N}}_\wp(M) = \cap\{Q : Q' \in \tau_\wp \text{ and } M \subseteq Q\} = cl_\wp(M) \text{ (represents the topological } \wp\text{-closure operator)},$$

$$\mathfrak{B}_\wp(M) = \overline{\mathfrak{N}}_\wp(M) - \underline{\mathfrak{N}}_\wp(M),$$

$$\mathfrak{A}_\wp(M) = \frac{|\underline{\mathfrak{N}}_\wp(M)|}{|\overline{\mathfrak{N}}_\wp(M)|}, \text{ where } M \text{ is nonempty.}$$

Definition 2.3. *[1, 2, 30] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS.* $M \subseteq V$ *is termed a* $\wp$*-exact set if* $\overline{\mathfrak{N}}_\wp(M) = \underline{\mathfrak{N}}_\wp(M)$*. Otherwise,* M *is known as a* $\wp$*-rough set.*

Definition 2.4. *[22, 26] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $\mathfrak{D}$ *be an ideal on* V*.* $M \subseteq V$ *is called*

(i) $\mathfrak{D}$-$P_\wp$*-open, if* $\exists \mathcal{G} \in \tau_\wp \cdot \ni \cdot (M - \mathcal{G}) \in \mathfrak{D}$ *and* $(\mathcal{G} - cl_\wp(M)) \in \mathfrak{D}$.

(ii) $\mathfrak{D}$-$S_\wp$*-open), if* $\exists \mathcal{G} \in \tau_\wp \cdot \ni \cdot (M - cl_\wp(\mathcal{G})) \in \mathfrak{D}$ *and* $(\mathcal{G} - M) \in \mathfrak{D}$.

(iii) $\mathfrak{D}$-$\beta_\wp$*-open, if* $\exists \mathcal{G} \in \tau_\wp \cdot \ni \cdot (M - cl_\wp(\mathcal{G})) \in \mathfrak{D}$ *and* $(\mathcal{G} - cl_\wp(M)) \in \mathfrak{D}$.

(iv) $\mathfrak{D}$-$\theta\beta_\wp$*-open if* $\exists\, \mathcal{G} \in \tau_\wp \cdot \ni \cdot (M - cl_{\mathbb{S}_\wp}(\mathcal{G})) \in \mathfrak{D}$ *and* $(\mathcal{G} - cl^\theta_\wp(M)) \in \mathfrak{D}, cl^\theta_\wp(M) = \{t \in V : M \cap cl_\wp(\mathcal{G}) \neq \emptyset, \mathcal{G} \in \tau_\wp \text{ and } t \in \mathcal{G}\}$.

These are named $\mathfrak{D}$-$\wp$*-nearly open, their complements are named* $\mathfrak{D}$-$\wp$*-nearly closed, all* $\mathfrak{D}$-$\wp$*-nearly open of* V *indicated by* $\mathfrak{D}$-$\xi_\wp O(V)$ *and all* $\mathfrak{D}$-$\wp$*-nearly closed of* V *indicated by* $\mathfrak{D}$-$\xi_\wp C(V)$, $\forall \xi \in \{P, S, \alpha, \beta, \theta\beta\}$.

Definition 2.5. *[22, 26] Let*$(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $\mathfrak{D}$ *be an ideal on* V *and* $M \subseteq V$*. The* $\mathfrak{D}$-$\wp$*-nearly lower,* $\mathfrak{D}$-$\wp$*-nearly upper approximations,* $\mathfrak{D}$-$\wp$*-nearly boundary regions and* $\mathfrak{D}$-$\wp$*-nearly accuracy of* M *are:*
$\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_\wp(M) = \cup\{\mathcal{G} \in \mathfrak{D}\text{-}\xi_\wp O(V) : \mathcal{G} \subseteq M\}$,
$\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_\wp(M) = \cap\{\mathcal{H} \in \mathfrak{D}\text{-}\xi_\wp C(V) : M \subseteq \mathcal{H}\}$,
$\mathfrak{B}^{\mathfrak{D}-\xi}_\wp(M) = \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_\wp(M) - \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_\wp(M)$.
$\mathfrak{A}^{\mathfrak{D}-\xi}_\wp(M) = \frac{|\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_\wp(M)|}{|\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_\wp(M)|}$, *where* $|\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_\wp(M)| \neq 0$.

Definition 2.6. *[22, 26] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $\mathfrak{D}$ *be an ideal on* V*. A subset* $M \subseteq V$ *is termed a* $\mathfrak{D}$*-*$\xi_\wp$*-exact set if* $\overline{\mathfrak{N}}_\wp^{\mathfrak{D}-\xi}(M) = \underline{\mathfrak{N}}_\wp^{\mathfrak{D}-\xi}(M)$*. Otherwise,* M *is known as a* $\mathfrak{D}$*-*$\xi_\wp$*-rough set.*

Definition 2.7. *[10] Take* Υ *as an arbitrary binary relation on a finite set* $V \neq \emptyset$ *and* $t \in V$*. Then the subset neighborhood of* $t \in V(briefly, \mathbb{S}_\wp(t)), \forall \wp$ *is defined by follows,*

(i) $\mathbb{S}_R(t) = \{s \in V : \aleph_R(t) \subseteq \aleph_R(s)\}$[18].

(ii) $\mathbb{S}_L(t) = \{s \in V : \aleph_L(t) \subseteq \aleph_L(s)\}$[10].

(iii) $\mathbb{S}_I(t) = \mathbb{S}_R(t) \cap \mathbb{S}_L(t)$[10].

(iv) $\mathbb{S}_U(t) = \mathbb{S}_R(t) \cup \mathbb{S}_L(t)$[10].

(v) $\mathbb{S}_{\langle R\rangle}(t) = \{s \in V : \aleph_{\langle R\rangle}(t) \subseteq \aleph_{\langle R\rangle}(t)\}$[10].

(vi) $\mathbb{S}_{\langle L\rangle}(t) = \{s \in V : \aleph_{\langle L\rangle}(t) \subseteq \aleph_{\langle L\rangle}(t)\}$[10].

(vii) $\mathbb{S}_{\langle I\rangle}(t) = \mathbb{S}_{\langle R\rangle}(t) \cap \mathbb{S}_{\langle L\rangle}(t)$[10].

(viii) $\mathbb{S}_{\langle U\rangle}(t) = \mathbb{S}_{\langle R\rangle}(t) \cup \mathbb{S}_{\langle L\rangle}(t)$[10].

Theorem 2.2. *[43] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS. Then* $\tau_{\mathbb{S}_\wp} = \{M \subseteq V : \mathbb{S}_\wp(t) \subseteq M, \forall t \in M\}, \forall \mathbb{S}_\wp$ *constitutes a topology on V. Sets in* $\tau_{\mathbb{S}_\wp}$ *are termed* $\mathbb{S}_\wp$*-open, their complements are* $\mathbb{S}_\wp$*-closed set and all* $\mathbb{S}_\wp$*-closed set is indicated by* $\urcorner_{\mathbb{S}_\wp}$*.*

Definition 2.8. *[43] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS. Then* $\mathbb{S}_\wp$*-lower and* $\mathbb{S}_\wp$*-upper approximations, boundary region and accuracy of a set* M *derived from a topological space* $(V, \tau_{\mathbb{S}_\wp})$ *are respectively determined by*

$$\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) = \cup\{N \in \tau_{\mathbb{S}_\wp} : N \subseteq M\} = int_{\mathbb{S}_\wp}(M) \text{ (represents the topological } \mathbb{S}_\wp\text{-interior operator),}$$

$$\overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) = \cap\{Q : Q' \in \tau_{\mathbb{S}_\wp} \text{ and } M \subseteq Q\} = cl_{\mathbb{S}_\wp}(M) \text{ (represents the topological } \mathbb{S}_\wp\text{-closure operator),}$$

$$\mathfrak{B}_{\mathbb{S}_\wp}(M) = \overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) - \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M),$$

$$\mathfrak{A}_{\mathbb{S}_\wp}(M) = \frac{|\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)|}{|\overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)|}, \text{ where } M \text{ is nonempty.}$$

Definition 2.9. *[43] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS. A subset* $M \subseteq V$ *is termed a* $\mathbb{S}_\wp$*-exact set if* $\overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)) = \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$*. Otherwise,* M *is known as a* $\mathbb{S}_\wp$*-rough set.*

Proposition 2.1. *[43] Let* $(V, \Upsilon, \Pi_{\mathbb{S}_\wp})$ *be a* $\wp$*-nbdS,* Υ *be a similarity relation,* $\wp \in \{R, L, I, U\}$ *and* $M \subseteq V$*. Then,* $\tau_\wp \subseteq \tau_{\mathbb{S}_\wp}$*.*

Theorem 2.3. *[43] Let $(V, \Upsilon, \Pi_{\mathbb{S}_\wp})$ be a $\wp$-nbdS, Υ be a similarity relation, $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then,*

(i) $\tau_\wp \subseteq \tau_{\mathbb{S}_\wp}$.

(ii) $\underline{\mathfrak{N}}_\wp(M) \subseteq \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$.

(iii) $\overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \subseteq \overline{\mathfrak{N}}_\wp(M)$.

(iv) $\mathfrak{B}_{\mathbb{S}_\wp}(M) \subseteq \mathfrak{B}_\wp(M)$.

(v) $\mathfrak{A}_\wp(M) \leq \mathfrak{A}_{\mathbb{S}_\wp}(M)$.

Definition 2.10. *[43] Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS. $M \subseteq V$ is*

(i) $\mathbb{S}_\wp$*-preopen ($P_{\mathbb{S}_\wp}$-open), if* $int_{\mathbb{S}_\wp}(cl_{\mathbb{S}_\wp}(M)) \supseteq M$.

(ii) $\mathbb{S}_\wp$*-semiopen ($S_{\mathbb{S}_\wp}$-open), if* $cl_{\mathbb{S}_\wp}(int_{\mathbb{S}_\wp}(M)) \supseteq M$.

(iii) $\alpha_{\mathbb{S}_\wp}$*-open, if* $M \subseteq int_{\mathbb{S}_\wp}[cl_{\mathbb{S}_\wp}(int_{\mathbb{S}_\wp}(M))]$.

(iv) $\beta_{\mathbb{S}_\wp}$*-open (semi preopen), if* $M \subseteq cl_{\mathbb{S}_\wp}[int_{\mathbb{S}_\wp}(cl_{\mathbb{S}_\wp}(M))]$.

(v) $\theta\beta_{\mathbb{S}_\wp}$*-open , if* $M \subseteq cl_{\mathbb{S}_\wp}[int_{\mathbb{S}_\wp}(cl^\theta_{\mathbb{S}_\wp}(M))]$*, where* $cl^\theta_{\mathbb{S}_\wp}(M) = \{t \in V : M \cap cl_{\mathbb{S}_\wp}(\mathcal{G}) \neq \emptyset, \mathcal{G} \in \tau_{\mathbb{S}_\wp}$ *and* $t \in \mathcal{G}\}$.

These are named $\mathbb{S}_\wp$-nearly open, all $\mathbb{S}_\wp$-nearly open of V indicated by $\xi_{\mathbb{S}_\wp}O(V)$, their complements are named $\mathbb{S}_\wp$-nearly closed and all $\mathbb{S}_\wp$-nearly closed of V indicated by $\xi_{\mathbb{S}_\wp}C(V)$, $\forall \xi \in \{P, S, \alpha, \beta, \theta\beta\}$.

Proposition 2.2. *[43] Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS. Then, the implications between $\tau_{\mathbb{S}_\wp}, \urcorner_{\mathbb{S}_\wp}, \xi_{\mathbb{S}_\wp}O(V)$ and $\xi_{\mathbb{S}_\wp}C(V)$ are*

$$\begin{array}{ccccccc} \tau_{\mathbb{S}_\wp}(\urcorner_{\mathbb{S}_\wp}) & \Rightarrow & \alpha_{\mathbb{S}_\wp}O(\alpha_{\mathbb{S}_\wp}C) & \Rightarrow & P_{\mathbb{S}_\wp}O(P_{\mathbb{S}_\wp}C) & & \\ & & \Downarrow & & \Downarrow & & \\ & & S_{\mathbb{S}_\wp}O(S_{\mathbb{S}_\wp}C) & \Rightarrow & \beta_{\mathbb{S}_\wp}O(\beta_{\mathbb{S}_\wp}C) & \Rightarrow & \theta\beta_{\mathbb{S}_\wp}O(\theta\beta_{\mathbb{S}_\wp}C). \end{array}$$

Definition 2.11. *[43] Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $M \subseteq V$. The $\mathbb{S}_\wp$-nearly lower, $\mathbb{S}_\wp$-nearly upper approximations, $\mathbb{S}_\wp$-nearly boundary regions and $\mathbb{S}_\wp$-nearly accuracy of M are:*
$\underline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M)$ is the union of all $\mathbb{S}_\wp$-nearly open sets which are subset of $M = \mathbb{S}_\wp$-nearly interior of M.
$\overline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M)$ is the intersection of all $\mathbb{S}_\wp$-nearly closed sets which are superset of $M = \mathbb{S}_\wp$-nearly closure of A.
$\mathfrak{B}^\xi_{\mathbb{S}_\wp}(M) = \overline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M) - \underline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M)$.
$\mathfrak{A}^\xi_{\mathbb{S}_\wp}(M) = \frac{|\underline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M)|}{|\overline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M)|}$*, where* $|\overline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M)| \neq 0$.

