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Advancements in Topological Approaches via Core Minimal Neighborhoods and Their Applications

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Abstract. Graph theory provides many topological systems for modelling blood circulation. The main object is determining the best topology for a successful correct diagnosis. This work illustrates the justification for using topology, rough sets, and graph analysis through neighborhoods. Generalization for an approximation space and a model of the topological graph is presented. Investigating core minimal neighborhoods is essential for categorizing subsets and computing, these techniques perform better than current techniques while maintaining Pawlakl's characteristics. This work presents a method for generalizing rough sets utilizing core minimal neighborhoods using binary relations. Moreover, we will construct four types of dual approximations concerning core minimal neighborhoods as lower and upper approximations. A comparison between different types of dual approximations is discussed. Core minimal neighborhoods induce certain types of topological structures. Finally, we compare different topologies that assist us in determining the main parts of a human heart's graph.

2020 Mathematics Subject Classifications: 60L90, 54C55, 54B10, 54D30, 54A05

Key Words and Phrases: Graphs, topological space, approximation space, rough set, neighborhood, core neighborhood, minimal neighborhood, human heart

1. Introduction

The use of powerful mathematical methods on medical models in recent years has yielded priceless insights into intricate datasets. The paper provides a clear and succinct explanation of the reasoning behind the use of neighborhood systems in conjunction with topological visualization and rough sets. The importance of this work is underscored by the abundance of medical models that are currently in use, each of which poses a different set of difficulties in terms of interdependencies and data complexity. Topological visualization provides a visually intuitive representation of complex data structures, surpassing

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the constraints of applied mathematics. Converting complex data into topological spaces allows for the discovery of hidden patterns and correlations. The flexibility of the analysis is increased through the incorporation of rough sets, and theoretical structure for handling imprecision and uncertainty. By combining the best features of both methods, this synergy enables a more thorough comprehension of complex medical data. The granularity of interactions between parts can be changed via the neighborhood systems lens to accommodate different levels of abstraction needed for medical applications. The importance of this undertaking is further highlighted by the context of medical models. Medical data is intrinsically complex, frequently involving complex relationships between factors.

In information systems, several math tools can be used to handle knowledge that is not exact or certain. Some of these tools include rough sets [29] and fuzzy sets [45]. Rough set theory (For short RST) was created by Pawlak [28] to help with incomplete and uncertain information. Many researchers in various fields have shown interest in RST and its applications [8, 9]. Moreover, Pawlak investigated the relationship between topology and its generalization. The indiscernibility relation is the basic idea of Pawlak, it was explained using the concept of equivalence relation. However, the need for something called equivalence relation like the rule of indiscernibility, makes things more difficult and puts limits on what can be done in many situations. So, the equivalence relation is used for different kinds of relations, like arbitrary relation [42], fuzzy relations [21], similarity relation [30], tolerance relation [43], and covering of the universal sets [11]. One of the most crucial and vital areas of mathematics is topology. In system analysis, topological structures and their generalizations are regarded as fundamental definitions and theorems [16]. Many of these structures have applications in analysis [34], chemistry [6], and physics [14]. Many academics have turned to topological methods in recent years to examine rough sets and their applications. Topics including the relationship between RST and topological spaces and the characteristics of topological rough sets are introduced [39]. Lin [18, 20] investigated approximations using neighborhood systems and topological concepts. Binary relations can also create neighborhood systems. The equivalence class of any element in the equivalence relation can be thought of as this element's neighborhood [27]. The minimal structure of RST and topology are investigated in [13] and various applications are presented in [4]. The basic concept of RST is that there are dual approximations, which are created utilizing right neighborhood, left neighborhood [41], minimal right neighborhood [2], and minimal left neighborhood [3]. Some types of neighborhoods termed E_j-neighborhoods are established [38]. Several types of neighborhoods are called C_j-neighborhoods which were investigated in applications for medicine by Al-Shami [36]. Moreover, Al-Shami [37] researched the features and applications of maximum neighborhoods in medicine. Shbair et al [35] investigate minimal structure as well as minimal right, minimal left, minimal intersection, and minimal union neighborhoods, and some application of the human heart is studied. In 2008, Hung conducted research on core neighborhood systems [15]. The notion of minimal neighborhoods by researching features of finite topological spaces [1]. Additionally, four types of neighborhoods, core neighborhood, minimal neighborhood, and core minimal neighborhood are established and his medical application using human heart data is discussed [31]. Today, the breadth of rough set applications is significantly broader than before, it can be used in various scientific and technical domains including computer networks [17], missing attribute values solution [33], decision-making problems [10], biology [26], economic fields [12], and decision-making for COVID-19 [22].

In this paper, the concept of the core minimal neighborhoods is used to provide a new generalization for RST according to general relations. Four types of core minimal neighborhoods are introduced. The attributes of the new RST are defined and compared with the characteristics of different methods. We examine the relation between four approximations and made a comparison between neighborhood, core neighborhood, minimal neighborhood, and core minimal neighborhood using four types of right, left, union, and intersection neighborhoods and we found a relationship between them. We also provide the relation between four types of dual approximation. The boundary region and accuracy are discussed and the relationship between them is presented. Additionally, four types of topologies were generated using core minimal neighborhood and compared them. Application of human heart was introduced, and some topologies generated using core minimal neighborhood were used in blood circulation. We suggest that our method is an extension of traditional RST. We will use $\mathfrak{X}$ to denote the universal set.

2. Preliminaries

In this study, we will review the definition of topology and RST by defining approximation space and dual approximations as upper and lower approximations, accuracy, four types of neighborhoods, four types of core neighborhoods, and four types of minimal neighborhoods.

Definition 1. *[16] Let τ be a family of subsets of $\mathfrak{X}$. τ is a topology on $\mathfrak{X}$ if it satisfies:(i) ϕ and $\mathfrak{X}$ are in τ, (ii) Let $\mathfrak{B}_i \in \tau$ for $i \in \mathcal{I}$. Then, $\bigcup_{i\in\mathcal{I}} \mathfrak{B}_i \in \tau$, and (iii) Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \tau$. Then, $\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \tau$.*

Pawlak [19, 29] defined the approximation space $\mathfrak{K} = (\mathfrak{X}, \aleph)$, where $\aleph$ is an equivalence relation. This approximation space constitutes a clopen topological space that arose due to the need to divide $\mathfrak{X}$ as a partition. We shall define the equivalence class containing ξ as $[\xi]$. In Definition 2, we will define upper and lower approximations.

Definition 2. *[29] Let $\mathfrak{K} = (\mathfrak{X}, \aleph)$ be an approximation space with $\mathfrak{B} \subseteq \mathfrak{X}$. The lower approximation is defined by $\underline{\aleph}(\mathfrak{B}) = \{\xi \in \mathfrak{X} : [\xi] \subseteq \mathfrak{B}\}$, and upper approximation is defined by $\overline{\aleph}(\mathfrak{B}) = \{\xi \in \mathfrak{X} : [\xi] \cap \mathfrak{B} \neq \phi\}$.*

In Definition 2, $\mathfrak{X}$ is partitioned into three disjoint regions in $\mathfrak{K} = (\mathfrak{X}, \aleph)$, boundary region $B_{\aleph}(\mathfrak{B}) = \overline{\aleph}(\mathfrak{B}) - \underline{\aleph}(\mathfrak{B})$, positive region $P_{\aleph}(\mathfrak{B}) = \underline{\aleph}(\mathfrak{B})$, and negative region $N_{\aleph}(\mathfrak{B}) = \mathfrak{X} - \overline{\aleph}(\mathfrak{B})$.

Definition 3. *[28] Let $\mathfrak{K} = (\mathfrak{X}, \aleph)$ be an approximation space with $\mathfrak{B} \subseteq \mathfrak{X}$. The accuracy of $\mathfrak{B}$ is defined by $\varkappa(\mathfrak{B}) = \frac{|\underline{\aleph}(\mathfrak{B})|}{|\overline{\aleph}(\mathfrak{B})|}$, where $|\overline{\aleph}(\mathfrak{B})| \neq 0$ and $|.|$ denotes the cardinality.*

Theorem 1. *[29] Let* $\mathfrak{K} = (\mathfrak{X}, \aleph)$ *and* $\mathfrak{A}$, $\mathfrak{B} \subseteq \mathfrak{X}$ *where* $\mathfrak{A}^c$ *is the complement of* $\mathfrak{A}$. *Then,*

