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An Application of Generalized Fuzzy Ideals in Ordered Semigroups

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Abstract. Ordered semigroups are algebraic systems consisting of a nonempty set, an associative binary operation, and a partial order compatible with this binary operation. This concept is a generalization of semigroups. One mathematical tool used to study ordered semigroups is the concept of ideals. It turns out that ordered semigroups can be decomposed based on their regularities using various kinds of ideals. The concept of fuzzy sets is one of the mathematical tools used to investigate ordered semigroups, specifically through so-called fuzzy ideals, which are more appropriate than set-theoretical ideals. It is known that the concepts of (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals are generalizations of various types of ideals and many kinds of fuzzy ideals in ordered semigroups. In this paper, we apply the notions of (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals to classify ordered semigroups into classes depended on their regularities, using the meaning of characteristic functions.

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1. Introduction

Ordered semigroups constitute an algebraic structure comprising a binary operation satisfying associative property and a partial order with the compatibility (see [2, 8]). This concept extends the notion of semigroups, prompting numerous researchers to explore various properties of semigroups in an ordered semigroup setting. Studying ideals is key to examining several properties inherent to ordered semigroups. Let us briefly outline the historical context of the set-theoretical ideals under consideration in this paper.

The concept of bi-ideals extends left and right ideals in ordered semigroups, started by Kehayopulu. The author studied ordered semigroups that do not contain proper bi-ideals (see [14]). This property resonates with group theory, emphasizing the notion of

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bi-ideals significance in ordered semigroups. Subsequently, in 2013, Saritha [45] considered prime and semiprime properties of bi-ideals. Moreover, bi-ideals satisfying prime and semiprime properties were characterized. Building upon Saritha's insights, Gu [9] introduced quasi-prime properties for bi-ideals, which were then utilized to characterize regular and intra-regular ordered semigroups. Hansda [10] expanded upon Gu's results, further characterizing regular and completely regular ordered semigroups based on the minimality of bi-ideals. Kehayopulu introduced the notion of interior ideals in ordered semigroups in 1999. It was demonstrated that interior ideals coincide with two-sided ideals in certain classes of ordered semigroups (see [15]).

Sanborisoot and Changphas initially generalized the concept of bi-ideals in ordered semigroups to (m, n)-ideals. Subsequently, the notion of (m, n)-ideals was employed to characterize a class of ordered semigroups known as (m, n)-regular ordered semigroups (see [44]). In 2015, Bussaban and Changphas [3] highlighted that in regular duo ordered semigroups, the concept of (m, n)-ideals coincides with two-sided ideals. Luangchaisri and Changphas [37] further extended bi-ideals results to (m, n)-ideals in 2019. Following this extension, Tiprachot et al. [49] generalized the notion of interior ideals to n-interior ideals in ordered semigroups in 2022. The authors characterized several classes of ordered semigroups by combining (m, n)-ideals and n-interior ideals. The characterization of ordered semigroups using (m, n)-ideals and n-interior ideals was further extended by (m, n)-ideal elements and n-interior ideal elements (see [36]). Additionally, they provided a comprehensive overview of the classification of ordered semigroups through α-ideals (see [48]).

In 1965, Zadeh [51] introduced the concept of fuzzy sets that can deal with problems involving uncertain conditions. Fuzzy sets have widely applied across various scientific and engineering investigations. Particularly in mathematics, they play as a tool to analyze the properties of several algebraic systems, including semigroups, ordered semigroups, groups, semirings, ordered semirings, and rings (see [1, 4, 5, 35, 38–41]). Given the importance of ideals in exploring ordered semigroups, we offer a brief overview of how their fuzzy counterparts are utilized in this process. In the early 2000s, fuzzy sets were applied to ordered groupoids, which led to their use in considering ordered semigroups (see [19]). In 2003, Kehayopulu and Tsingelis [20] showed that any ordered semigroup embeds into a fuzzy ordered semigroup, which was a significant result. Two years later, they introduced the concept of fuzzy bi-ideals in ordered semigroups. The authors [21] used them to characterize different classes of ordered semigroups, for example, left (right) simple, completely regular, and strongly regular. In 2006, fuzzy left (right) ideals [23] were used to study left (right) regular and intra-regular ordered semigroups. Over two decades, several studies have characterized special classes of ordered semigroups using fuzzy ideals in different combinations (see [16, 24, 25, 46, 50, 52]).

Khan and Shabir [31] developed the concept of fuzzy ideals in ordered semigroups by introducing the "belong to" ($\in$) and "quasi-coincident" (q) relations. They introduced various notions of $(\in, \in \vee q)$-fuzzy ideals in ordered semigroups, including $(\in, \in \vee q)$-fuzzy left (right, two-sided, interior) ideals. Subsequently, Jun et al. [11] defined the concept of $(\in, \in \vee q)$-fuzzy bi-ideals, which generalize $(\in, \in \vee q)$-fuzzy left and right ideals. These concepts were applied to characterize ordered semigroups satisfying regular and intra-regular conditions. In 2012, Khan et al. [28] further investigated the properties of $(\in, \in \vee q)$-fuzzy left and right ideals in-depth and utilized them to characterize regular ordered semigroups. The concepts of $(\in, \in \vee q)$-fuzzy ideals in ordered semigroups were extended to $(\in, \in \vee q_k)$-fuzzy ideals, where $k \in [0, 1)$ initially by Khan et al. They defined the notions of $(\in, \in \vee q)$-fuzzy left (right and generalized bi-) ideals in ordered semigroups and utilized them to characterize several classes of ordered semigroups (see [27]). The applications of such fuzzy ideals for classifying ordered semigroups appeared in at least two papers by Khan et al. and Khan et al. (see [29, 32]) Later, Tang and Xie [47] examined prime property of $(\in, \in \vee q_k)$-fuzzy ideals. The concepts of $(\in, \in \vee q_k)$-fuzzy ideals in ordered semigroups were independently extended to $(\in, \in \vee (k^*, q_k))$-fuzzy ideals and $(\in, \in \vee q_k^\delta)$-fuzzy ideals by Khan et al. [34] and Ali Khan et al. [33], respectively. Interestingly, it was discovered that these two concepts coincide. In a recent development, Muhiuddin et al. [42] generalized the notion of $(\in, \in \vee q_k)$-fuzzy ideals to $(\in, \in \vee q_k)$-fuzzy (m, n)-ideals. A class of ordered semigroups, (m, n)-regular ordered semigroups, were characterized by this extension.

Another generalization of the above-mentioned fuzzy ideals in ordered semigroups is the notion of (α, β)-fuzzy ideals, where $0 \leq \alpha < \beta \leq 1$. This notion was initiated by Feng and Corsini in 2012. They defined the concepts of (α, β)-fuzzy left (right, interior, quasi-, bi-) ideals and studied the fundamental properties of these ideals (see [7]). Independently, Khan et al. [30] introduced the notion of (α, β)-fuzzy bi-ideals and characterized completely regular ordered semigroups using this notion. Later, Feng and Corsini [6] discovered that in certain classes of ordered semigroups, for example regular and intra-regular ordered semigroups, the concepts of (α, β)-fuzzy two-sided ideals and (α, β)-fuzzy interior ideals coincide. Jun et al. [12] also characterized some particular classes of ordered semigroups using (α, β)-fuzzy ideals. Directly studying the structural properties of (α, β)-fuzzy ideals is complex. By applying the operation defined in [30], Lekkoksung et al. recently demonstrated that the algebraic structure comprising the set of all fuzzy sets on an ordered semigroup, along with this operation and a binary relation, forms a representation of an ordered semigroup. This result emphasizes the significance of investigating (α, β)-fuzzy ideals in ordered semigroups (see [35]). Subsequently, Davvaz et al. [4] introduced the concepts of (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals in ordered semigroups. It was discovered that several notions of ideals mentioned earlier are a particular case of these concepts. The authors also studied the properties of (α, β)-fuzzy (m, n)- (n-interior) ideals. Furthermore, the concepts of (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals were characterized by the operation $\circ_I$.