Definition 2.12. *[43] Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS. $M \subseteq V$ is $\mathbb{S}_\wp$-nearly exact if $\overline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M) = \underline{\mathfrak{N}}^\xi_{\mathbb{S}_\wp}(M)$. Otherwise, M is $\mathbb{S}_\wp$-nearly rough.*

Theorem 2.4. *[43] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $M \subseteq V$*. Then,* $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \subseteq \underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M) \subseteq M \subseteq \overline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$*.*

Definition 2.13. *[1] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $t \in V$ *and* $M \subseteq V$

(i) if $t \in \underline{\mathfrak{N}}_\wp(M)$*, then* t *is* $\wp$*-surely belongs to* M*, denoted by* $t \,\underline{\in}_\wp M$

(ii) if $t \in \overline{\mathfrak{N}}_\wp(M)$*, then* t *is* $\wp$*-possibly belongs to* M*, denoted by* $t \,\overline{\in}_\wp M$

Definition 2.14. *[1] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $M \subseteq V$ *and* $t \in V$*. The* $\wp$*-rough membership functions of* M *are presented by* $\omega^{\wp}_M : V \to [0, 1]$*, with* $\omega^{\wp}_M(t) = \frac{|\cap \aleph_\wp(t) \cap M|}{|\cap \aleph_\wp(t)|}, \cap \aleph_\wp(t) \neq \emptyset$ *and* $|M|$ *is cardinality of* M*.*

Definition 2.15. *[22, 26] The subsequent features valid for each subset* M*.*

(i) *If* $t \in \underline{\mathcal{R}}^{\mathfrak{D}-\xi}_\wp(M)$*, then* t *is* $\mathfrak{D} - \xi_\wp$*-certainly belongs to* M*, symbolized by* $t \,\underline{\in}^{\mathfrak{D}-\xi}_\wp M$*.*

(ii) *If* $t \in \overline{\mathcal{R}}^{\mathfrak{D}-\xi}_\wp(M)$*, then* t *is* $\mathfrak{D} - \xi_\wp$*-probably belongs to* M*, symbolized by* $t \,\overline{\in}^{\mathfrak{D}-\xi}_\wp M$*.*

Definition 2.16. *[22, 26] Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $\mathfrak{D}$ *be an ideal on* $V, M \subseteq V$ *and* $t \in V$*. The* $\mathfrak{D}$*-*$\mathbb{S}_\wp$*-nearly rough membership functions of a* $\wp$*-nbdS on* V *for a* M *are defines by* $\omega^{\mathfrak{D}-\xi_\wp}_M : V \to [0, 1]$*, where*

$$\omega^{\mathfrak{D}-\xi_\wp}_M(t) = \{^{1}_{min(\chi^{\mathfrak{D}-\xi_\wp}_M(t))} \quad ^{if\ 1 \in \chi^{\mathfrak{D}-\xi_\wp}_M(t).}_{otherwise.}\}.$$
$$and\ \chi^{\mathfrak{D}-\xi_\wp}_M(t) = \frac{|\mathfrak{D}-\xi_\wp(t) \cap M|}{|\mathfrak{D}-\xi_\wp(t)|}, t \in \mathfrak{D} - \xi_\wp(t),\ \mathfrak{D} - \xi_\wp(t) \in \mathfrak{D}\text{-}\xi_\wp O(V).$$

3. $\mathbb{S}_\wp$-nearly open sets via ideals and comparisons with the prior studies

This section introduces a new nearly open sets, namely $\mathfrak{D}$-$\alpha_{\mathbb{S}_\wp}$-open sets. These sets are defined using subset neighbourhood and ideal to be a preliminary step toward developing rough set paradigms. Moreover, the key characteristics of these classes are outlined and clarified how it relates to the previously discussed classes.

3.1. $\mathbb{S}_\wp$-nearly open sets via ideals

Definition 3.1. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $\mathfrak{D}$ *be an ideal on* V*.* $M \subseteq V$ *is called*

(i) $\mathfrak{D}$*-*$\alpha_{\mathbb{S}_\wp}$*-open, if* $\exists \mathcal{G} \in \tau_{\mathbb{S}_\wp} \cdot \ni \cdot (M - int_{\mathbb{S}_\wp}(cl_{\mathbb{S}_\wp}((\mathcal{G})) \in \mathfrak{D}$ *and* $(\mathcal{G} - M) \in \mathfrak{D}$*.*

(ii) $\mathfrak{D}$*-*$\mathbb{S}_\wp$*-Preopen (shortly* $\mathfrak{D}$*-*$P_{\mathbb{S}_\wp}$*-open), if* $\exists \mathcal{G} \in \tau_{\mathbb{S}_\wp} \cdot \ni \cdot (M - \mathcal{G}) \in \mathfrak{D}$ *and* $(\mathcal{G} - cl_{\mathbb{S}_\wp}(M)) \in \mathfrak{D}$*.*

(iii) $\mathfrak{D}$*-*$\mathbb{S}_\wp$*-Semi open (shortly* $\mathfrak{D}$*-*$S_{\mathbb{S}_\wp}$*-open), if* $\exists \mathcal{G} \in \tau_{\mathbb{S}_\wp} \cdot \ni \cdot (M - cl_{\mathbb{S}_\wp}(\mathcal{G})) \in \mathfrak{D}$ *and* $(\mathcal{G} - M) \in \mathfrak{D}$*.*

(iv) $\mathfrak{D}$*-*$\beta_{\mathbb{S}_\wp}$*-open, if* $\exists \mathcal{G} \in \tau_{\mathbb{S}_\wp} \cdot \ni \cdot (M - cl_{\mathbb{S}_\wp}(\mathcal{G})) \in \mathfrak{D}$ *and* $(\mathcal{G} - cl_{\mathbb{S}_\wp}(M)) \in \mathfrak{D}$*.*

(v) $\mathfrak{D}$-$\theta\beta_{\mathbb{S}_\wp}$*-open if* $\exists\, \mathcal{G} \in \tau_{\mathbb{S}_\wp} \cdot \ni \cdot (M - cl_{\mathbb{S}_\wp}(\mathcal{G})) \in \mathfrak{D}$ *and* $(\mathcal{G} - cl^{\theta}_{\mathbb{S}_\wp}(M)) \in \mathfrak{D}$.

These are named $\mathfrak{D}$-$\mathbb{S}_\wp$*-nearly open, their complements are named* $\mathfrak{D}$-$\mathbb{S}_\wp$*-nearly closed, all* $\mathfrak{D}$-$\mathbb{S}_\wp$*-nearly open of* V *indicated by* $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}O(V)$ *and all* $\mathfrak{D}$-$\mathbb{S}_\wp$*-nearly closed of* V *indicated by* $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}C(V)$, $\forall \xi \in \{P, S, \alpha, \beta, \theta\beta\}$.

Example 3.1. *Let* $V = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$, $\Upsilon = \{(\mathfrak{l}_1, \mathfrak{l}_1), (\mathfrak{l}_2, \mathfrak{l}_1), (\mathfrak{l}_2, \mathfrak{l}_4), (\mathfrak{l}_3, \mathfrak{l}_1), (\mathfrak{l}_3, \mathfrak{l}_3), (\mathfrak{l}_4, \mathfrak{l}_1), (\mathfrak{l}_4, \mathfrak{l}_3)\}$, *and* $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_1\}, \{\mathfrak{l}_2\}, \{\mathfrak{l}_1, \mathfrak{l}_2\}\}$. *Then,* $\tau_{\mathbb{S}_R} = \{V, \phi, \{\mathfrak{l}_2\}, \{\mathfrak{l}_3, \mathfrak{l}_4\}, \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}\}$, *and* $M = \{\mathfrak{l}_1\} \in \mathfrak{D}$-$\beta_{\mathbb{S}_R}O(V)$ *(respectively,* $\mathfrak{D}$-$S_{\mathbb{S}_R}O(V)$, $\mathfrak{D}$-$P_{\mathbb{S}_R}O(V)$, $\mathfrak{D}$-$\alpha_{\mathbb{S}_R}O(V)$ *), but* $M=\{\mathfrak{l}_1\} \notin \beta_{\mathbb{S}_R}O(V)$ *(respectively,* $S_{\mathbb{S}_R}O(V)$, $P_{\mathbb{S}_R}O(V)$, $\alpha_{\mathbb{S}_R}O(V)$*).*

The following finding highlights the rapports among the $\mathfrak{D}$-$\mathbb{S}_\wp$-nearly open sets.

Proposition 3.1. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $\mathfrak{D}$ *be an ideal on* V*. Then*

$$\begin{array}{ccccc} \mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}\text{-open} & & \mathfrak{D}\text{-}P_{\mathbb{S}_\wp}\text{-open} & & \\ \Downarrow & & \Downarrow & & \\ \mathfrak{D}\text{-}S_{\mathbb{S}_\wp}\text{-open} & \Rightarrow & \mathfrak{D}\text{-}\beta_{\mathbb{S}_\wp}\text{-open} & \Rightarrow & \mathfrak{D}\text{-}\theta\beta_{\mathbb{S}_\wp}\text{-open.} \end{array}$$

Proof. Straightforward by Definition 3.1.

Remark 3.1. *In Example 3.1*

(i) $\{\mathfrak{l}_3\} \in \mathfrak{D}$-$\beta_{\mathbb{S}_R}O(V), \notin \mathfrak{D}$-$\alpha_{\mathbb{S}_R}O(V), \notin \mathfrak{D}$-$S_{\mathbb{S}_R}O(V)$.

(ii) $\{\mathfrak{l}_3\} \in \mathfrak{D}$-$P_{\mathbb{S}_R}O(V), \notin \mathfrak{D}$-$\alpha_{\mathbb{S}_R}O(V)$.

(iii) *if* $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_2\}\}$, *then*

- **(a)** $\{\mathfrak{l}_1, \mathfrak{l}_2\} \in \mathfrak{D}$-$S_{\mathbb{S}_R}O(V) \notin \mathfrak{D}$-$\alpha_{\mathbb{S}_R}O(V)$.
- **(b)** $\{\mathfrak{l}_1, \mathfrak{l}_2\} \in \mathfrak{D}$-$\beta_{\mathbb{S}_R}O(V)$, *but it is not* $\mathfrak{D}$-$P_{\mathbb{S}_R}O(V)$.
- **(c)** $\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\} \in \mathfrak{D}$-$S_{\mathbb{S}_R}O(V) \notin \mathfrak{D}$-$P_{\mathbb{S}_R}O(V)$.
- **(d)** $\{\mathfrak{l}_4\} \in \mathfrak{D}$-$P_{\mathbb{S}_R}O(V) \notin \mathfrak{D}$-$P_{\mathbb{S}_R}O(V)$.
- **(e)** $\{\mathfrak{l}_1\} \in \mathfrak{D}$-$\alpha_{\mathbb{S}_R}O(V) \notin \mathfrak{D}$-$P_{\mathbb{S}_R}O(V)$.
- **(f)** $\{\mathfrak{l}_3\} \in \mathfrak{D}$-$P_{\mathbb{S}_R}O(V) \notin \mathfrak{D}$-$\alpha_{\mathbb{S}_R}O(V)$.
- **(g)** $\{\mathfrak{l}_1\} \in \mathfrak{D}$-$\theta\beta_{\mathbb{S}_R}O(V) \notin \mathfrak{D}$-$\beta_{\mathbb{S}_R}O(V)$. *Accordingly, it is not* $\mathbb{S}_RO(V)$, $\mathfrak{D}$-$\alpha_{\mathbb{S}_R}O(V)$, $\mathfrak{D}$-$S_{\mathbb{S}_R}O(V)$ *and* $\mathfrak{D}$-$P_{\mathbb{S}_R}O(V)$.

Remark 3.2. *Example 3.1 clarifies that*

(i) $\mathfrak{D}$-$\alpha_{\mathbb{S}_\wp}O(V)$ *and* $\mathfrak{D}$-$P_{\mathbb{S}_\wp}O(V)$ *are distinct even though every element in* $\alpha_{\mathbb{S}_\wp}O(V)$ *is in* $P_{\mathbb{S}_\wp}O(V)$ *[43].*

(ii) $\mathfrak{D}$-$S_{\mathbb{S}_\wp}$*-open sets and* $\mathfrak{D}$-$P_{\mathbb{S}_\wp}$*-open sets are incomparable.*

Proposition 3.2. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and let* $\mathfrak{D}$ *be an ideal on* V*. Then*

$$\begin{array}{ccc} \tau_{\mathbb{S}_\wp}(\Gamma_{\mathbb{S}_\wp}) \Rightarrow \mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}C) & \mathfrak{D}\text{-}P_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}P_{\mathbb{S}_\wp}C) & \\ \Downarrow & \Downarrow & \\ \mathfrak{D}\text{-}S_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}S_{\mathbb{S}_\wp}C) \Rightarrow & \mathfrak{D}\text{-}\beta_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}\beta_{\mathbb{S}_\wp}C) \Rightarrow & \mathfrak{D}\text{-}\theta\beta_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}\theta\beta_{\mathbb{S}_\wp}C). \end{array}$$

Proof. By Proposition 2.2 [43], and Propositions 3.4,3.1, the proof is evident.