(L1)	$\underline{\aleph}(\mathfrak{X}) = \mathfrak{X}$,	*(L1*)*	$\overline{\aleph}(\mathfrak{X}) = \mathfrak{X}$,
(L2)	$\underline{\aleph}(\phi) = \phi$,	*(L2*)*	$\overline{\aleph}(\phi) = \phi$,
(L3)	$\underline{\aleph}(\mathfrak{A}) \subseteq \mathfrak{A}$,	*(L3*)*	$\mathfrak{A} \subseteq \overline{\aleph}(\mathfrak{A})$,
(L4)	$\underline{\aleph}(\mathfrak{A}) \cap \underline{\aleph}(\mathfrak{B}) = \underline{\aleph}(\mathfrak{A} \cap \mathfrak{B})$,	*(L4*)*	$\overline{\aleph}(\mathfrak{A} \cup \mathfrak{B}) = \overline{\aleph}(\mathfrak{A}) \cup \overline{\aleph}(\mathfrak{B})$,
(L5)	$\underline{\aleph}(\mathfrak{A}^c) = [\overline{\aleph}(\mathfrak{A})]^c$,		
(L6)	$\underline{\aleph}(\underline{\aleph}(\mathfrak{A})) = \underline{\aleph}(\mathfrak{A})$,	*(L6*)*	$\overline{\aleph}(\overline{\aleph}(\mathfrak{A})) = \overline{\aleph}(\mathfrak{A})$,
(L7)	*If* $\mathfrak{A} \subseteq \mathfrak{B}$, *then* $\underline{\aleph}(\mathfrak{A}) \subseteq \underline{\aleph}(\mathfrak{B})$,	*(L7*)*	*If* $\mathfrak{A} \subseteq \mathfrak{B}$, *then* $\overline{\aleph}(\mathfrak{A}) \subseteq \overline{\aleph}(\mathfrak{B})$,
(L8)	$\underline{\aleph}([\underline{\aleph}(\mathfrak{A})]^c) = [\underline{\aleph}(\mathfrak{A})]^c$,	*(L8*)*	$\overline{\aleph}([\overline{\aleph}(\mathfrak{A})]^c) = [\overline{\aleph}(\mathfrak{A})]^c$,
(L9)	$\underline{\aleph}(\mathfrak{A}) \cup \underline{\aleph}(\mathfrak{B}) \subseteq \underline{\aleph}(\mathfrak{A} \cup \mathfrak{B})$,	*(L9*)*	$\overline{\aleph}(\mathfrak{A} \cap \mathfrak{B}) \subseteq \overline{\aleph}(\mathfrak{A}) \cap \overline{\aleph}(\mathfrak{B})$.

Definition 4. *[7] The general relation* $\aleph$ *is called*

i) Reflexive: $\forall \xi \in \mathfrak{X}$, $\xi \aleph \xi$.

ii) Symmetric: $\forall \xi, \gamma \in \mathfrak{X}$, *if* $\xi \aleph \gamma$, *then* $\gamma \aleph \xi$.

iii) If (i) and (ii) are hold, then the relation is called tolerance relation.

Definition 5. *[40] Let* $\aleph$ *be a general relation and* $\xi, \gamma \in \mathfrak{X}$. *The right neighborhood of* ξ *is defined by* $N_r(\xi) = \{\gamma \in \mathfrak{X} : \xi \aleph \gamma\}$, *and the left neighborhood of* ξ *is defined by* $N_l(\xi) = \{\gamma \in \mathfrak{X} : \gamma \aleph \xi\}$.

Definition 6. *[1] Let* $\aleph$ *be a general relation and* $\xi \in \mathfrak{X}$. *Then, minimal right neighborhood of* ξ *is* $MN_r(\xi) = \bigcap\{N_r(\gamma) : \gamma \aleph \xi\}$.

Definition 7. *Let* $\aleph$ *be a general relation. The right [7], left [7], union [25], and intersection [25] neighborhoods are defined by*
$N_r(\xi) = \{\gamma \in \mathfrak{X} : \xi \aleph \gamma\}$,
$N_l(\xi) = \{\gamma \in \mathfrak{X} : \gamma \aleph \xi\}$,
$N_u(\xi) = N_r(\xi) \cup N_l(\xi)$, *and*
$N_i(\xi) = N_r(\xi) \cap N_l(\xi)$, *respectively.*

Definition 8. *[24] Let* $\aleph$ *be a general relation. Then, core right, core left, core union, and core intersection neighborhoods are defined by*
$CN_r(\xi) = \{\gamma \in \mathfrak{X} : N_r(\xi) = N_r(\gamma)\}$,
$CN_l(\xi) = \{\gamma \in \mathfrak{X} : N_l(\xi) = N_l(\gamma)\}$,
$CN_u(\xi) = CN_r(\xi) \cup CN_l(\xi)$, *and*
$CN_i(\xi) = CN_r(\xi) \cap CN_l(\xi)$, *respectively.*

Definition 9. *[35] Let* $\aleph$ *be a general relation. The minimal right, minimal left, minimal union, and minimal intersection neighborhoods are defined by*
$MN_r(\xi) = \bigcap\{N_r(\gamma) : \gamma \aleph \xi\}$,
$MN_l(\xi) = \bigcap\{N_l(\gamma) : \xi \aleph \gamma\}$,
$MN_u(\xi) = MN_r(\xi) \cup MN_l(\xi)$, *and*
$MN_i(\xi) = MN_r(\xi) \cap MN_l(\xi)$, *respectively.*

3. Generalization for rough sets via core minimal neighborhoods

The present section views a generalization of RST using core minimal neighborhood systems with four types of upper and lower approximations. Relationships between neighborhood, core neighborhood, minimal neighborhood, and core minimal neighborhood using four types of right, left, union, and intersection neighborhoods are studied. Furthermore, a comparison between the current study and other studies is investigated.

Definition 10. *Let $\aleph$ be a general relation. The core minimal right, core minimal left, core minimal union, and core minimal intersection neighborhoods are defined by*
$CM_r(\xi) = \{\gamma \in \mathfrak{X} : MN_r(\xi) = MN_r(\gamma)\}$,
$CM_l(\xi) = \{\gamma \in \mathfrak{X} : MN_l(\xi) = MN_l(\gamma)\}$,
$CM_u(\xi) = CM_r(\xi) \cup CM_l(\xi)$, *and*
$CM_i(\xi) = CM_r(\xi) \cap CM_l(\xi)$, *respectively.*

Definition 11. *Let $\aleph$ be a general relation on $\mathfrak{X}$ and $CM_j : \mathfrak{X} \longrightarrow P(\mathfrak{X})$ be a mapping which assigns for each ξ in $\mathfrak{X}$ its core minimal neighborhoods in the power set of $\mathfrak{X}$ ($P(\mathfrak{X})$). The triple $(\mathfrak{X}, \aleph, CM_j)$ is called the core minimal approximation space (briefly, CM_j-approximation space) where $j \in \mathfrak{J} = \{r, l, u, i\}$.*

Corollary 1. *Let CM_j-approximation space with $\xi, \gamma \in \mathfrak{X}$. Then,*

i) $\xi \in CM_j(\xi)$, *where* $j \in \mathfrak{J}$.

ii) $\xi \in CM_j(\gamma) \iff \gamma \in CM_j(\xi)$, *where* $j \in \mathfrak{J}$.

iii) Let $\gamma \in CM_j(\xi)$. *Then,* $CM_j(\gamma) = CM_j(\xi)$, *where* $j \in \{r, l, i\}$.

Part (iii) is not true for $j = u$, in general.

Example 1. *If $\mathfrak{X} = \{\xi, \gamma, \zeta, \eta\}$ with $\aleph = \{(\xi, \eta), (\gamma, \zeta), (\gamma, \eta), (\zeta, \eta), (\eta, \xi), (\eta, \gamma)\}$, then $N_r(\mathfrak{X}, \aleph) = \{\{\eta\}, \{\zeta, \eta\}, \{\xi, \gamma\}\}$, $N_l(\mathfrak{X}, \aleph) = \{\{\eta\}, \{\gamma\}, \{\xi, \gamma, \zeta\}\}$, $MN_r(\xi) = MN_r(\gamma) = \{\xi, \gamma\}$, $MN_r(\zeta) = \{\zeta, \eta\}$, $MN_r(\eta) = \{\eta\}$, $MN_l(\xi) = MN_l(\zeta) = \{\xi, \gamma, \zeta\}$, $MN_l(\gamma) = \{\gamma\}$, $MN_l(\eta) = \{\eta\}$. Then, $CM_r(\xi) = CM_r(\gamma) = \{\xi, \gamma\}$, $CM_r(\zeta) = \{\zeta\}$, $CM_r(\eta) = \{\eta\}$, $CM_l(\xi) = CM_l(\zeta) = \{\xi, \zeta\}$, $CM_l(\gamma) = \{\gamma\}$, $CM_l(\eta) = \{\eta\}$, $CM_i(\xi) = \{\xi\}$, $CM_i(\gamma) = \{\gamma\}$, $CM_i(\zeta) = \{\zeta\}$, $CM_i(\eta) = \{\eta\}$, $CM_u(\xi) = \{\xi, \gamma, \zeta\}$, $CM_u(\gamma) = \{\xi, \gamma\}$, $CM_u(\zeta) = \{\xi, \zeta\}$, and $CM_u(\eta) = \{\eta\}$. Clearly, $\xi \in CM_u(\gamma)$ but $CM_u(\xi) \neq CM_u(\gamma)$.*

Corollary 2. *Let $\aleph$ be a reflexive relation with $\xi, \gamma \in \mathfrak{X}$ and $\gamma \in CM_j(\xi)$. Then, $CM_j(\gamma) = CM_j(\xi)$, $\forall j \in \mathfrak{J}$.*

Lemma 1. *Let $\aleph$ be a reflexive relation with $\xi \in \mathfrak{X}$. Then, $CM_j(\xi) \subseteq MN_j(\xi)$, $\forall j \in \mathfrak{J}$.*

Proof. Let $\aleph$ be a reflexive relation. Then, $\xi \in MN_j(\xi)$, $\forall \xi \in \mathfrak{X}$. If $\gamma \in CM_j(\xi)$, then $MN_j(\xi) = MN_j(\gamma)$ and since $\gamma \in MN_j(\gamma)$, then $\gamma \in MN_j(\xi)$. Therefore, $CM_j(\xi) \subseteq MN_j(\xi)$.

