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Conformable Sumudu Transform Based Adomian Decomposition Method for Linear and Nonlinear Fractional-Order Schrödinger Equations

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Abstract. Fractional-order Schrödinger differential equations extend the classical Schrödinger equation by incorporating fractional calculus to describe more complex physical phenomena. In the literature, the Schrödinger equation is mostly solved using fractional derivatives expressed through the Caputo derivative. However, there is limited research on exact and approximate solutions involving conformable fractional derivatives. This study aims to fill this gap by employing a hybrid approach that combines the Sumudu transform with the decomposition technique to solve the Schrödinger equation with conformable fractional derivatives, considering zero and nonzero trapping potentials. The efficiency of this approach is evaluated through the analysis of relative and absolute errors, confirming its accuracy. Moreover, the obtained results are compared with other techniques, including the homotopy analysis method (HAM) and the residual power series method (RPSM). The comparison demonstrates strong consistency with these methods, suggesting that our approach is a viable alternative to Caputo derivative-based methods for solving time-fractional Schrödinger equations. Furthermore, we can conclude that the conformable fractional derivative serves as a suitable substitute for the Caputo derivative in modeling Schrödinger equations.

2020 Mathematics Subject Classifications: 74S40, 65D15

Key Words and Phrases: Fractional calculus, approximate solutions, exact solutions, conformable fractional derivatives

1. Introduction

Fractional derivatives form the foundation of fractional calculus (FC), providing an effective means to describe and analyze systems with non-local or memory-driven behaviors. In contrast to integer-order derivatives, which focus solely on the instantaneous rate

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of change at a single point, fractional derivatives account for the function's behavior over an interval, thereby incorporating memory into the system's dynamics. There are several forms of fractional-order derivatives, each defined and characterized in its own way. Some of the commonly used types include the Riemann–Liouville, Caputo, Grünwald–Letnikov, Caputo-Fabrizio, and conformable fractional operators [1, 4, 16, 19, 32, 34]. The choice of which definition to use depends on the specific problem at hand and the desired mathematical properties of the derivative.

Recently, a large number of scholars have been actively investigating different kinds of fractional-order derivatives. In the context of Caputo-Fabrizio, for instance, the authors [30, 31] established highly significant results. Conformable fractional operators are a novel method for solving fractional-order differential equations (FODEs) that was presented by the authors [17]. Zhang *et al.* [33] developed solutions to differential equations (DEs) within the framework of Caputo fractional derivatives (Cap-FD). Using the framework of the Caputo derivative, Zhang and Xiong [35] showed that the periodic solution of FODEs with semilinear impulses has unique solutions and global exponential stability. Atangana-Baleanu derivatives are used by Syam and Al-Refai [24] to obtain numerous valuable results about solutions to FODEs. Using Grünwald-Letnikov fractional derivatives, Li and Wang [27] discovered solutions to fractional Rössler chaotic systems. For more study, refer to [6, 9, 20, 21].

Conformable fractional derivatives (Con-FrD) provide a useful and intuitive way to incorporate memory effects into differential equations (DEs). Con-FrD retains many properties of classical derivatives, making it easier to work with while offering the flexibility of FC. For this reason, Con-FrD is very useful for simulating real-world phenomena in physics, engineering, control systems, biology, and other fields.

The Con-FrD of a function $\delta(\varphi)$ of order μ is defined as [12]:

$$\mathfrak{T}_{\varphi}^{\mu}\delta(\varphi) = \lim_{\epsilon \to 0} \frac{\delta^{\lceil \mu \rceil - 1}(\varphi + \epsilon \varphi^{\lceil \mu \rceil - \mu}) - \delta^{\lceil \mu \rceil - 1}(\varphi)}{\epsilon}, \tag{1}$$

where the Con-FrD with regard to time is denoted by $\mathfrak{T}_{\varphi}^{\mu}$, and the smallest integer that is equal to or greater than μ is $\lceil \mu \rceil$, $\varkappa \in \mathscr{N}$, and $\varkappa - 1 < \mu \leq \varkappa,\ \varphi > 0$.

In the specific case where $0 < \mu \leq 1$, we obtain the following [18]:

$$\mathfrak{T}_{\varphi}^{\mu}\delta(\varphi) = \lim_{\epsilon \to 0} \frac{\delta(\varphi + \epsilon \varphi^{1-\mu}) - \delta(\varphi)}{\epsilon},\ \varphi > 0. \tag{2}$$

The conformable fractional integral is defined as follows:

$$\mathfrak{I}_{\mu}^{\tilde{\alpha}}(\delta)(\varphi) = \int_{\tilde{\alpha}}^{\varphi} \frac{\delta(\wp)}{\left(\wp - \tilde{\alpha}\right)^{1-\mu}} d\wp,\ \mu \in (0,\ 1]. \tag{3}$$

Con-FrD provides an alternative framework for fractional differentiation that simplifies the computation of fractional derivatives and maintains some key properties of classical derivatives. The Con-FrD offers several advantages compared to other types of fractional derivatives. Here are some of the key benefits [5, 15, 26, 28]:

i. Simplicity and Intuition: Con-FrD are designed to be a straightforward generalization of the classical derivative, making them easier to understand and apply. They retain many of the properties of integer-order derivatives, which can simplify the transition from classical to FC.

ii. Consistency with Classical Calculus: Unlike some other fractional derivatives, the conformable derivative maintains a closer relationship to classical calculus. It satisfies properties like the product rule, chain rule, and power rule in a manner that is more consistent with classical derivatives, making it more intuitive for those familiar with traditional calculus.

iii. Clear Physical Interpretation: Conformable derivatives offer a clearer physical interpretation compared to some other fractional derivatives, as they are often more closely related to the original physical quantities and processes. This makes them useful in modeling real-world phenomena where a straightforward interpretation is desired.

iv. Ease of Application: Due to their simpler form and consistency with classical calculus rules, Con-FrD are often easier to apply in various mathematical and engineering problems. They can be used in solving DEs, modeling systems, and analyzing dynamic behaviors without requiring complex modifications to existing methods.

v. Flexibility in Modeling: Conformable derivatives provide a flexible framework for modeling systems that exhibit non-integer order dynamics, such as memory effects or anomalous diffusion, without the complexity that comes with some other fractional derivatives. This makes them a practical choice for applications in physics, engineering, biology, and finance.

vi. Compatibility with Numerical Methods: The simpler and more intuitive nature of conformable derivatives often leads to easier implementation in numerical methods. This can be beneficial for computational modeling and simulation, where the goal is to approximate solutions to FODEs.

The FODEs are an advanced mathematical tool that extends the concept of traditional DEs to non-integer orders of differentiation and integration. This generalization provides a more flexible and accurate framework for modeling complex systems that exhibit memory, hereditary properties, or anomalous diffusion, which are not adequately captured by traditional integer-order models. The power of FODEs lies in their ability to model processes where the rate of change is not merely dependent on the current state but also influenced by the history of the system. This is particularly relevant in fields like physics, engineering, biology, and finance, where systems often have memory effects, such as viscoelastic materials, anomalous transport phenomena, and financial markets with long-term dependencies.

In the field of FODEs, several well-known models have been developed to describe complex systems with memory and hereditary properties. Among these models, the fractional Schrödinger differential equation (FSDEs) stands out as particularly important.

This equation extends the classical Schrödinger differential equation (SDE) by incorporating FC, allowing for a more accurate description of quantum systems with anomalous diffusion and other non-standard behaviors. Typically, the fractional derivative is introduced into either the time or spatial components of the SDE, leading to modifications in the wave function's behavior, with potential applications in quantum mechanics and other fields such as statistical mechanics and condensed matter physics.

FSDEs have diverse applications in various fields of science and engineering, primarily due to their ability to model systems with memory and non-local interactions. Some of the key applications include [8, 23]:

i. Quantum Mechanics: FSDEs are used to model quantum systems where the standard SDEs may not be sufficient, such as systems with fractal geometries, complex potentials, or in scenarios involving anomalous diffusion. They help in understanding the quantum behavior in disordered or complex materials.

ii. Condensed Matter Physics: In systems with fractal or irregular structures, like certain types of semiconductors, polymers, or amorphous materials, FSDEs can provide more accurate descriptions of the electronic properties, enabling better predictions of material behavior.

iii. Quantum Optics: FSDEs are applied in the study of light-matter interactions, particularly in media with non-standard refractive indices or in systems exhibiting anomalous dispersion. This can lead to new insights into photon transport and light localization.

iv. Statistical Mechanics: In statistical mechanics, FSDEs describe systems with long-range interactions or memory effects. This is particularly useful in modeling complex systems like glasses, spin systems, or in understanding anomalous transport phenomena.

v. Signal Processing and Image Analysis: FSDEs have been employed in signal processing, particularly in the analysis of signals with fractal characteristics or in the context of time-series analysis. Similarly, they are used in image processing, especially for edge detection or texture analysis in images with irregular patterns.

vi. Biophysics: In biological systems where anomalous diffusion or memory effects are present, such as in the transport of molecules within cellular environments or in modeling neuron dynamics, FSDEs can provide more accurate models.

vii. Financial Mathematics: Fractional order models, including those derived from the SDEs, are used in financial mathematics to describe markets with memory or to model the dynamics of prices and options in more complex financial environments.

The solutions to FSDEs are crucial for understanding, modeling, and predicting the behavior of complex systems that exhibit memory effects, non-local interactions, anomalous diffusion, and other phenomena. In the literature, the SDE has been solved using fractional derivatives represented in the form of the Cap-FD. For example, FSDEs have been solved using various methods [7, 11, 13, 22, 25, 29, 36]. However, the algorithms of each of these methods have been applied in the sense of Caputo derivatives. No research work has yet solved FSDEs using the Adomian decomposition method (ADM) with the Sumudu transform (ST) in the sense of Con-FrD. In this research, we addressed this gap by solving linear and nonlinear FSDEs using a hybrid approach that combines ADM and ST, applied in the sense of Con-FrD. We refer to it as the Sumudu Adomian decomposition method (SADM). Moreover, the correctness of SADM is assessed through an examination of absolute errors (Abs-E) and relative errors (Rel-E) presented in both numerical and graphical representations. It is observed that the approximate solution (App-S) rapidly approaches the exact solution (Ex-S), as evidenced by the evaluation of 2D and 3D graphs across various fractional-order values. The numerical and graphical findings confirm the notable precision and effectiveness of SADM. In addition, the results obtained using SADM are compared with other methods, such as the RPSM [37] and HAM [14]. The comparison shows excellent agreement with these methods, indicating that SADM is a suitable alternative to Cap-FD-based approaches for solving FSDEs. Furthermore, we can conclude that Con-FD is a suitable alternative to Cap-FD in modeling FSDEs.