Remark 3.3. *" $\tau_{\mathbb{S}_\wp}(\Gamma_{\mathbb{S}_\wp}) \Leftarrow \mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}C)$" is false. In Example 3.1, $M = \{\mathfrak{l}_1\} \in \mathfrak{D}\text{-}\alpha_{\mathbb{S}_R}O(V), \notin \tau_{\mathbb{S}_R}$.*

Proposition 3.3. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, Υ be a similarity relation, $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then*

$$\begin{array}{ccc} \tau_\wp(\Gamma_\wp) \Rightarrow \tau_{\mathbb{S}_\wp}(\Gamma_{\mathbb{S}_\wp}) \Rightarrow \mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}C) & \mathfrak{D}\text{-}P_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}P_{\mathbb{S}_\wp}C) & \\ \Downarrow & \Downarrow & \\ \mathfrak{D}\text{-}S_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}S_{\mathbb{S}_\wp}C) \Rightarrow & \mathfrak{D}\text{-}\beta_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}\beta_{\mathbb{S}_\wp}C) \Rightarrow & \mathfrak{D}\text{-}\theta\beta_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}\theta\beta_{\mathbb{S}_\wp}C). \end{array}$$

Proof. By Theorem 2.3 and Proposition 3.2 the proof is uncomplicated.

Remark 3.4. *It should be noted that*

(i) the similarity relation in Proposition 3.3 is indispensable as

Example 3.2. *Let $V = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$ and $\Upsilon = \{(\mathfrak{l}_1, \mathfrak{l}_1), (\mathfrak{l}_2, \mathfrak{l}_2), (\mathfrak{l}_1, \mathfrak{l}_3), (\mathfrak{l}_2, \mathfrak{l}_3), (\mathfrak{l}_2, \mathfrak{l}_4)\}$. Then, $\tau_R = \{V, \emptyset, \{\mathfrak{l}_3\}, \{\mathfrak{l}_4\}, \{\mathfrak{l}_3, \mathfrak{l}_4\}, \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}\}$ and $\tau_{\mathbb{S}_R} = \{V, \emptyset, \{\mathfrak{l}_1\}, \{\mathfrak{l}_2\}, \{\mathfrak{l}_1, \mathfrak{l}_2\}\}$ are incomparable.*

(ii) "$\tau_\wp(\Gamma_\wp) \Leftarrow \mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}O(\mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}C)$" is incorrect as:

Example 3.3. *Let $V = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$, $\Upsilon = \{(\mathfrak{l}_1, \mathfrak{l}_1), (\mathfrak{l}_2, \mathfrak{l}_2), (\mathfrak{l}_3, \mathfrak{l}_3), (\mathfrak{l}_4, \mathfrak{l}_4), (\mathfrak{l}_1, \mathfrak{l}_2), (\mathfrak{l}_2, \mathfrak{l}_1), (\mathfrak{l}_2, \mathfrak{l}_4), (\mathfrak{l}_4, \mathfrak{l}_2), (\mathfrak{l}_3, \mathfrak{l}_4), (\mathfrak{l}_4, \mathfrak{l}_3)\}$, and $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_3\}\}$. Then it is clear that, $\{\mathfrak{l}_2\} \in \mathfrak{D}\text{-}\alpha_{\mathbb{S}_R}, \notin \tau_R$*

(iii) Example 3.3 shows also that $\tau_{\mathbb{S}_\wp}, \tau_\wp$ are not comparable if $\wp \in \{<R>, <L>, <I>, <U>\}$. Then, $\tau_{\mathbb{S}_{<R>}} = \{V, \emptyset, \{\mathfrak{l}_1\}, \{\mathfrak{l}_3\}, \{\mathfrak{l}_1, \mathfrak{l}_2\}, \{\mathfrak{l}_1, \mathfrak{l}_3\}, \{\mathfrak{l}_3, \mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}, \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}\}$ and $\tau_{<R>} = \{V, \emptyset, \{\mathfrak{l}_2\}, \{\mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_2\}, \{\mathfrak{l}_2, \mathfrak{l}_4\}, \{\mathfrak{l}_3, \mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\}, \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}\}$. So, Proposition 3.3 applies only for $\wp \in \{R, L, I, U\}$.

Theorem 3.1. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS and $\mathfrak{D}$ be an ideal on V. Then, the union of two $\mathfrak{D}$-$\alpha_{\mathbb{S}_\wp}$-open (respectively, $\mathfrak{D}$-$S_{\mathbb{S}_\wp}$-open, $\mathfrak{D}$-$P_{\mathbb{S}_\wp}$-open, $\mathfrak{D}$-$\beta_{\mathbb{S}_\wp}$-open) sets is also $\mathfrak{D}$-$\alpha_{\mathbb{S}_\wp}$-open (respectively, $\mathfrak{D}$-$S_{\mathbb{S}_\wp}$-open, $\mathfrak{D}$-$P_{\mathbb{S}_\wp}$-open, $\mathfrak{D}$-β_J-open) set.*

Proof. We prove in the case of $\mathfrak{D}$-$\alpha_{\mathbb{S}_\wp}$-open sets and the others cases are similarly. Let $M, N \in \mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}O(V)$. Then, $\exists \mathcal{G}, \mathcal{H} \in \tau_{\mathbb{S}_\wp}$ such that $(M - int_{\mathbb{S}_\wp}cl_{\mathbb{S}_\wp}(\mathcal{G})) \in \mathfrak{D}, (\mathcal{G} - M) \in \mathfrak{D}, (N - int_{\mathbb{S}_\wp}cl_{\mathbb{S}_\wp}(\mathcal{H})) \in \mathfrak{D}$ and $(\mathcal{H} - N) \in \mathfrak{D}$. Since, $(\mathcal{G} - (M \cup N)) \subseteq (\mathcal{G} - M)$ and

$(\mathcal{G} - M) \in \mathfrak{D}$, so $(\mathcal{G} - (M \cup N)) \in \mathfrak{D}$. Similarly, $(\mathcal{H} - (M \cup N)) \in \mathfrak{D}$, and hence $(\mathcal{G} - (M \cup N)) \cup (\mathcal{H} - (M \cup N)) \in \mathfrak{D}$. Let $W = \mathcal{G} \cup \mathcal{H}$, then $(W - (M \cup N)) \in \mathfrak{D}$. Also, $(M - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(W)) \subseteq (M - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(\mathcal{G})) \in \mathfrak{D}$ and $(N - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(W)) \subseteq (N - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(\mathcal{H})) \in \mathfrak{D}$. Then, $(M - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(W)) \cup (N - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(W)) \in \mathfrak{D}$ and so $((M \cup N) - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(W)) \subseteq (M - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(\mathcal{G})) \cup (N - int_{\mathbb{S}_\wp} cl_{\mathbb{S}_\wp}(\mathcal{H})) \in \mathfrak{D}$. Thus, $M \cup N \in \mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp} O(V)$.

Remark 3.5. *$\mathfrak{D}\text{-}\xi_{\mathbb{S}_\wp} O(V)$ do not generate a topology as in Example 3.1, take $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_1\}, \{\mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_4\}\}$, then $M = \{\mathfrak{l}_1, \mathfrak{l}_2\}, N = \{\mathfrak{l}_1, \mathfrak{l}_4\} \in \mathfrak{D}\text{-}\alpha_{\mathbb{S}_R} O(V)$, but $M \cap N = \{\mathfrak{l}_1\} \notin \mathfrak{D}\text{-}\alpha_{\mathbb{S}_R} O(V)$. Additionally, if $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_2\}\}$, then*

(i) *$M = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}, N = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\} \in \mathfrak{D}\text{-}P_{\mathbb{S}_R} O(V)$, but $M \cap N = \{\mathfrak{l}_1, \mathfrak{l}_2\} \notin \mathfrak{D}\text{-}P_{\mathbb{S}_R} O(V)$.*

(ii) *$M = \{\mathfrak{l}_1, \mathfrak{l}_2\}, N = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\} \in \mathfrak{D}\text{-}S_{\mathbb{S}_R} O(V)$, but $M \cap N = \{\mathfrak{l}_1\} \notin \mathfrak{D}\text{-}S_{\mathbb{S}_R} O(V)$.*

(iii) *$M = \{\mathfrak{l}_1, \mathfrak{l}_2\}, N = \{\mathfrak{l}_1, \mathfrak{l}_3\} \in \mathfrak{D}\text{-}\beta_{\mathbb{S}_R} O(X)$, but $M \cap N = \{\mathfrak{l}_1\} \notin \mathfrak{D}\text{-}\beta_{\mathbb{S}_R} O(V)$.*

Remark 3.6. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS and $\mathfrak{D}$ be an ideal on V. Hence, the statements below are not true in most cases:*

(i) $\mathfrak{D}\text{-}\xi_U O(V) \subseteq \mathfrak{D}\text{-}\xi_R O(V) \subseteq \mathfrak{D}\text{-}\xi_I O(V)$.

(ii) $\mathfrak{D}\text{-}\xi_U O(V) \subseteq \mathfrak{D}\text{-}\xi_L O(V) \subseteq \mathfrak{D}\text{-}\xi_I O(V)$.

(iii) $\mathfrak{D}\text{-}\xi_{<U>} O(V) \subseteq \mathfrak{D}\text{-}\xi_{<R>} O(V) \subseteq \mathfrak{D}\text{-}\xi_{<I>} O(V)$.

(iv) $\mathfrak{D}\text{-}\xi_{<U>} O(V) \subseteq \mathfrak{D}\text{-}\xi_{<L>} O(V) \subseteq \mathfrak{D}\text{-}\xi_{<I>} O(V)$.

(v) *$\mathfrak{D}\text{-}\xi_R O(V)$ is the dual of $\mathfrak{D}\text{-}\xi_L O(V)$.*

(vi) *$\mathfrak{D}\text{-}\xi_{<R>} O(V)$ is the dual of $\mathfrak{D}\text{-}\xi_{<L>} O(V)$.*

Example 3.4. *Let $V = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$, $\Upsilon = \{(\mathfrak{l}_1, \mathfrak{l}_1), (\mathfrak{l}_1, \mathfrak{l}_2), (\mathfrak{l}_1, \mathfrak{l}_3), (\mathfrak{l}_2, \mathfrak{l}_4), (\mathfrak{l}_2, \mathfrak{l}_3)\}$, and if $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_3\}\}$. Then, $\mathfrak{D}\text{-}\beta_{\mathbb{S}_R} O(V) = \{\emptyset, V, \{\mathfrak{l}_1\}, \{\mathfrak{l}_2\}, \{\mathfrak{l}_1, \mathfrak{l}_2\}, \{\mathfrak{l}_1, \mathfrak{l}_3\}, \{\mathfrak{l}_1, \mathfrak{l}_4\}, \{\mathfrak{l}_2, \mathfrak{l}_3\}, \{\mathfrak{l}_2, \mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}, \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}, \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}\}, \mathfrak{D}\text{-}\beta_{\mathbb{S}_U} O(V) = P(V)$. So, $\mathfrak{D}\text{-}\beta_{\mathbb{S}_U} O(V) \nsubseteq \mathfrak{D}\text{-}\beta_{\mathbb{S}_R} O(V)$. Additionally, if $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_2\}\}$. Then,*

(i) $\mathfrak{D}\text{-}\beta_{\mathbb{S}_R} O(V) = \mathfrak{D}\text{-}\beta_{\mathbb{S}_U} O(V) = \mathfrak{D}\text{-}\beta_{\mathbb{S}_{<L>}} O(V) = \mathfrak{D}\text{-}\beta_{\mathbb{S}_{<U>}} O(V) = P(V)$.

(ii) $\mathfrak{D}\text{-}\beta_{\mathbb{S}_L} O(V) = \{\emptyset, V, \{\mathfrak{l}_2\}, \{\mathfrak{l}_1, \mathfrak{l}_3\}, \{\mathfrak{l}_2, \mathfrak{l}_3\}, \{\mathfrak{l}_3, \mathfrak{l}_4\}, , \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}, \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}, \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}\}$.

(iii) $\mathfrak{D}\text{-}\beta_{\mathbb{S}_I} O(V) = \{\emptyset, V, \{\mathfrak{l}_1\}, \{\mathfrak{l}_2\}, \{\mathfrak{l}_3\}, \{\mathfrak{l}_1, \mathfrak{l}_2\}, \{\mathfrak{l}_1, \mathfrak{l}_3\}, \{\mathfrak{l}_2, \mathfrak{l}_3\}, \{\mathfrak{l}_3, \mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}, \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}, \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}\}$.

(iv) $\mathfrak{D}\text{-}\beta_{\mathbb{S}_{<R>}} O(V) = P(V) - \{\mathfrak{l}_1\}$.

(v) $\mathfrak{D}\text{-}\beta_{\mathbb{S}_{<I>}} O(V) = \{\emptyset, V, \{\mathfrak{l}_1\}, \{\mathfrak{l}_2\}, \{\mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_2\}, \{\mathfrak{l}_1, \mathfrak{l}_4\}, \{\mathfrak{l}_2, \mathfrak{l}_4\}, \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\}\}$.