The equality in Lemma 1 is not true, in general.

Example 2. *Let* $\mathfrak{X} = \{\xi, \gamma, \zeta, \eta\}$ *with* $\aleph = \{(\xi,\xi), (\gamma,\gamma), (\zeta,\zeta), (\eta,\eta), (\xi,\zeta), (\gamma,\zeta), (\gamma,\eta), (\zeta,\xi), (\eta,\gamma)\}$. *Then,* $N_r(\mathfrak{X},\aleph) = \{\{\xi,\zeta\},\{\gamma,\zeta,\eta\},\{\gamma,\eta\}\}$, $N_l(\mathfrak{X},\aleph) = \{\{\xi,\zeta\}, \{\gamma,\eta\}, \{\xi,\gamma,\zeta\}\}$, $MN_r(\xi) = \{\xi,\zeta\}$, $MN_r(\gamma) = MN_r(\eta) = \{\gamma,\eta\}$, $MN_r(\zeta) = \{\zeta\}$, $MN_l(\xi) = MN_l(\zeta) = \{\xi,\zeta\}$, $MN_l(\gamma) = \{\gamma\}$, $MN_l(\eta) = \{\gamma,\eta\}$, $MN_i(\xi) = \{\xi,\zeta\}$, $MN_i(\gamma) = \{\gamma\}$, $MN_i(\zeta) = \{\zeta\}$ $MN_i(\eta) = \{\gamma,\eta\}$, $MN_u(\xi) = \{\xi,\zeta\}$, $MN_u(\gamma) = \{\gamma,\eta\}$, $MN_u(\zeta) = \{\xi,\zeta\}$, *and* $MN_u(\eta) = \{\gamma,\eta\}$. *Then,* $CM_r(\xi) = \{\xi\}$, $CM_r(\gamma) = CM_r(\eta) = \{\gamma,\eta\}$, $CM_r(\zeta) = \{\zeta\}$, $CM_l(\xi) = CM_l(\zeta) = \{\xi,\zeta\}$, $CM_l(\gamma) = \{\gamma\}$, $CM_l(\eta) = \{\eta\}$, $CM_i(\xi) = \{\xi\}$, $CM_i(\gamma) = \{\gamma\}$, $CM_i(\zeta) = \{\zeta\}$, $CM_i(\eta) = \{\eta\}$, $CM_u(\xi) = CM_u(\zeta) = \{\xi,\zeta\}$, *and* $CM_u(\gamma) = CM_u(\eta) = \{\gamma,\eta\}$. *But,* $CM_r(\xi) \neq MN_r(\xi)$, $CM_l(\eta) \neq MN_l(\eta)$, *and* $CM_i(\xi) \neq MN_i(\xi)$.

Lemma 2. *Let* $\aleph$ *be a reflexive relation. Then,* $CN_j(\xi) \subseteq N_j(\xi)$, $\forall \xi \in \mathfrak{X}$ *and* $\forall j \in \mathfrak{J}$.

Proof. Let $\aleph$ be a reflexive relation. Then, $\xi \in N_j(\xi)$, $\forall \xi \in \mathfrak{X}$. Now, let $\gamma \in CN_j(\xi)$. Then, $N_j(\xi) = N_j(\gamma)$. Hence, $\gamma \in N_j(\xi)$. Therefore, $CN_j(\xi) \subseteq N_j(\xi)$.

The equality in Lemma 2 is not true, in general.

Example 3. *In Example 2,* $CN_r(\eta) = \{\eta\}$, $N_r(\eta) = \{\gamma,\eta\}$, $CN_l(\zeta) = \{\zeta\}$, $N_l(\zeta) = \{\xi,\gamma,\zeta\}$, $CN_i(\zeta) = \{\zeta\}$, $N_i(\zeta) = \{\xi,\zeta\}$, $CN_u(\zeta) = \{\xi,\zeta\}$, *and* $N_u(\zeta) = \{\xi,\gamma,\zeta\}$. *But,* $CN_r(\eta) \neq N_r(\eta)$, $CN_l(\zeta) \neq N_l(\zeta)$, $CN_u(\zeta) \neq N_u(\zeta)$, *and* $CN_i(\zeta) \neq N_i(\zeta)$.

The $CM_j(\xi)$ and $CN_j(\xi)$ are independent with general relation for $j \in \mathfrak{J}$, in general.

Example 4. *In Example 2,* $CM_r(\gamma) \neq CN_r(\gamma)$ *and* $CN_l(\xi) \neq CM_l(\xi)$.

Lemma 3. *Let* $\aleph$ *be a tolerance relation. Then,* $CM_j(\xi) \subseteq CN_j(\xi)$, $\forall \xi \in \mathfrak{X}$.

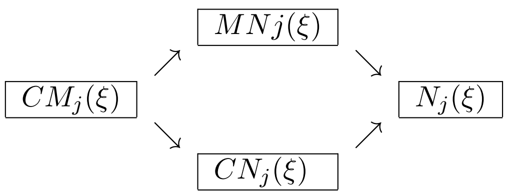
Lemma 4. *[35] Let* $\aleph$ *be a tolerance relation. Then,* $MN_j(\xi) \subseteq N_j(\xi)$, $\forall \xi \in \mathfrak{X}$.

The equality in Lemma 4 is not true, in general.

Example 5. *In Example 2,* $MN_r(\zeta) \neq N_r(\zeta)$, $MN_r(\gamma) \neq N_r(\gamma)$, $MN_u(\gamma) \neq N_u(\gamma)$, *and* $MN_i(\gamma) \neq N_i(\gamma)$.

In Remark 1, a relationship between neighborhood, core neighborhood, minimal neighborhood, and core minimal neighborhood using the four types of right, left, union, and intersection neighborhoods is demonstrated when the relation is tolerance.

Remark 1. *Let* $\aleph$ *be a tolerance relation. Then, for each* $\xi \in \mathfrak{X}$:



The equality of these implications is not true in general. This can be shown in Examples 2, 3, 4, and 5.

In the following, we study RST by studying $\underline{\aleph}_j(\mathfrak{B})$ and $\overline{\aleph}_j(\mathfrak{B})$, we give results that compare with Pawlak.

Definition 12. *Let* $(\mathfrak{X}, \aleph, CM_j)$ *be an approximation space with* $\mathfrak{B} \subseteq \mathfrak{X}$. *Then,* CM_j-*lower and* CM_j-*upper approximations of* $\mathfrak{B}$ *are defined by* $\underline{\aleph}_j(\mathfrak{B}) = \bigcup\{CM_j(\xi) : CM_j(\xi) \subseteq \mathfrak{B}\}$, *and* $\overline{\aleph}_j(\mathfrak{B}) = \bigcup\{CM_j(\xi) : CM_j(\xi) \cap \mathfrak{B} \neq \phi\}$, *respectively.*

Definition 13. *In Definition 12,* $\mathfrak{B}$ *is called* CM_j-*exact if* $\underline{\aleph}_j(\mathfrak{B}) = \overline{\aleph}_j(\mathfrak{B})$, $\forall j \in \mathfrak{J}$. *Otherwise,* $\mathfrak{B}$ *is* CM_j-*rough.*

Definition 14. *For each* $j \in \mathfrak{J}$, CM_j-*boundary,* CM_j-*positive, and* CM_j-*negative sets are* $B_j(\mathfrak{B}) = \overline{\aleph}_j(\mathfrak{B}) - \underline{\aleph}_j(\mathfrak{B})$, $P_j(\mathfrak{B}) = \underline{\aleph}_j(\mathfrak{B})$, *and* $N_j(\mathfrak{B}) = \mathfrak{X} - \overline{\aleph}_j(\mathfrak{B})$, *respectively.*

