

An Invitation to Proofs Without Words

Claudi Alsina¹ and Roger B. Nelsen^{2*}

¹ *Universitat Politècnica de Catalunya, Secció de Matemàtiques i Informàtica, Barcelona, Spain*

² *Lewis & Clark College, Department of Mathematical Sciences, Portland Oregon, USA*

Abstract. Proofs without words are pictures or diagrams that help the reader see *why* a particular mathematical statement may be true, and also see *how* one might begin to go about proving it true. In some instances a proof without words may include an equation or two to guide the reader, but the emphasis is clearly on providing visual clues to stimulate mathematical thought. While proofs without words can be employed in many areas of mathematics (geometry, number theory, trigonometry, calculus, inequalities, and so on) in our “invitation” we examine only one area: elementary combinatorics. In this article we use combinatorial proof methods based on two simple counting principles (the Fubini principle and the Cantor principle) to wordlessly prove several simple theorems about the natural numbers.

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1. Introduction

What are “proofs without words”? As you will see from this article, the question does not have a simple, concise answer. Generally, proofs without words are pictures or diagrams that help the reader see *why* a particular mathematical statement may be true, and also to see *how* one might begin to go about proving it true. As Yuri Ivanovich Manin said, “A good proof is one that makes us wiser,” a sentiment echoed by Andrew Gleason: “Proofs really aren’t there to convince you that something is true - they’re there to show you why it is true.”

Proofs without words (PWWs) are regular features in two journals published by the Mathematical Association of America. PWWs began to appear in *Mathematics Magazine* about 1975, and in the *College Mathematics Journal* about ten years later. Many of these appear in two collections of PWWs published by the Mathematical Association of America [8, 9].

*Corresponding author.

Email address: claudio.alsina@upc.edu (C. Alsina), nelsen@clark.edu (R. Nelsen)

But PWWs are not recent innovations—they have been around for a very long time, perhaps first appearing in ancient Greece and China, and later in tenth century Arabia and renaissance Italy. Today PWWs regularly appear in journals published around the world and on the World Wide Web.

Some argue that PWWs are not really “proofs,” nor, for that matter, are they “without words,” on account of equations which often accompany a PWW. Martin Gardner, in his popular “Mathematical Games” column in the October 1973 issue of *Scientific American*, discussed PWWs as “look - see” diagrams. He said “in many cases a dull proof can be supplemented by a geometric analog so simple and beautiful that the truth of a theorem is almost seen at a glance.” It is in that spirit that we write this “invitation” to PWWs. In some instances we include an equation or two to guide the reader, but the emphasis is clearly on providing visual clues to stimulate mathematical thought. We encourage the reader to think about how the picture “proves” the theorem before reading on. However, in each case we have included a short description of what we hope the reader sees in each picture.

We believe there is a role for PWWs in mathematics classrooms from elementary schools to universities. The ability to visualize is essential for success in mathematics, and George Pólya’s “Draw a figure. . .” is classic pedagogical advice.

2. Combinatorial Proofs

PWWs can be employed in many areas of mathematics, to prove theorems in geometry, number theory, trigonometry, calculus, inequalities, and so on. In our “invitation” we will examine only one area: elementary combinatorics. In many theorems concerning the natural numbers $\{1, 2, \dots\}$, insight can be gained by representing the numbers as sets of objects. Since the particular choice of object is unimportant, in PWWs we usually use dots, squares, balls, cubes, and other easily drawn objects.

In this article we will use combinatorial proof methods based on two simple counting principles that can be applied to representations of natural numbers by sets of objects. The principles are:

1. If you count the objects in a set in two different ways, you will obtain the same result; and
2. If two sets are in one-to-one correspondence, then they have the same number of elements.

The first principle has been called the *Fubini principle* [11] after the theorem in multivariable calculus concerning exchanging the order of integration in iterated integrals. We call the second the *Cantor principle*, after Georg Cantor (1845 - 1918), who used it extensively in his investigations into the cardinality of infinite sets. The two proof techniques are also known as the *double - counting method* and the *bijection method*, respectively.

3. Figurate Numbers

The idea of representing a number by a set of objects (perhaps as pebbles on the beach) dates back at least to the ancient Greeks. When that representation takes the shape of a polygon such as a triangle or a square, the number is often called a *figurate number*. We begin with some theorems and proofs about the simplest figurate numbers: triangular numbers and squares.