So, $\mathfrak{D}\text{-}\beta_{\mathbb{S}_R} O(V) \nsubseteq \mathfrak{D}\text{-}\beta_{\mathbb{S}_I} O(V), \mathfrak{D}\text{-}\beta_{\mathbb{S}_U} O(V) \nsubseteq \mathfrak{D}\text{-}\beta_{\mathbb{S}_L} O(V) \nsubseteq \mathfrak{D}\text{-}\beta_{\mathbb{S}_I} O(V), \mathfrak{D}\text{-}\beta_{\mathbb{S}_{<U>}} O(V) \nsubseteq \mathfrak{D}\text{-}\beta_{\mathbb{S}_{<R>}} O(V) \nsubseteq \mathfrak{D}\text{-}\beta_{\mathbb{S}_{<I>}} O(V), \mathfrak{D}\text{-}\beta_{\mathbb{S}_{<L>}} O(V) \nsubseteq \mathfrak{D}\text{-}\beta_{\mathbb{S}_{<I>}} O(V), \mathfrak{D}\text{-}\beta_{\mathbb{S}_R} O(V)$ is not the dual of $\mathfrak{D}\text{-}\beta_{\mathbb{S}_R} O(V)$ and $\mathfrak{D}\text{-}\beta_{\mathbb{S}_{<R>}} O(V)$ is not the dual of $\mathfrak{D}\text{-}\beta_{\mathbb{S}_{<L>}} O(V)$.

3.2. Comparisons with the prior studies

The following findings confirm that the suggested Definition 3.1 is superior than Yildirim's Definition 2.10 [43].

Proposition 3.4. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS and $\mathfrak{D}$ be an ideal on V. Then*

$$\alpha_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}\alpha_{\mathbb{S}_\wp}\text{-}open.$$
$$P_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}P_{\mathbb{S}_\wp}\text{-}open.$$
$$S_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}S_{\mathbb{S}_\wp}\text{-}open.$$
$$\beta_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}\beta_{\mathbb{S}_\wp}\text{-}open.$$
$$\theta\beta_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}\theta\beta_{\mathbb{S}_\wp}\text{-}open.$$

Proof. By applying Definitions 2.10 and 3.1.

Example 3.1 confirms that the reverse implications of Proposition 3.4 is not guaranteed to be true as $M = \{\mathfrak{l}_1\} \in \mathfrak{D}\text{-}\beta_{\mathbb{S}_R}O(V)$ (respectively, $\mathfrak{D}\text{-}S_{\mathbb{S}_R}O(V)$, $\mathfrak{D}\text{-}P_{\mathbb{S}_R}O(V)$, $\mathfrak{D}\text{-}\alpha_{\mathbb{S}_R}O(V)$), but $M = \{\mathfrak{l}_1\} \notin \beta_{\mathbb{S}_R}O(V)$ (respectively, $S_{\mathbb{S}_R}O(V)$, $P_{\mathbb{S}_R}O(V)$, $\alpha_{\mathbb{S}_R}O(V)$).

Theorem 3.2. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS and $\mathfrak{D}$ be an ideal on V. If $\mathfrak{D} = \{\emptyset\}$ in the present manner 3.1, then Yildirim's Definition is obtained 2.10 [43].*

Proof. Straightforward.

Theorem 3.2 emphasizes that Yildirim's definitions [43] can be interpreted as a special case of the current ones. As, when $\mathfrak{D} = \{\emptyset\}$ in the current definitions, we see that the resulting definitions equivalent to those put forth by Yildirim [43]. This equivalence suggests that the specific case in the current framework aligns with Yildirim's definitions.

Proposition 3.5. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, Υ be a similarity relation, $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then*

$$\mathfrak{D}\text{-}P_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}P_\wp\text{-}open.$$
$$\mathfrak{D}\text{-}S_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}S_\wp\text{-}open.$$
$$\mathfrak{D}\text{-}\beta_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}\beta_\wp\text{-}open.$$
$$\mathfrak{D}\text{-}\theta\beta_{\mathbb{S}_\wp}\text{-}open \Rightarrow \mathfrak{D}\text{-}\theta\beta_\wp\text{-}open.$$

Proof. By applying Definitions 2.4 and 3.1.

Example 3.3 clarifies that the converse of Proposition 3.5 does not always apply, consequently the prior manners in [22, 26] are preferable than the present one in the case of similarity relation.

4. Approximations by using $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly open sets and comparisons to the prior ones

In this section, new rough paradigms inspired by the family $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly open sets are introduced. Additionally, the proposed models for all cases of $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly open sets are compared using counterexamples to illustrate their distinctions. More importantly, it is showed that how these novel paradigms contribute to decision-making. As, it improves the accuracy of the knowledge extracted, compared to existing ones.

4.1. Approximations by using $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly open sets

Definition 4.1. *Let$(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. The $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly lower, $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly upper approximations, $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly boundary regions and $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly accuracy of M are:*
$\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) = \cup\{\mathcal{G} \in \mathfrak{D}\text{-}\xi_{\mathbb{S}_{\wp}}O(V) : \mathcal{G} \subseteq M\} = \mathfrak{D}$-$\mathbb{S}_{\wp}$*-nearly interior of* M.
$\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) = \cap\{\mathcal{H} \in \mathfrak{D}\text{-}\xi_{\mathbb{S}_{\wp}}C(V) : M \subseteq \mathcal{H}\} = \mathfrak{D}$-$\mathbb{S}_{\wp}$*-nearly closure of* M.
$\mathfrak{B}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) = \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) - \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)$.
$\mathfrak{A}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) = \frac{|\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)|}{|\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)|}$, *where* $|\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)| \neq 0$.

Proposition 4.1. *Let $(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M, N \subseteq V$. Then,*

(i) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) \subseteq M \subseteq \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)$ *equality hold if* $M = \emptyset$ *or* V.

(ii) $M \subseteq N \Rightarrow \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) \subseteq \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(N)$.

(iii) $M \subseteq N \Rightarrow \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(N)$.

(iv) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M \cap N) \subseteq \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) \cap \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(N)$.

(v) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M \cup N) \supseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) \cup \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(N)$.

(vi) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M \cup N) \supseteq \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) \cup \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(N)$.

(vii) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M \cap N) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) \cap \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(N)$.

(viii) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) = (\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M'))'$, $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M) = (\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M'))'$.

(ix) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)) = \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)$.

(x) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)) = \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)$.

(xi) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M)) \subseteq \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_{\wp}}(M))$.

(xii) $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(\overline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M)) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(\overline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M))$.

Straightforward using $\mathfrak{D}$-$\mathbb{S}\wp$-nearly interior and $\mathfrak{D}$-$\mathbb{S}\wp$-nearly closure, so it is not be detailed here.

Remark 4.1. *In Example 3.1, take $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_2\}\}$. It shows that*

(i) *if $M = \{\mathfrak{l}_1\}$, then $\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) = \phi$, so, $A \nsubseteq \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)$. Additionally, take $M = \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) = V$, then $\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) \nsubseteq M$.*

(ii) *if $M = \{\mathfrak{l}_1, \mathfrak{l}_2\}, N = \{\mathfrak{l}_2, \mathfrak{l}_3\}, M \cap N = \{\mathfrak{l}_2\}, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M) = M, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(N) = V, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M \cap N) = \{\mathfrak{l}_2\}$, then $\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M) \cap \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(N) = \{\mathfrak{l}_1, \mathfrak{l}_2\} \nsubseteq \{\mathfrak{l}_2\} = \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M \cap N)$.*

(iii) *if $M = \{\mathfrak{l}_1, \mathfrak{l}_4\}, N = \{\mathfrak{l}_3, \mathfrak{l}_4\}, M \cup N = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M) = \emptyset, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(N) = \{\mathfrak{l}_3, \mathfrak{l}_4\}, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M \cup N) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$, then $\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M \cup N) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\} \nsubseteq \{\mathfrak{l}_3, \mathfrak{l}_4\} = \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M) \cup \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-S}(N)$.*

(iv) *if $M = \{\mathfrak{l}_2, \mathfrak{l}_3\}, N = \{\mathfrak{l}_2, \mathfrak{l}_4\}, M \cup N = \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) = \{\mathfrak{l}_2, \mathfrak{l}_3\}, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(N) = \{\mathfrak{l}_2, \mathfrak{l}_4\}, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M \cup N) = V$, then $\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M \cup N) = V \nsubseteq \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\} = \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) \cup \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(N)$.*

(v) *if $M = \{\mathfrak{l}_1, \mathfrak{l}_3\}, N = \{\mathfrak{l}_1, \mathfrak{l}_4\}, M \cap N = \{\mathfrak{l}_1\}, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) = \{\mathfrak{l}_1, \mathfrak{l}_3\}, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(N) = \{\mathfrak{l}_1, \mathfrak{l}_4\}, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M \cap N) = \emptyset$, then $\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) \cap \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(N) = \{\mathfrak{l}_1\} \nsubseteq \emptyset = \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M \cap N)$.*

(vi) *if $M = \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)) = M, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)) = V$, then $\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)) \nsubseteq \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M))$.*

(vii) *for part 12, if $M = \{\mathfrak{l}_1\}, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)) = M, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)) = \emptyset$, then $\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)) \nsubseteq \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M))$.*

(viii) *if $M = \{\mathfrak{l}_1\}, N = \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$, then $\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) = \{\mathfrak{l}_1\}, \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(N)) = V$. Therefore, $\overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(N)$, but $M \nsubseteq N$.*

(ix) *if $M = \{\mathfrak{l}_1\}, N = \{\mathfrak{l}_3\}$, then $\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) = \emptyset, \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(N) = \{\mathfrak{l}_3\}$. Therefore, $\underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M) \subseteq \underline{\mathfrak{N}}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(N)$, but $M \nsubseteq N$.*

Definition 4.2. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, and $M \subseteq V$. M is $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly definable ($\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly exact) set if $\overline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M) = \underline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M)$. Otherwise, M is $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly rough set.*

In Example 3.1, take $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_2\}\}$ and $M = \{\mathfrak{l}_2\}$ is $\mathfrak{D}$-$\beta_{\mathbb{S}_R}$-exact, while $N = \{\mathfrak{l}_1\}$ is $\mathfrak{D}$-$\beta_{\mathbb{S}_R}$-rough.

Remark 4.2. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V. Then the intersection of two $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-rough sets need not to be $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-rough set as in Example 3.1 $\{\mathfrak{l}_1, \mathfrak{l}_3\}$ and $\{\mathfrak{l}_1, \mathfrak{l}_4\}$, are $\mathfrak{D}$-$S_{\mathbb{S}_R}$-rough sets, $\{\mathfrak{l}_1, \mathfrak{l}_3\} \cap \{\mathfrak{l}_1, \mathfrak{l}_4\} = \{\mathfrak{l}_1\}$ is not $\mathfrak{D}$-$S_{\mathbb{S}_R}$-rough set.*

4.2. Relationships among the proposed approximations and comparisons to the prior ones

The following results underscore the advantages of the present manner 4.1 with the comparison of the prior one in Definitions 2.8 and 2.11 [43].

Theorem 4.1. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. Then*

(i) $\underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

(ii) $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

(iii) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) \subseteq \overline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M)$.

(iv) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$.

Proof.

(1) $\underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M) = \cup\{G \in \xi_{\mathbb{S}_\wp}O(V) : G \subseteq M\} \subseteq \cup\{G \in \mathfrak{D}\text{-}\xi_{\mathbb{S}_\wp}O(V) : G \subseteq M\} = \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$ (by Proposition 3.1).

(2) By Theorem 2.4, $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \subseteq \underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M)$, and by (1) $\underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$. Hence, $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

(3) and (4) Similar to (1) and (2).

Corollary 4.1. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. Then*

(i) $\mathfrak{B}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) \subseteq \mathfrak{B}^{\xi}_{\mathbb{S}_\wp}(M)$.

(ii) $\mathfrak{B}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) \subseteq \mathfrak{B}_{\mathbb{S}_\wp}(M)$.

(iii) $\mathfrak{A}^{\xi}_{\mathbb{S}_\wp}(M) \leqslant \mathfrak{A}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

(iv) $\mathfrak{A}_{\mathbb{S}_\wp}(M) \leqslant \mathfrak{A}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

Corollary 4.2. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V. Then*

(i) Every $\xi_{\mathbb{S}_\wp}$-nearly exact subset in V is $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly exact.

(ii) Every $\mathbb{S}_\wp$-exact subset in V is $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly exact.

(iii) Every $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly rough subset in V is $\xi_{\mathbb{S}_\wp}$-nearly rough.

(iv) Every $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly rough subset in V is $\mathbb{S}_\wp$-rough.

Table 1: The boundary regions and accuracy by the prior manner in 2.8 [43] for $\wp = R$ and the present manner in 4.1 for $\xi = \{\alpha, P, S, \beta, \theta\beta\}, \wp = R$.

M	The prior manner in 2.8 [43] for $\wp = R$		The present manner in 4.1 for $\xi = \alpha, \wp = R$		The present manner in 4.1 for $\xi = P, \wp = R$		The present manner in 4.1 for $\xi = S, \wp = R$		The present manner in 4.1 for $\xi = \beta, \wp = R$		The present manner in 4.1 for $\xi = \theta\beta, \wp = R$	
	$\mathfrak{B}_{\mathbb{S}_R}(M)$	$\mathfrak{A}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-\alpha}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-\alpha}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-P}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-P}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-S}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-S}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-\theta\beta}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-\theta\beta}_{\mathbb{S}_R}(M)$
$\{\mathfrak{l}_1\}$	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\emptyset$	1
$\{\mathfrak{l}_2\}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2\}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\emptyset$	1	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{2}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\emptyset$	1	$\{\mathfrak{l}_2, \mathfrak{l}_3\}$	$\frac{1}{2}$	$\emptyset$	1	$\emptyset$	1
$\{a\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\emptyset$	1

Table 2: The boundary regions and accuracy by the prior manner in 2.11 [43] and the present manner in 4.1 for $\xi = \{\alpha, P\}, \wp = R$.