The above motivations provide a brief overview of the investigation into fuzzy ideals in ordered semigroups. Moreover, the importance of separating ordered semigroups into classes using these various fuzzy ideals is shown. In this paper, we build upon the investigations in [4, 35]. We apply the concepts of (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals to characterize all classes of ordered semigroups based on their regularities through characteristic functions. The outline of the paper is as follows. Section 2 presents the preliminary concepts used in this study. We revisit the definitions of (α, β)-

fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals, along with their significant results in Section 3. In Section 4, we apply these notions of fuzzy ideals to classify ordered semigroups.

2. Preliminaries

This section recalls the concept of ordered semigroups and fuzzy sets. Additionally, their preliminary results are provided.

Definition 1. *An* ordered semigroup $\langle S; \cdot, \leq \rangle$ *is an algebraic system consisting of a nonempty set* S*, an associative binary operation* $\cdot$ *on* S *and a partial order relation on* S *satisfying the compatibility. That is, for any* $a, b \in S$ $a \leq b$ *implies* $a \cdot c \leq b \cdot c$ *and* $c \cdot a \leq c \cdot b$ *for all* $c \in S$*.*

We observe that any semigroup $\langle S; \cdot \rangle$ can be regarded as an ordered semigroup that defines the equality relation as the partial order compatible with the operation of such a semigroup.

The product $x \cdot y$ of any elements x and y of S is usually denoted by xy, and the n-product $\underbrace{a \cdots a}_{n\text{-time}}$ of any element a of S is denoted by a^n, where n is a positive integer. For convenience, we denote an ordered semigroup $\langle S; \cdot, \leq \rangle$ by $\mathbf{S}$ the boldface letter of its underlying set.

Let $\mathbf{S}$ be an ordered semigroup, and A, B subsets of S. We define the sets AB and $(A]$ by $AB := \{ab : a \in A \text{ and } b \in B\}$ and $(A] := \{x \in S : x \leq a \text{ for some } a \in A\}$. The importance of the operator $(\cdot]$ is shown as follows.

Lemma 1 ([13])**.** *Let* $\mathbf{S}$ *be an ordered semigroup,* A, B, A_i *subsets of* S*, where* $i \in I$*. Then, we obtain the following statements.*

(i) $A \subseteq (A]$.

(ii) $A \subseteq B$ *implies* $(A] \subseteq (B]$.

(iii) $(A](B] \subseteq (AB]$.

(iv) $\left(\bigcup_{i \in I} A_i\right] = \bigcup_{i \in I} (A_i]$.

(v) $\left(\bigcap_{i \in I} A_i\right] \subseteq \bigcap_{i \in I} (A_i]$.

(vi) $((A]] = (A]$.

Let $\mathbf{S}$ be an ordered semigroup, A a nonempty subset of S such that $(A] \subseteq A$, and m, n nonnegative integers not all zero. Then, A is said to be:

(i) a *subsemigroup* of $\mathbf{S}$ if $AA \subseteq A$;

(ii) an (m, n)*-ideal* [44] of $\mathbf{S}$ if A is a subsemigroup of $\mathbf{S}$ and $A^m S A^n \subseteq A$;

(iii) an n*-interior ideal* [49] of $\mathbf{S}$ if A is a subsemigroup of $\mathbf{S}$ and $S A^n S \subseteq A$.

We note that $A^0 S = S = S A^0$. The above definition shows that any left, right, bi-, and interior ideal can be considered as a $(0, 1)$-, $(1, 0)$-, $(1, 1)$-, and 1-interior ideal, respectively.

We note that for any positive integers m and n with $m \leq n$, we obtain that any m-interior ideal is an n-interior ideal. The converse of this statement is not true to the evidence provided in [49, Example 3.2]. Similarly, for any positive integers m_1, m_2, n_1, n_2 with $m_1 \leq m_2$ and $n_1 \leq n_2$, we obtain that any (m_1, n_1)-ideal is an (m_2, n_2)-ideal. The converse of this statement does not generally hold, as shown by the following example.

Example 1. *Let $S = \{0, 1, 2, 3, 4, 5, 6\}$. Define a binary operation $\cdot$ and a partial order $\leq$ on S as follows.*

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	1
5	0	0	0	0	0	0	2
6	0	0	1	0	0	4	0

and $\leq := \Delta_S \cup \{(0, 1)\}$, where Δ_S is the equality relation on S. Then, $\mathbf{S} := \langle S; \cdot, \leq \rangle$ is an ordered semigroup. Let $A = \{0, 6\}$. We can see that A is a $(2, 2)$-ideal of $\mathbf{S}$ but not a $(1, 1)$-ideal of $\mathbf{S}$.

Now, we recall the concept of fuzzy sets. Let X be a nonempty set. A *fuzzy set* f in X is a mapping $f : X \to [0, 1]$. We denote by $F(X)$ the set of all fuzzy sets in X.

Any subset A of X can be regarded as a fuzzy set χ_A in X defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $x \in A$. The fuzzy set χ_A is called the *characteristic function of A in X*. Since the codomain of a fuzzy set lies in the unit closed interval $[0, 1]$, for any $\alpha \in [0, 1]$ can be considered as a fuzzy set in X defined by $\alpha(x) := \alpha$ for all $x \in X$. For any fuzzy sets f and g in X, we define the operations $\cup$ and $\cap$ on $F(X)$ by $(f \cup g)(x) := f(x) \vee g(x)$ and $(f \cap g)(x) := f(x) \wedge g(x)$ for all $x \in X$, where $\vee$ and $\wedge$ is the supremum operation and infimum operation, respectively. A relation $\subseteq$ on $F(X)$ is defined by $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Now, we combine the concepts of ordered semigroups and fuzzy sets. In what follows, the notation $F(\mathbf{S})$ stands for the set of all fuzzy sets in S, where S is the underlying set of the ordered semigroups $\mathbf{S}$.

Let $\mathbf{S}$ be an ordered semigroup. For any $a \in S$, we define $\mathbf{S}_a := \{(u, v) \in S \times S : a \leq uv\}$. Given fuzzy sets $f, g \in F(\mathbf{S})$, we define

$$(f \circ g)(x) := \begin{cases} \bigvee_{(u,v) \in \mathbf{S}_a} \{f(u) \wedge g(v)\} & \text{if } \mathbf{S}_a \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

for all $x \in S$. It was proved in [20] by Kehayopulu and Tsingelis that $\langle F(\mathbf{S}); \circ, \subseteq\rangle$ is an ordered semigroup. Moreover, $\langle F(\mathbf{S}); \circ, \subseteq\rangle$ is a representation of an ordered semigroup $\mathbf{S}$. This means that any ordered semigroup $\mathbf{K}$ can be embedded into $\langle F(\mathbf{K}); \circ, \subseteq\rangle$. This finding emphasizes the significance of ordered semigroups induced by fuzzy sets, so-called fuzzy ordered semigroups.

Efforts have been made to construct ordered semigroups induced by fuzzy sets, which aim to represent ordered semigroups. As a result, the operation $\circ$ defined in (1) can be extended accordingly. Let $\mathbf{S}$ be an ordered semigroup and $\alpha, \beta \in [0, 1]$ with $\alpha < \beta$. For any $f \in F(\mathbf{S})$, the fuzzy set f_I with restricted range I in S is defined by $f_I(x) := [f(x) \wedge \beta] \vee \alpha$ for all $x \in S$. We note here that, for any $x \in S$, $f_I(x) \in [\alpha, \beta]$. Therefore, $(f_I)_I = f_I$. It is not difficult to observe that, for any $f, g \in F(\mathbf{S})$:

(i) $(f \cap g)_I = f_I \cap g_I$;

(ii) $f_I(x) \leq f_I(y)$ if $f(x) \leq f(y)$ for all $x, y \in S$;

(iii) $f_I(x) \leq g_I(y)$ if $f(x) \vee \alpha \leq g(y) \wedge \beta$ for all $x, y \in S$.