Our method also has an assumption. To obtain the solution in the original space, SADM first requires finding the ST of the target equations and then performing the inverse ST. Consequently, for nonhomogeneous equations, the source functions must be piecewise continuous and of exponential order, and the inverse ST must exist after computations.

The ST is defined for exponentially ordered functions. We examine functions that are defined within the set $\mathbb{Q}$.

$$\mathbb{Q} = \{\delta(\varphi)|\exists \mathbb{P}, \mathbb{H}_1, \mathbb{H}_2 > 0, |\delta(\varphi) < \mathbb{P} \exp^{\frac{|\varphi|}{\mathbb{H}_\ell}} \ if \ \delta \in (-1)^\ell \times [0, \infty)\}.$$

The constant $\mathbb{P}$ for a given function in the set $\mathbb{Q}$ must be a finite value.

The ST is defined as follows [10]:

$$\mathbb{S}_\mu[\delta(\varphi)] = \delta^*(\Omega) = \frac{1}{\Omega} \int_0^\infty \delta(\varphi) \exp^{-\frac{\varphi^\mu}{\mu\Omega}} d_\mu\varphi, \ \Omega \in (-\mathbb{H}_1, \mathbb{H}_2).$$

The ADM is a powerful technique for solving DEs, including FODEs. It is particularly useful for handling non-linear DEs and can be adapted to the fractional-order case. ADM involves decomposing the solution of a DE into a series of simpler functions, making it easier to solve the equation iteratively. The main idea is to break down the problem into manageable components. The ADM holds significant importance in solving FODEs due to several key advantages and features. FODEs often include non-linear terms that can be challenging to solve using traditional methods. ADM effectively decomposes these non-linear terms into simpler components using Adomian polynomials, making it easier to find solutions even when the equations are highly non-linear. ADM provides a systematic framework for solving FODEs by breaking down the problem into a series of simpler problems. This approach allows for an iterative solution where each term in the series is computed sequentially, making it easier to handle complex DEs. The method is flexible and can be applied to a wide range of FODEs, including those with complex boundary conditions and variable coefficients. This adaptability makes ADM suitable for a diverse set of problems across different scientific and engineering disciplines.

We take into account the following: nonlinear SDE with respect to a conformable operator and a non-zero trapping potential that is time-fractional.

$$\iota\mathfrak{T}^{\mu}_{\varphi}\delta(\varrho,\varphi)+\zeta\delta_{\varrho\varrho}(\varrho,\varphi)+\beth(\varrho)\delta(\varrho,\varphi)+\aleph|\delta(\varrho,\varphi)|^{2}\delta(\varrho,\varphi)=0, \tag{4}$$

with the initial condition (I-C):

$$\delta(\varrho,0)=\delta_{0}(\varrho), \tag{5}$$

here $\zeta,\aleph\in\mathbb{R}$, $0<\mu\leq 1$, $\iota^{2}=-1$; $\mathfrak{T}^{\mu}_{\varphi}$ is Con-FrD; $\delta(\varrho,\varphi)$ complex-valued function, $\varrho\in\mathbb{R}$; $\varphi\geq 0$; $\beth(\varrho)$ represents trapping potential; $|\delta(\varrho,\varphi)|$ is the modulus of $\delta(\varrho,\varphi)$.

The subsequent sections are structured as follows: Section 2 presents the main characteristics and results that form the foundation of our study. Section 3 provides the algorithm of the SADM. In Section 4, we solve three types of FSDEs using the SADM. In Section 5, we discuss the results obtained in Section 4 through graphical and numerical outcomes and analyze the correctness of our approach. Finally, Section 6 presents the conclusion.

2. Preliminaries

In this section, we go over some useful characteristics of Con-FrD and ST that will be useful in this paper.

Theorem 1. *[2] Let $0<\mu\leq 1$, $\mathbb{A}_{1}(\varrho,\varphi)$ and $\mathbb{A}_{2}(\varrho,\varphi)$ be μ-differentiable at a point $\varphi>0$. Then,*

i. $\mathfrak{T}^{\mu}_{\varphi}(\nu_{1}\mathbb{A}_{1}(\varrho,\varphi)+\nu_{2}\mathbb{A}_{2}(\varrho,\varphi))=\nu_{1}\mathfrak{T}^{\mu}_{\varphi}\mathbb{A}_{1}(\varrho,\varphi)+\nu_{2}\mathfrak{T}^{\mu}_{\varphi}\mathbb{A}_{2}(\varrho,\varphi),\ \forall\ \nu_{1},\ \nu_{2}\in\mathbb{R}.$

ii. $\mathfrak{T}^{\mu}_{\varphi}(\varphi^{\nu})=\nu\varphi^{\nu-\mu},\ \forall\,\nu\in\mathbb{R}.$

iii. $\mathfrak{T}^{\mu}_{\varphi}(\nu)=0,\ \nu\in\mathbb{R}.$

iv. $\mathfrak{T}^{\mu}_{\varphi}\mathbb{A}_{1}(\varrho,\varphi)\mathbb{A}_{2}(\varrho,\varphi))=\mathbb{A}_{1}(\varrho,\varphi)\mathfrak{T}^{\mu}_{\varphi}\mathbb{A}_{2}(\varrho,\varphi)+\mathbb{A}_{2}(\varrho,\varphi)\mathfrak{T}^{\mu}_{\varphi}\mathbb{A}_{1}(\varrho,\varphi).$

v. $\mathfrak{T}^{\mu}_{\varphi}\dfrac{\mathbb{A}_{1}(\varrho,\varphi)}{\mathbb{A}_{2}(\varrho,\varphi)}=\dfrac{\mathbb{A}_{2}(\varrho,\varphi)\mathfrak{T}^{\mu}_{\varphi}\mathbb{A}_{1}(\varrho,\varphi)-\mathbb{A}_{1}(\varrho,\varphi)\mathfrak{T}^{\mu}_{\varphi}\mathbb{A}_{2}(\varphi)}{\mathbb{A}_{2}^{2}(\varphi)}.$

Lemma 1. *[3] Assume that $\mathbb{A}_{1}(\varrho,\varphi)$ and $\mathbb{A}_{2}(\varrho,\varphi)$ satisfies the axioms of existence of ST, $\mathbb{S}_{\mu}[\mathbb{A}_{1}(\varrho,\varphi)]=\mathbb{A}_{1}^{*}(\varrho,\Omega),\mathbb{S}_{\mu}[\mathbb{A}_{2}(\varrho,\varphi)]=\mathbb{A}_{2}^{*}(\varrho,\Omega)$ and $\mathfrak{B}_{1},\mathfrak{B}_{2}$ are constants. Then, the following axioms hold:*

i. $\mathbb{S}_{\mu}[\mathfrak{B}_{1}\mathbb{A}_{1}(\varrho,\varphi)+\mathfrak{B}_{2}\mathbb{A}_{2}(\varrho,\varphi)]=\mathfrak{B}_{1}\mathbb{A}_{1}^{*}(\varrho,\Omega)+\mathfrak{B}_{2}\mathbb{A}_{2}^{*}(\varrho,\Omega).$

ii. $\mathbb{S}_{\mu}^{-1}[\mathfrak{B}_{1}\mathbb{A}_{1}^{*}(\varrho,\Omega)+\mathfrak{B}_{2}\mathbb{A}_{2}^{*}(\varrho,\Omega)]=\mathfrak{B}_{1}\mathbb{A}_{1}(\varrho,\varphi)+\mathfrak{B}_{2}\mathbb{A}_{2}(\varrho,\varphi).$

iii. $\mathbb{S}_{\mu}[\mathfrak{T}^{\mu}_{\varphi}\mathbb{A}(\varrho,\varphi)]=\frac{\mathbb{S}[\mathbb{A}(\varrho,\varphi)]}{\Omega}-\frac{\mathbb{A}(\varrho,0)}{\Omega}.$

iv. $\mathbb{S}_{\mu}[\mathfrak{T}^{\mathrm{n}\mu}_{\varphi}\mathbb{A}(\varrho,\varphi)]=\frac{\mathbb{S}[\mathbb{A}(\varrho,\varphi)]}{\Omega^{\mathrm{n}}}-\frac{\mathbb{A}(\varrho,0)}{\Omega^{\mathrm{n}}}.$

3. Analysis of the SADM

This section presents the main steps for solving FSDEs using ST and ADM. We apply these steps to solve FSDEs in the standard structure. For this, take $\delta(\varrho,\varphi) = \delta_1(\varrho,\varphi) + \iota\delta_2(\varrho,\varphi)$ in Eq. (4) and we get:

$$\begin{aligned}\mathfrak{T}^{\mu}_{\varphi}\delta_1(\varrho,\varphi) =& -\zeta\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi) - \beth(\varrho)\delta_2(\varrho,\varphi) - \aleph\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big),\\ \mathfrak{T}^{\mu}_{\varphi}\delta_2(\varrho,\varphi) =& \zeta\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi) + \beth(\varrho)\delta_1(\varrho,\varphi) + \aleph\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big),\end{aligned} \tag{6}$$

with I-Cs:

$$\delta_1(\varrho,0) = \delta_{1,0}(\varrho),\ \delta_2(\varrho,0) = \delta_{2,0}(\varrho), \tag{7}$$

here, $\delta(\varrho,0) = \delta_{1,0}(\varrho) + \iota\delta_{2,0}(\varrho)$.
By applying $\mathbb{S}_\mu$ on the above system.

$$\begin{aligned}\mathbb{S}_\mu[\mathfrak{T}^{\mu}_{\varphi}\delta_1(\varrho,\varphi)] =& -\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi) + \beth(\varrho)\delta_2(\varrho,\varphi)+\\ &\aleph\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big)\big],\\ \mathbb{S}_\mu[\mathfrak{T}^{\mu}_{\varphi}\delta_2(\varrho,\varphi)] =& \mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi) + \beth(\varrho)\delta_1(\varrho,\varphi)+\\ &\aleph\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big)\big].\end{aligned} \tag{8}$$