M	The prior manner in 2.11 [43] for $\xi = \alpha, \wp = R$		The present manner in 4.1 for $\xi = \alpha, \wp = R$		The prior manner in 2.11 [43] for $\xi = P, \wp = R$		The present manner in 4.1 for $\xi = P, \wp = R$	
	$\mathfrak{B}^{\alpha}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\alpha}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-\alpha}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-\alpha}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{P}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{P}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-P}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-P}_{\mathbb{S}_R}(M)$
$\{\mathfrak{l}_1\}$	$\{\mathfrak{l}_1\}$	0	$\emptyset$	1	$\{\mathfrak{l}_1\}$	0	$\emptyset$	1
$\{\mathfrak{l}_2\}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_3\mathfrak{l}_4\}$	0	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2\}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{2}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{2}$	$\emptyset$	1	$\emptyset$	1
$\{a\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\emptyset$	1

Table 3: The boundary regions and accuracy by the prior manner in 2.11 [43] and the present manner in 4.1 for $\xi = \{S, \beta\}, \wp = R$.

M	The prior manner in 2.11 [43] for $\xi = S, \wp = R$		The present manner in 4.1 for $\xi = S, \wp = R$		The prior manner in 2.11 [43] for $\xi = \beta, \wp = R$		The present manner in 4.1 for $\xi = \beta, \wp = R$	
	$\mathfrak{B}^{S}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{S}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-S}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-S}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\beta}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\beta}_{\mathbb{S}_R}(M)$	$\mathfrak{B}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(M)$	$\mathfrak{A}^{\mathfrak{D}-\alpha}_{\mathbb{S}_R}(M)$
$\{\mathfrak{l}_1\}$	$\{\mathfrak{l}_1\}$	0	$\emptyset$	1	$\{\mathfrak{l}_1\}$	0	$\emptyset$	1
$\{\mathfrak{l}_2\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_3\}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_4\}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{3}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	0	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{3}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_3\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{3}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{4}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{3}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{2}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{2}$	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{2}$	$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\frac{1}{2}$	$\emptyset$	1	$\emptyset$	1
$\{a\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{3}{4}$	$\emptyset$	1

The boundary regions and accuracy by the prior manner in 2.8 [43] and the present manner in 4.1 are calculated in Table 1 by using Example 3.1 when $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_2\}\}$. Whereas, the boundary regions and accuracy by the prior manner in 2.11 [43] and the present manner in 4.1 are computed in Tables 2, 3 by using Example 3.1.

Definition 4.1 is superior to Definition 2.2 [1, 2, 30], as exhibited by the subsequent findings.

Theorem 4.2. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $\mathfrak{D}$ *be an ideal on* V, Υ *be a similarity relation,* $\wp \in \{R, L, I, U\}$ *and* $M \subseteq V$*. Then*

(i) $\underline{\mathfrak{N}}_\wp(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

(ii) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) \subseteq \overline{\mathfrak{N}}_\wp(M)$.

Proof.

(1) By Theorem 2.3, $\underline{\mathfrak{N}}_\wp(M) \subseteq \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$, and by (2) in Theorem 4.1 $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$. So, $\underline{\mathfrak{N}}_\wp(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

(2) Similar to (1).

Corollary 4.3. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $\mathfrak{D}$ *be an ideal on* V, Υ *be a similarity relation* $\wp \in \{R, L, I, U\}$ *and* $M \subseteq V$*. Then*

(i) $\mathfrak{B}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) \subseteq \mathfrak{B}_\wp(M)$.

(ii) $\mathfrak{A}_\wp(M) \leqslant \mathfrak{A}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

Corollary 4.4. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $\mathfrak{D}$ *be an ideal on* V, Υ *be a similarity relation,* $\wp \in \{R, L, I, U\}$ *and* $M \subseteq V$*. Then*

(i) Every $\wp$*-exact subset in* V *is* $\mathfrak{D}$*-*$\xi_{\mathbb{S}_\wp}$*-nearly exact.*

(ii) Every $\mathfrak{D}$*-*$\xi_{\mathbb{S}_\wp}$*-nearly rough subset in* V *is* $\wp$*-rough.*

The prior approximations in Definition 2.5 [22, 26] are outperform those in Definition 4.1 in the case of similarity relation, as displayed in the subsequent results.

Theorem 4.3. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $\mathfrak{D}$ *be an ideal on* V, Υ *be a similarity relation,* $\wp \in \{R, L, I, U\}$ *and* $M \subseteq V$*. Then*

(i) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) \subseteq \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\wp}(M)$.

(ii) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\wp}(M) \subseteq \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$.

Proof. By Proposition 3.5, the proof is evident.

Corollary 4.5. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, Υ be a similarity relation $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then*

(i) $\mathfrak{B}_{\wp}^{\mathfrak{D}-\xi}(M) \subseteq \mathfrak{B}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M)$.

(ii) $\mathfrak{A}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M) \leqslant \mathfrak{A}_{\wp}^{\mathfrak{D}-\xi}(M)$.

Corollary 4.6. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, Υ be a similarity relation, $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then*

(i) Every $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly exact in V is $\mathfrak{D}$-$\xi_\wp$-nearly exact.

(ii) Every $\mathfrak{D}$-$\xi_\wp$-nearly rough in V is $\mathfrak{D}$-$\xi_{\mathbb{S}_\wp}$-nearly rough.

Remark 4.3. *It should be noted that*

(i) the similarity relation in Theorems 4.2,4.3, Corollaries 4.3, 4.44.5, 4.6 is not dispensable as shown in Example 3.2 that $\tau_\wp$ and $\tau_{\mathbb{S}_\wp}$ are not comparable. Consequently, it is meant that we can not apply Theorems 4.2,4.3, Corollaries 4.3, 4.44.5, 4.6.

(ii) the boundary regions and accuracy by the prior manner in 2.2 [1, 2, 30], 2.5 [22, 26] and the present manner in 4.1 are calculated in Tables 4, 5 by using Example 3.3. The results underscore the advantages of the present manner compared to 2.2 [1, 2, 30] and the superiority of the old ones 2.5 [22, 26] compared to the current ones in the presence of the similarity relation.

(iii) Example 3.3 shows also that $\tau_{\mathbb{S}_\wp}, \tau_\wp$ are not comparable if $\wp \in \{< R >, < L >, < I >, < U >\}$. So, Theorems 4.2,4.3, Corollaries 4.3, 4.44.5, 4.6 apply only for $\wp \in \{R, L, I, U\}$.

Table 4: The boundary regions and accuracy by the prior manner in 2.2 [1, 2, 30] for $\wp = R$ and the present manner in 4.1 for $\xi = \{\alpha, P, S, \beta, \theta\beta\}, \wp = R$.

M	Definition 2.2 [1, 2, 30] for $\wp = R$		Definitions 4.1 for $\xi = \alpha, \wp = R$		Definitions 4.1 for $\xi = P, \wp = R$		Definitions 4.1 for $\xi = S, \wp = R$		Definitions 4.1 for $\xi = \beta, \wp = R$		Definitions 4.1 for $\xi = \theta\beta, \wp = R$	
	$\mathfrak{B}_R(M)$	$\mathfrak{A}_R(M)$	$\mathfrak{B}_{\mathbb{S}_R}^{\mathfrak{D}-\alpha}(M)$	$\mathfrak{A}_{\mathbb{S}_R}^{\mathfrak{D}-\alpha}(M)$	$\mathfrak{B}_{\mathbb{S}_R}^{\mathfrak{D}-P}(M)$	$\mathfrak{A}_{\mathbb{S}_R}^{\mathfrak{D}-P}(M)$	$\mathfrak{B}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M)$	$\mathfrak{A}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M)$	$\mathfrak{B}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)$	$\mathfrak{A}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)$	$\mathfrak{B}_{\mathbb{S}_R}^{\mathfrak{D}-\theta\beta}(M)$	$\mathfrak{A}_{\mathbb{S}_R}^{\mathfrak{D}-\theta\beta}(M)$
$\{\mathfrak{l}_1\}$	V	0	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\emptyset$	1
$\{\mathfrak{l}_2\}$	V	0	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\mathfrak{l}_3\}$	V	0	$\{\mathfrak{l}_3\}$	0	$\{\mathfrak{l}_3\}$	0	$\{\mathfrak{l}_3\}$	0	$\{\mathfrak{l}_3\}$	0	$\emptyset$	1
$\{\mathfrak{l}_4\}$	V	0	$\{\mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_3\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2\}$	V	0	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	V	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	0	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_4\}$	V	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_3\}$	V	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_4\}$	V	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	V	0	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}$	V	0	$\{\mathfrak{l}_3\}$	$\frac{2}{3}$	$\{\mathfrak{l}_3\}$	$\frac{2}{3}$	$\{\mathfrak{l}_3\}$	$\frac{2}{3}$	$\{\mathfrak{l}_3\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\}$	V	0	$\{\mathfrak{l}_3\}$	$\frac{3}{4}$	$\{\mathfrak{l}_3\}$	$\frac{3}{4}$	$\{\mathfrak{l}_3\}$	$\frac{3}{4}$	$\{\mathfrak{l}_3\}$	$\frac{3}{4}$	$\emptyset$	1
$\{a\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	V	0	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$	V	0	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1

Table 5: The boundary regions and accuracy by the prior manner in 2.5 [22, 26] and the present manner in 4.1 for $\xi = \{P, S, \beta\}, \wp = R$.

M	Definition 2.5 [22, 26] for $\xi = P, \wp = R$		Definitions 4.1 for $\xi = P, \wp = R$		Definition 2.5 [22, 26] for $\xi = S, \wp = R$		Definitions 4.1 for $\xi = S, \wp = R$		Definition 2.5 [22, 26] for $\xi = \beta, \wp = R$		Definitions 4.1 for $\xi = \beta, \wp = R$	
	$\mathfrak{B}_R^{\mathfrak{D}-P}(M)$	$\mathfrak{A}_R^{\mathfrak{D}-P}(M)$	$\mathfrak{B}_{\mathbb{S}_R}^{\mathfrak{D}-P}(M)$	$\mathfrak{A}_{\mathbb{S}_R}^{\mathfrak{D}-P}(M)$	$\mathfrak{B}_R^{\mathfrak{D}-S}(M)$	$\mathfrak{A}_R^{\mathfrak{D}-S}(M)$	$\mathfrak{B}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M)$	$\mathfrak{A}_{\mathbb{S}_R}^{\mathfrak{D}-S}(M)$	$\mathfrak{B}_R^{\mathfrak{D}-\beta}(M)$	$\mathfrak{A}_R^{\mathfrak{D}-\beta}(M)$	$\mathfrak{B}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)$	$\mathfrak{A}_{\mathbb{S}_R}^{\mathfrak{D}-\beta}(M)$
$\{\mathfrak{l}_1\}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\{\mathfrak{l}_1\}$	0	$\emptyset$	1	$\{\mathfrak{l}_1\}$	0
$\{\mathfrak{l}_2\}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$
$\{\mathfrak{l}_3\}$	$\emptyset$	1	$\{\mathfrak{l}_3\}$	0	$\{\mathfrak{l}_3\}$	0	$\{\mathfrak{l}_3\}$	0	$\emptyset$	1	$\{\mathfrak{l}_3\}$	0
$\{\mathfrak{l}_4\}$	$\emptyset$	1	$\{\mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_3\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\mathfrak{l}_3\}$	$\frac{1}{2}$
$\{\mathfrak{l}_1, \mathfrak{l}_2\}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$
$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\emptyset$	1	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	0	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	0	$\emptyset$	1	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	0
$\{\mathfrak{l}_1, \mathfrak{l}_4\}$	$\emptyset$	1	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\emptyset$	1	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$
$\{\mathfrak{l}_2, \mathfrak{l}_3\}$	$\emptyset$	1	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$	$\emptyset$	1	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{3}$
$\{\mathfrak{l}_2, \mathfrak{l}_4\}$	$\emptyset$	1	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\mathfrak{l}_1, \mathfrak{l}_3\}$	$\frac{1}{2}$
$\{\mathfrak{l}_3, \mathfrak{l}_4\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3\}$	$\emptyset$	1	$\{\mathfrak{l}_3\}$	$\frac{2}{3}$	$\{\mathfrak{l}_3\}$	$\frac{2}{3}$	$\{\mathfrak{l}_3\}$	$\frac{2}{3}$	$\emptyset$	1	$\{\mathfrak{l}_3\}$	$\frac{2}{3}$
$\{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4\}$	$\emptyset$	1	$\{\mathfrak{l}_3\}$	$\frac{3}{4}$	$\{\mathfrak{l}_3\}$	$\frac{3}{4}$	$\{\mathfrak{l}_3\}$	$\frac{3}{4}$	$\emptyset$	1	$\{\mathfrak{l}_3\}$	$\frac{3}{4}$
$\{a\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{2}{3}$
$\{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\mathfrak{l}_1\}$	$\frac{1}{2}$

The connections among the $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly lower (upper) approximations, $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly boundary regions and $\mathfrak{D}$-$\mathbb{S}_{\wp}$-nearly accuracy are introduced in the following results.