In [30], Khan et al. introduced operations $\cap_I, \cup_I$ and $\circ_I$ on $F(\mathbf{S})$ by $f \cap_I g := (f \cap g)_I, f \cup_I g := (f \cup g)_I$, and $f \circ_I g := (f \circ g)_I$. It is routine to verify that the following statements hold for all $f, g \in F(\mathbf{S})$.

(i) $f \cap_I g = f_I \cap g_I = f_I \cap_I g_I$.

(ii) $f \cup_I g = f_I \cup g_I = f_I \cup_I g_I$.

(iii) $f_I \circ_I g_I = f \circ_I g = (f \circ g)_I$.

(iv) $(f \circ_I g)(x) \geq (f_I \circ g_I)(x)$ for all $x \in S$. The equality is valid if $\mathbf{S}_x \neq \emptyset$.

By the product of fuzzy sets and the properties of fuzzy sets with restricted range I, we obtain the following result.

Proposition 1. *Let $\mathbf{S}$ be an ordered semigroup, f, g fuzzy sets in S, and $a \in S$. Suppose that $(u, v) \in \mathbf{S}_a$ for some $u, v \in S$. Then, we have*

$$(f \circ_I g)(a) \geq f_I(u) \wedge g_I(v).$$

In particular, $(\beta \circ_I f)(a) \geq f_I(v)$ and $(f \circ_I \beta)(a) \geq f_I(u)$.

By the associativity of $\circ$ and $\circ_I$, Proposition 1 can be naturally extended to yield the following corollary.

Corollary 1. *Let $\mathbf{S}$ be an ordered semigroup, $f_1, \ldots, f_n$ fuzzy sets in S, and $a \in S$. Suppose that $a \leq u_1 \cdots u_n$ for some $u_1, \ldots, u_n \in S$. Then, we have*

$$(f_1 \circ_I \cdots \circ_I f_n)(a) \geq (f_1)_I(u_1) \wedge \cdots \wedge (f_n)_I(u_n).$$

In 2023, Lekkoksung et al. [35] illustrated that any ordered semigroup $\mathbf{S}$ can be embedded into an ordered semigroup $\mathbf{F}^I(\mathbf{S}) = \langle F(\mathbf{S}); \circ_I, \subseteq\rangle$. With a slight modification of $\subseteq$ on $F(\mathbf{S})$, we obtain a new quasi-order relation $\subseteq_I$ defined by $f \subseteq_I g$ if and only if $f_I \subseteq g_I$. It is natural to obtain an equivalence relation $\equiv_I$ on $F(\mathbf{S})$ given by $f \equiv_I g$ if and only if $f \subseteq_I g$ and $g \subseteq_I f$. With the help of the equivalence relation $\equiv_I$, we obtain an associative operation $\diamond_I$ and a partial order $\sqsubseteq_I$ on $F(\mathbf{S})/{\equiv_I}$ defined by

$$f/{\equiv_I} \diamond_I g/{\equiv_I} := (f \circ_I g)/{\equiv_I}$$

and

$$f/{\equiv_I} \sqsubseteq_I g/{\equiv_I} \qquad \text{if and only if} \qquad f \subseteq_I g.$$

Then, the algebraic system $\mathbf{F}_I(\mathbf{S}) := \langle F(\mathbf{S})/{\equiv_I}; \diamond_I, \sqsubseteq_I\rangle$ is an ordered semigroup which is a representation of an ordered semigroup $\mathbf{S}$.

We conclude that the algebraic systems we obtained above are representations of any ordered semigroup, as presented by the following theorem.

Theorem 1 ([35]). *Any ordered semigroup* $\mathbf{S}$ *can be embedded into the algebraic systems* $\mathbf{F}^I(\mathbf{S})$ *and* $\mathbf{F}_I(\mathbf{S})$.

The above result demonstrates the importance of such ordered semigroups induced by fuzzy sets. Therefore, it is worth studying the algebraic properties of ordered semigroups by these algebraic systems.

3. The Concepts of (α, β)-Fuzzy (m, n)-Ideals and (α, β)-Fuzzy n-Interior Ideals

As we recall the concepts of (m, n)-ideals and n-interior ideals in ordered semigroups in the previous section, we recall the concepts of such ideals applied by fuzzy sets in this section.

Let $\mathbf{S}$ be an ordered semigroup, and $x_1, \ldots, x_n \in S$. We simply denote the product $x_1 \cdots x_n$ by x_1^n. A fuzzy set $f \in F(\mathbf{S})$ is said to be (α, β)-*strongly convex* if $f(x) \vee \alpha \geq f(y) \wedge \beta$ whenever $x \leq y$ for all $x, y \in S$. An (α, β)-*fuzzy subsemigroup* of $\mathbf{S}$ is a fuzzy set $f \in F(\mathbf{S})$ such that $f(xy) \vee \alpha \geq f(x) \wedge f(y) \wedge \beta$ for all $x, y \in S$.

Definition 2 ([4]). *Let* $\mathbf{S}$ *be an ordered semigroup. An* (α, β)*-strongly convex* (α, β)*-fuzzy subsemigroup* f *of* $\mathbf{S}$ *is called:*

(i) an (α, β)-fuzzy (m, n)-ideal *of* $\mathbf{S}$ *if*

$$f(x_1^m y z_1^n) \vee \alpha \geq \left(\bigwedge_{i=1}^{m} f(x_i)\right) \wedge \left(\bigwedge_{i=1}^{n} f(z_i)\right) \wedge \beta$$

for any $x_1, \ldots, x_m, y, z_1, \ldots, z_n \in S$;

(ii) an (α, β)-fuzzy n-interior ideal *of* $\mathbf{S}$ *if*

$$f(xy_1^n z) \vee \alpha \geq \left(\bigwedge_{i=1}^{n} f(y_i)\right) \wedge \beta$$

for any $x, y_1, \ldots, y_n, z \in S$.

By putting an appropriate α, β, m, and n, the above notions generalize several kinds of fuzzy ideals in ordered semigroups investigated in some literature described as follows. Any fuzzy left (resp., right, bi-, interior) ideal is a $(0,1)$-fuzzy $(0,1)$- (resp., $(1,0)$-, $(1,1)$-, 1-interior) ideal (see [19, 21, 22]). Any $(\in, \in \vee q)$-fuzzy left (resp., right, bi-, interior) ideal is a $(0, 0.5)$-fuzzy $(0,1)$- (resp., $(1,0)$-, $(1,1)$-, 1-interior) ideal (see [11, 31]). Any $(\in, \in \vee q_k)$-fuzzy left (resp., right, bi-, interior) ideal is a $(0, \frac{1-k}{2})$-fuzzy $(0,1)$- (resp., $(1,0)$-, $(1,1)$-, 1-interior) ideal, where $0 \leq k < 1$ (see [29, 47]). Any $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp., right, bi-, interior) ideal is a $(0, \frac{k^*-k}{2})$-fuzzy $(0,1)$- (resp., $(1,0)$-, $(1,1)$-, 1-interior) ideal, where $0 < k^* \leq 1$ and $0 \leq k < 1$ (see [33, 34]). Any (α, β)-fuzzy left (resp., right, bi-, interior) ideal is a (α, β)-fuzzy $(0,1)$- (resp., $(1,0)$-, $(1,1)$-, 1-interior) ideal (see [6, 7, 30]).