Through Lemma 1(iii) and performing various computations, we acquire the following:

$$\begin{aligned}\mathbb{S}_\mu[\delta_1(\varrho,\varphi)] =& \delta_1(0) - \Omega\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi)\big] - \Omega\mathbb{S}_\mu\big[\beth(\varrho)\delta_2(\varrho,\varphi)\big]\\ &- \Omega\mathbb{S}_\mu\big[\aleph\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big)\big],\\ \mathbb{S}_\mu[\delta_2(\varrho,\varphi)] =& \delta_2(0) + \Omega\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi)\big] + \Omega\mathbb{S}_\mu\big[\beth(\varrho)\delta_1(\varrho,\varphi)\big]\\ &+ \Omega\mathbb{S}_\mu\big[\aleph\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big)\big].\end{aligned} \tag{9}$$

Implementing the $\mathbb{S}_\mu^{-1}$ on the system mentioned above:

$$\begin{aligned}\delta_1(\varrho,\varphi) =& \mathbb{S}_\mu^{-1}\big[\delta_1(0)\big] - \mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi)\big]\big] - \mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\beth(\varrho)\delta_2(\varrho,\varphi)\big]\big]\\ &- \mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\aleph\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big)\big]\big],\\ \delta_2(\varrho,\varphi) =& \mathbb{S}_\mu^{-1}\big[\delta_2(0)\big] + \mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi)\big]\big] + \mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\beth(\varrho)\delta_1(\varrho,\varphi)\big]\big]\\ &+ \mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\aleph\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big)\big)\big].\end{aligned} \tag{10}$$

Consider the solution of Eq. (6) in the following fractional power series (FPS) using the idea of ADM.

$$\begin{aligned}\delta_1(\varrho,\varphi) &= \sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi),\\ \delta_2(\varrho,\varphi) &= \sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi).\end{aligned} \tag{11}$$

The nonlinear terms $\mathscr{Z}_1(\delta_1,\delta_2)$ and $\mathscr{Z}_2(\delta_1,\delta_2)$ represented as

$$\mathscr{Z}_1(\delta_1,\delta_2)=\sum_{\rho=0}^{\infty}\mathscr{A}_{1,\rho}(\delta_1,\delta_2),$$
$$\mathscr{Z}_2(\delta_1,\delta_2)=\sum_{\rho=0}^{\infty}\mathscr{A}_{2,\rho}(\delta_1,\delta_2), \tag{12}$$

here, $\mathscr{A}_{1,\rho}(\delta_1,\delta_2)$ and $\mathscr{A}_{2,\rho}(\delta_1,\delta_2)$ are the Adomian polynomials (A-Ps) for the nonlinear term.

$$\mathscr{A}_{1,\rho}(\delta_1,\delta_2)=\frac{1}{\Gamma(\rho+1)}\frac{\partial^\rho}{\partial\chi^\rho}\big[\mathscr{Z}_1\big(\sum_{\sigma=0}^{\rho}\chi^\sigma\delta_{1,\sigma}(\varrho,\varphi)\big)\big]_{\chi=0},\ \rho=0,\ 1,\ 2,\ \ldots,$$
$$\mathscr{A}_{2,\rho}(\delta_1,\delta_2)=\frac{1}{\Gamma(\rho+1)}\frac{\partial^\rho}{\partial\chi^\rho}\big[\mathscr{Z}_2\big(\sum_{\sigma=0}^{\rho}\chi^\sigma\delta_{2,\sigma}(\varrho,\varphi)\big)\big]_{\chi=0},\ \rho=0,\ 1,\ 2,\ \ldots. \tag{13}$$

From Eqs. (11), (12), and (10).

$$\begin{aligned}\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi)=&\mathbb{S}_\mu^{-1}\big[\delta_1(0)\big]-\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi)\big]\big]-\\&\aleph_\varpi^{-1}\big[\Omega\aleph_\varpi\big[\beth(\varrho)\sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi)\big]\big]-\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\aleph\sum_{\rho=0}^{\infty}\mathscr{A}_{1,\rho}(\delta_1,\delta_2)\big],\\\sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi)=&\mathbb{S}_\mu^{-1}\big[\delta_2(0)\big]+\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi)\big]\big]+\\&\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\beth(\varrho)\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi)\big]\big]+\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\aleph\sum_{\rho=0}^{\infty}\mathscr{A}_{2,\rho}(\delta_1,\delta_2)\big].\end{aligned} \tag{14}$$

From Eq. (14), we have as

$$\delta_{1,0}(\varrho,\varphi)=\mathbb{S}_\mu^{-1}\big[\delta_1(0)\big],$$
$$\delta_{2,0}(\varrho,\varphi)=\mathbb{S}_\mu^{-1}\big[\delta_2(0)\big]. \tag{15}$$

The 2nd term of the FPS is as

$$\begin{aligned}\delta_{1,1}(\varrho,\varphi)=&-\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\delta_{2,0}(\varrho,\varphi)\big]\big]-\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\beth(\varrho)\delta_{2,0}(\varrho,\varphi)\big]\big]-\\&\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\aleph\mathscr{A}_{1,0}(\delta_1,\delta_2)\big],\\\delta_{2,1}(\varrho,\varphi)=&\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\zeta\mathscr{D}\varrho\varrho\delta_{1,0}(\varrho,\varphi)\big]\big]+\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\beth(\varrho)\delta_{1,0}(\varrho,\varphi)\big]\big]+\\&\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\aleph\mathscr{A}_{2,0}(\delta_1,\delta_2)\big].\end{aligned} \tag{16}$$

The following are the 3rd and 4th terms of the FPS:

$$\delta_{1,2}(\varrho,\varphi)=-\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\varrho\big[\zeta\mathscr{D}\varrho\varrho\delta_{2,1}(\varrho,\varphi)\big]\big]-\mathbb{S}_\mu^{-1}\big[\Omega\mathbb{S}_\mu\big[\beth(\varrho)\delta_{2,1}(\varrho,\varphi)\big]\big]-$$

$$\begin{aligned}&\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\aleph\mathscr{A}_{1,1}(\delta_1,\delta_2)\big],\\ \delta_{2,2}(\varrho,\varphi)=&\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\zeta\mathscr{D}_{\varrho\varrho}\delta_{1,1}(\varrho,\varphi)\big]\big]+\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\beth(\varrho)\delta_{1,1}(\varrho,\varphi)\big]\big]+\\ &\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\aleph\mathscr{A}_{2,1}(\delta_1,\delta_2)\big].\end{aligned}\tag{17}$$

$$\begin{aligned}\delta_{1,3}(\varrho,\varphi)=&-\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\zeta\mathscr{D}_{\varrho\varrho}\delta_{2,2}(\varrho,\varphi)\big]\big]-\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\beth(\varrho)\delta_{2,2}(\varrho,\varphi)\big]\big]-\\ &\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\aleph\mathscr{A}_{1,2}(\delta_1,\delta_2)\big],\\ \delta_{2,3}(\varrho,\varphi)=&\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\zeta\mathscr{D}_{\varrho\varrho}\delta_{1,2}(\varrho,\varphi)\big]\big]+\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\beth(\varrho)\delta_{1,2}(\varrho,\varphi)\big]\big]+\\ &\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\aleph\mathscr{A}_{2,2}(\delta_1,\delta_2)\big].\end{aligned}\tag{18}$$

By making generalizing we get the following:

$$\begin{aligned}\delta_{1,\rho+1}(\varrho,\varphi)=&-\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\zeta\mathscr{D}_{\varrho\varrho}\delta_{2,\rho}(\varrho,\varphi)\big]\big]-\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\beth(\varrho)\delta_{2,\rho}(\varrho,\varphi)\big]\big]-\\ &\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\aleph\mathscr{A}_{1,\rho}(\delta_1,\delta_2)\big],\\ \delta_{2,\rho+1}(\varrho,\varphi)=&\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\zeta\mathscr{D}_{\varrho\varrho}\delta_{1,\rho}(\varrho,\varphi)\big]\big]+\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\beth(\varrho)\delta_{1,\rho}(\varrho,\varphi)\big]\big]+\\ &\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\aleph\mathscr{A}_{2,\rho}(\delta_1,\delta_2)\big].\end{aligned}\tag{19}$$

Lastly, the FPS is obtained in the following ways:

$$\begin{aligned}\delta_1(\varrho,\varphi)&=\lim_{\sigma\to\infty}\sum_{\rho=0}^{\sigma}\delta_{1,\rho}(\varrho,\varphi),\\ \delta_2(\varrho,\varphi)&=\lim_{\sigma\to\infty}\sum_{\rho=0}^{\sigma}\delta_{2,\rho}(\varrho,\varphi).\end{aligned}\tag{20}$$

4. Numerical Problems

Using the method outlined in the preceding section, we find the App-Ss and Ex-Ss of linear and nonlinear FSDEs in this section. The solutions to the Schrödinger equations encompass a wealth of information regarding the probabilistic nature of quantum systems, their energy states, and their dynamic behavior. This information is foundational for understanding the principles of quantum mechanics and their applications in various fields, including quantum chemistry, condensed matter physics, and quantum information science.

Problem 4.1. Examine the subsequent linear FSDEs:

$$\iota\mathfrak{T}_{\varphi}^{\mu}\delta(\varrho,\varphi)+\mathscr{D}_{\varrho\varrho}\delta(\varrho,\varphi)=0,\ \varphi\geq 0,\ 0<\mu\leq 1,\tag{21}$$

with the I-C:

$$\delta(\varrho,0)=1+\cosh(2\varrho),\ \varrho\in\mathbb{R}.$$

By utilizing $\delta(\varrho,\varphi)=\delta_1(\varrho,\varphi)+\iota\delta_2(\varrho,\varphi)$ and $\delta(\varrho,0)=\delta_{1,0}(\varrho)+\iota\delta_{2,0}(\varrho)$, we have the following:

$$\mathfrak{T}_{\varphi}^{\mu}\delta_1(\varrho,\varphi)=\mathscr{D}_{\varrho\varrho}\delta_2(\varrho,\varphi),$$

$$\mathfrak{T}_{\varphi}^{\mu}\delta_2(\varrho,\varphi) = -\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi), \tag{22}$$

with I-Cs:

$$\begin{aligned}\delta_{1,0}(\varrho,0) =& 1+\cosh(2\varrho),\\ \delta_{2,0}(\varrho,0) =& 0.\end{aligned} \tag{23}$$

By applying $\mathbb{S}_{\mu}$ to the above system and implementing some calculations, we get the subsequent:

$$\begin{aligned}\mathbb{S}_{\mu}[\delta_1(\varrho,\varphi)] &= \delta_1(0) + \Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi)\big],\\ \mathbb{S}_{\mu}[\delta_2(\varrho,\varphi)] &= \delta_2(0) - \Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi)\big].\end{aligned} \tag{24}$$