Definition 15. *If* $\aleph$ *is a general relation with* $\mathfrak{B} \subseteq \mathfrak{X}$ *and* $j \in \mathfrak{J}$, *the* CM_j-*accuracy of approximation of the subset* $\mathfrak{B}$ *is* $\varkappa_j(\mathfrak{B}) = \frac{|\underline{\aleph}_j(\mathfrak{B})|}{|\overline{\aleph}_j(\mathfrak{B})|}$. *Where,* $|\overline{\aleph}_j(\mathfrak{B})| \neq 0$ *and* $|.|$ *denotes the cardinality.*

Remark 2. *From Definition 15, we deduce that with a relation* $\aleph$:

i) $0 \leq \varkappa_j(\mathfrak{B}) \leq 1$.

ii) Let $\varkappa_j(\mathfrak{B}) = 1$. *Then,* $\mathfrak{B}$ *is* CM_j-*exact . Otherwise,* $\mathfrak{B}$ *is* CM_j-*rough.*

Theorem 2. *Let* $\aleph$ *be a general relation and* $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{X}$. *Then, the following are the properties of a generalization of RST, with* $\mathfrak{A}^c$ *representing the complement.*

(L1)	$\underline{\aleph}_j(\mathfrak{X}) = \mathfrak{X}$,	*(L1*)*	$\overline{\aleph}_j(\mathfrak{X}) = \mathfrak{X}$,
(L2)	$\underline{\aleph}_j(\phi) = \phi$,	*(L2*)*	$\overline{\aleph}_j(\phi) = \phi$,
(L3)	$\underline{\aleph}_j(\mathfrak{A}) \subseteq \mathfrak{A}$,	*(L3*)*	$\mathfrak{A} \subseteq \overline{\aleph}_j(\mathfrak{A})$,
(L4)	$\underline{\aleph}_j(\mathfrak{A}) \cap \underline{\aleph}_j(\mathfrak{B}) = \underline{\aleph}_j(\mathfrak{A} \cap \mathfrak{B})$,	*(L4*)*	$\overline{\aleph}_j(\mathfrak{A} \cup \mathfrak{B}) = \overline{\aleph}_j(\mathfrak{A}) \cup \overline{\aleph}_j(\mathfrak{B})$,
(L5)	$\underline{\aleph}_j(\mathfrak{A}^c) = [\overline{\aleph}_j(\mathfrak{A})]^c$,		
(L6)	$\underline{\aleph}_j(\underline{\aleph}_j(\mathfrak{A})) = \underline{\aleph}_j(\mathfrak{A})$,	*(L6*)*	$\overline{\aleph}_j(\overline{\aleph}_j(\mathfrak{A})) = \overline{\aleph}_j(\mathfrak{A})$,
(L7)	*If* $\mathfrak{A} \subseteq \mathfrak{B}$, *then* $\underline{\aleph}_j(\mathfrak{A}) \subseteq \underline{\aleph}_j(\mathfrak{B})$,	*(L7*)*	*If* $\mathfrak{A} \subseteq \mathfrak{B}$, *then* $\overline{\aleph}_j(\mathfrak{A}) \subseteq \overline{\aleph}_j(\mathfrak{B})$,
(L8)	$\underline{\aleph}_j([\underline{\aleph}_j(\mathfrak{A})]^c) = [\underline{\aleph}_j(\mathfrak{A})]^c$,	*(L8*)*	$\overline{\aleph}_j([\overline{\aleph}_j(\mathfrak{A})]^c) = [\overline{\aleph}_j(\mathfrak{A})]^c$,
(L9)	$\underline{\aleph}_j(\mathfrak{A}) \cup \underline{\aleph}_j(\mathfrak{B}) \subseteq \underline{\aleph}_j(\mathfrak{A} \cup \mathfrak{B})$,	*(L9*)*	$\overline{\aleph}_j(\mathfrak{A} \cap \mathfrak{B}) \subseteq \overline{\aleph}_j(\mathfrak{A}) \cap \overline{\aleph}_j(\mathfrak{B})$.

Proof. Properties (L1), (L1*), (L2), (L2*), (L3), (L3*), (L6), and (L6*) are obvious. Hence, the remainder of the properties can be proven as follows:

(L4) $\underline{\aleph}_j(\mathfrak{A} \cap \mathfrak{B}) = \bigcup\{CM_j(\xi) : CM_j(\xi) \subseteq \mathfrak{A} \cap \mathfrak{B}\} = [\bigcup\{CM_j(\xi) : CM_j(\xi) \subseteq \mathfrak{A}\}] \cap [\bigcup\{CM_j(\xi) : CM_j(\xi) \subseteq \mathfrak{B}\}] = \underline{\aleph}_j(\mathfrak{A}) \cap \underline{\aleph}_j(\mathfrak{B})$.

(L4*) Similar to the proof of (L4).

(L5) $\underline{\aleph}_j(\mathfrak{A}^c) = \bigcup\{CM_j(\xi) : CM_j(\xi) \subseteq \mathfrak{A}^c\}$=$\bigcup\{CM_j(\xi) : CM_j(\xi) \cap \mathfrak{A} = \phi\}$. Since $\xi \in CM_(\xi)$, for all $\xi \in \mathfrak{X}$, then $\underline{\aleph}_j(\mathfrak{A}^c) = \bigcup\{\xi \in \mathfrak{X} : CM_j(\xi) \cap \mathfrak{A} \neq \phi\}^c$=$[\overline{\aleph}_j(\mathfrak{A})]^c$.

(L7) Let $\mathfrak{A} \subseteq \mathfrak{B}$. Then, $\underline{\aleph}_j(\mathfrak{A}) = \bigcup\{CM_j(\xi) : CM_j(\xi)) \subseteq \mathfrak{A}\} \subseteq \bigcup\{CM_j(\xi) : CM_j(\xi) \subseteq \mathfrak{B}\} = \underline{\aleph}_j(\mathfrak{B})$.

(L7*) Similar to the proof of (L7).

(L8) By using (L7), we have $\underline{\aleph}_j([\underline{\aleph}_j(\mathfrak{A})]^c) \subseteq [\underline{\aleph}_j(\mathfrak{A})]^c$. Conversely, let $\gamma \in [\underline{\aleph}_j(\mathfrak{A})]^c$. Then, $\gamma \in [\bigcup\{CM_j(\xi) : CM_j(\xi)) \subseteq \mathfrak{A}\}]^c = \bigcup\{CM_j(\xi) : CM_j(\xi)) \cap \mathfrak{A} = \phi\}$. So, $\gamma \in \bigcup\{CM_j(\xi) : CM_j(\xi)) \cap \underline{\aleph}_j(\mathfrak{A}) = \phi\}$. Then, $\gamma \in \bigcup\{CM_j(\xi) : CM_j(\xi)) \subseteq [\underline{\aleph}_j(\mathfrak{A})]^c\}$. This implies that, $\gamma \in \underline{\aleph}_j([\underline{\aleph}_j(\mathfrak{A})]^c)$. Therefore, $[\underline{\aleph}_j(\mathfrak{A})]^c \subseteq \underline{\aleph}_j([\underline{\aleph}_j(\mathfrak{A})]^c)$.

(L8*) Similar to the proof of (L8).

(L9) Since $\mathfrak{A} \subseteq \mathfrak{A} \cup \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{A} \cup \mathfrak{B}$. Then, $\underline{\aleph}_j(\mathfrak{A}) \subseteq \underline{\aleph}_j(\mathfrak{A} \cup \mathfrak{B})$ and $\underline{\aleph}_j(\mathfrak{B}) \subseteq \underline{\aleph}_j(\mathfrak{A} \cup \mathfrak{B})$. Therefore, $\underline{\aleph}_j(\mathfrak{A}) \cup \underline{\aleph}_j(\mathfrak{B}) \subseteq \underline{\aleph}_j(\mathfrak{A} \cup \mathfrak{B})$.

(L9*) Similar to the proof of (L9).

The equality of L8 and L9 in Theorem 2 is not true, in general.