Nearly every biography of the great mathematician Carl Friedrich Gauss (1777 - 1855) relates the following story. When Gauss was about ten years old, his arithmetic teacher asked the students in class to compute the sum $1 + 2 + 3 + \dots + 100$, anticipating this would keep the students busy for some time. He barely finished stating the problem when young Carl came forward and placed his slate on the teacher's desk, void of calculation, with the correct answer: 5050. When asked to explain, Gauss admitted he recognized the pattern $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, and so on to $50 + 51 = 101$. Since there are fifty such pairs, the sum must be $50 \times 101 = 5050$. The pattern for the sum (adding the largest number to the smallest, the second largest to the second smallest, and so on) is illustrated in Figure 1, where the rows of balls represent positive integers.

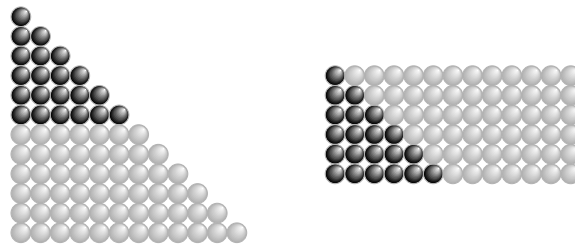


Figure 1

The number $t_n = 1 + 2 + 3 + \dots + n$ for a positive integer n is called the n^{th} *triangular number*, from the pattern of the dots on the left in Figure 1. Young Carl correctly computed $t_{100} = 5050$. However, this solution works only for n even, so we first prove

Theorem 1. For all $n \geq 1$, $t_n = \frac{n(n+1)}{2}$.

Proof. See Figure 2.

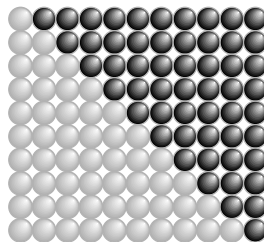


Figure 2

We arrange two copies of t_n to form a rectangular array of balls in n rows and $n + 1$ columns. Then we have $2t_n = n(n + 1)$, or $t_n = n(n + 1)/2$.

The counting procedure in the preceding combinatorial proof is the Fubini principle. We employ the same procedure to prove that sums of odd numbers are squares.

Theorem 2. For all $n \geq 1$, $1 + 3 + 5 + \dots + (2n - 1) = n^2$.

Proof. We give two combinatorial proofs in Figure 3.

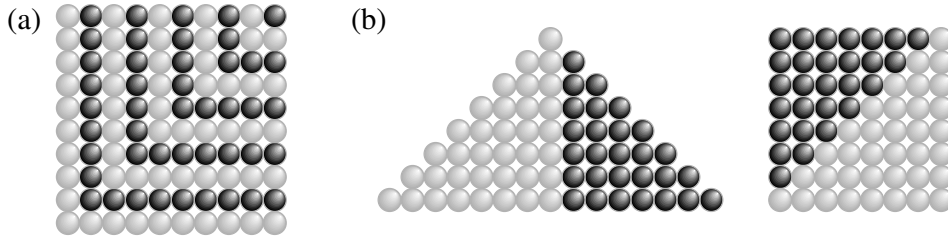


Figure 3

In Figure 3a, we count the balls in two ways, first as a square array of balls, and then by the number of balls in each L-shaped region of similarly colored balls (the Fubini principle). In Figure 3b, we see a one-to one correspondence (illustrated by the color of the balls) between a triangular array of balls in rows with $1, 3, 5, \dots, (2n - 1)$ balls, and a square array of balls (the Cantor principle).

The same idea can be employed in three dimensions to establish the following sequence of identities:

$$\begin{aligned}
 1 + 2 &= 3, \\
 4 + 5 + 6 &= 7 + 8, \\
 9 + 10 + 11 + 12 &= 13 + 14 + 15, \text{ etc.}
 \end{aligned}$$

Note that each row begins with a square number. The general pattern

$$n^2 + (n^2 + 1) + \dots + (n^2 + n) = (n^2 + n + 1) + \dots + (n^2 + 2n)$$

can be proved by induction, but the following visual proof is much nicer.