Taking $\mathbb{S}_{\mu}^{-1}$.

$$\begin{aligned}\delta_1(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_1(0)\big] + \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi)\big]\big],\\ \delta_2(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_2(0)\big] - \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi)\big]\big].\end{aligned} \tag{25}$$

We derive the following result:

$$\begin{aligned}\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_1(0)\big] + \mathbb{S}_{\mu}^{-1}\Big[\Omega\mathbb{S}_{\mu}\Big[\mathscr{D}\varrho\varrho\sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi)\Big]\Big],\\ \sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_2(0)\big] - \mathbb{S}_{\mu}^{-1}\Big[\Omega\mathbb{S}_{\mu}\Big[\mathscr{D}\varrho\varrho\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi)\Big]\Big].\end{aligned} \tag{26}$$

We extract the following from the above:

$$\begin{aligned}\delta_{1,0}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_{1,0}(\varrho,0)\big],\\ \delta_{2,0}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_{2,0}(\varrho,0)\big].\end{aligned} \tag{27}$$

Below is the first term:

$$\begin{aligned}\delta_{1,0}(\varrho,\varphi) =& 1+\cosh(2\varrho),\\ \delta_{2,0}(\varrho,\varphi) =& 0.\end{aligned} \tag{28}$$

The next term expressions are as follows:

$$\begin{aligned}\delta_{1,1}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_{2,0}(\varrho,\varphi)\big]\big],\\ \delta_{2,1}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_{1,0}(\varrho,\varphi)\big]\big].\end{aligned} \tag{29}$$

From the above system, we have

$$\begin{aligned}\delta_{1,1}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho(0)\big]\big],\\ \delta_{2,1}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho(1+\cosh(2\varrho))\big]\big].\end{aligned} \tag{30}$$

As a result,

$$\begin{aligned}\delta_{1,1}(\varrho,\varphi) =&0,\\ \delta_{2,1}(\varrho,\varphi) =&-\frac{4}{\Gamma(2)}\cosh(2\varrho)\frac{\varphi^{\mu}}{\mu}.\end{aligned} \tag{31}$$

The third-term expressions are as follows:

$$\begin{aligned}\delta_{1,2}(\varrho,\varphi) =&\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_{2,1}(\varrho,\varphi)\big]\big],\\ \delta_{2,2}(\varrho,\varphi) =&\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_{1,1}(\varrho,\varphi)\big]\big].\end{aligned} \tag{32}$$

We have the following from above:

$$\begin{aligned}\delta_{1,2}(\varrho,\varphi) =&\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho(\frac{4}{\Gamma(2)}\cosh(2\varrho)\frac{\varphi^{\mu}}{\mu})\big]\big],\\ \delta_{2,2}(\varrho,\varphi) =&\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho(0)\big]\big].\end{aligned} \tag{33}$$

The third term is as follows:

$$\begin{aligned}\delta_{1,2}(\varrho,\varphi) =&-\frac{8}{\Gamma(3)}\cosh(2\varrho)\frac{\varphi^{2\mu}}{\mu^{2}},\\ \delta_{2,2}(\varrho,\varphi) =&0.\end{aligned} \tag{34}$$

The next terms are as follows:

$$\begin{aligned}\delta_{1,3}(\varrho,\varphi) =&0,\\ \delta_{2,3}(\varrho,\varphi) =&\frac{64}{\Gamma(4)}\cosh(2\varrho)\frac{\varphi^{3\mu}}{\mu^{3}}.\end{aligned} \tag{35}$$

$$\begin{aligned}\delta_{1,4}(\varrho,\varphi) =&\frac{256}{\Gamma(5)}\cosh(2\varrho)\frac{\varphi^{4\mu}}{\mu^{4}},\\ \delta_{2,4}(\varrho,\varphi) =&0.\end{aligned} \tag{36}$$

The procedure is repeated to achieve the results for the sixth term.

$$\begin{aligned}\delta_{1,5}(\varrho,\varphi) =&0,\\ \delta_{2,5}(\varrho,\varphi) =&-\frac{1024}{\Gamma(6)}\cosh(2\varrho)\frac{\varphi^{5\mu}}{\mu^{5}}.\end{aligned} \tag{37}$$

As a result,

$$\delta(\varrho,\varphi) = 1+\cosh(2\varrho)\bigg(\sum_{\rho=0}^{\infty}\frac{(-1)^{\rho}}{\Gamma(2\rho+1)}\big(\frac{4\varphi^{\mu}}{\mu}\big)^{2\rho}-\iota\sum_{\rho=0}^{\infty}\frac{(-1)^{\rho}}{\Gamma(2\rho+2)}\big(\frac{4\varphi^{\mu}}{\mu}\big)^{2\rho+1}\bigg). \tag{38}$$

The Ex-S when $\mu = 1.0$ is $1 + \cosh(2\varrho)\exp^{-4\iota\varphi}$. A similar result was obtained by [10].
Problem 4.2. Take into consideration the subsequent nonlinear FSDE:

$$\iota\mathfrak{T}^{\mu}_{\varphi}\delta(\varrho,\varphi) + \delta_{\varrho\varrho}(\varrho,\varphi) + 2|\delta(\varrho,\varphi)|^2\delta(\varrho,\varphi) = 0, \tag{39}$$

with the I-C:

$$\delta(\varrho,0) = \exp^{i\varrho},\ \varrho \in \mathbb{R}. \tag{40}$$

Suppose $\delta(\varrho,\varphi) = \delta_1(\varrho,\varphi) + \iota\delta_2(\varrho,\varphi)$ so that $\delta(\varrho,0) = \delta_{1,0}(\varrho) + \iota\delta_{2,0}(\varrho)$. As a result,

$$\begin{aligned}\mathfrak{T}^{\mu}_{\varphi}\delta_1(\varrho,\varphi) &= -\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi) - 2\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big),\\ T^{\mu}_{\tau}\delta_2(\varrho,\varphi) &= \mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi) + 2\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big).\end{aligned} \tag{41}$$

with the I.Cs:

$$\begin{aligned}\delta_1(\varrho,0) &= \cos(\varrho),\\ \delta_2(\varrho,0) &= \sin(\varrho),\end{aligned} \tag{42}$$

Applying ST to Eq. (41), using the linear property of ST and Lemma 1(iii) and making some calculations, we get

$$\begin{aligned}\mathbb{S}_{\mu}[\delta_1(\varrho,\varphi)] =& \delta_1(0) - \Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi)\big]\\ &- \Omega\mathbb{S}_{\mu}\big[2\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big)\big],\\ \mathbb{S}_{\mu}[\delta_2(\varrho,\varphi)] =& \delta_2(0) + \Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi)\big]\\ &+ \Omega\mathbb{S}_{\mu}\big[2\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big)\big].\end{aligned} \tag{43}$$

Through $\mathbb{S}_{\mu}^{-1}$ on the system mentioned before.

$$\begin{aligned}\delta_1(\upsilon,\tau) =& \mathbb{S}_{\mu}^{-1}\big[\delta_1(0)\big] - \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\upsilon\upsilon\delta_2(\varrho,\varphi)\big]\big]\\ &- \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[2\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big)\big]\big],\\ \delta_2(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_2(0)\big] + \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi)\big]\big]\\ &+ \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[2\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big)\big]\big].\end{aligned} \tag{44}$$

We get the following:

$$\begin{aligned}\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_1(0)\big] - \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi)\big]\big]-\\ &\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[2\sum_{\rho=0}^{\infty}\mathscr{A}_{1,\rho}(\delta_1,\delta_2)\big],\\ \sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi) =& \mathbb{S}_{\mu}^{-1}\big[\delta_2(0)\big] + \mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\mathscr{D}\varrho\varrho\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi)\big]\big]+\end{aligned}$$

$$\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[2\sum_{\rho=0}^{\infty}\mathscr{A}_{2,\rho}(\delta_1,\delta_2)\big]\big]. \tag{45}$$

From the above system, we obtain the first terms of the FPS solutions.

$$\begin{aligned}\delta_{1,0}(\varrho,\varphi) &= \cos(\varrho),\\ \delta_{2,0}(\varrho,\varphi) &= \sin(\varrho).\end{aligned} \tag{46}$$

We extract the following terms of the FPS as follows:

$$\begin{aligned}\delta_{1,1}(\varrho,\varphi) &= -\sin(\varrho)\frac{\varphi^{\mu}}{\mu\Gamma(2)},\\ \delta_{2,1}(\varrho,\varphi) &= \cos(\varrho)\frac{\varphi^{\mu}}{\mu\Gamma(2)}.\end{aligned} \tag{47}$$

$$\begin{aligned}\delta_{1,2}(\varrho,\varphi) &= -\frac{\varphi^{2\mu}}{\mu^2 2!}\cos(\varrho),\\ \delta_{2,2}(\varrho,\varphi) &= -\sin(\varrho)\frac{\varphi^{2\mu}}{\mu^2\Gamma(3)}.\end{aligned} \tag{48}$$

$$\begin{aligned}\delta_{1,3}(\varrho,\varphi) &= \sin(\varrho)\frac{\varphi^{3\mu}}{\mu^3\Gamma(4)},\\ \delta_{2,3}(\varrho,\varphi) &= -\cos(\varrho)\frac{\varphi^{3\mu}}{\mu^3\Gamma(4)}.\end{aligned} \tag{49}$$

$$\begin{aligned}\delta_{1,4}(\varrho,\varphi) &= \sin(\varrho)\frac{\varphi^{3\mu}}{\mu^3\Gamma(5)},\\ \delta_{2,4}(\varrho,\varphi) &= \sin(\varrho)\frac{\varphi^{3\mu}}{\mu^3\Gamma(5)}.\end{aligned} \tag{50}$$

The procedure is repeated to achieve the results for the sixth terms.

$$\begin{aligned}\delta_{1,5}(\varrho,\varphi) &= -\sin(\varrho)\frac{\varphi^{5\mu}}{\mu^5\Gamma(6)},\\ \delta_{2,5}(\varrho,\varphi) &= \cos(\varrho)\frac{\varphi^{5\mu}}{\mu^5\Gamma(6)}.\end{aligned} \tag{51}$$

The FPS solution is as follows:

$$\delta(\varrho,\varphi) = (\cos(\varrho)+\iota\sin(\varrho))\left(\sum_{\rho=0}^{\infty}\frac{1}{\Gamma(\rho+1)}\big(\frac{\iota\varphi^{\mu}}{\mu}\big)^{\rho}\right). \tag{52}$$