Example 6. *If* $\mathfrak{X} = \{\xi, \gamma, \zeta, \eta\}$ *with* $\aleph = \{(\xi, \xi), (\gamma, \gamma), (\zeta, \zeta), (\xi, \zeta), (\gamma, \eta), (\zeta, \xi), (\eta, \zeta)\}$, *then* $N_r(\mathfrak{X}, \aleph) = \{\{\xi, \zeta\}, \{\gamma, \eta\}, \{\zeta\}\}$, $N_l(\mathfrak{X}, \aleph) = \{\{\xi, \zeta\}, \{\gamma\}, \{\xi, \zeta, \eta\}\}$, $MN_r(\xi) = \{\xi, \zeta\}$, $MN_r(\gamma) = MN_r(\eta) = \{\gamma, \eta\}$, $MN_r(\zeta) = \{\zeta\}$, $MN_l(\xi) = MN_l(\zeta) = \{\xi, \zeta\}$, $MN_l(\gamma) = \{\gamma\}$, $MN_l(\eta) = \{\xi, \zeta, \eta\}$. *Then,* $CM_r(\xi) = \{\xi\}$, $CM_r(\gamma) = CM_r(\eta) = \{\gamma, \eta\}$, $CM_r(\zeta) = \{\zeta\}$, $CM_l(\xi) = CM_l(\zeta) = \{\xi, \zeta\}$, $CM_l(\gamma) = \{\gamma\}$, $CM_l(\eta) = \{\eta\}$, $CM_i(\xi) = \{\xi\}$, $CM_i(\gamma) = \{\gamma\}$, $CM_i(\zeta) = \{\zeta\}$, $CM_i(\eta) = \{\eta\}$, $CM_u(\xi) = CM_u(\zeta) = \{\xi, \zeta\}$, *and* $CM_u(\gamma) = CM_u(\eta) = \{\gamma, \eta\}$. *Let* $\mathfrak{A} = \{\gamma\}$, $\mathfrak{B} = \{\eta\}$, $\mathfrak{C} = \{\xi\}$, *and* $\mathfrak{D} = \{\zeta\}$. *Then,* $\underline{\aleph}_r(\mathfrak{A}) = \underline{\aleph}_r(\mathfrak{B}) = \underline{\aleph}_l(\mathfrak{C}) = \underline{\aleph}_l(\mathfrak{D}) = \underline{\aleph}_u(\mathfrak{C}) = \underline{\aleph}_u(\mathfrak{D}) = \phi$, $\underline{\aleph}_r(\mathfrak{A} \cup \mathfrak{B}) = \{\gamma, \eta\}$, $\underline{\aleph}_l(\mathfrak{C} \cup \mathfrak{D}) = \underline{\aleph}_u(\mathfrak{C} \cup \mathfrak{D}) = \{\xi, \zeta\}$, $\overline{\aleph}_r(\mathfrak{A}) = \overline{\aleph}_r(\mathfrak{B}) = \overline{\aleph}_u(\mathfrak{A}) = \overline{\aleph}_u(\mathfrak{B}) = \{\gamma, \eta\}, \overline{\aleph}_l(\mathfrak{C}) = \overline{\aleph}_l(\mathfrak{D}) = \{\xi, \zeta\}$, $\overline{\aleph}_r(\mathfrak{C} \cap \mathfrak{D}) = \overline{\aleph}_l(\mathfrak{A} \cap \mathfrak{B}) = \overline{\aleph}_u(\mathfrak{A} \cap \mathfrak{B}) = \phi$. *But,* $\overline{\aleph}_r(\mathfrak{C} \cap \mathfrak{D}) \neq \overline{\aleph}_r(\mathfrak{C}) \cap \overline{\aleph}_r(\mathfrak{D})$, $\overline{\aleph}_l(\mathfrak{A} \cap \mathfrak{B}) \neq \overline{\aleph}_l(\mathfrak{A}) \cap \overline{\aleph}_l(\mathfrak{B})$, $\overline{\aleph}_u(\mathfrak{C} \cap \mathfrak{D}) \neq \overline{\aleph}_u(\mathfrak{C}) \cap \overline{\aleph}_u(\mathfrak{D})$, $\underline{\aleph}_r(\mathfrak{A}) \cup \underline{\aleph}_r(\mathfrak{B}) \neq \underline{\aleph}_r(\mathfrak{A} \cup \mathfrak{B})$, $\underline{\aleph}_l(\mathfrak{C}) \cup \underline{\aleph}_l(\mathfrak{D}) \neq \underline{\aleph}_l(\mathfrak{C} \cup \mathfrak{D})$, *and* $\underline{\aleph}_u(\mathfrak{C}) \cup \underline{\aleph}_u(\mathfrak{D}) \neq \underline{\aleph}_u(\mathfrak{C} \cup \mathfrak{D})$.

Remark 3. *Theorem 2 shows that our method has the same characteristics as Pawlak's method. In our method,* $\aleph$ *is an arbitrary relation. As a result, we believe that our method is a generalization for RST. Table 1 shows a comparison between our method and others.*

Pawlak's properties	Yao's [40]	Yun et al [44]	Shbair et al [35]	Our method
(L1)	√	√		√
(L2)		√		√
(L3)		√		√
(L4)	√		√	√
(L5)	√		√	√
(L6)		√		√
(L7)	√	√	√	√
(L8)				√
(L9)	√	√	√	√
(L1*)		√	√	√
(L2*)		√		√
(L3*)		√	√	√
(L4*)	√	√	√	√
(L6*)				√
(L7*)	√	√	√	√
(L8*)				√
(L9*)	√	√	√	√

Table 1: A comparison between different methods of rough set with our method.

4. Relationship between several types of $CMj-$approximations operators

This section aims to compare several types of $CMj-$approximations. Also, the boundary and accuracy of $CMj-$approximations are discussed.

In Table 2, 3 by using Example 6, we compare different types of CM_j-approximations, CM_j-boundary, and CM_j-accuracy.

$\mathfrak{B}$	$\underline{\aleph}_r(\mathfrak{B})$	$\overline{\aleph}_r(\mathfrak{B})$	$B_r(\mathfrak{B})$	$\varkappa_r(\mathfrak{B})$	$\underline{\aleph}_l(\mathfrak{B})$	$\overline{\aleph}_l(\mathfrak{B})$	$B_l(\mathfrak{B})$	$\varkappa_l(\mathfrak{B})$
$\{\xi\}$	$\{\xi\}$	$\{\xi\}$	ϕ	1	ϕ	$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	0
$\{\gamma\}$	ϕ	$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	0	$\{\gamma\}$	$\{\gamma\}$	ϕ	1
$\{\zeta\}$	$\{\zeta\}$	$\{\zeta\}$	ϕ	1	ϕ	$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	0
$\{\eta\}$	ϕ	$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	0	$\{\eta\}$	$\{\eta\}$	ϕ	1
$\{\xi,\gamma\}$	$\{\xi\}$	$\{\xi,\gamma,\eta\}$	$\{\gamma,\eta\}$	1/3	$\{\gamma\}$	$\{\xi,\gamma,\zeta\}$	$\{\xi,\zeta\}$	1/3
$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	ϕ	1	$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	ϕ	1
$\{\xi,\eta\}$	$\{\xi\}$	$\{\xi,\gamma,\eta\}$	$\{\gamma,\eta\}$	1/3	$\{\eta\}$	$\{\xi,\zeta,\eta\}$	$\{\xi,\zeta\}$	1/3
$\{\gamma,\zeta\}$	$\{\zeta\}$	$\{\gamma,\zeta,\eta\}$	$\{\gamma,\eta\}$	1/3	$\{\gamma\}$	$\{\xi,\gamma,\zeta\}$	$\{\xi,\zeta\}$	1/3
$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	ϕ	1	$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	ϕ	1
$\{\zeta,\eta\}$	$\{\zeta\}$	$\{\gamma,\zeta,\eta\}$	$\{\gamma,\eta\}$	1/3	$\{\eta\}$	$\{\xi,\zeta,\eta\}$	$\{\xi,\zeta\}$	1/3
$\{\xi,\gamma,\zeta\}$	$\{\xi,\zeta\}$	$\mathfrak{X}$	$\{\gamma,\eta\}$	1/2	$\{\xi,\gamma,\zeta\}$	$\{\xi,\gamma,\zeta\}$	ϕ	1
$\{\xi,\gamma,\eta\}$	$\{\xi,\gamma,\eta\}$	$\{\xi,\gamma,\eta\}$	ϕ	1	$\{\gamma,\eta\}$	$\mathfrak{X}$	$\{\xi,\zeta\}$	1/2
$\{\xi,\zeta,\eta\}$	$\{\xi,\zeta\}$	$\mathfrak{X}$	$\{\gamma,\eta\}$	1/2	$\{\xi,\zeta,\eta\}$	$\{\xi,\zeta,\eta\}$	ϕ	1
$\{\gamma,\zeta,\eta\}$	$\{\gamma,\zeta,\eta\}$	$\{\gamma,\zeta,\eta\}$	ϕ	1	$\{\gamma,\eta\}$	$\mathfrak{X}$	$\{\xi,\zeta\}$	1/2
$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$	ϕ	1	$\mathfrak{X}$	$\mathfrak{X}$	ϕ	1

Table 2: A comparison between several types of CM_j- approximations.