Proposition 4.2. *Let $(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. Then*

(i) $\underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-P}(M) \subseteq \underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\beta}(M) \subseteq \underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\theta\beta}(M)$.

(ii) $\underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\alpha}(M) \subseteq \underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-S}(M) \subseteq \underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\beta}(M) \subseteq \underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\theta\beta}(M)$.

(iii) $\overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\theta\beta}(M) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\beta}(M) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-P}(M)$.

(iv) $\overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\theta\beta}(M) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\beta}(M) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-S}(M) \subseteq \overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\alpha}(M)$.

Proof. Proposition 3.1 renders the proof clear.

Corollary 4.7. *Let $(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. Then*

(i) $\mathfrak{B}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\theta\beta}(M) \subseteq \mathfrak{B}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\beta}(M) \subseteq \mathfrak{B}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-P}(M)$.

(ii) $\mathfrak{B}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\theta\beta}(M) \subseteq \mathfrak{B}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\beta}(M) \subseteq \mathfrak{B}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-S}(M) \subseteq \mathfrak{B}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\alpha}(M)$.

(iii) $\mathfrak{A}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-P}(M) \leqslant \mathfrak{A}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\beta}(M) \leqslant \mathfrak{A}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\theta\beta}(M)$.

(iv) $\mathfrak{A}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\alpha}(M) \leqslant \mathfrak{A}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-S}(M) \leqslant \mathfrak{A}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\beta}(M) \leqslant \mathfrak{A}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\theta\beta}(M)$.

Corollary 4.8. *Let $(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. Then*

(i) M is $\mathbb{S}_{\wp}$-exact $\Rightarrow$ M is $\mathfrak{D}$-$\alpha_{\mathbb{S}_{\wp}}$-exact $\Rightarrow$ M $\mathfrak{D}$-$S_{\mathbb{S}_{\wp}}$-exact $\Rightarrow$ M $\mathfrak{D}$-$\beta_{\mathbb{S}_{\wp}}$-exact $\Rightarrow$ M $\mathfrak{D}$-$\theta\beta_{\mathbb{S}_{\wp}}$-exact.

(ii) M is $\mathfrak{D}$-$P_{\mathbb{S}_{\wp}}$-exact $\Rightarrow$ M $\mathfrak{D}$-$\beta_{\mathbb{S}_{\wp}}$-exact $\Rightarrow$ M $\mathfrak{D}$-$\theta\beta_{\mathbb{S}_{\wp}}$-exact.

Table 1 illustrates that, in general, the opposite of Corollaries 4.7, 4.8, as well as Proposition 4.2, does not hold.

Remark 4.4. *It is evident that various methods exist for approximating. Among these, the most effective approach involves using $\mathfrak{D}$-$\theta\beta\mathbb{S}\wp$ for constructing set approximations. This family reduces or eliminates boundary regions. $\mathfrak{D}$-$\theta\beta\mathbb{S}\wp$-accuracy shows to be more precise compared to other families.*

5. $\mathbb{S}_{\wp}$-rough, $\mathbb{S}_{\wp}$-nearly rough membership functions and generalization via ideals

Various types of rough membership functions are defined to characterize the approximation operators. Their core properties are studied and the relationships among them are given. Additionally, it is proved that they extended the traditional rough membership functions.

5.1. $\mathbb{S}_\wp$-rough membership functions

Definition 5.1. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $t \in V$ *and* $M \subseteq V$*.*

(i) if $t \in \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$*, then* t *is* $\mathbb{S}_\wp$*-nearly surely (*$\mathbb{S}_\wp$*-surely) belongs to* M*, denoted by* $t \underline{\in}_{\mathbb{S}_\wp} M$

(ii) if $t \in \overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$*, then* t *is* $\mathbb{S}_\wp$*-nearly possibly (*$\mathbb{S}_\wp$*-possibly) belongs to* M*, denoted by* $t \overline{\in}_{\mathbb{S}_\wp} M$

It is known as $\mathbb{S}_\wp$*-strong and* $\mathbb{S}_\wp$*-weak membership relations.*

Remark 5.1. *The* $\mathbb{S}_\wp$*-approximations 2.8 [43] are redefined for any* $M \subseteq V$ *by using* $\underline{\in}_{\mathbb{S}_\wp}$ *and* $\overline{\in}_{\mathbb{S}_\wp}$ *as follows:*

(i) $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) = \{t \in V : t \underline{\in}_{\mathbb{S}_\wp} M\}$.

(ii) $\overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) = \{t \in V : t \overline{\in}_{\mathbb{S}_\wp} M\}$.

Lemma 5.1. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $M \subseteq V$*. Then*

(i) if $t \underline{\in}_{\mathbb{S}_\wp} M$*, then* $t \in M$*.*

(ii) if $t \in M$*, then* $t \overline{\in}_{\mathbb{S}_\wp} M$*.*

Proof. Straightforward.

Remark 5.2. *In Example 3.3*

(i) if $M = \{\mathfrak{l}_3\}$*, then* $\mathfrak{l}_3 \in M$*, but* $\mathfrak{l}_3 \underline{\notin}_{\mathbb{S}_R} M$*.*

(ii) if $M = \{\mathfrak{l}_2\}$*, then* $\mathfrak{l}_1 \overline{\in}_{\mathbb{S}_R} A$*, but* $\mathfrak{l}_1 \notin M$*.*

Definition 5.2. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $M \subseteq V$ *and* $t \in V$*. The* $\mathbb{S}_\wp$*-rough membership functions of* M *are symbolized by* $\omega_M^{\mathbb{S}_\wp} : V \to [0, 1]$*, with*

$$\omega_M^{\mathbb{S}_\wp}(t) = \{ {}^{1}_{min(\chi_M^{\mathbb{S}_\wp}(t))} \quad {}^{if\ 1 \in \chi_M^{\mathbb{S}_\wp}(t).}_{otherwise.} \},$$
$$and\ \chi_M^{\mathbb{S}_\wp}(t) = \frac{|\cap \mathbb{S}_\wp(t) \cap M|}{|\cap \mathbb{S}_\wp(t)|}, \cap \mathbb{S}_\wp(t) \neq \emptyset.$$

Remark 5.3. *The* $\mathbb{S}\wp$*-rough membership functions serve to establish the* $\mathbb{S}\wp$*-lower (upper) approximations in the following manner:*

(i) $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) = \{t \in V : \omega_M^{\mathbb{S}_\wp}(t) = 1\}$.

(ii) $\overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) = \{t \in V : \omega_M^{\mathbb{S}_\wp}(t) > 0\}$.

(iii) $\mathfrak{B}_{\mathbb{S}_\wp}(M) = \{t \in V : 0 < \omega_M^{\mathbb{S}_\wp}(t) < 1\}$.

Proposition 5.1. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $M, N \subseteq V$*. Then,* *(i)* *if* $\omega_M^{\mathbb{S}_\wp}(t) = 1 \Leftrightarrow t \underline{\in}_{\mathbb{S}_\wp} M$.

(ii) *if* $\omega_M^{\mathbb{S}_\wp}(t) = 0 \Leftrightarrow t \in V - \overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$.

(iii) *if* $0 < \omega_M^{\mathbb{S}_\wp}(t) < 1 \Leftrightarrow t \in \mathfrak{B}_{\mathbb{S}_\wp}(M)$.

(iv) *if* $\omega_{M'}^{\mathbb{S}_\wp}(t) = 1 - \omega_M^{\mathbb{S}_\wp}(t), \forall\, t \in V$.

(v) *if* $\omega_{M\cup N}^{\mathbb{S}_\wp}(t) \geq max(\omega_M^{\mathbb{S}_\wp}(t), \omega_N^{\mathbb{S}_\wp}(t)), \forall\, t \in V$.

(vi) *if* $\omega_{M\cap N}^{\mathbb{S}_\wp}(t) \leq min(\omega_M^{\mathbb{S}_\wp}(t), \omega_N^{\mathbb{S}_\wp}(t)), \forall\, t \in V$.

Proof. We prove (1) and handle the others in a similar manner. $t \underline{\in}_{\mathbb{S}_\wp} M \Leftrightarrow t \in \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$. Since $\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$ is $\mathbb{S}_\wp$-open contained in M, thus $\frac{|\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)\cap A|}{|\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)|} = \frac{|\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)|}{|\underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)|} = 1$. Then, $1 \in \chi_M^{\mathbb{S}_\wp}(t)$ so $\omega_M^{\mathbb{S}_\wp}(t) = 1$.

5.2. $\mathbb{S}_\wp$-nearly rough membership functions

Definition 5.3. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS,* $t \in V$ *and* $M \subseteq V$.

(i) *if* $t \in \underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M)$, *then* t *is* $\mathbb{S}_\wp$*-nearly surely* ($\xi_{\mathbb{S}_\wp}$*-surely) belongs to* M, *denoted by* $t \,\underline{\in}^{\xi}_{\mathbb{S}_\wp} M$

(ii) *if* $t \in \overline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M)$, *then* t *is* $\mathbb{S}_\wp$*-nearly possibly* ($\xi_{\mathbb{S}_\wp}$*-possibly) belongs to* M, *denoted by* $t \,\overline{\in}^{\xi}_{\mathbb{S}_\wp} M$

It is known as $\mathbb{S}_\wp$*-nearly strong and* $\mathbb{S}_\wp$*-nearly weak membership relations.*

Remark 5.4. *The* $\mathbb{S}_\wp$*-nearly approximations are redefined for any* $M \subseteq V$ *by using* $\underline{\in}^{\xi}_{\mathbb{S}_\wp}$ *and* $\overline{\in}^{\xi}_{\mathbb{S}_\wp}$ *as follows:*

(i) $\underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M) = \{t \in V : t \,\underline{\in}^{\xi}_{\mathbb{S}_\wp} M\}$.

(ii) $\overline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M) = \{t \in V : t \,\overline{\in}^{\xi}_{\mathbb{S}_\wp} M\}$.

Lemma 5.2. *Let* $(V, \Upsilon, \Pi_\wp)$ *be a* $\wp$*-nbdS and* $M \subseteq V$*. Then*

(i) *if* $t \,\underline{\in}^{\xi}_{\mathbb{S}_\wp} M$, *then* $t \in M$.

(ii) *if* $t \in M$, *then* $t \,\overline{\in}^{\xi}_{\mathbb{S}_\wp} M$.

Proof. Straightforward.

Remark 5.5. *In Example 3.1*

(i) if $M = \{\mathfrak{l}_1\}$, then $\mathfrak{l}_1 \in M$, but $\mathfrak{l}_1 \underline{\notin}^{\beta}_{\mathbb{S}_R} M$.

(ii) if $M = \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$, then $\mathfrak{l}_1 \overline{\in}^{\beta}_{\mathbb{S}_R} M$, but $\mathfrak{l}_1 \notin M$.

Definition 5.4. *Let $(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS, $M \subseteq V$ and $t \in V$. The $\mathbb{S}_{\wp}$-rough nearly membership functions of M are defined by $\omega_M^{\xi_{\mathbb{S}_{\wp}}} : V \to [0,1]$, where*

$$\omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) = \{ {}^{1}_{min(\chi_M^{\xi_{\mathbb{S}_{\wp}}}(t))} \quad {}^{if\ 1 \in \chi_M^{\xi_{\mathbb{S}_{\wp}}}(t).}_{otherwise.} \}.$$

$$and\ \chi_M^{\xi_{\mathbb{S}_{\wp}}}(t) = \frac{|\xi_{\mathbb{S}_{\wp}}(t) \cap M|}{|\xi_{\mathbb{S}_{\wp}}(t)|}, t \in \xi_{\mathbb{S}_{\wp}}(t), \xi_{\mathbb{S}_{\wp}}(t) \in \xi_{\mathbb{S}_{\wp}} O(V).$$

Remark 5.6. *The $\mathbb{S}\wp$-rough nearly membership functions are employed to define the $\mathbb{S}\wp$-nearly lower (upper) approximations as outlined below:*

(i) $\underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_{\wp}}(M) = \{t \in V : \omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) = 1\}$.

(ii) $\overline{\mathfrak{N}}^{\xi}_{\mathbb{S}_{\wp}}(M) = \{t \in V : \omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) > 0\}$.

(iii) $\mathfrak{B}^{\xi}_{\mathbb{S}_{\wp}}(M) = \{t \in V : 0 < \omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) < 1\}$.

Lemma 5.3. *Let $(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS and $M \subseteq V$. Then,*

(i) $\omega_M^{\mathbb{S}_{\wp}}(t) = 1 \Rightarrow \omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) = 1, \forall\, t \in V$.

(ii) $\omega_M^{\mathbb{S}_{\wp}}(t) = 0 \Rightarrow \omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) = 0, \forall\, t \in V$.

Proposition 5.2. *Let $(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS and $M, N \subseteq V$. Then,*

(i) if $\omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) = 1 \Leftrightarrow t \in \underline{\in}^{\xi_{\mathbb{S}_{\wp}}}_{\mathbb{S}_{\wp}} M$.