The Ex-S when $\mu = 1$ is $\delta(\varrho,\varphi) = \exp^{\iota\left(\varrho+\varphi\right)}$. The same solution was achieved by [2].
Problem 4.3. Consider the following nonlinear FSDE that follows:

$$\iota\mathfrak{T}^{\mu}_{\varphi}\delta(\varrho,\varphi) + \frac{1}{2}\delta_{\varrho\varrho}(\varrho,\varphi) - \cos^2(\varrho)\delta(\varrho,\varphi) - |\delta(\varrho,\varphi)|^2\delta(\varrho,\varphi) = 0, \tag{53}$$

with the I-C:

$$\delta(\varrho,0) = \sin(\varrho),\ \ \varrho \in \mathbb{R}. \tag{54}$$

Suppose $\delta(\varrho,\varphi) = \delta_1(\varrho,\varphi) + \iota\delta_2(\varrho,\varphi)$ then $\delta(\varrho,0) = \delta_{1,0}(\varrho) + \iota\delta_{2,0}(\varrho)$. As a result,

$$\begin{aligned}\mathfrak{T}^{\mu}_{\varphi}\delta_1(\varrho,\varphi) =& -\frac{1}{2}\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi) + \cos^2(\varrho)\delta_2(\varrho,\varphi) + \left(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\right),\\ \mathfrak{T}^{\mu}_{\varphi}\delta_2(\varrho,\varphi) =& \frac{1}{2}\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi) - \cos^2(\varrho)\delta_1(\varrho,\varphi) - \left(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\right),\end{aligned} \tag{55}$$

with the I-Cs:

$$\begin{aligned}\delta_1(\varrho,0) =& \sin(\varrho),\\ \delta_2(\varrho,0) =& 0.\end{aligned} \tag{56}$$

After applying ST to Eq. (55), utilizing Lemma 1(iii) and the linear property of ST, and performing certain computations, we obtain

$$\begin{aligned}\mathbb{S}[\delta_1(\varrho,\varphi)] =& \delta_1(0) - \Omega\mathbb{S}_{\mu}\big[\frac{1}{2}\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi)\big] + \Omega\mathbb{S}_{\mu}\big[\cos^2(\varrho)\delta_2(\varrho,\varphi)\big]\\ &+ \Omega\mathbb{S}_{\mu}\big[\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big)\big],\\ \mathbb{S}_{\mu}[\delta_2(\varrho,\varphi)] =& \delta_2(0) + \Omega\mathbb{S}_{\mu}\big[\frac{1}{2}\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi)\big] - \Omega\mathbb{S}_{\mu}\big[\cos^2(\varrho)\delta_1(\varrho,\varphi)\big]\\ &- \Omega\mathbb{S}_{\mu}\big[\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big)\big].\end{aligned} \tag{57}$$

Assess $\mathbb{S}^{-1}_{\mu}$ in the system mentioned earlier.

$$\begin{aligned}\delta_1(\varrho,\varphi) =& \mathbb{S}^{-1}_{\mu}\big[\delta_1(0)\big] - \mathbb{S}^{-1}_{\mu}\big[\Omega\mathbb{S}\big[\frac{1}{2}\mathscr{D}\varrho\varrho\delta_2(\varrho,\varphi)\big]\big] + \mathbb{S}^{-1}\big[\Omega\mathbb{S}\big[\cos^2(\varrho)\delta_2(\varrho,\varphi)\big]\big]\\ &+ \mathbb{S}^{-1}_{\mu}\big[\Omega\mathbb{S}_{\mu}\big[\big(\delta_1^2(\varrho,\varphi)\delta_2(\varrho,\varphi) + \delta_2^3(\varrho,\varphi)\big)\big]\big],\\ \delta_2(\varrho,\varphi) =& \mathbb{S}^{-1}_{\mu}\big[\delta_2(0)\big] + \mathbb{S}^{-1}_{\mu}\big[\Omega\mathbb{S}_{\mu}\big[\frac{1}{2}\mathscr{D}\varrho\varrho\delta_1(\varrho,\varphi)\big]\big] - \mathbb{S}^{-1}_{\mu}\big[\Omega\mathbb{S}_{\mu}\big[\cos^2(\varrho)\delta_1(\varrho,\varphi)\big]\big]\\ &- \mathbb{S}^{-1}_{\mu}\big[\Omega\mathbb{S}\big[\big(\delta_2^2(\varrho,\varphi)\delta_1(\varrho,\varphi) + \delta_1^3(\varrho,\varphi)\big)\big]\big].\end{aligned} \tag{58}$$

Using the method outlined in Section 3, we derive the following result from Eq. (58):

$$\begin{aligned}\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi) =& \mathbb{S}^{-1}_{\mu}\big[\delta_1(0)\big] - \mathbb{S}^{-1}_{\mu}\big[\Omega\mathbb{S}_{\mu}\big[\frac{1}{2}\mathscr{D}\varrho\varrho\sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi)\big]\big] +\\ &\mathbb{S}^{-1}_{\mu}\big[\Omega\mathbb{S}_{\mu}\big[\cos^2(\varrho)\sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi)\big]\big] + \mathbb{S}^{-1}_{\mu}\big[\Omega\mathbb{S}_{\mu}\big[\sum_{\rho=0}^{\infty}\mathscr{A}_{1,\rho}(\delta_1,\delta_2)\big],\end{aligned}$$

$$\sum_{\rho=0}^{\infty}\delta_{2,\rho}(\varrho,\varphi)=\mathbb{S}_{\mu}^{-1}\big[\delta_2(0)\big]+\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}\big[\frac{1}{2}\mathscr{D}\varrho\varrho\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi)\big]\big]-$$
$$\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\cos^2(\varrho)\sum_{\rho=0}^{\infty}\delta_{1,\rho}(\varrho,\varphi)\big]\big]-\mathbb{S}_{\mu}^{-1}\big[\Omega\mathbb{S}_{\mu}\big[\sum_{\rho=0}^{\infty}\mathscr{A}_{2,\rho}(\delta_1,\delta_2)\big].\tag{59}$$

We extracted the first term of the FPS solution for the Eq. (55) via the equivalent on the two ends of Eq. (59).

$$\begin{aligned}\delta_{1,0}(\varrho,\varphi)&=\sin(\varrho),\\ \delta_{2,0}(\varrho,\varphi)&=0.\end{aligned}\tag{60}$$

We extract the following second term of the FPS.

$$\begin{aligned}\delta_{1,1}(\varrho,\varphi)&=0,\\ \delta_{2,1}(\varrho,\varphi)&=-\frac{3}{2}\sin(\varrho)\frac{\varphi^{\mu}}{\mu\Gamma(2)}.\end{aligned}\tag{61}$$

Similarly, we found the third, fourth, and fifth terms.

$$\begin{aligned}\delta_{1,2}(\varrho,\varphi)&=-\frac{9}{4}\sin(\varrho)\frac{\varphi^{2\mu}}{\mu^2\Gamma(3)},\\ \delta_{2,2}(\varrho,\varphi)&=0.\end{aligned}\tag{62}$$

$$\begin{aligned}\delta_{1,3}(\varrho,\varphi)&=0,\\ \delta_{2,3}(\varrho,\varphi)&=\frac{27}{8}\sin(\varrho)\frac{\varphi^{3\mu}}{\mu^3\Gamma(4)}.\end{aligned}\tag{63}$$

$$\begin{aligned}\delta_{1,4}(\varrho,\varphi)&=\frac{81}{16}\sin(\varrho)\frac{\varphi^{4\mu}}{\mu^4\Gamma(5)},\\ \delta_{2,4}(\varrho,\varphi)&=0.\end{aligned}\tag{64}$$

The procedure is repeated to achieve the results for the sixth term.

$$\begin{aligned}\delta_{1,5}(\varrho,\varphi)&=0,\\ \delta_{2,5}(\varrho,\varphi)&=-\frac{243}{32}\sin(\varrho)\frac{\varphi^{5\mu}}{\mu^5\Gamma(6)}.\end{aligned}\tag{65}$$

Hence, the system described in Eq. (55) has an FPS solution, which is outlined below.

$$\delta(\varrho,\varphi)=\sin(\varrho)\Big(\sum_{\rho=0}^{\infty}\frac{(-1)^{\rho}}{\Gamma(2\rho+1)}(\frac{3\varphi^{\mu}}{2\mu})^{2\rho}-\iota\sum_{\rho=0}^{\infty}\frac{(-1)^{\rho}}{\Gamma(2\rho+2)}(\frac{3\varphi^{\mu}}{2\mu})^{2\rho+1}\Big).\tag{66}$$

The Ex-S when $\mu=1.0$ is $\delta(\varrho,\varphi)=\sin(\varrho)\exp^{-\frac{3\iota\varphi}{2}}$. An identical result was obtained by [10].

5. Graphical and Numerical Outcome

In this section, we analyze the numerical and graphical results of the Ex-Ss and App-Ss for the linear and nonlinear problems presented in the fourth section of this research study. To assess the SADM's accuracy, we use two error functions, the Abs-E and Rel-E functions. Delineating the errors in the App-Ss is essential since SADM provides an approximation expressed in terms of an infinite FPS.

First, we present some useful notation for Ex-S and App-S, along with formulas for error functions, which we utilize in this section to analyze the reliability and correctness of our approach.

$$\delta(\varrho,\varphi) \approx \delta^{\kappa}(\varrho,\varphi),\ \kappa = 1,\ 2,\ 3,\cdots,$$

where $\delta(\varrho,\varphi)$ and $\delta^{\kappa}(\varrho,\varphi)$ denote the Ex-S and App-S of the nonlinear problems 4.2 and 4.3 obtained by SADM.

Abs-E plays a crucial role in quantifying the accuracy of these App-Ss and is indispensable for validating and refining numerical techniques. The Abs-E represents the magnitude of the absolute difference between the App-S ($\delta^{\kappa}(\varrho,\varphi)$) obtained through approximate methods and the Ex-S ($\delta(\varrho,\varphi)$), if available. Smaller Abs-E values indicate higher accuracy in the approximation. The Abs-E is defined as follows:

$$Abs.E^{\kappa}(\varrho,\varphi) = |\delta(\varrho,\varphi) - \delta^{\kappa}(\varrho,\varphi)|,\ \kappa = 1,\ 2,\ 3,\cdots,$$

when κ increases to infinity, it frequently happens that Abs.E$^{\kappa}(\varrho,\varphi)$ gets decreasing, eventually decreasing almost to zero.