$\mathfrak{B}$	$\underline{\aleph}_u(\mathfrak{B})$	$\overline{\aleph}_u(\mathfrak{B})$	$B_u(\mathfrak{B})$	$\varkappa_u(\mathfrak{B})$	$\underline{\aleph}_i(\mathfrak{B})$	$\overline{\aleph}_i(\mathfrak{B})$	$B_i(\mathfrak{B})$	$\varkappa_i(\mathfrak{B})$
$\{\xi\}$	ϕ	$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	0	$\{\xi\}$	$\{\xi\}$	ϕ	1
$\{\gamma\}$	ϕ	$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	0	$\{\gamma\}$	$\{\gamma\}$	ϕ	1
$\{\zeta\}$	ϕ	$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	0	$\{\zeta\}$	$\{\zeta\}$	ϕ	1
$\{\eta\}$	ϕ	$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	0	$\{\eta\}$	$\{\eta\}$	ϕ	1
$\{\xi,\gamma\}$	ϕ	$\mathfrak{X}$	$\mathfrak{X}$	0	$\{\xi,\gamma\}$	$\{\xi,\gamma\}$	ϕ	1
$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	ϕ	1	$\{\xi,\zeta\}$	$\{\xi,\zeta\}$	ϕ	1
$\{\xi,\eta\}$	ϕ	$\mathfrak{X}$	$\mathfrak{X}$	0	$\{\xi,\eta\}$	$\{\xi,\eta\}$	ϕ	1
$\{\gamma,\zeta\}$	ϕ	$\mathfrak{X}$	$\mathfrak{X}$	0	$\{\gamma,\zeta\}$	$\{\gamma,\zeta\}$	ϕ	1
$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	ϕ	1	$\{\gamma,\eta\}$	$\{\gamma,\eta\}$	ϕ	1
$\{\zeta,\eta\}$	ϕ	$\mathfrak{X}$	$\mathfrak{X}$	0	$\{\zeta,\eta\}$	$\{\zeta,\eta\}$	ϕ	1
$\{\xi,\gamma,\zeta\}$	$\{\xi,\zeta\}$	$\mathfrak{X}$	$\{\gamma,\eta\}$	1/2	$\{\xi,\gamma,\zeta\}$	$\{\xi,\gamma,\zeta\}$	ϕ	1
$\{\xi,\gamma,\eta\}$	$\{\gamma,\eta\}$	$\mathfrak{X}$	$\{\xi,\zeta\}$	1/2	$\{\xi,\gamma,\eta\}$	$\{\xi,\gamma,\eta\}$	ϕ	1
$\{\xi,\zeta,\eta\}$	$\{\xi,\zeta\}$	$\mathfrak{X}$	$\{\gamma,\eta\}$	1/2	$\{\xi,\zeta,\eta\}$	$\{\xi,\zeta,\eta\}$	ϕ	1
$\{\gamma,\zeta,\eta\}$	$\{\gamma,\eta\}$	$\mathfrak{X}$	$\{\xi,\zeta\}$	1/2	$\{\gamma,\zeta,\eta\}$	$\{\gamma,\zeta,\eta\}$	ϕ	1
$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$	ϕ	1	$\mathfrak{X}$	$\mathfrak{X}$	ϕ	1

Table 3: A comparison between several types of CM_j- approximations.

Theorem 3. *Let $\aleph$ be a general relation and $\mathfrak{B} \subseteq \mathfrak{X}$. Then,*

i) $\underline{\aleph}_u(\mathfrak{B}) \subseteq \underline{\aleph}_r(\mathfrak{B}) \subseteq \underline{\aleph}_i(\mathfrak{B}) \subseteq \mathfrak{B} \subseteq \overline{\aleph}_i(\mathfrak{B}) \subseteq \overline{\aleph}_r(\mathfrak{B}) \subseteq \overline{\aleph}_u(\mathfrak{B})$.

ii) $\underline{\aleph}_u(\mathfrak{B}) \subseteq \underline{\aleph}_l(\mathfrak{B}) \subseteq \underline{\aleph}_i(\mathfrak{B}) \subseteq \mathfrak{B} \subseteq \overline{\aleph}_i(\mathfrak{B}) \subseteq \overline{\aleph}_l(\mathfrak{B}) \subseteq \overline{\aleph}_u(\mathfrak{B})$.

Proof. Let $\xi \in \underline{\aleph}_u(\mathfrak{B}) = \bigcup\{CM_u(\xi) : CM_u(\xi) \subseteq \mathfrak{B}\}$. But, $CM_u(\xi) = [CM_r(\xi) \cup CM_l(\xi)] \subseteq \mathfrak{B}$. Thus, either $\xi \in \bigcup\{CM_r(\xi) : CM_r(\xi) \subseteq \mathfrak{B}\}$ or $\xi \in \bigcup\{CM_l(\xi) : CM_l(\xi) \subseteq \mathfrak{B}\}$. Hence, $\xi \in \underline{\aleph}_r(\mathfrak{B})$ or $\xi \in \underline{\aleph}_l(\mathfrak{B})$. Therefore, $\underline{\aleph}_u(\mathfrak{B}) \subseteq \underline{\aleph}_r(\mathfrak{B})$ or $\underline{\aleph}_u(\mathfrak{B}) \subseteq \underline{\aleph}_l(\mathfrak{B})$. Now, let $\xi \in \underline{\aleph}_r(\mathfrak{B}) = \bigcup\{CM_r(\xi) : CM_r(\xi) \subseteq \mathfrak{B}\}$. But, $CM_i(\xi) = [CM_r(\xi) \cap CM_l(\xi)] \subseteq \mathfrak{B}$, Thus, $\xi \in \bigcup\{CM_i(\xi) : CM_i(\xi) \subseteq \mathfrak{B}\}$. Hence, $\xi \in \underline{\aleph}_i(\mathfrak{B})$. Therefore, $\underline{\aleph}_r(\mathfrak{B}) \subseteq \underline{\aleph}_i(\mathfrak{B})$. Similarly, $\underline{\aleph}_l(\mathfrak{B}) \subseteq \underline{\aleph}_i(\mathfrak{B})$. By Theorem 2, we have $\underline{\aleph}_i(\mathfrak{B}) \subseteq \mathfrak{B} \subseteq \overline{\aleph}_i(\mathfrak{B})$. Now, let $\xi \in \overline{\aleph}_i(\mathfrak{B}) = \bigcup\{CM_i(\xi) : CM_i(\xi) \cap \mathfrak{B} \neq \phi\}$. But, $CM_i(\xi) = CM_r(\xi) \cap CM_l(\xi)$. Hence, $\xi \in \bigcup\{CM_r(\xi) : CM_r(\xi) \cap \mathfrak{B} \neq \phi\}$ and $\xi \in \bigcup\{CM_l(\xi) : CM_l(\xi) \cap \mathfrak{B} \neq \phi\}$. Therefore, $\overline{\aleph}_i(\mathfrak{B}) \subseteq \overline{\aleph}_r(\mathfrak{B})$ and $\overline{\aleph}_i(\mathfrak{B}) \subseteq \overline{\aleph}_l(\mathfrak{B})$. Now, let $\xi \in \overline{\aleph}_r(\mathfrak{B}) = \bigcup\{CM_r(\xi) : CM_r(\xi) \cap \mathfrak{B} \neq \phi\}$. But, $CM_u(\xi) = CM_r(\xi) \cup CM_l(\xi)$. Hence, $\xi \in \bigcup\{CM_u(\xi) : CM_u(\xi) \cap \mathfrak{B} \neq \phi\}$. Therefore, $\overline{\aleph}_r(\mathfrak{B}) \subseteq \overline{\aleph}_u(\mathfrak{B})$. Similarly, $\overline{\aleph}_l(\mathfrak{B}) \subseteq \overline{\aleph}_u(\mathfrak{B})$.

The equality of parts (i) and (ii) in Theorem 3 is not true, in general.

Example 7. *In Example 6 by using Table 2, 3,* $\underline{\aleph}_u(\{\xi,\eta\}) \neq \underline{\aleph}_r(\{\xi,\eta\}) \neq \underline{\aleph}_l(\{\xi,\eta\}) \neq \underline{\aleph}_i(\{\xi,\eta\})$ *and* $\overline{\aleph}_i(\{\xi,\gamma\}) \neq \overline{\aleph}_r(\{\xi,\gamma\}) \neq \overline{\aleph}_l(\{\xi,\gamma\}) \neq \overline{\aleph}_u(\{\xi,\gamma\})$.

Remark 4. *In Figure 1, we compare between several types of CM_j- approximations operators with general relation $\aleph$ and $\mathfrak{B} \subseteq \mathfrak{X}$.*

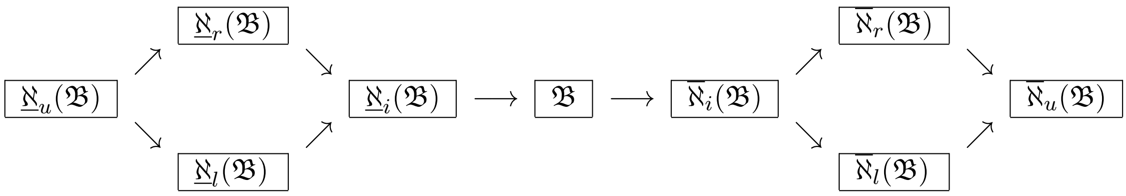


Figure 1: Relationship between CM_j-approximations operators.