(ii) if $\omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) = 0 \Leftrightarrow t \in V - \overline{\mathfrak{N}}^{\xi}_{\mathbb{S}_{\wp}}(M)$.

(iii) if $0 < \omega_M^{\xi_{\mathbb{S}_{\wp}}}(t) < 1 \Leftrightarrow x \in \mathfrak{B}^{\xi}_{\mathbb{S}_{\wp}}(M)$.

(iv) if $\omega_{M'}^{\xi_{\mathbb{S}_{\wp}}}(t) = 1 - \omega_M^{\xi_{\mathbb{S}_{\wp}}}(x), \forall\, t \in V$.

(v) if $\omega_{M \cup N}^{\xi_{\mathbb{S}_{\wp}}}(t) \geq max(\omega_M^{\xi_{\mathbb{S}_{\wp}}}(t), \omega_N^{\xi_{\mathbb{S}_{\wp}}}(t)), \forall\, t \in V$.

(vi) if $\omega_{M \cap N}^{\xi_{\mathbb{S}_{\wp}}}(t) \leq min(\omega_M^{\xi_{\mathbb{S}_{\wp}}}(t), \omega_N^{\xi_{\mathbb{S}_{\wp}}}(t)), \forall\, t \in V$.

Proof. It resembles Proposition 5.1.

5.3. $\mathbb{S}_\wp$-nearly rough membership functions via ideals

Definition 5.5. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $t \in V$.*

(i) if $t \in \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$, then t is $\mathbb{S}_\wp$-nearly surely with respect to $\mathfrak{D}$ ($\mathfrak{D}-\xi_{\mathbb{S}_\wp}$-surely) belongs to M, denoted by $t \,\underline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M$.

(ii) if $t \in \overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$, then t is $\mathbb{S}_\wp$-nearly possibly with respect to $\mathfrak{D}$ ($\mathfrak{D}-\xi_{\mathbb{S}_\wp}$-possibly) belongs to M, denoted by $t \,\overline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M$.

It is known as $\mathbb{S}_\wp$-nearly strong and $\mathbb{S}_\wp$-nearly weak membership relations with respect to $\mathfrak{D}$ respectively.

Remark 5.7. *Based on Definition 5.5 the $\mathbb{S}_\wp$-nearly approximations via ideal for any $M \subseteq V$ can be expressed as:*

(i) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) = \{t \in V : t \,\underline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M\}$.

(ii) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) = \{t \in V : t \,\overline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M\}$.

Lemma 5.4. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. Then*

(i) if $t \,\underline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M$, then $t \in M$.

(ii) if $t \in M$, then $t \,\overline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M$.

Proof. Straightforward.

Remark 5.8. *In Example 3.1, if $\mathfrak{D} = \{\emptyset, \mathfrak{l}_2\}$, then*

(i) $\mathfrak{l}_1 \in \{\mathfrak{l}_1\}$, but $\mathfrak{l}_1 \underline{\notin}^{\mathfrak{D}-\beta}_{\mathbb{S}_R} M$.

(ii) $\mathfrak{l}_1 \,\overline{\in}^{\mathfrak{D}-\beta}_{R} M$, but $\mathfrak{l}_1 \notin \{\mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4\}$.

Proposition 5.3. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. Then*

(i) if $t \,\underline{\in}_{\mathbb{S}_\wp} M \Rightarrow t \,\underline{\in}^{\xi}_{\mathbb{S}_\wp} M \Rightarrow t \,\underline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M$.

(ii) if $t \,\overline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M \Rightarrow t \,\overline{\in}^{\xi}_{\mathbb{S}_\wp} M \Rightarrow t \,\overline{\in}_{\mathbb{S}_\wp} M$.

Proof. We demonstrate (1) and address the others similarly. $t \,\underline{\in}_{\mathbb{S}_\wp} M \Rightarrow t \in \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \Rightarrow t \in \underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M)$ by Theorem 2.3 [43]. Hence, $t \,\underline{\in}^{\xi}_{\mathbb{S}_\wp} M$, so, $t \in \underline{\mathfrak{N}}^{\xi}_{\mathbb{S}_\wp}(M) \Rightarrow t \in \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M)$ by Theorem 4.1. Therefore, $t \,\underline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M$.

Remark 5.9. *In Example 3.1*

(i) if $M = \{\mathfrak{l}_1\}$, then $\mathfrak{l}_1 \,\underline{\in}^{\mathfrak{D}-\beta}_{\mathbb{S}_R} M$, but $\mathfrak{l}_1 \,\underline{\notin}^{\beta}_{\mathbb{S}_R} M$ and $\mathfrak{l}_1 \,\underline{\notin}_{R} M$.

(ii) if $M = \{\mathfrak{l}_2\}$, then $\mathfrak{l}_1 \,\overline{\in}_{R} M$, but $\mathfrak{l}_1 \,\overline{\notin}^{\beta}_{\mathbb{S}_R} M$ and $\mathfrak{l}_1 \,\overline{\notin}^{\mathfrak{D}-\beta}_{\mathbb{S}_R} M$.

The present membership relations is more accurate than the previous ones in [1, 22, 26] as it illustrated in the following consequences.

Proposition 5.4. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, Υ be a similarity relation, $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then*

(i) if $t \,\underline{\in}_\wp M \Rightarrow t \,\underline{\in}_{\mathbb{S}_\wp} M \Rightarrow t \,\underline{\in}^{\xi}_{\mathbb{S}_\wp} M \Rightarrow t \,\underline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M$.

(ii) if $t \,\overline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M \Rightarrow t \,\overline{\in}^{\xi}_{\mathbb{S}_\wp} M \Rightarrow t \,\overline{\in}_{\mathbb{S}_\wp} M \Rightarrow t \,\overline{\in}_\wp M$.

Proof.

(i) We only prove $t \,\underline{\in}_\wp M \Rightarrow t \,\underline{\in}_{\mathbb{S}_\wp} M$ and the other cases directly by Proposition 5.3. Let $t \,\underline{\in}_\wp M \Rightarrow t \in \underline{\mathfrak{N}}_\wp(M) \Rightarrow t \in \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M)$ by Theorem 2.3. Therefore, $t \,\underline{\in}_{\mathbb{S}_\wp} M$.

(ii) Similar to (1).

Proposition 5.5. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, Υ be a similarity relation, $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then*

(i) if $t \,\underline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M \Rightarrow t \,\underline{\in}^{\mathfrak{D}-\xi}_{\wp} M$.

(ii) if $t \,\overline{\in}^{\mathfrak{D}-\xi}_{\wp} M \Rightarrow t \,\overline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M$.

Proof.

(i) Let $t \,\underline{\in}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp} M \Rightarrow t \in \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) \Rightarrow t \in \underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\wp}(M)$ by Theorem 4.3. Therefore, $t \,\underline{\in}^{\mathfrak{D}-\xi}_{\wp} M$.

(ii) Similar to (1).

Definition 5.6. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on $V, M \subseteq V$ and $t \in V$. The $\mathfrak{D}$-$\mathbb{S}_\wp$-nearly rough membership functions of a $\wp$-nbdS on V for a M are symbolized by $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}} : V \to [0, 1]$, where*

$$\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = \{ {}^{1}_{min(\chi_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t))} \quad {}^{if\ 1 \in \chi_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t).}_{otherwise.} \}.$$

$$\text{and } \chi_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = \frac{|\mathfrak{D}-\xi_{\mathbb{S}_\wp}(t) \cap M|}{|\mathfrak{D}-\xi_{\mathbb{S}_\wp}(t)|}, t \in \mathfrak{D} - \xi_{\mathbb{S}_\wp}(t),\ \mathfrak{D} - \xi_{\mathbb{S}_\wp}(t) \in \mathfrak{D}\text{-}\xi_{\mathbb{S}_\wp}O(V).$$

Remark 5.10. *The $\mathfrak{D}$-$\mathbb{S}\wp$-nearly rough membership functions are utilized to present the $\mathfrak{D}$-$\mathbb{S}\wp$-nearly lower (upper) approximations as:*

(i) $\underline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) = \{t \in V : \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 1\}$.

(ii) $\overline{\mathfrak{N}}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) = \{t \in V : \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) > 0\}$.

(iii) $\mathfrak{B}^{\mathfrak{D}-\xi}_{\mathbb{S}_\wp}(M) = \{t \in V : 0 < \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) < 1\}$.

Proposition 5.6. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M, N \subseteq V$. Then*

(i) *if* $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 1 \Leftrightarrow t \in \underline{\in}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}} M$.

(ii) *if* $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(x) = 0 \Leftrightarrow t \in V - \overline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M)$.

(iii) *if* $0 < \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) < 1 \Leftrightarrow t \in \mathfrak{B}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M)$.

(iv) *if* $\omega_{M'}^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 1 - \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t), \forall\, t \in V$.

(v) *if* $\omega_{M\cup N}^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) \geq max(\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t), \omega_N^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t)), \forall\, t \in V$.

(vi) *if* $\omega_{M\cap N}^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) \leq min(\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t), \omega_N^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t)), \forall\, t \in V$.

Proof. It is similar to Proposition 5.1.

Lemma 5.5. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V and $M \subseteq V$. Then*

(i) $\omega_M^{\mathbb{S}_\wp}(t) = 1 \Rightarrow \omega_M^{\xi_{\mathbb{S}_\wp}}(t) = 1 \Rightarrow \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 1, \forall\, t \in V$.

(ii) $\omega_M^{\mathbb{S}_\wp}(t) = 0 \Rightarrow \omega_M^{\xi_{\mathbb{S}_\wp}}(t) = 0 \Rightarrow \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 0, \forall\, t \in V$.

Proof.

(i) $\omega_M^{\mathbb{S}_\wp}(t) = 1 \Rightarrow t \in \underline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \Rightarrow t \in \underline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\xi}(M)$ by Theorem 2.4 [43]. Therefore, $\omega_M^{\xi_{\mathbb{S}_\wp}}(t) = 1$. Let $\omega_M^{\xi_{\mathbb{S}_\wp}}(x) = 1$, then $t \in \underline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\xi}(M) \Rightarrow t \in \underline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M)$ by Theorem 4.1. Hence, $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 1, \forall\, t \in V$.

(ii) $\omega_M^{\mathbb{S}_\wp}(t) = 0 \Rightarrow t \in V - \overline{\mathfrak{N}}_{\mathbb{S}_\wp}(M) \Rightarrow t \in V - \overline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\xi}(M)$ by Theorem 2.4 [43]. Hence, $\omega_M^{\xi_{\mathbb{S}_\wp}}(t) = 0$. Let $\omega_M^{\xi_{\mathbb{S}_\wp}}(t) = 0$, then $t \in V - \overline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\xi}(M) \Rightarrow t \in V - \overline{\mathfrak{N}}_{\mathbb{S}_\wp}^{\mathfrak{D}-\xi}(M)$ by Theorem 4.1. Hence, $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 0, \forall\, t \in V$.

Remark 5.11. *The opposite of Lemma 5.5 is incorrect, as shown in Example 3.1.*

The different types of membership functions defined in this manuscript are more precise and general than the last ones in [1] as it is presented in the following consequences.

Lemma 5.6. *Let $(V, \Upsilon, \Pi_\wp)$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, Υ be a similarity relation, $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then*

(i) $\omega_M^{\wp}(t) = 1 \Rightarrow \omega_M^{\mathbb{S}_\wp}(t) = 1 \Rightarrow \omega_M^{\xi_{\mathbb{S}_\wp}}(t) = 1 \Rightarrow \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 1, \forall\, t \in V$.

(ii) $\omega_M^{\wp}(t) = 0 \Rightarrow \omega_M^{\mathbb{S}_\wp}(t) = 0 \Rightarrow \omega_M^{\xi_{\mathbb{S}_\wp}}(t) = 0 \Rightarrow \omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_\wp}}(t) = 0, \forall\, t \in V$.

Proof.

(i) $\omega_M^{\wp}(t) = 1 \Rightarrow t \in \underline{\mathfrak{N}}_{\wp}(M) \Rightarrow t \in \underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}(M)$ by Theorem 2.3. Therefore, $\omega_M^{\mathbb{S}_{\wp}}(t) = 1$.

(ii) $\omega_M^{\wp}(t) = 0 \Rightarrow t \in V - \overline{\mathfrak{N}}_{\wp}(M) \Rightarrow t \in V - \overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}(M)$ by Theorem 2.3. Hence, $\omega_M^{\mathbb{S}_{\wp}}(t) = 0$.

The other cases directly by Lemma 5.5.

Lemma 5.7. *Let $(V, \Upsilon, \Pi_{\wp})$ be a $\wp$-nbdS, $\mathfrak{D}$ be an ideal on V, Υ be a similarity relation, $\wp \in \{R, L, I, U\}$ and $M \subseteq V$. Then*

(i) $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_{\wp}}}(t) = 1 \Rightarrow \omega_M^{\mathfrak{D}-\xi_{\wp}}(t) = 1, \forall\, t \in V.$

(ii) $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_{\wp}}}(t) = 0 \Rightarrow \omega_M^{\mathfrak{D}-\xi_{\wp}}(t) = 0, \forall\, t \in V.$

Proof.