The Rel-E serves as a powerful tool for evaluating the effectiveness of the approach that generates App-Ss. It is calculated as the ratio of the Abs-E to the magnitude of the Ex-S at each point within the solution domain. This provides valuable insights into the degree of alignment between the App-S and the behavior of the Ex-S. A smaller Rel-E indicates higher accuracy in the approximation. Mathematically, it is defined as follows:

$$Rel.E^{\kappa}(\varrho,\varphi) = \frac{|\delta(\varrho,\varphi) - \delta^{\kappa}(\varrho,\varphi)|}{|\delta(\varrho,\varphi)|},\ \kappa = 1,\ 2,\ 3,\cdots,$$

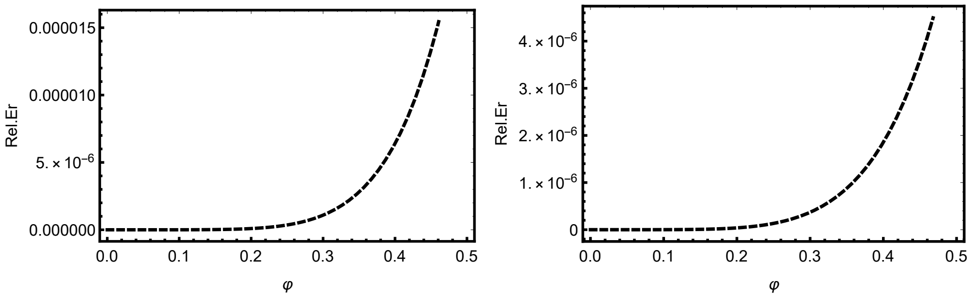
where the Rel-E for the κth-step App-S is represented by Rel.E$^{\kappa}(\varrho,\varphi)$ for the Ex-S ($\delta(\varrho,\varphi)$). In fact, it often happens that as κ goes to infinity, Rel.E$^{\kappa}(\varrho,\varphi)$ gets ever smaller until it almost reaches zero.

In Figures 1-4 for Problems 4.2 and 4.3, 2D curves are employed to compare the App-Ss and Ex-Ss in terms of Rel-E and Abs-E. The comparative analysis reveals a high degree of similarity between the fifth-step App-Ss and the Ex-Ss. The Abs-E is presented on the graphs to demonstrate the excellent precision of SADM.

The 2D graphs of the App-Ss obtained from five iterations and the Ex-Ss derived by SADM for $\mu = 0.6, 0.7, 0.8, 0.9$, and 1.0 are depicted in Figures 5 and 6 for Problems 4.2 and 4.3. These graphs illustrate how, as $\mu \to 1.0$, the App-Ss converge to the Ex-Ss. The interaction between the App-Ss and Ex-Ss when $\mu = 1.0$ demonstrates the accuracy of the proposed approach.

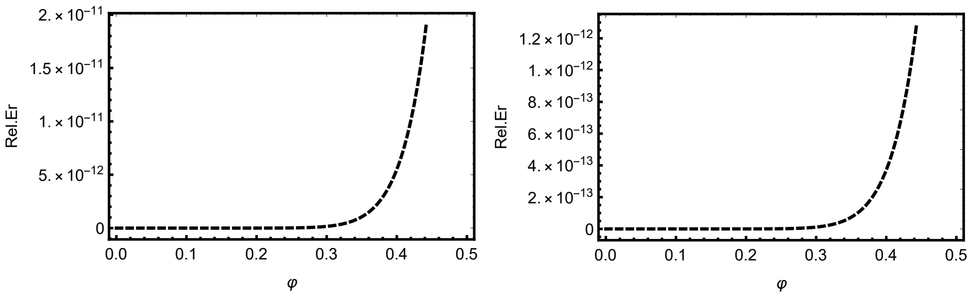
Figures 7-14 for Problems 4.2 and 4.3 display the 3D graphs of the App-Ss obtained from five iterations and the Ex-Ss determined by SADM for $\mu = 0.7, 0.8, 0.9, 1.0$ and Ex-Ss. These graphs demonstrate how the App-Ss converge to the Ex-Ss as μ approaches 1.0. The interaction between the App-Ss and Ex-Ss when $\mu = 1.0$ illustrates the accuracy of the proposed method.

The Abs-E and Rel-E for specified locations between the Ex-Ss and fifth-order App-Ss derived by SADM in Problems 4.2 and 4.3 at $\mu = 1.0$ are presented in Tables 1 and 2. These tables demonstrate that the App-Ss and Ex-Ss are nearly in agreement, confirming the accuracy of SADM. From these tables, it is observed that the Abs-E and Rel-E for all problems in the fifth-step App-Ss is very small. The findings presented in this section, depicted in both graphs and tables, demonstrate that SADM is a useful and effective technique for solving FSDEs, requiring fewer calculations and iterations.



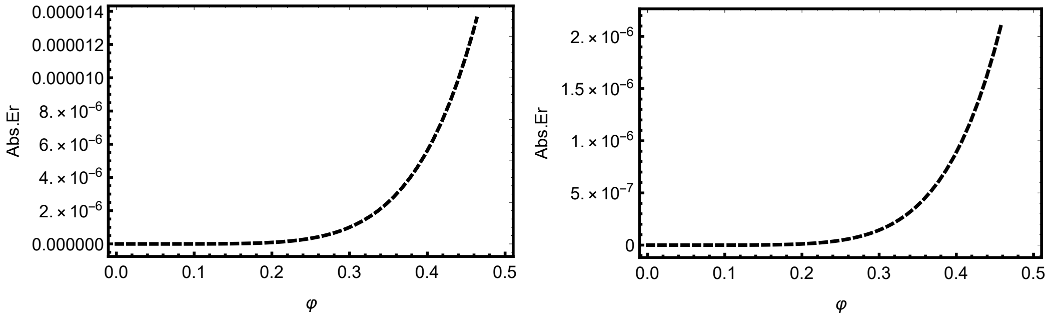
(a) (b)

Figure 1: 2D graphs of Rel-E in the range $\varphi \in [0, 0.5]$ comparing the fifth-step App-S and Ex-S for $\varrho = 0.1$ with $\mu = 1.0$ in problem 4.2: (a) $\delta_1(\varrho, \varphi)$; (b) $\delta_2(\varrho, \varphi)$.



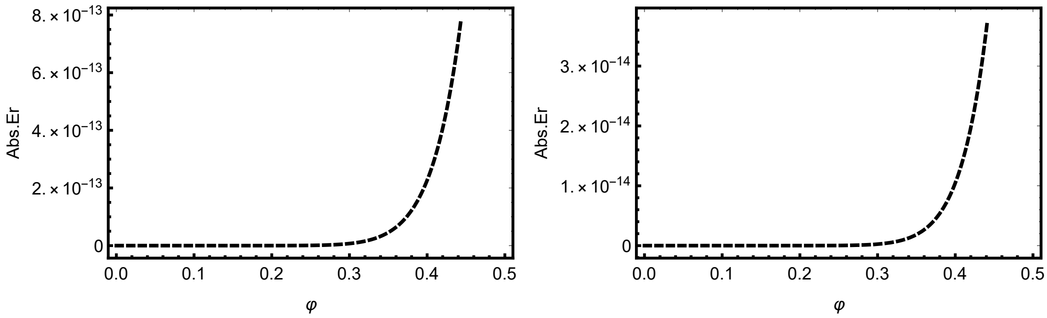
(a) (b)

Figure 2: 2D graphs of Rel-E in the range $\varphi \in [0, 0.5]$ comparing the fifth-step App-S and Ex-S for $\varrho = 0.1$ with $\mu = 1.0$ in problem 4.3: (a) $\delta_1(\varrho, \varphi)$; (b) $\delta_2(\varrho, \varphi)$.



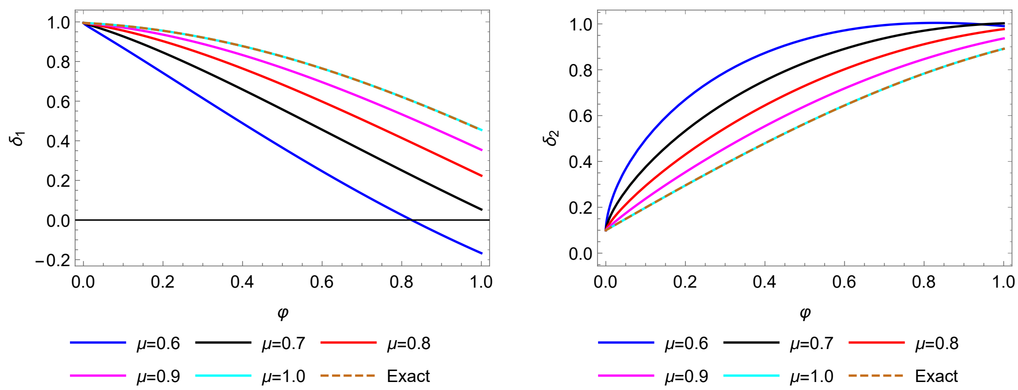
(a) (b)

Figure 3: 2D graphs of Abs-E in the range $\varphi \in [0, 0.5]$ comparing the fifth-step App-S and Ex-S for $\varrho = 0.1$ with $\mu = 1.0$ in problem 4.2: (a) $\delta_1(\varrho, \varphi)$; (b) $\delta_2(\varrho, \varphi)$.



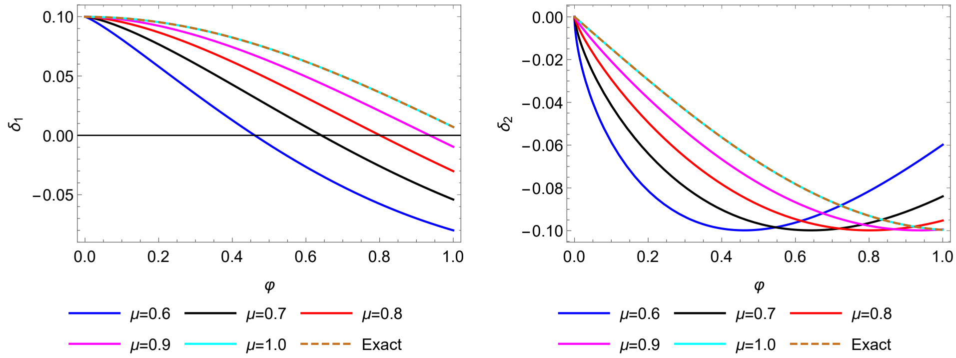
(a) (b)

Figure 4: 2D graphs of Abs-E in the range $\varphi \in [0, 0.5]$ comparing the fifth-step App-S and Ex-S for $\varrho = 0.1$ with $\mu = 1.0$ in problem 4.3: (a) $\delta_1(\varrho, \varphi)$; (b) $\delta_2(\varrho, \varphi)$.