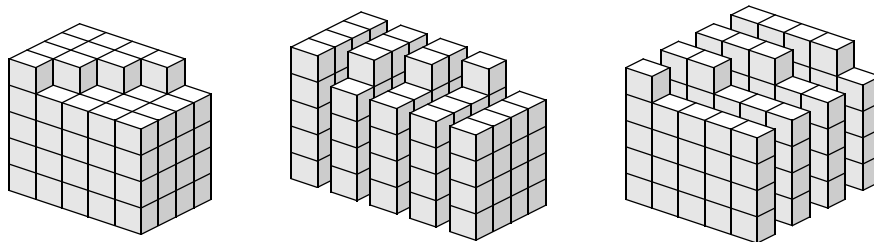


Figure 4

In Figure 4, we see the $n = 4$ version of the identity where counting the number of small cubes in the pile in two different ways yields $16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$.

There are many nice relationships between triangular and square numbers. The simplest is perhaps the one illustrated in the right side of Figure 3b: $t_{n-1} + t_n = n^2$. Two more are given in the following lemma (setting $t_0 = 0$ for convenience):

Lemma 1. For all $n \geq 0$, (a) $8t_n + 1 = (2n + 1)^2$, and (b) $9t_n + 1 = t_{3n+1}$.

Proof. See Figure 5 (where we have replaced balls by squares).

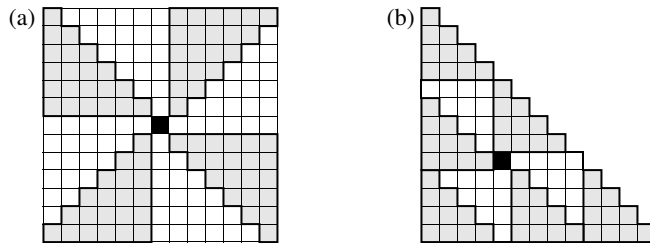


Figure 5

Lemma 1 enables us to prove the following two theorems.

Theorem 3. There are infinitely many numbers that are simultaneously square and triangular.

Proof. Observe that

$$t_{8t_n} = \frac{8t_n(8t_n + 1)}{2} = 4t_n(2n + 1)^2,$$

so if t_n is square, then so is t_{8t_n} . Since $t_1 = 1$, this relation generates an infinite sequence of square triangular numbers, e.g., $t_8 = 6^2$ and $t_{288} = 204^2$. However, there are square triangular numbers such as $t_{49} = 35^2$ and $t_{1681} = 1189^2$ that are not in this sequence.

Theorem 4. Sums of powers of 9 are triangular numbers, i.e., for all $n \geq 0$,

$$1 + 9 + 9^2 + \dots + 9^n = t_{1+3+3^2+\dots+3^n}$$

Proof. See Figure 6.

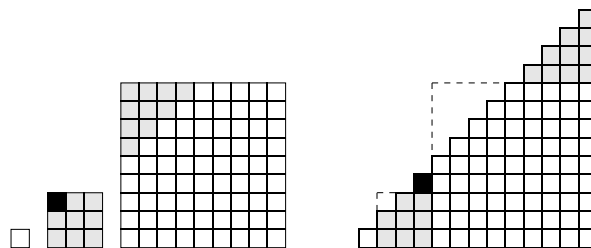


Figure 6

As a consequence, in base 9 the numbers 1, 11, 111, 1111, ... are all triangular.

The next theorem presents a companion to the identity $t_{n-1} + t_n = n^2$.

Theorem 5. *The sum of the squares of consecutive triangular numbers is a triangular number, i.e., $t_{n-1}^2 + t_n^2 = t_{n^2}$ for all $n \geq 1$.*

Proof. See Figure 7, where we illustrate the square of a triangular number as a triangular array of triangular numbers.