Theorem 4. *Let $\aleph$ be a general relation and $\mathfrak{B} \subseteq \mathfrak{X}$. Then,*

i) $B_i(\mathfrak{B}) \subseteq B_r(\mathfrak{B}) \subseteq B_u(\mathfrak{B})$.

ii) $B_i(\mathfrak{B}) \subseteq B_l(\mathfrak{B}) \subseteq B_u(\mathfrak{B})$.

Proof. (i) If $\gamma \in B_i(\mathfrak{B})$, then $\gamma \in \overline{\aleph}_i(\mathfrak{B})$ and $\gamma \notin \underline{\aleph}_i(\mathfrak{B})$. By Theorem 3, $\gamma \in \overline{\aleph}_r(\mathfrak{B})$ and $\gamma \notin \underline{\aleph}_r(\mathfrak{B})$. Hence, $\gamma \in B_r(\mathfrak{B})$. Therefore, $B_i(\mathfrak{B}) \subseteq B_r(\mathfrak{B})$. Now, if $\gamma \in B_r(\mathfrak{B})$, then $\gamma \in \overline{\aleph}_r(\mathfrak{B})$ and $\gamma \notin \underline{\aleph}_r(\mathfrak{B})$. By Theorem 3, $\gamma \in \overline{\aleph}_u(\mathfrak{B})$ and $\gamma \notin \underline{\aleph}_u(\mathfrak{B})$. Hence, $\gamma \in B_u(\mathfrak{B})$. Therefore, $B_r(\mathfrak{B}) \subseteq B_u(\mathfrak{B})$. Part (ii) is similar to the proof of part (i).

Theorem 5. *Let $\aleph$ be a general relation and $\mathfrak{B} \subseteq \mathfrak{X}$. Then,*

i) $\varkappa_u(\mathfrak{B}) \leqslant \varkappa_r(\mathfrak{B}) \leqslant \varkappa_i(\mathfrak{B})$.

ii) $\varkappa_u(\mathfrak{B}) \leqslant \varkappa_l(\mathfrak{B}) \leqslant \varkappa_i(\mathfrak{B})$.

Proof. Obvious.

The equality in Theorem 4 and Theorem 5 are not true, in general.

Example 8. *In Example 6 and Table 2, 3, $B_i(\{\xi, \eta\}) \neq B_r(\{\xi, \eta\}) \neq B_l(\{\xi, \eta\}) \neq B_u(\{\xi, \eta\})$ and $\varkappa_u(\{\xi, \eta\}) \neq \varkappa_r(\{\xi, \eta\}) \neq \varkappa_l(\{\xi, \eta\} \neq \varkappa_i(\{\xi, \eta\}$.*

Theorem 6. *Let $\aleph$ be a general relation and $\mathfrak{B} \subseteq \mathfrak{X}$. Then,*

i) $\mathfrak{B}$ *is* CM_u-*exact* $\Longrightarrow$ $\mathfrak{B}$ *is* CM_r-*exact* $\Longrightarrow$ $\mathfrak{B}$ *is* CM_i-*exact.*

ii) $\mathfrak{B}$ *is* CM_u-*exact* $\Longrightarrow$ $\mathfrak{B}$ *is* CM_l-*exact* $\Longrightarrow$ $\mathfrak{B}$ *is* CM_i-*exact.*

Proof. Obvious.

The converse of Theorem 6 is not true, in general.

Example 9. *In Example 6 and Table 2, 3, $\{\zeta, \eta\}$ is CM_i-exact, but $\{\zeta, \eta\}$ is not CM_r-exact or CM_l-exact or CM_u-exact. $\{\xi\}$ is CM_r-exact, but $\{\xi\}$ is not CM_u-exact. $\{\gamma\}$ is CM_l-exact, but $\{\gamma\}$ is not CM_u-exact.*

5. Topological spaces generated by core minimal neighborhoods

This part uses the fundamental concept of core minimal neighborhoods to generate topologies by using general relations. A comparison of different kinds of topologies is explored.

Theorem 7. *Let $(\mathfrak{X}, \aleph, CM_j)$ be core minimal approximation space and $\aleph$ be a general relation. Then, the families $\tau_j = \{\mathfrak{B} \subseteq \mathfrak{X} : CM_j(\xi) \subseteq \mathfrak{B}, \xi \in \mathfrak{B}\}$ are topologies on $\mathfrak{X}$, for all $j \in \mathfrak{J}$.*

Proof. (i) Clearly, $\mathfrak{X}, \phi \in \tau_j$.
(ii) Let $\mathfrak{A}_i \in \tau_j$ where $i \in \mathcal{I}$ and $\xi \in \bigcup_{i\in\mathcal{I}} \mathfrak{A}_i$. Then, there exists $\mathfrak{A}_{i_0} \in \tau_j$ such that $\xi \in \mathfrak{A}_{i_0} \in \bigcup_{i\in\mathcal{I}} \mathfrak{A}_i$. This implies that $CM_j(\xi) \subseteq \mathfrak{A}_{i_0}$. Hence, $CM_j(\xi) \subseteq \bigcup_{\xi\in\mathcal{I}} \mathfrak{A}_i$. Therefore, $\bigcup_{i\in\mathcal{I}} \mathfrak{A}_i \in \tau_j$.
(iii) If $\mathfrak{A}_1, \mathfrak{A}_2 \in \tau_j$ and $\xi \in \mathfrak{A}_1 \cap \mathfrak{A}_2$, then $\xi \in \mathfrak{A}_1$ and $\xi \in \mathfrak{A}_2$. Hence, $CM_j(\xi) \subseteq \mathfrak{A}_1$ and $CM_j(\xi) \subseteq \mathfrak{A}_2$. So, $CM_j(\xi) \subseteq \mathfrak{A}_1 \cap \mathfrak{A}_2$. Therefore, $\mathfrak{A}_1 \cap \mathfrak{A}_2 \in \tau_j$.

Example 10. *In Example 2, we have:*
$\tau_r = \{\mathfrak{X}, \phi, \{\xi\}, \{\zeta\}, \{\xi,\zeta\}, \{\gamma,\eta\}, \{\xi,\gamma,\eta\}, \{\gamma,\zeta,\eta\}\}$,
$\tau_l = \{\mathfrak{X}, \phi, \{\gamma\}, \{\eta\}, \{\xi,\zeta\}, \{\gamma,\eta\}, \{\xi,\gamma,\zeta\}, \{\xi,\zeta,\eta\}\}$,
$\tau_u = \{\mathfrak{X}, \phi, \{\xi,\zeta\}, \{\gamma,\eta\}\}$, *and*
$\tau_i = \tau_{discrete}$.

Theorem 8. *If τ_j are topologies, then*

i) $\tau_u \subseteq \tau_r \subseteq \tau_i$.

ii) $\tau_u \subseteq \tau_l \subseteq \tau_i$.

Proof. Let $\mathfrak{B} \in \tau_u$. Then, $\forall \xi \in \mathfrak{B}$, $CM_u(\xi) \subseteq \mathfrak{B}$. But, $CM_u(\xi) = CM_r(\xi) \cup CM_l(\xi)$, then $CM_r(\xi) \subseteq \mathfrak{B}$ for all $\xi \in \mathfrak{B}$. Hence, $\mathfrak{B} \in \tau_r$. Therefore, $\tau_u \subseteq \tau_r$. Now, let $\mathfrak{B} \in \tau_r$. Then, $CM_r(\xi) \subseteq \mathfrak{B}$, $\forall \xi \in \mathfrak{B}$. But, $CM_i(\xi) = CM_r(\xi) \cap CM_l(\xi)$, then $CM_i(\xi) \subseteq \mathfrak{B}$ for all $\xi \in \mathfrak{B}$. Hence, $\mathfrak{B} \in \tau_i$. Therefore, $\tau_r \subseteq \tau_i$. similarly, the proof of part (ii).

The equality of parts (i) and (ii) in Theorem 8 is not true, in general.

Example 11. *In Example 10, $\tau_r \neq \tau_l \neq \tau_u \neq \tau_i$.*

Theorem 9. *Let $\aleph$ be a symmetric relation and τ_j are topologies. Then, $\tau_r = \tau_l = \tau_u = \tau_i$.*

Proof. Let $\aleph$ be a symmetric relation and $\mathfrak{B} \subseteq \mathfrak{X}$. Then, $N_r(\mathfrak{B}) = N_l(\mathfrak{B}) = N_u(\mathfrak{B}) = N_i(\mathfrak{B})$, $MN_r(\mathfrak{B}) = MN_l(\mathfrak{B}) = MN_u(\mathfrak{B}) = MN_i(\mathfrak{B})$, and $CM_r(\mathfrak{B}) = CM_l(\mathfrak{B}) = CM_u(\mathfrak{B}) = CM_i(\mathfrak{B})$. Therefore, $\tau_r = \tau_l = \tau_u = \tau_i$.