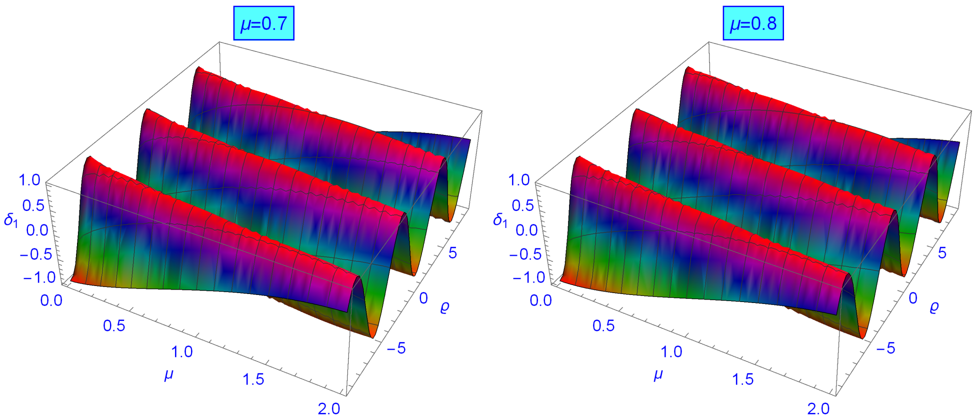
(a) (b)

Figure 5: For problem 4.2, the 2D diagrams of App-S for various levels of μ and Ex-S in the range $\varphi \in [0, 1.0]$ at $\varrho = 0.1$ are shown for: (a) $\delta_1(\varrho, \varphi)$; (b) $\delta_2(\varrho, \varphi)$.

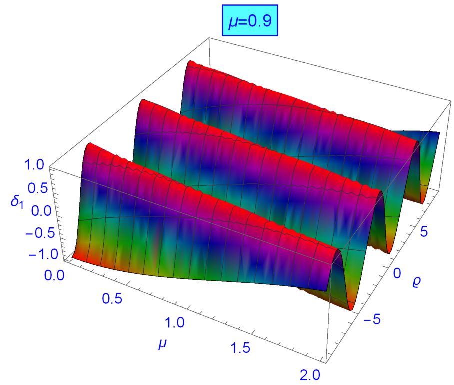


(a) (b)

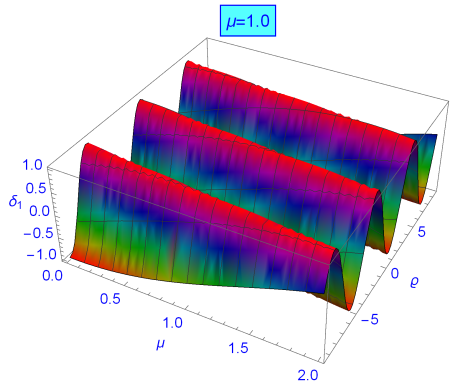
Figure 6: For problem 4.3, the 2D diagrams of App-Ss for various levels of μ and Ex-Ss in the range $\varphi \in [0, 1.0]$ at $\varrho = 0.1$ are shown for: (a) $\delta_1(\varrho, \varphi)$; (b) $\delta_2(\varrho, \varphi)$.



(a) (b)



(c)



(d)

Figure 7: The 3D diagrams of App-S for various levels of μ in the range $\varphi \in [0, 2.0]$ and $-3\pi \leq \varrho \leq 3\pi$ for Problem 4.2 of $\delta_1(\varrho, \varphi)$ are as follows: (a) $\mu = 0.7$; (b) $\mu = 0.8$; (c) $\mu = 0.9$; and (d) $\mu = 1.0$.

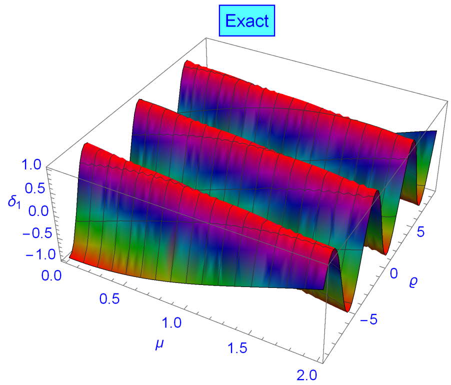
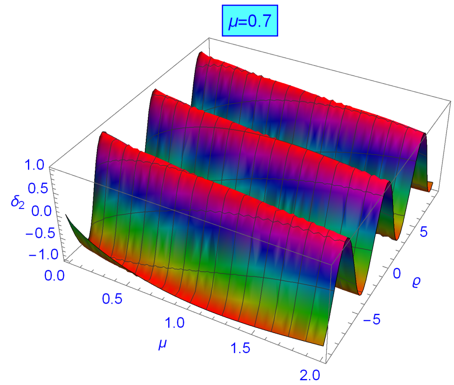
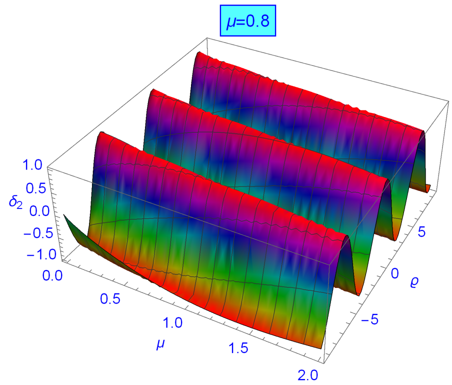


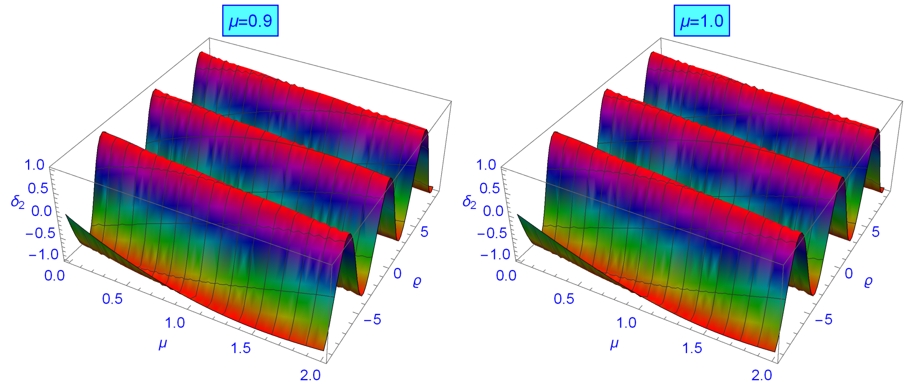
Figure 8: For problem 4.2, the 3D diagrams of Ex-S in the range $\varphi \in [0, 2.0]$ and $-3\pi \leq \varrho \leq 3\pi$ for $\delta_1(\varrho, \varphi)$ are shown.



(a)



(b)



(c) (d)

Figure 9: The 3D diagrams of App-S for various levels of μ in the range $\varphi \in [0, 2.0]$ and $-3\pi \leq \varrho \leq 3\pi$ for Problem 4.2 of $\delta_2(\varrho, \varphi)$ are as follows: (a) $\mu = 0.7$; (b) $\mu = 0.8$; (c) $\mu = 0.9$; and (d) $\mu = 1.0$.

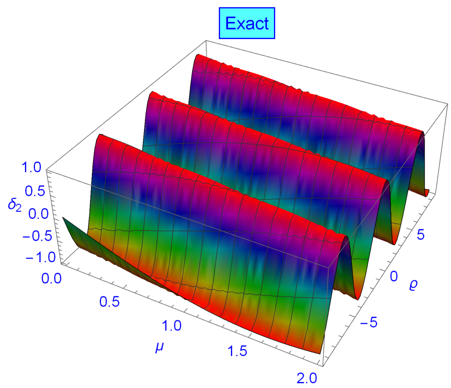
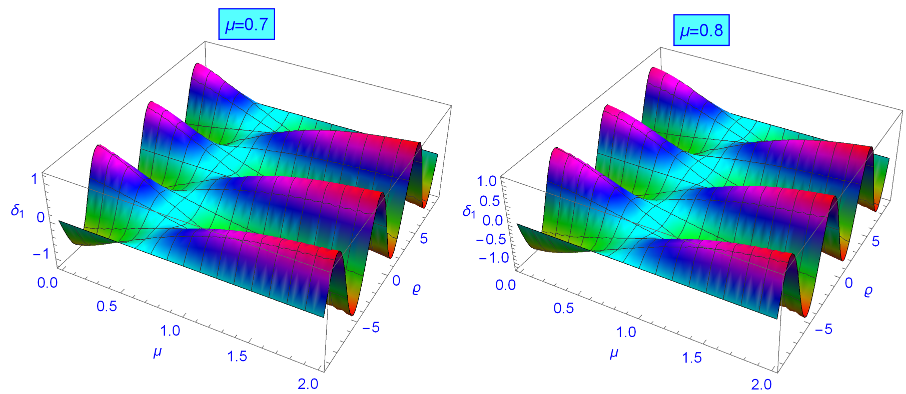
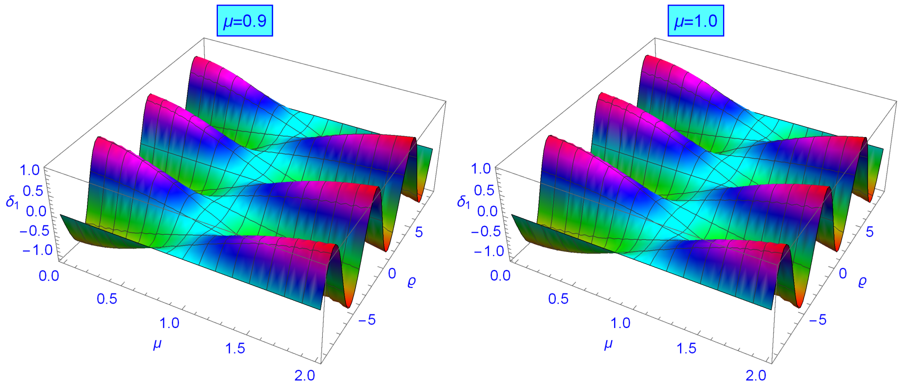


Figure 10: For problem 4.2, the 3D diagrams of Ex-S in the range $\varphi \in [0, 2.0]$ and $-3\pi \leq \varrho \leq 3\pi$ for $\delta_2(\varrho, \varphi)$ are shown.