Example 2 ([4])**.** *Let* $S = \{0, 1, 2, 3, 4\}$. *We define a binary operation* $\cdot$ *and a partial order* $\leq$ *on* S *as follows.*

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	3
2	0	0	0	0	3
3	3	3	3	3	3
4	4	4	4	4	4

and $\leq := \{(0,3)\} \cup \Delta_S$. *Then,* $\mathbf{S} := \langle S; \cdot, \leq \rangle$ *is an ordered semigroup. Define a fuzzy set* f *in* S *by* $f(0) = 0.9, f(1) = 1, f(2) = 0.2, f(3) = 0$ *and* $f(4) = 0.2$. *We can calculate that* f *is a* $(0.3, 0.6)$*-fuzzy* $(2,2)$*-ideal of* $\mathbf{S}$*, but* f *is not a fuzzy* $(2,2)$*-ideal of* $\mathbf{S}$.

Example 3 ([4])**.** *Let* $S = \{0, 1, 2, 3, 4, 5\}$. *We define a binary operation* $\cdot$ *and a partial order* $\leq$ *on* S *as follows.*

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	3	1
2	0	0	0	0	3	1
3	0	0	0	0	0	3
4	0	3	3	0	0	3
5	0	3	3	3	4	5

and $\leq := \{(0,3)\} \cup \Delta_S$. *Then,* $\mathbf{S} := \langle S; \cdot, \leq \rangle$ *is an ordered semigroup. Define a fuzzy set* f *in* S *by* $f(0) = 0.9, f(1) = 0, f(2) = 0.4, f(3) = 0, f(4) = 0.8$ *and* $f(5) = 0.2$. *We can calculate that* f *is a* $(0.4, 0.6)$*-fuzzy* 2*-interior ideal of* $\mathbf{S}$*, but* f *is not a fuzzy* 2*-interior ideal of* $\mathbf{S}$.

The above examples demonstrate the distinct concepts of (α,β)-fuzzy (m,n)-ideals and (α,β)-fuzzy n-interior ideals whenever α,β,m, and n differ in ordered semigroups.

Remark 1. *By the property of fuzzy sets with restricted range I, we observe that:*

(i) if f is an (α,β)-fuzzy (m,n)-ideal of $\mathbf{S}$, then

$$f_I(x_1^m y z_1^n) \geq \left(\bigwedge_{i=1}^{m} f_I(x_i)\right) \wedge \left(\bigwedge_{i=1}^{n} f_I(z_i)\right)$$

for any $x_1,\ldots,x_m,y,z_1,\ldots,z_n \in S$;

(ii) if f is an (α,β)-fuzzy n-interior ideal of $\mathbf{S}$, then

$$f_I(xy_1^n z) \geq \bigwedge_{i=1}^{n} f_I(y_i)$$

for any $x,y_1,\ldots,y_n,z \in S$.

The subsequent result is essential for categorizing ordered semigroups into classes.

Proposition 2. *Let $\mathbf{S}$ be an ordered semigroup, A and B subsets of S. Then, the following statements hold.*

(i) $(\chi_A)_I \subseteq (\chi_B)_I$ if and only if $A \subseteq B$.

(ii) $\chi_A \circ_I \chi_B = (\chi_{(AB]})_I$.

(iii) $\chi_A \cap_I \chi_B = (\chi_{A\cap B})_I$.

(iv) $(\chi_A)_I$ is an (α,β)-fuzzy (m,n)- (resp., n-interior ideal) ideal of $\mathbf{S}$ if and only if A is an (m,n)- (resp., n-interior ideal) ideal of $\mathbf{S}$.

Proof. (1). Let $x \in A$. Then, $\beta = (\chi_A)_I(x) \leq (\chi_B)_I(x) \leq \beta$. This means that $x \in B$. On the other hand, let $x \in S$. If $x \notin A$, then $(\chi_A)_I(x) = \alpha \leq (\chi_B)_I(x)$. If $x \in A$, then $x \in B$. This implies $(\chi_A)_I(x) = \beta = (\chi_B)_I(x)$.

(2). It was shown in [26, Lemma 2.5] that $\chi_A \circ \chi_B = \chi_{(AB]}$. Thus, we have $\chi_A \circ_I \chi_B = (\chi_A \circ \chi_B)_I = (\chi_{(AB]})_I$.

(3). In [24, Proposition 9], the authors illustrated that $\chi_A \cap \chi_B = \chi_{A\cap B}$. Then, we obtain $\chi_A \cap_I \chi_B = (\chi_A \cap \chi_B)_I = (\chi_{A\cap B})_I$ as required.

(4). The proof can be found in [4].

By the associativity of $\circ_I$ and Lemma 1, Proposition 2(ii) and 2(iii) can be extended as follows.

Corollary 2. *Let $\mathbf{S}$ be an ordered semigroup, and $A_1,\ldots,A_n$ subsets of S. Then, we have $\chi_{A_1} \circ_I \cdots \circ_I \chi_{A_n} = (\chi_{(A_1\cdots A_n]})_I$ and $\chi_{A_1} \cap_I \cdots \cap_I \chi_{A_n} = (\chi_{A_1\cap\cdots\cap A_n})_I$.*

4. Main Results

In this section, we delineate the classification of ordered semigroups based on their regularities, employing the concepts of (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals. According to their regularities, ordered semigroups can be categorized into 16 classes, as outlined below. An ordered semigroup **S** satisfies:

(C1) if $A \subseteq (SAS]$ for all $\emptyset \neq A \subseteq S$;

(C2) if $A \subseteq (SA]$ for all $\emptyset \neq A \subseteq S$;

(C3) if $A \subseteq (AS]$ for all $\emptyset \neq A \subseteq S$;

(C4) if $A \subseteq (SASAS]$ for all $\emptyset \neq A \subseteq S$;

(C5) if $A \subseteq (SASA]$ for all $\emptyset \neq A \subseteq S$;

(C6) if $A \subseteq (ASAS]$ for all $\emptyset \neq A \subseteq S$;

(C7) if $A \subseteq (ASA]$ for all $\emptyset \neq A \subseteq S$;

(C8) of degree n if $A \subseteq (SA^nS]$ for all $\emptyset \neq A \subseteq S$ and $n \in \mathbb{N} \setminus \{1\}$;

(C9) of degree n if $A \subseteq (SA^nSA]$ for all $\emptyset \neq A \subseteq S$ and $n \in \mathbb{N} \setminus \{1\}$;

(C10) of degree n if $A \subseteq (ASA^nS]$ for all $\emptyset \neq A \subseteq S$ and $n \in \mathbb{N} \setminus \{1\}$;

(C11) of degree n if $A \subseteq (ASA^nSA]$ for all $\emptyset \neq A \subseteq S$ and $n \in \mathbb{N} \setminus \{1\}$;

(C12) if $A \subseteq (SA^2]$ for all $\emptyset \neq A \subseteq S$;

(C13) if $A \subseteq (A^2S]$ for all $\emptyset \neq A \subseteq S$;

(C14) if $A \subseteq (A^2SA^2]$ for all $\emptyset \neq A \subseteq S$;

(C15) if $A \subseteq (ASA^2]$ for all $\emptyset \neq A \subseteq S$;

(C16) if $A \subseteq (A^2SA]$ for all $\emptyset \neq A \subseteq S$.

The relation of such classes of ordered semigroups can be represented by Figure 1.

Example 4. *By Example 1, 2 and 3, we can see that* **S** *does not satisfy* (C1)–(C16) *since* $\{1\}$ *does not meet any regularity condition.*

Example 5. *Let* $S = \{0, 1, 2, 3, 4\}$. *We define a binary operation* $\cdot$ *and a partial order* $\leq$ *on* S *as follows.*

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	3	0	1
2	0	4	0	2	0
3	0	1	0	3	0
4	0	0	2	0	4

and $\leq := \{(0,1),(0,2),(0,3),(0,4)\} \cup \Delta_S$. *Then,* $\mathbf{S} := \langle S; \cdot, \leq\rangle$ *is an ordered semigroup. We can see that:*

- $0 \leq 000$;
- $1 \leq 314$;
- $2 \leq 423$;
- $3 \leq 333$;
- $4 \leq 444$.

This means that $\mathbf{S}$ *satisfies* (C7).