(i) $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_{\wp}}}(t) = 1 \Rightarrow t \in \underline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\xi}(M) \Rightarrow t \in \underline{\mathfrak{N}}_{\wp}^{\mathfrak{D}-\xi}(M)$. by Theorem 4.3. Therefore, $\omega_M^{\mathfrak{D}-\xi_{\wp}}(t) = 1$.

(ii) $\omega_M^{\mathfrak{D}-\xi_{\mathbb{S}_{\wp}}}(t) = 0 \Rightarrow t \in V - \overline{\mathfrak{N}}_{\mathbb{S}_{\wp}}^{\mathfrak{D}-\xi}(M) \Rightarrow t \in V - \overline{\mathfrak{N}}_{\wp}^{\mathfrak{D}-\xi}(M)$ by Theorem 4.3. Hence, $\omega_M^{\mathfrak{D}-\xi_{\wp}}(t) = 0$.

Example 3.2 confirms that the similarity relation in Propositions 5.4, 5.5 and Lemma 5.6 is not dispensable. Additionally, Example 3.3 illustrates that, in general, the opposite of Propositions 5.4, 5.5 and Lemma 5.6 dose not hold and $\tau_{\mathbb{S}_{\wp}}, \tau_{\wp}$ are incomparable if $\wp \in \{< R >, < L >, < I >, < U >\}$. So, Propositions 5.4, 5.5 and Lemma 5.6 apply only for $\wp \in \{R, L, I, U\}$.

6. Medical example: Chikungunya disease

This section primarily seeks to evaluate the suggested technique by comparing it with the prior approach in [1, 2, 30, 43], using the Chikungunya disease information system. Common symptoms include joint pain, fever, while joint swelling, headache, and rash may also occur but vary among individuals. Until now, there is no specific treatment or vaccine for chikungunya. However, some symptoms relief can be obtained through fluids, rest, and over-the-counter pain relievers. Although most patients recover within a week, the riskiness of this sickness is that joint pain can be intense and long-lasting, potentially persisting for months. Chikungunya poses a significant medical challenge in many regions

In the following analysis assist the decision-makers in making a precise decision for the specific subset of patients which are considered. Table 6 lists patients $V = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4, \mathfrak{l}_5, \mathfrak{l}_6, \mathfrak{l}_7\}$ in rows, and the details of symptoms (attributes)in columns: T_1 represents a fever, T_2 represents joint pains, T_3 represents joint swelling, and T_4 represents a headache, T_5 is rash, where $\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3, \mathsf{T}_4$ represented by two values: "$\top^{\bullet}$" and "$\top^{\circ}$" which respectively show whether each symptom is present or absent for the patients. While, T_5 represented by three values: $(\mathcal{F}_1)$, $(\mathcal{F}_2)$ and $(\mathcal{F}_3)$. The disease decision D is displayed in the seventh column by two possible values "yes" or "no".

V	T_1	T_2	T_3	T_4	T_5	Chikungunya disease
$\mathfrak{l}_1$	$\top^\bullet$	$\top^\circ$	$\top^\circ$	$\top^\circ$	$(\mathcal{F}_1)$	no
$\mathfrak{l}_2$	$\top^\bullet$	$\top^\bullet$	$\top^\circ$	$\top^\circ$	$(\mathcal{F}_3)$	yes
$\mathfrak{l}_3$	$\top^\circ$	$\top^\circ$	$\top^\bullet$	$\top^\bullet$	$(\mathcal{F}_1)$	no
$\mathfrak{l}_4$	$\top^\bullet$	$\top^\bullet$	$\top^\bullet$	$\top^\circ$	$(\mathcal{F}_3)$	yes
$\mathfrak{l}_5$	$\top^\circ$	$\top^\circ$	$\top^\bullet$	$\top^\bullet$	$(\mathcal{F}_1)$	no
$\mathfrak{l}_6$	$\top^\circ$	$\top^\bullet$	$\top^\circ$	$\top^\bullet$	$(\mathcal{F}_2)$	no
$\mathfrak{l}_7$	$\top^\bullet$	$\top^\circ$	$\top^\bullet$	$\top^\bullet$	$(\mathcal{F}_2)$	yes

Table 6: information system of Chikungunya disease

$\varpi(\mathfrak{l}_i, \mathfrak{l}_j)$ represents the similarity between two patients $\mathfrak{l}_i, \mathfrak{l}_j$. It is computed in Table 7 by

$$\varpi(\mathfrak{l}_i, \mathfrak{l}_j) = \frac{\sum_{k=1}^{n}(\mathsf{T}_k(\mathfrak{l}_i) = \mathsf{T}_k(\mathfrak{l}_j))}{n} \tag{1}$$

where, n the number of symptoms.

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{l}_3$	$\mathfrak{l}_4$	$\mathfrak{l}_5$	$\mathfrak{l}_6$	$\mathfrak{l}_7$
$\mathfrak{l}_1$	1	0.6	0.4	0.4	0.4	0.2	0.4
$\mathfrak{l}_2$	0.6	1	0	0.8	0	0.4	0.2
$\mathfrak{l}_3$	0.4	0	1	0.2	1	0.4	0.6
$\mathfrak{l}_4$	0.4	0.8	0.2	1	0.2	0.2	0.6
$\mathfrak{l}_5$	0.4	0	1	0.2	1	0.4	0.6
$\mathfrak{l}_6$	0.2	0.4	0.4	0.2	0.4	1	0.4
$\mathfrak{l}_7$	0.4	0.2	0.6	0.4	0.6	0.4	1

Table 7: Similarities of the patients' symptoms

The relation between patients are based on their shared symptoms and represented by $\mathfrak{l}_i \Upsilon \mathfrak{l}_j \iff \varpi(\mathfrak{l}_i, \mathfrak{l}_j) \geq 0.6$. Relation is introduced by the system's experts, therefore it may be changed under their considerations. Hence, $\Upsilon = \{(\mathfrak{l}_1, \mathfrak{l}_1), (\mathfrak{l}_2, \mathfrak{l}_2), (\mathfrak{l}_3, \mathfrak{l}_3), (\mathfrak{l}_4, \mathfrak{l}_4), (\mathfrak{l}_5, \mathfrak{l}_5), (\mathfrak{l}_6, \mathfrak{l}_6), (\mathfrak{l}_7, \mathfrak{l}_7), (\mathfrak{l}_1, \mathfrak{l}_2), (\mathfrak{l}_2, \mathfrak{l}_1), (\mathfrak{l}_2, \mathfrak{l}_4), (\mathfrak{l}_3, \mathfrak{l}_7), (\mathfrak{l}_4, \mathfrak{l}_2), (\mathfrak{l}_4, \mathfrak{l}_7), (\mathfrak{l}_5, \mathfrak{l}_7), (\mathfrak{l}_7, \mathfrak{l}_3), (\mathfrak{l}_7, \mathfrak{l}_5)\}$ and let $\mathfrak{D} = \{\emptyset, \{\mathfrak{l}_7\}\}$.

The uninfected patients with Chikungunya are represented by the set $M = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5, \mathfrak{l}_6\}$ while the infected $N = \{\mathfrak{l}_2, \mathfrak{l}_4, \mathfrak{l}_7\}$. Their approximation, boundary regions and accuracy by the prior manner in [1, 43] and the present manner are calculated as follows.

(i) The uninfected patients $M = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5, \mathfrak{l}_6\}$

(i) The prior manner in 2.2 [1, 2, 30]:

- $\underline{\mathfrak{N}}_R(M) = \{\mathfrak{l}_6\}$;
- $\overline{\mathfrak{N}}_R(M) = V$;
- $\mathfrak{B}_R(M) = V \setminus \{\mathfrak{l}_6\}$;
- $\mathfrak{A}_R(M) = \frac{1}{7}$.

(ii) Yildirim's manner in Definition 2.8 [43]:

- $\underline{\mathfrak{N}}_{\mathbb{S}_R}(M) = \{\mathfrak{l}_6\}$;
- $\overline{\mathfrak{N}}_{\mathbb{S}_R}(M) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5, \mathfrak{l}_6\}$;
- $\mathfrak{B}_{\mathbb{S}_R}(M) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5\}$;
- $\mathfrak{A}_{\mathbb{S}_R}(M) = \frac{1}{4}$.

(iii) Yildirim's manner in Definition 2.11 [43]:

- $\underline{\mathfrak{N}}^{\beta}_{\mathbb{S}_R}(M) = \{\mathfrak{l}_6\}$;
- $\overline{\mathfrak{N}}^{\beta}_{\mathbb{S}_R}(M) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5, \mathfrak{l}_6\}$;
- $\mathfrak{B}^{\beta}_{\mathbb{S}_R}(M) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5\}$;
- $\mathfrak{A}^{\beta}_{\mathbb{S}_R}(M) = \frac{1}{4}$.

(iv) The present manner in Definition 4.1

- $\underline{\mathfrak{N}}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(M) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5, \mathfrak{l}_6\}$;
- $\overline{\mathfrak{N}}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(M) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5, \mathfrak{l}_6\}$;
- $\mathfrak{B}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(M) = \emptyset$;
- $\mathfrak{A}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(M) = 1$.

(ii) The infection patients $N = \{\mathfrak{l}_2, \mathfrak{l}_4, \mathfrak{l}_7\}$

(i) The prior manner 2.2 [1, 2, 30]:

- $\underline{\mathfrak{N}}_R(M) = \emptyset$;
- $\overline{\mathfrak{N}}_R(M) = V$;
- $\mathfrak{B}_R(M) = V$;
- $\mathfrak{A}_R(M) = 0$.

(ii) Yildirim's manner in Definition 2.8 [43]:

- $\underline{\mathfrak{N}}_{\mathbb{S}_R}(N) = \{\mathfrak{l}_2, \mathfrak{l}_4, \mathfrak{l}_7\}$;
- $\overline{\mathfrak{N}}_{\mathbb{S}_R}(N) = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4, \mathfrak{l}_5, \mathfrak{l}_7\}$;
- $\mathfrak{B}_{\mathbb{S}_R}(N) = \{\mathfrak{l}_1, \mathfrak{l}_3, \mathfrak{l}_5\}$;
- $\mathfrak{A}_{\mathbb{S}_R}(N) = \frac{1}{2}$.

(iii) Yildirim's manner in Definition 2.11 [43]:

- $\underline{\mathfrak{N}}^{\beta}_{\mathbb{S}_R}(N) = \{\mathfrak{l}_2, \mathfrak{l}_4, \mathfrak{l}_7\}$;
- $\overline{\mathfrak{N}}^{\beta}_{\mathbb{S}_R}(N) = \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4, \mathfrak{l}_7\}$;

- $\mathfrak{B}^{\beta}_{\mathbb{S}_R}(N) = \{\mathfrak{l}_1\}$;
- $\mathfrak{A}^{\beta}_{\mathbb{S}_R}(N) = \frac{3}{4}$.

(iv) The present manner in Definition 4.1

- $\underline{\mathfrak{N}}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(N) = \{\mathfrak{l}_2, \mathfrak{l}_4, \mathfrak{l}_7\}$;
- $\overline{\mathfrak{N}}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(N) = \{\mathfrak{l}_2, \mathfrak{l}_4, \mathfrak{l}_7\}$;
- $\mathfrak{B}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(N) = \emptyset$;
- $\mathfrak{A}^{\mathfrak{D}-\beta}_{\mathbb{S}_R}(N) = 1$.

Based on this calculations, the boundary regions for the uninfected and infected sets according to the manners in [1, 2, 30, 43] are $V, \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3, \mathfrak{l}_4, \mathfrak{l}_5, \mathfrak{l}_7\}, \{\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_4, \mathfrak{l}_7\}$, respectively. This indicates that, in this case, it is difficult to determine whether individuals are infected or not. Therefore, the vagueness is increased and accordingly the accuracy of made-decision loses. The boundary relied on the present manner is $\emptyset$. It is indicated to a successful reduction in vagueness for the two sets which achieved an enhanced accuracy.

7. Conclusions

The primary goal of this theory is to minimize the boundary by reducing upper and increasing lower, thereby maximizing the accuracy measure. Rough set theory is a vast domain with numerous innovations with various branches. One of these branches is the derivation of rough sets from topology, highlighting a strong homogeneity between rough set theory and topology. Topological concepts are widely recognized as essential for understanding rough set theory, with ideals being particularly important. Ideals have significantly contributed to the generalization of rough sets. Specifically, ideals had proven effective in enhancing lower approximations and reducing upper approximations, thereby narrowing the boundary region and improving accuracy. This process effectively addressed vagueness which is a crucial objective in rough set theory.

In the present results, new topological concepts utilizing ideals were explored. Moreover, the characteristics of the proposed concepts were analyzed, and their features were discussed. Relationships among different types of these notions were conducted. After that, new operators were presented relying on ideals. Ideal increased the data which derived from the information systems by using rough set. Comparisons between the current and previous versions were provided, demonstrating that the current approach was both more accurate and more general. As, the current approach helped in reducing vagueness and, as a result, improved accuracy. Additionally, three types of rough membership functions were introduced. Relationships among them were also highlighted. Furthermore, an example from the medical field was given to prove the utility of the present concepts in a practical context. The proposed manners through this medical example had proven to be effective and robust in reducing the boundary and enhancing the accuracy.

A promising direction for future work will focus on investigating new approximations using distinct and innovative neighborhoods via two ideals and expanding the current rough set paradigms to rough multiset with multiset ideals.

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