(a) (b)



(c) (d)

Figure 11: The 3D diagrams of App-S for various levels of μ in the range $\varphi \in [0, 2.0]$ and $-3\pi \leq \varrho \leq 3\pi$ for Problem 4.3 of $\delta_1(\varrho, \varphi)$ are as follows: (a) $\mu = 0.7$; (b) $\mu = 0.8$; (c) $\mu = 0.9$; and (d) $\mu = 1.0$.

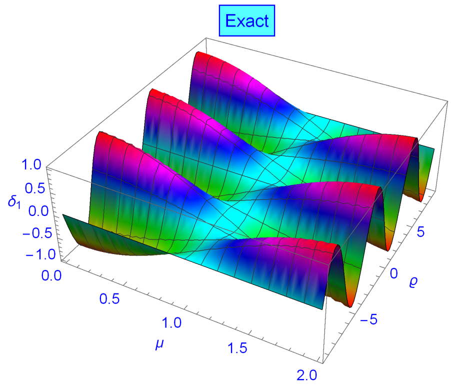
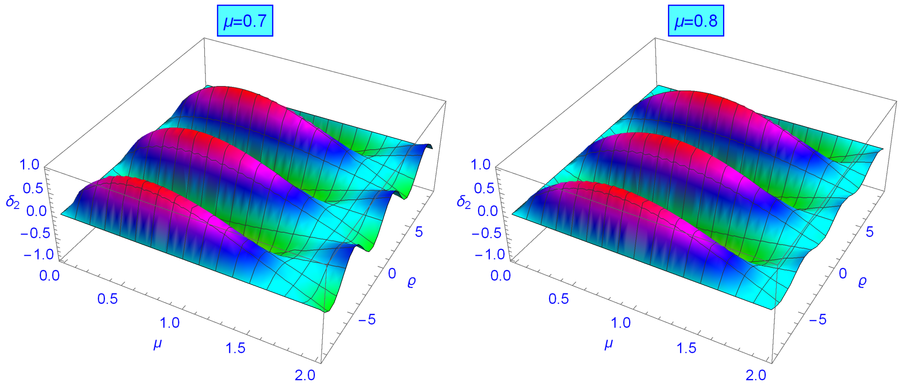
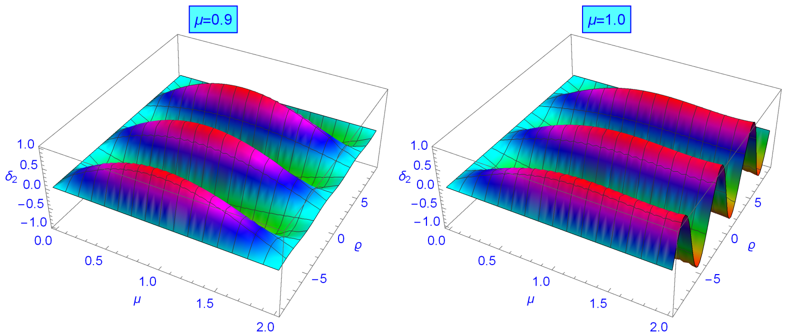


Figure 12: For problem 4.3, the 3D diagrams of Ex-S in the range $\varphi \in [0, 2.0]$ and $-3\pi \leq \varrho \leq 3\pi$ for $\delta_1(\varrho, \varphi)$ are shown.



(a) (b)



(c) (d)

Figure 13: The 3D diagrams of App-S for various levels of μ in the range $\varphi \in [0, 2.0]$ and $-3\pi \leq \varrho \leq 3\pi$ for Problem 4.3 of $\delta_2(\varrho, \varphi)$ are as follows: (a) $\mu = 0.7$; (b) $\mu = 0.8$; (c) $\mu = 0.9$; and (d) $\mu = 1.0$.

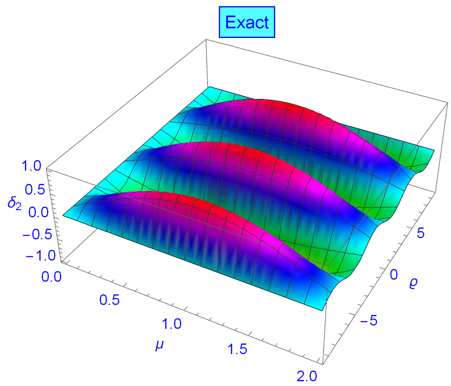


Figure 14: For problem 4.3, the 3D diagrams of Ex-S in the range $\varphi \in [0, 2.0]$ and $-3\pi \leq \varrho \leq 3\pi$ for $\delta_2(\varrho, \varphi)$ are shown.

The numerical values of Abs-E and Rel-E are shown in Tables 1 and 2 for the 5th-step App-S and Ex-S of the real $(\delta_1(\varrho, \varphi))$ and imaginary $(\delta_2(\varrho, \varphi))$ parts of $\delta(\varrho, \varphi)$ for problems 4.2 and 4.3, respectively, with $\mu = 1$ and $\varrho = 0.2$. These tables demonstrate that the App-Ss and Ex-Ss are nearly in agreement, confirming the accuracy of SADM.

Table 1: The Abs-E and Rel-E for $\delta_1(\varrho,\varphi)$ and $\delta_2(\varrho,\varphi)$ at $\mu=1$ for problem 4.2.

φ	$\|\delta_1-\delta_1^5\|$	$\frac{\|\delta_1-\delta_1^5\|}{\|\delta_1\|}$	$\|\delta_2-\delta_2^5\|$	$\frac{\|\delta_2-\delta_2^5\|}{\|\delta_2\|}$
0.1	1.35702×10^{-9}	1.42046×10^{-9}	2.95323×10^{-10}	9.99334×10^{-10}
0.2	8.65506×10^{-8}	9.39683×10^{-8}	2.01346×10^{-8}	5.17042×10^{-8}
0.3	9.82114×10^{-7}	1.11911×10^{-6}	2.43305×10^{-7}	5.07492×10^{-7}
0.4	5.49515×10^{-6}	6.65808×10^{-6}	1.44488×10^{-6}	2.55892×10^{-6}
0.5	2.08672×10^{-5}	2.72831×10^{-5}	5.80614×10^{-6}	9.01270×10^{-6}
0.6	6.20037×10^{-5}	8.89954×10^{-5}	1.82078×10^{-5}	2.53818×10^{-5}
0.7	1.55526×10^{-4}	2.50199×10^{-4}	4.80863×10^{-5}	6.13872×10^{-6}
0.8	3.44589×10^{-4}	6.37771×10^{-4}	1.11933×10^{-4}	1.33020×10^{-4}
0.9	6.94386×10^{-4}	1.53085×10^{-3}	2.36508×10^{-4}	2.65379×10^{-4}
1.0	1.29829×10^{-3}	3.58289×10^{-3}	4.62838×10^{-6}	4.96586×10^{-4}

Table 2: The Abs-E and Rel-E for $\delta_1(\varrho,\varphi)$ and $\delta_2(\varrho,\varphi)$ at $\mu=1$ for problem 4.3.

φ	$\|\delta_1-\delta_1^5\|$	$\frac{\|\delta_1-\delta_1^5\|}{\|\delta_1\|}$	$\|\delta_2-\delta_2^5\|$	$\frac{\|\delta_2-\delta_2^5\|}{\|\delta_2\|}$
0.1	2.77556×10^{-17}	1.41294×10^{-16}	3.46945×10^{-18}	1.16861×10^{-16}
0.2	2.22045×10^{-16}	1.16991×10^{-15}	0.00000	0.00000
0.3	2.85605×10^{-14}	1.59653×10^{-13}	9.99201×10^{-16}	1.15629×10^{-14}
0.4	9.01057×10^{-13}	5.49529×10^{-12}	4.16195×10^{-14}	3.71016×10^{-13}
0.5	1.30975×10^{-11}	7.98778×10^{-11}	7.55923×10^{-13}	5.58203×10^{-12}
0.6	1.16620×10^{-10}	9.44332×10^{-10}	8.07848×10^{-12}	5.19106×10^{-11}
0.7	7.40353×10^{-10}	7.48950×10^{-9}	5.98459×10^{-11}	3.47274×10^{-10}
0.8	3.66893×10^{-9}	5.09649×10^{-8}	3.39027×10^{-10}	1.83092×10^{-9}
0.9	1.50474×10^{-8}	3.45838×10^{-7}	1.56469×10^{-9}	8.07183×10^{-9}
1.0	5.31541×10^{-8}	3.78232×10^{-6}	6.14324×10^{-9}	3.09996×10^{-8}

6. Conclusions

To the best of the authors' knowledge, no research has yet solved FSDEs using the ADM with the ST in the sense of Con-FrD. In this research, we addressed this gap by solving linear and nonlinear FSDEs using a hybrid approach that combines ADM and ST, applied in the sense of Con-FrD. The effectiveness of the SADM is demonstrated through graphical and numerical results, showing that the App-Ss obtained via the SADM are in complete agreement with the Ex-Ss. Tables 1 and 2 provide numerical evidence of the correctness of our approach by comparing the App-Ss and Ex-Ss through Abs-E and Rel-E. Furthermore, a comparison is made between the results obtained from SADM and those from alternative techniques, including the HAM and RPSM. The comparison demonstrates a high degree of agreement with these techniques, suggesting that SADM is a good substitute for Cap-FD-based methods in solving FSDEs. Additionally, we can conclude that Con-FD serves as a suitable alternative to Cap-FD for modeling FSDEs.

The key features of the SADM distinguish it from other methods that provide approximate solutions. This method eliminates the need for assumptions about physical parameters, making it applicable to both weakly and strongly nonlinear problems and addressing some limitations of perturbation techniques. The SADM allows for the derivation of FPS solutions for FODEs without requiring perturbation, linearization, or discretization, unlike other approximate solution methods. The efficiency of the SADM underscores its computational strength, making it a valuable alternative to Cap-FD-based methods for solving FSDEs. We also found that the Con-FD effectively replaces the Cap-FD for modeling time-FSDEs. Overall, the SADM proves to be user-friendly, accurate, and efficient. Moreover, this method is versatile and can be applied to a range of ordinary and partial FODEs.

In the future, we intend to employ SADM to solve various nonlinear fractional models arising in biological systems.

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