For more details about the regularities of ordered semigroups, the readers can refer to [43].

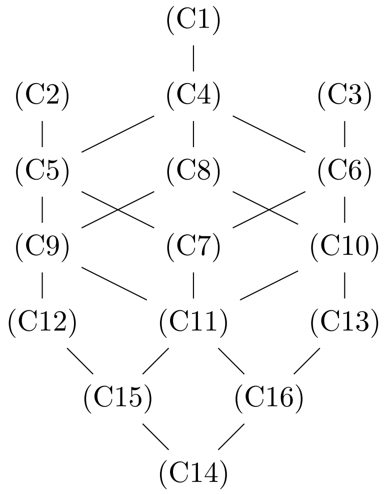

Figure 1: Hesse diagram of ordered semigroups classified by regularities under the inclusion.

In 2022, Lekkoksung et al. [36] considered ordered semigroups with the greatest element, commonly known as poe-semigroups. Notably, any ordered semigroup can be embedded into a poe-semigroup. Consequently, the authors comprehensively investigated and characterized poe-semigroups based on their regularities. Furthermore, they extended the implications of their findings to semigroups and hypersemigroups. In this section, we use the results outlined in [36] in conjunction with the methodology given by Kehayopulu, which involves transposing results from poe-semigroups to ordered semigroups and ordered hypersemigroups (see [17, 18]). This strategic approach enables us to characterize ordered semigroups by (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals effectively. Consequently, we will present these facts of poe-semigroups in terms of ordered semigroups.

We begin categorizing ordered semigroups into distinct classes based on their regularities through (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals as follows.

Lemma 2 ([36])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then,* $\mathbf{S}$ *satisfies* (C1) *if and only if* $A \subseteq (SAS]$ *for any* 1*-interior ideal* A *of* $\mathbf{S}$.

Theorem 2. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) **S** *satisfies* (C1).

(ii) $f_I \subseteq \beta \circ_I f \circ_I \beta$ *for any* (α, β)*-fuzzy* 1*-interior ideal* f *of* **S**.

Proof. (1)⇒(2). Let f be an (α, β)-fuzzy 1-interior ideal of **S**. Let $a \in S$. Since **S** satisfies (C1), there exist $x_1, x_2 \in S$ such that $a \leq x_1 a x_2 \leq x_1 x_1 a x_2 x_2$. That is, $(x_1 x_1 a x_2, x_2) \in \mathbf{S}_a$. Then,

$$(\beta \circ_I f \circ_I \beta)(a) \geq (\beta \circ_I f)_I(x_1 x_1 a x_2) = (\beta \circ f)_I(x_1 x_1 a x_2) = f_I(x_1 a x_2) \geq f_I(a).$$

(2)⇒(1). Let A be a 1-interior ideal of **S**. Then, by Proposition 2, $(\chi_A)_I$ is an (α, β)-fuzzy 1-interior ideal of **S**. By our presumption, we have $((\chi_A)_I)_I \subseteq \beta \circ_I (\chi_A)_I \circ_I \beta$. This implies that

$$\begin{aligned}(\chi_A)_I &= ((\chi_A)_I)_I \\ &\subseteq \beta \circ_I (\chi_A)_I \circ_I \beta \\ &= (\chi_S)_I \circ_I (\chi_A)_I \circ_I (\chi_S)_I \\ &= (\chi_S \circ \chi_A \circ \chi_S)_I \\ &= (\chi_{(SAS]})_I.\end{aligned}$$

By Proposition 2, we have $A \subseteq (SAS]$. Hence, by Lemma 2, **S** satisfies (C1).

Lemma 3 ([36])**.** *Let* **S** *be an ordered semigroup. Then,* **S** *satisfies* (C2) *if and only if* $A \subseteq (SA]$ *for any* $(0, 1)$*-ideal* A *of* **S**.

Theorem 3. *Let* **S** *be an ordered semigroup. Then, the following statements are equivalent.*

(i) **S** *satisfies* (C2).

(ii) $f_I \subseteq \beta \circ_I f$ *for any* (α, β)*-fuzzy* $(0, 1)$*-ideal* f *of* **S**.

Proof. (1)⇒(2). Let f be an (α, β)-fuzzy $(0, 1)$-ideal of **S**. Let $a \in S$. Since **S** satisfies (C2), there exist $x \in S$ such that $a \leq xa$. That is, $(x, a) \in \mathbf{S}_a$. Then, $(\beta \circ_I f)(a) \geq f_I(a)$.

(2)⇒(1). Let A be a $(0, 1)$-ideal of **S**. Then, by Proposition 2, $(\chi_A)_I$ is an (α, β)-fuzzy $(0, 1)$-ideal of **S**. By our presumption, we have $((\chi_A)_I)_I \subseteq \beta \circ_I (\chi_A)_I$. This implies that

$$(\chi_A)_I = ((\chi_A)_I)_I \subseteq \beta \circ_I (\chi_A)_I = (\chi_S)_I \circ_I (\chi_A)_I = \chi_S \circ_I \chi_A = (\chi_{(SA]})_I.$$

By Proposition 2, we have $A \subseteq (SA]$. Hence, by Lemma 3, **S** satisfies (C2).

Similarly, we obtain a characterization of ordered semigroups satisfying (C3) as follows.

Theorem 4. *Let* **S** *be an ordered semigroup. Then, the following statements are equivalent.*

(i) **S** *satisfies* (C3).

(ii) $f_I \subseteq f \circ_I \beta$ *for any* (α, β)*-fuzzy* $(1,0)$*-ideal* f *of* $\mathbf{S}$*.*

Lemma 4 ([49])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then,* $\mathbf{S}$ *satisfies* (C4) *if and only if* $A \cap B \subseteq (ABA]$ *for any* 1*-interior ideal* A *and* $(1,1)$*-ideal* B *of* $\mathbf{S}$*.*

Theorem 5. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) $\mathbf{S}$ *satisfies* (C4)*.*

(ii) $f \cap_I g \subseteq f \circ_I g \circ_I f$ *for any* (α, β)*-fuzzy* 1*-interior ideal* f *and* (α, β)*-fuzzy* $(1,1)$*-ideal* g *of* $\mathbf{S}$*.*

Proof. (1)⇒(2). Let f and g be an (α, β)-fuzzy 1-interior ideal and an (α, β)-fuzzy $(1,1)$-ideal of $\mathbf{S}$, respectively. Let $a \in S$. Since $\mathbf{S}$ satisfies (C4), there exist $y_1, y_2, y_3 \in S$ such that $a \leq y_1ay_2ay_3 \leq y_1ay_2y_1ay_2ay_3y_3 \leq y_1ay_2y_1ay_2y_1ay_2ay_3y_3y_3$. That is, $(x_1ax_2ax_3a, x_4ax_5) \in \mathbf{S}_a$ for some $x_1, x_2, x_3, x_4, x_5 \in S$. Then,

$$\begin{aligned}
(f \circ_I g \circ_I f)(a) &= [(f \circ_I g) \circ f]_I (a) \\
&\geq (f \circ_I g)_I(x_1ax_2ax_3a) \wedge f_I(x_4ax_5) \\
&\geq (f \circ_I g)_I(x_1ax_2ax_3a) \wedge f_I(a) \\
&= (f \circ g)_I(x_1ax_2ax_3a) \wedge f_I(a) \\
&= f_I(x_1ax_2) \wedge g_I(ax_3a) \wedge f_I(a) \\
&\geq f_I(a) \wedge g_I(a) \wedge f_I(a) \\
&= (f \cap_I g)(a).
\end{aligned}$$

(2)⇒(1). Let A and B be a 1-interior ideal and a $(1,1)$-ideal of $\mathbf{S}$, respectively. Then, by Proposition 2, $(\chi_A)_I$ and $(\chi_B)_I$ is an (α, β)-fuzzy 1-interior ideal and an (α, β)-fuzzy $(1,1)$-ideal of $\mathbf{S}$, respectively. By our presumption, we have $(\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I \circ_I (\chi_A)_I$. This implies that

$$\begin{aligned}
(\chi_{A\cap B})_I &= \chi_A \cap_I \chi_B \\
&= (\chi_A)_I \cap_I (\chi_B)_I \\
&\subseteq (\chi_A)_I \circ_I (\chi_B)_I \circ_I (\chi_A)_I \\
&= \chi_A \circ_I \chi_B \circ_I \chi_A \\
&= (\chi_{(ABA]})_I.
\end{aligned}$$

By Proposition 2, we have $A \cap B \subseteq (ABA]$. Hence, by Lemma 4, $\mathbf{S}$ satisfies (C4).