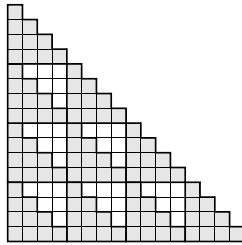


Figure 7

You may have noticed that the n^{th} triangular number is a binomial coefficient, i.e., $t_n = \binom{n+1}{2}$. One explanation for this is that each is equal to $n(n+1)/2$, but this answer sheds little light on *why* it is true. Here is a better explanation using the Cantor principle:

Theorem 6. *There exists a one-to-one correspondence between a set of t_n objects and the set of two-element subsets of a set with $n + 1$ objects.*

Proof. See Figure 8 [6], and recall that the binomial coefficient $\binom{k}{2}$ is the number of ways to choose 2 elements from a set of k elements. The arrows denote the correspondence between an element of the set with t_n elements and a pair of elements from a set of $n + 1$ elements.

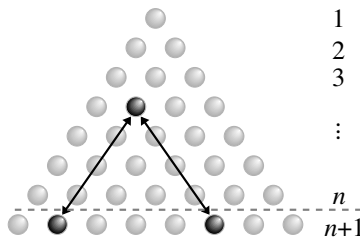


Figure 8

4. Sums of Squares, Triangular Numbers, and Cubes

Having examined triangular numbers and squares as sums of integers and sums of odd integers, we now consider sums of triangular numbers and sums of squares.

Theorem 7. For all $n \geq 1$, $1^2 + 2^2 + 3^2 + \dots + n^2 = n(n + 1)(2n + 1)/6$.

Proof. We give two proofs. The first is in Figure 9.

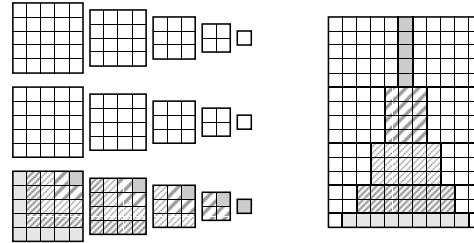


Figure 9

We exhibit a one-to-one correspondence between *three* copies of $1^2 + 2^2 + 3^2 + \dots + n^2$ and a rectangle whose dimensions are $2n + 1$ and $1 + 2 + \dots + n = n(n + 1)/2$ [4]. Hence $3(1^2 + 2^2 + 3^2 + \dots + n^2) = (2n + 1)(1 + 2 + \dots + n)$ from which the result now follows.

For a second proof, see Figure 10 [5].

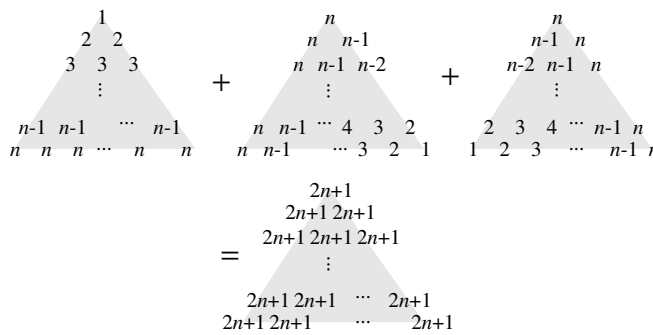


Figure 10

Here we write each square k^2 as a sum of k k s, then place those numbers in a triangular array, create two more arrays by rotating the triangular array by 120° and 240° , and add corresponding entries in each triangular array.

Theorem 8. For all $n \geq 1$, $t_1 + t_2 + t_3 + \dots + t_n = n(n + 1)(n + 2)/6$.

Proof. See Figure 11.

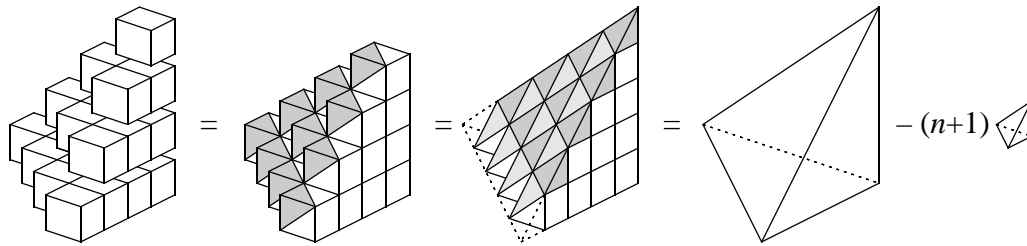


Figure 11

Here we stack layers of unit cubes to represent the triangular numbers. The sum of the triangular numbers is total number of cubes, which is the same as the total volume of the cubes. To compute the volume, we “slice” off small pyramids (shaded gray) and place each small pyramid on the top of the cube from which it came. The result is a large right triangular pyramid minus some smaller right triangular pyramids along one edge of the base. Thus

$$t_1 + t_2 + t_3 + \dots + t_n = \frac{1}{6}(n + 1)^3 - (n + 1)\frac{1}{6} = \frac{n(n + 1)(n + 2)}{6}.$$