Lemma 5 ([49])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then,* $\mathbf{S}$ *satisfies* (C5) *if and only if* $A \cap B \subseteq (AB]$ *for any* 1*-interior ideal* A *and* $(1,1)$*-ideal* B *of* $\mathbf{S}$*.*

Theorem 6. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) **S** *satisfies* (C5).

(ii) $f \cap_I g \subseteq f \circ_I g$ *for any* (α, β)*-fuzzy* 1*-interior ideal* f *and* (α, β)*-fuzzy* $(1,1)$*-ideal* g *of* **S**.

Proof. (1)⇒(2). Let f and g be an (α, β)-fuzzy 1-interior ideal and an (α, β)-fuzzy $(1,1)$-ideal of **S**, respectively. Let $a \in S$. Since **S** satisfies (C5), there exist $y_1, y_2 \in S$ such that $a \leq y_1ay_2a \leq y_1ay_2y_1ay_2a$. That is, $(x_1ax_2, ax_3a) \in \mathbf{S}_a$ for some $x_1, x_2, x_3 \in S$. Then,

$$(f \circ_I g)(a) \geq f_I(x_1ax_2) \wedge g_I(ax_3a) \geq f_I(a) \wedge g_I(a) = (f \cap_I g)(a).$$

(2)⇒(1). Let A and B be a 1-interior ideal and a $(1,1)$-ideal of **S**, respectively. Then, by Proposition 2, $(\chi_A)_I$ and $(\chi_B)_I$ is an (α, β)-fuzzy 1-interior ideal and an (α, β)-fuzzy $(1,1)$-ideal of **S**, respectively. By our presumption, we have $(\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I$. This implies that

$$(\chi_{A\cap B})_I = \chi_A \cap_I \chi_B = (\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I = \chi_A \circ_I \chi_B = (\chi_{(AB]})_I.$$

By Proposition 2, we have $A \cap B \subseteq (AB]$. Hence, by Lemma 5, **S** satisfies (C5).

Similarly to the above result, we obtain a characterization of ordered semigroups satisfying (C6) as follows.

Theorem 7. *Let* **S** *be an ordered semigroup. Then, the following statements are equivalent.*

(i) **S** *satisfies* (C6).

(ii) $f \cap_I g \subseteq f \circ_I g$ *for any* (α, β)*-fuzzy* $(1,1)$*-ideal* f *and* (α, β)*-fuzzy* 1*-interior ideal* g *of* **S**.

Lemma 6 ([36])**.** *Let* **S** *be an ordered semigroup. Then,* **S** *satisfies* (C7) *if and only if* $A \cap B \cap C \subseteq (ABC]$ *for any* $(1,0)$*-ideal* A*,* 1*-interior ideal* B*, and* $(0,1)$*-ideal* C *of* **S**.

Theorem 8. *Let* **S** *be an ordered semigroup. Then, the following statements are equivalent.*

(i) **S** *satisfies* (C7).

(ii) $f \cap_I g \cap_I h \subseteq f \circ_I g \circ_I h$ *for any* (α, β)*-fuzzy* $(1,0)$*-ideal* f*,* (α, β)*-fuzzy* 1*-interior ideal* g*, and* (α, β)*-fuzzy* $(0,1)$*-ideal* h *of* **S**.

Proof. (1)⇒(2). Let f, g and h be an (α, β)-fuzzy $(1,0)$-ideal, an (α, β)-fuzzy 1-interior ideal, and an (α, β)-fuzzy $(0,1)$-ideal of **S**, respectively. Let $a \in S$. Since **S** satisfies (C7), there exist $y_1 \in S$ such that $a \leq ay_1a \leq ay_1ay_1a \leq ay_1ay_1ay_1a \leq ay_1ay_1ay_1ay_1a$. That is, $(ax_1x_2ax_3, x_4a) \in \mathbf{S}_a$ for some $x_1, x_2, x_3, x_4 \in S$. Then,

$$(f \circ_I g \circ_I h)(a) \quad = \quad [(f \circ_I g) \circ h]_I\,(a)$$

$$\begin{aligned}
&\geq (f \circ_I g)_I(ax_1x_2ax_3) \wedge h_I(x_4a)\\
&\geq (f \circ_I g)_I(ax_1x_2ax_3) \wedge h_I(a)\\
&= (f \circ g)_I(ax_1x_2ax_3) \wedge h_I(a)\\
&= f_I(ax_1) \wedge g_I(x_2ax_3) \wedge h(a)\\
&\geq f_I(a) \wedge g_I(a) \wedge h_I(a)\\
&= (f \cap_I g \cap_I h)(a).
\end{aligned}$$

(2)⇒(1). Let A, B and C be a $(1,0)$-ideal, a 1-interior ideal and a $(0,1)$-ideal of $\mathbf{S}$, respectively. Then, by Proposition 2, $(\chi_A)_I, (\chi_B)_I$ and $(\chi_C)_I$ is an (α,β)-fuzzy $(1,0)$-ideal, an (α,β)-fuzzy 1-interior ideal and an (α,β)-fuzzy $(0,1)$-ideal of $\mathbf{S}$, respectively. By our presumption, we have $(\chi_A)_I \cap_I (\chi_B)_I \cap_I (\chi_C)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I \circ_I (\chi_C)_I$. This implies that

$$\begin{aligned}
(\chi_{A\cap B\cap C})_I &= \chi_A \cap_I \chi_B \cap_I \chi_C\\
&= (\chi_A)_I \cap_I (\chi_B)_I \cap_I (\chi_C)_I\\
&\subseteq (\chi_A)_I \circ_I (\chi_B)_I \circ_I (\chi_C)_I\\
&= \chi_A \circ_I \chi_B \circ_I \chi_C\\
&= (\chi_{(ABC]})_I.
\end{aligned}$$

By Proposition 2, we have $A \cap B \cap C \subseteq (ABC]$. Hence, by Lemma 6, $\mathbf{S}$ satisfies (C7).

Lemma 7 ([49])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then,* $\mathbf{S}$ *satisfies* (C8) *of degree* n *if and only if* $A \subseteq (A^2]$ *for any* n*-interior ideal* A *of* $\mathbf{S}$.

Theorem 9. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) $\mathbf{S}$ *satisfies* (C8) *of degree* n.

(ii) $f_I \subseteq f \circ_I f$ *for any* (α,β)*-fuzzy* n*-interior ideal* f *of* $\mathbf{S}$.

Proof. (1)⇒(2). Let f be an (α,β)-fuzzy n-interior ideal of $\mathbf{S}$. Let $a \in S$. Since $\mathbf{S}$ satisfies (C8) of degree n, there exist $y_1, y_2 \in S$ such that $a \leq y_1a^ny_2 = y_1a^{n-1}aay_2 \leq y_1a^{n-2}y_1a^ny_2y_1a^ny_2y_2$. That is, $(x_1a^nx_2, x_3a^nx_4) \in \mathbf{S}_a$ for some $x_1, x_2, x_3, x_4 \in S$. Then,

$$(f \circ_I f)(a) \geq f_I(x_1a^nx_2) \wedge f_I(x_3a^nx_4) \geq f_I(a) \wedge f_I(a) = f_I(a).$$

(2)⇒(1). Let A be an n-interior ideal of $\mathbf{S}$. Then, by Proposition 2, $(\chi_A)_I$ is an (α,β)-fuzzy n-interior ideal of $\mathbf{S}$. By our presumption, we have $((\chi_A)_I)_I \subseteq (\chi_A)_I \circ_I (\chi_A)_I$. This implies that

$$(\chi_A)_I = ((\chi_A)_I)_I \subseteq (\chi_A)_I \circ_I (\chi_A)_I = \chi_A \circ_I \chi_A = (\chi_{(A^2]})_I.$$

By Proposition 2, we have $A \subseteq (A^2]$. Hence, by Lemma 7, $\mathbf{S}$ satisfies (C8) of degree n.