In the above proof we evaluated the sum of the first n triangular numbers by computing volumes of pyramids. This is actually an extension of the Fubini principle from simple enumeration of objects to additive measures such as length, area and volume. The volume version of the Fubini principle is: *computing the volume of an object in two different ways yields the same number*; and similarly for length and area. We cannot, however, extend the Cantor principle to additive measures - for example, one can construct a one-to-one correspondence between the points on two line segments with different lengths.

Theorem 9. For all $n \geq 1$, $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2 = t_n^2$.

Proof. Again, we give two proofs. In the first, we represent k^3 as k copies of a square with area k^2 to establish the identity [3, 7].

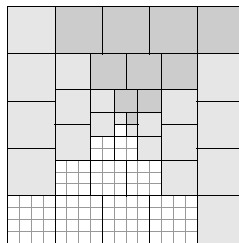


Figure 12

In Figure 12, we have $4(1^3 + 2^3 + 3^3 + \dots + n^3) = [n(n + 1)]^2$ (for $n = 4$).

For the second proof, we use the fact that $1+2+3+\dots+(n-1)+n+(n-1)+\dots+2+1 = n^2$ (we leave it as an exercise for the reader to draw a picture with balls in a square array and

count the balls by diagonals in the square array) and consider a square array of *numbers* (rather than balls) in which the element in row i and column j is ij , and sum the numbers in two different ways. See Figure 13 [10].

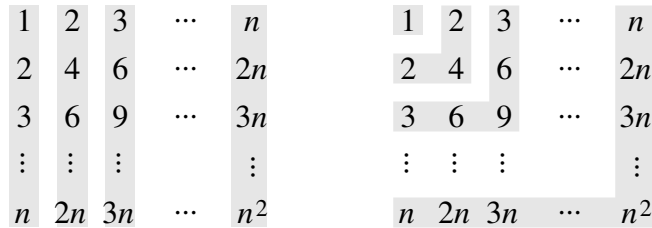


Figure 13

Summing by columns yields

$$\sum_{i=1}^n i + 2 \left(\sum_{i=1}^n i \right) + \dots + n \left(\sum_{i=1}^n i \right) = \left(\sum_{i=1}^n i \right)^2,$$

while summing by the L-shaped shaded regions yields $1 \cdot 1^2 + 2 \cdot 2^2 + \dots + n \cdot n^2 = \sum_{i=1}^n i^3$.

We conclude this section with a theorem representing a cube as a double sum of integers.

Theorem 10. For all $n \geq 1$, $\sum_{i=1}^n \sum_{j=1}^n (i + j + 1) = n^3$.

Proof. We represent the double sum as a collection of unit cubes and compute the volume of a rectangular box composed of two copies of the collection. See Figure 14.

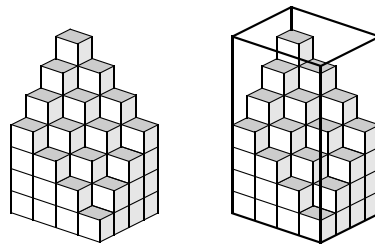


Figure 14

Observe that two copies of the sum $S = \sum_{i=1}^n \sum_{j=1}^n (i + j + 1)$ fit into a rectangular box with base n^2 and height $2n$, hence computing the volume of the box in two ways yields $2S = 2n^3$, or $S = n^3$.

5. Conclusion

In this short survey we have looked at visual proofs using representations of numbers by sets of objects. Of course, there are many other ways to represent numbers (not necessarily

natural numbers), including representing them as lengths of segments, areas of plane figures, volumes of objects. With such representations, other techniques can be employed in the proofs, such as rotations, translations, reflections, and other transformations that preserve length, area, or volume. For a comprehensive survey of these techniques (and many others), see the books *Math Made Visual: Creating Images for Understanding Mathematics* and *When Less Is More: Visualizing Basic Inequalities* [1, 2].

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