Lemma 8 ([49]). *Let* $\mathbf{S}$ *be an ordered semigroup. Then,* $\mathbf{S}$ *satisfies* (C9) *of degree* n *if and only if* $A \cap B \subseteq (AB]$ *for any* n*-interior ideal* A *and* $(n, 1)$*-ideal* B *of* $\mathbf{S}$.

Theorem 10. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) $\mathbf{S}$ *satisfies* (C9) *of degree* n.

(ii) $f \cap_I g \subseteq f \circ_I g$ *for any* (α, β)*-fuzzy* n*-interior ideal* f *and* (α, β)*-fuzzy* $(n, 1)$*-ideal* g *of* $\mathbf{S}$.

Proof. (1)⇒(2). Let f and g be an (α, β)-fuzzy n-interior ideal and an (α, β)-fuzzy $(n, 1)$-ideal of $\mathbf{S}$, respectively. Let $a \in S$. Since $\mathbf{S}$ satisfies (C9) of degree n, there exist $y_1, y_2 \in S$ such that $a \leq y_1 a^n y_2 a = y_1 a^{n-2} a a y_2 a \leq y_1 a^{n-2} y_1 a^n y_2 a y_1 a^n y_2 a y_2 a$. That is, $(x_1 a^n x_2, a^n x_3 a) \in \mathbf{S}_a$ for some $x_1, x_2, x_3 \in S$. Then,

$$(f \circ_I g)(a) \geq f_I(x_1 a^n x_2) \wedge g_I(a^n x_3 a) \geq f_I(a) \wedge g_I(a) = (f \cap_I g)(a).$$

(2)⇒(1). Let A and B be an n-interior ideal and an $(n, 1)$-ideal of $\mathbf{S}$, respectively. Then, by Proposition 2, $(\chi_A)_I$ and $(\chi_B)_I$ is an (α, β)-fuzzy n-interior ideal and (α, β)-fuzzy $(n, 1)$-ideal of $\mathbf{S}$, respectively. By our presumption, we have $(\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I$. This implies that

$$(\chi_{A\cap B})_I = \chi_A \cap_I \chi_B = (\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I = \chi_A \circ_I \chi_B = (\chi_{(AB]})_I.$$

By Proposition 2, we have $A \cap B \subseteq (AB]$. Hence, by Lemma 8, $\mathbf{S}$ satisfies (C9) of degree n.

We can characterize ordered semigroups satisfying (C10) in the same way of the above result using the concepts of (α, β)-fuzzy $(1, n)$-ideals and (α, β)-fuzzy n-interior ideals as follows.

Theorem 11. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) $\mathbf{S}$ *satisfies* (C10) *of degree* n.

(ii) $f \cap_I g \subseteq f \circ_I g$ *for any* (α, β)*-fuzzy* $(1, n)$*-ideal* f *and* (α, β)*-fuzzy* n*-interior ideal* g *of* $\mathbf{S}$.

Lemma 9 ([36]). *Let* $\mathbf{S}$ *be an ordered semigroup. Then,* $\mathbf{S}$ *satisfies* (C11) *of degree* n *if and only if* $A \cap B \cap C \subseteq (ABC]$ *for any* $(1, 0)$*-ideal* A*,* n*-interior ideal* B *and* $(0, 1)$*-ideal* C *of* $\mathbf{S}$.

Theorem 12. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) $\mathbf{S}$ *satisfies* (C11) *of degree* n.

(ii) $f \cap_I g \cap_I h \subseteq f \circ_I g \circ_I h$ *for any* (α, β)*-fuzzy* $(1,0)$*-ideal* f*,* (α, β)*-fuzzy* n*-interior ideal* g *and* (α, β)*-fuzzy* $(0,1)$*-ideal* h *of* $\mathbf{S}$*.*

Proof. (1)⇒(2). Let f, g and h be an (α, β)-fuzzy $(1,0)$-ideal, an (α, β)-fuzzy n-interior ideal and an (α, β)-fuzzy $(0,1)$-ideal of $\mathbf{S}$, respectively. Let $a \in S$. Since $\mathbf{S}$ satisfies (C11) of degree n, there exist $y_1, y_2 \in S$ such that $a \leq ay_1a^ny_2a \leq ay_1a^ny_2ay_1a^ny_2a$. That is, $(ax_1x_2a^nx_3, x_4a) \in \mathbf{S}_a$ for some $x_1, x_2, x_3, x_4 \in S$. Then,

$$\begin{aligned}(f \circ_I g \circ_I h)(a) &= [(f \circ_I g) \circ h]_I(a) \\ &\geq (f \circ_I g)_I(ax_1x_2a^nx_3) \wedge h_I(x_4a) \\ &\geq (f \circ_I g)_I(ax_1x_2a^nx_3) \wedge h_I(a) \\ &= (f \circ g)_I(ax_1x_2a^nx_3) \wedge h_I(a) \\ &= f_I(ax_1) \wedge g_I(x_2a^nx_3) \wedge h_I(a) \\ &\geq f_I(a) \wedge g_I(a) \wedge h_I(a) \\ &= (f \cap_I g \cap_I h)(a).\end{aligned}$$

(2)⇒(1). Let A, B and C be a $(1,0)$-ideal, an n-interior ideal and a $(0,1)$-ideal of $\mathbf{S}$, respectively. Then, by Proposition 2, $(\chi_A)_I, (\chi_B)_I$ and $(\chi_C)_I$ is an (α, β)-fuzzy $(1,0)$-ideal, (α, β)-fuzzy n-interior ideal and (α, β)-fuzzy $(0,1)$-ideal of $\mathbf{S}$, respectively. By our presumption, we have $(\chi_A)_I \cap_I (\chi_B)_I \cap_I (\chi_C)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I \circ_I (\chi_C)_I$. This implies that

$$\begin{aligned}(\chi_{A\cap B\cap C})_I &= \chi_A \cap_I \chi_B \cap_I \chi_C \\ &= (\chi_A)_I \cap_I (\chi_B)_I \cap_I (\chi_C)_I \\ &\subseteq (\chi_A)_I \circ_I (\chi_B)_I \circ_I (\chi_C)_I \\ &= \chi_A \circ_I \chi_B \circ_I \chi_C \\ &= (\chi_{(ABC]})_I.\end{aligned}$$

By Proposition 2, we have $A \cap B \cap C \subseteq (ABC]$. Hence, by Lemma 9, $\mathbf{S}$ satisfies (C11) of degree n.

Lemma 10 ([36])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then,* $\mathbf{S}$ *satisfies* (C12) *if and only if* $A \subseteq (SA^2]$ *for any* $(0,2)$*-ideal* A *of* $\mathbf{S}$*.*

Theorem 13. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) $\mathbf{S}$ *satisfies* (C12)*.*

(ii) $f_I \subseteq \beta \circ_I f \circ_I f$ *for any* (α, β)*-fuzzy* $(0,2)$*-ideal* f *of* $\mathbf{S}$*.*

Proof. (1)⇒(2). Let f be an (α, β)-fuzzy $(0,2)$-ideal of $\mathbf{S}$. Let $a \in S$. Since $\mathbf{S}$ satisfies (C12), there exist $y_1 \in S$ such that $a \leq y_1a^2 \leq y_1^2a^3 \leq y_1^3a^4 \leq y_1^4a^5$. That is, $(x_1x_2a^2, x_3a^2) \in \mathbf{S}_a$ for some $x_1, x_2, x_3 \in S$. Then,

$$(\beta \circ_I f \circ_I f)(a) = [(\beta \circ_I f) \circ f]_I(a)$$

$$
\begin{aligned}
&\geq (\beta \circ_I f)_I(x_1x_2a^2) \wedge f_I(x_3a^2) \\
&\geq (\beta \circ_I f)_I(x_1x_2a^2) \wedge f_I(a) \\
&= (\beta \circ f)_I(x_1x_2a^2) \wedge f_I(a) \\
&= f_I(x_2a^2) \wedge f_I(a) \\
&= f_I(a) \wedge f_I(a) \\
&= f_I(a).
\end{aligned}
$$

(2)⇒(1). Let A be a $(0,2)$-ideal of $\mathbf{S}$. Then, by Proposition 2, $(\chi_A)_I$ is an (α,β)-fuzzy $(0,2)$-ideal of $\mathbf{S}$. By our presumption, we have $(\chi_A)_I = ((\chi_A)_I)_I \subseteq \beta \circ_I (\chi_A)_I \circ_I (\chi_A)_I = (\chi_S)_I \circ_I (\chi_A)_I \circ_I (\chi_A)_I$. This implies that

$$
\begin{aligned}
(\chi_A)_I &= (\chi_{S\cap A\cap A})_I \\
&= \chi_S \cap_I \chi_A \cap_I \chi_A \\
&= (\chi_S)_I \cap_I (\chi_A)_I \cap_I (\chi_A)_I \\
&\subseteq (\chi_S)_I \circ_I (\chi_A)_I \circ_I (\chi_A)_I \\
&= \chi_S \circ_I \chi_A \circ_I \chi_A \\
&= (\chi_{(SA^2]})_I.
\end{aligned}
$$

By Proposition 2, we have $A \subseteq (SA^2]$. Hence, by Lemma 9, $\mathbf{S}$ satisfies (C12).

The following theorem, we provide a characterization of ordered semigroups satisfying (C13) by (α,β)-fuzzy $(2,0)$-ideals. The proof is similar to the above theorem, so we skip the proof.

Theorem 14. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) $\mathbf{S}$ *satisfies* (C13).

(ii) $f_I \subseteq f \circ_I f \circ_I \beta$ *for any* (α,β)*-fuzzy* $(2,0)$*-ideal* f *of* $\mathbf{S}$.

Lemma 11 ([36])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then,* $\mathbf{S}$ *satisfies* (C14) *if and only if* $A \cap B \subseteq (AB]$ *for any* $(2,0)$*-ideal* A *and* $(0,2)$*-ideal* B *of* $\mathbf{S}$.

Theorem 15. *Let* $\mathbf{S}$ *be an ordered semigroup. Then, the following statements are equivalent.*

(i) $\mathbf{S}$ *satisfies* (C14).

(ii) $f \cap_I g \subseteq f \circ_I g$ *for any* (α,β)*-fuzzy* $(2,0)$*-ideal* f *and* (α,β)*-fuzzy* $(0,2)$*-ideal* g *of* $\mathbf{S}$.

Proof. (1)⇒(2). Let f and g be an (α,β)-fuzzy $(2,0)$-ideal and an (α,β)-fuzzy $(0,2)$-ideal of $\mathbf{S}$, respectively. Let $a \in S$. Since $\mathbf{S}$ satisfies (C14), there exist $y_1 \in S$ such that $a \leq a^2y_1a^2 \leq a^2y_1a^3y_1a^2$. That is, $(a^2x_1, x_2a^2) \in \mathbf{S}_a$ for some $x_1, x_2 \in S$. Then,

$$(f \circ_I g)(a) \geq f_I(a^2x_1) \wedge g_I(x_2a^2) \geq f_I(a) \wedge g_I(a) = (f \cap_I g)(a).$$

(2)⇒(1). Let A and B be a $(2,0)$-ideal and a $(0,2)$-ideal of **S**, respectively. Then, by Proposition 2, $(\chi_A)_I$ and $(\chi_B)_I$ is an (α,β)-fuzzy $(2,0)$-ideal and an (α,β)-fuzzy $(0,2)$-ideal of **S**, respectively. By our presumption, we have $(\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I$. This implies that

$$(\chi_{A\cap B})_I = \chi_A \cap_I \chi_B = (\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I = \chi_A \circ_I \chi_B = (\chi_{(AB]})_I.$$

By Proposition 2, we have $A \subseteq (AB]$. Hence, by Lemma 11, **S** satisfies (C14).

Lemma 12 ([36])**.** *Let* **S** *be an ordered semigroup. Then,* **S** *satisfies* (C15) *if and only if* $A \cap B \subseteq (AB]$ *for any* $(1,0)$*-ideal* A *and* $(0,2)$*-ideal* B *of* **S**.

Theorem 16. *Let* **S** *be an ordered semigroup. Then, the following statements are equivalent.*

(i) **S** *satisfies* (C15).

(ii) $f \cap_I g \subseteq f \circ_I g$ *for any* (α,β)*-fuzzy* $(1,0)$*-ideal* f *and* (α,β)*-fuzzy* $(0,2)$*-ideal* g *of* **S**.

Proof. (1)⇒(2). Let f and g be an (α,β)-fuzzy $(1,0)$-ideal and an (α,β)-fuzzy $(0,2)$-ideal of **S**, respectively. Let $a \in S$. Since **S** satisfies (C15), there exist $y_1 \in S$ such that $a \leq ay_1a^2 \leq ay_1a^2y_1a^2$. That is, $(ax_1, x_2a^2) \in \mathbf{S}_a$ for some $x_1, x_2 \in S$. Then,

$$(f \circ_I g)(a) \geq f_I(ax_1) \wedge g_I(x_2a^2) \geq f_I(a) \wedge g_I(a) = (f \cap_I g)(a).$$

(2)⇒(1). Let A and B be a $(1,0)$-ideal and a $(0,2)$-ideal of **S**, respectively. Then, by Proposition 2, $(\chi_A)_I$ and $(\chi_B)_I$ is an (α,β)-fuzzy $(1,0)$-ideal and an (α,β)-fuzzy $(0,2)$-ideal of **S**, respectively. By our presumption, we have $(\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I$. This implies that

$$(\chi_{A\cap B})_I = \chi_A \cap_I \chi_B = (\chi_A)_I \cap_I (\chi_B)_I \subseteq (\chi_A)_I \circ_I (\chi_B)_I = \chi_A \circ_I \chi_B = (\chi_{(AB]})_I.$$

By Proposition 2, we have $A \subseteq (AB]$. Hence, by Lemma 12, **S** satisfies (C15).

Similarly, we obtain the following theorem.

Theorem 17. *Let* **S** *be an ordered semigroup. Then, the following statements are equivalent.*

(i) **S** *satisfies* (C16).

(ii) $f \cap_I g \subseteq f \circ_I g$ *for any* (α,β)*-fuzzy* $(2,0)$*-ideal* f *and* (α,β)*-fuzzy* $(0,1)$*-ideal* g *of* **S**.

5. Conclusion

This paper uses the concepts of (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals to categorize ordered semigroups into sixteen distinct classes based on the regularities of ordered semigroups. An advantage of the results in the current study is that, by carefully selecting appropriate parameters, we derive several characterizations of ordered semigroups utilizing fuzzy ideals that several authors studied. However, in the theorems we obtained, in practice, there is a challenging problem: Are there any easier conditions to obtain the set of all (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals? This question can be stated as an open problem. Furthermore, in future research, we aim to delve deeper into the properties of (α, β)-fuzzy (m, n)-ideals and (α, β)-fuzzy n-interior ideals, exploring their pure and prime properties.

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