6. Medical applications: human blood circulation

Humans depend on blood circulation to deliver nutrients and oxygen to all cells of the body. The pulmonary circulation is part of the circulatory system, which includes the cardiovascular system, which consists of blood vessels that carry deoxygenated blood from the heart to the lungs, and then return oxygenated blood to the heart through the right ventricle again. This is contrary to what happens in the greater blood circulation. Deoxygenated blood leaves the right part (right ventricle) of the heart through the pulmonary arteries, which take blood to the lungs, where red blood cells release carbon dioxide and combine with oxygen during breathing. The oxygenated blood leaves the lungs through the pulmonary veins, which drain into the left part, or what is called the left atrium of the heart, thus completing the pulmonary circulation. The blood is then distributed to all parts of the body through the greater blood circulation before returning again to the pulmonary circulation. This effective circulation system makes sure every cell receives the nutrients and oxygen they require while also eliminating waste, promoting general health and organ function.

Graph operators were used to investigate the topology of the human heart [5, 23]. Nada et al. [32] advanced their study by separating the heart into vertices and edges, look at the shown Figure 2. Using this graph, they created a topological structure.

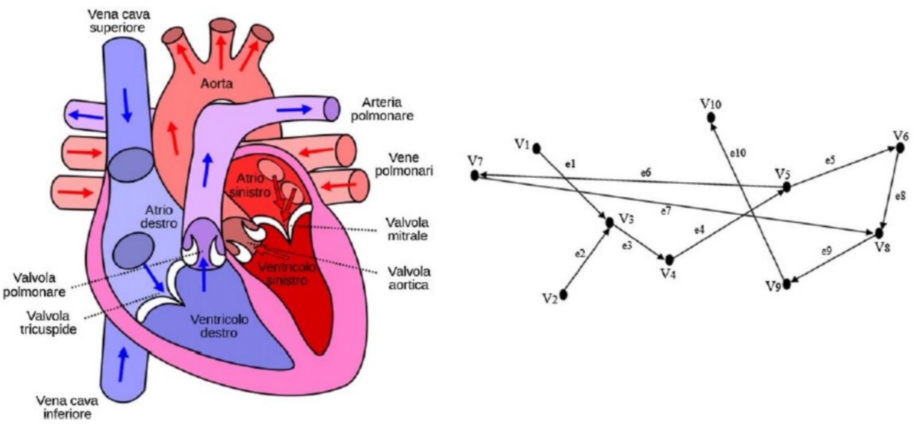


Figure 2: A digraph representation of the heart in humans.

We are exploring additional cardiac taxa using core minimal right, core minimal left, core minimal union, and core minimal intersection neighborhoods. These four types can be used to generate topologies that can provide a decision. The graph $G = (V, E)$ has vertices representing regions of blood flow and edges representing paths throughout the heart. Specifically, vertices ζ_1 = Superior vena cavae, ζ_2 = Inferior vena cavae, ζ_3 = Right atrium, ζ_4 = Right ventricle, ζ_5 = Pulmonary trunk, ζ_6 = Right lung, ζ_7 = Left lung, ζ_8 = Left atrium, ζ_9 =Left ventricle, and ζ_{10} =Aorta.

Now, take a set $\mathfrak{X} = \{\zeta_i : 1 \leqslant i \leqslant 10\}$ and find core minimal right, core minimal left, core minimal union, and core minimal intersection neighborhoods for each vertex in Figure 2. These neighborhoods are presented in Tables 4, 5. Choose a subgraph

$\mathfrak{B} = \{\zeta_2, \zeta_3, \zeta_7 \zeta_8, \zeta_9\}$ of a graph say G a human heart.

ξ	$N_r(\xi)$	$N_l(\xi)$	$MN_r(\xi)$	$MN_l(\xi)$	$CM_r(\xi)$	$CM_l(\xi)$	$CM_u(\xi)$	$CM_i(\xi)$
ζ_1	$\{\zeta_3\}$	ϕ	ϕ	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$
ζ_2	$\{\zeta_3\}$	ϕ	ϕ	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_1, \zeta_2\}$
ζ_3	$\{\zeta_4\}$	$\{\zeta_1, \zeta_2\}$	$\{\zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_3\}$	$\{\zeta_3\}$
ζ_4	$\{\zeta_5\}$	$\{\zeta_3\}$	$\{\zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_4\}$	$\{\zeta_4\}$
ζ_5	$\{\zeta_6, \zeta_7\}$	$\{\zeta_4\}$	$\{\zeta_5\}$	$\{\zeta_5\}$	$\{\zeta_5\}$	$\{\zeta_5\}$	$\{\zeta_5\}$	$\{\zeta_5\}$
ζ_6	$\{\zeta_8\}$	$\{\zeta_5\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$
ζ_7	$\{\zeta_8\}$	$\{\zeta_5\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_6, \zeta_7\}$
ζ_8	$\{\zeta_9\}$	$\{\zeta_6, \zeta_7\}$	$\{\zeta_8\}$	$\{\zeta_8\}$	$\{\zeta_8\}$	$\{\zeta_8\}$	$\{\zeta_8\}$	$\{\zeta_8\}$
ζ_9	$\{\zeta_{10}\}$	$\{\zeta_8\}$	$\{\zeta_9\}$	$\{\zeta_9\}$	$\{\zeta_9\}$	$\{\zeta_9\}$	$\{\zeta_9\}$	$\{\zeta_9\}$
ζ_{10}	ϕ	$\{\zeta_9\}$	$\{\zeta_{10}\}$	ϕ	$\{\zeta_{10}\}$	$\{\zeta_{10}\}$	$\{\zeta_{10}\}$	$\{\zeta_{10}\}$

Table 4: CM_j for $\zeta_i \in \mathfrak{X}$ and $j \in \mathfrak{J}$.

ξ	$N_r(\xi)$	$N_l(\xi)$	$MN_r(\xi)$	$MN_l(\xi)$	$CM_r(\xi)$	$CM_l(\xi)$	$CM_u(\xi)$	$CM_i(\xi)$
ζ_2	$\{\zeta_3\}$	ϕ	ϕ	$\{\zeta_2\}$	$\{\zeta_2, \zeta_7\}$	$\{\zeta_2\}$	$\{\zeta_2, \zeta_7\}$	$\{\zeta_2\}$
ζ_3	ϕ	$\{\zeta_2\}$	$\{\zeta_3\}$	ϕ	$\{\zeta_3\}$	$\{\zeta_3, \zeta_9\}$	$\{\zeta_3, \zeta_9\}$	$\{\zeta_3\}$
ζ_7	$\{\zeta_8\}$	ϕ	ϕ	$\{\zeta_7\}$	$\{\zeta_2, \zeta_7\}$	$\{\zeta_7\}$	$\{\zeta_2, \zeta_7\}$	$\{\zeta_7\}$
ζ_8	$\{\zeta_9\}$	$\{\zeta_7\}$	$\{\zeta_8\}$	$\{\zeta_8\}$	$\{\zeta_8\}$	$\{\zeta_8\}$	$\{\zeta_8\}$	$\{\zeta_8\}$
ζ_9	ϕ	$\{\zeta_8\}$	$\{\zeta_9\}$	ϕ	$\{\zeta_9\}$	$\{\zeta_3, \zeta_9\}$	$\{\zeta_3, \zeta_9\}$	$\{\zeta_9\}$

Table 5: CM_j for $\zeta_i \in \mathfrak{B}$ and $j \in \mathfrak{J}$.

We examine the topologies on a subgraph $\mathfrak{B}$ as follows:

i) $\tau_r = \{\mathfrak{B}, \phi, \{\zeta_3\}, \{\zeta_8\}, \{\zeta_9\}, \{\zeta_2, \zeta_7\}, \{\zeta_3, \zeta_8\}, \{\zeta_3, \zeta_9\}, \{\zeta_8, \zeta_9\}, \{\zeta_2, \zeta_3, \zeta_7\}, \{\zeta_2, \zeta_7, \zeta_8\}, \{\zeta_2, \zeta_7, \zeta_9\}, \{\zeta_3, \zeta_8, \zeta_9\}, \{\zeta_2, \zeta_3, \zeta_7, \zeta_8\}, \{\zeta_2, \zeta_3, \zeta_7, \zeta_9\}, \{\zeta_2, \zeta_7, \zeta_8, \zeta_9\}\}$.

ii) $\tau_l = \{\mathfrak{B}, \phi, \{\zeta_2\}, \{\zeta_7\}, \{\zeta_8\}, \{\zeta_2, \zeta_7\}, \{\zeta_2, \zeta_8\}, \{\zeta_3, \zeta_9\}, \{\zeta_7, \zeta_8\}, \{\zeta_2, \zeta_3, \zeta_9\}, \{\zeta_2, \zeta_7, \zeta_8\}, \{\zeta_3, \zeta_7, \zeta_9\}, \{\zeta_3, \zeta_8, \zeta_9\}, \{\zeta_2, \zeta_3, \zeta_7, \zeta_9\}, \{\zeta_2, \zeta_3, \zeta_8, \zeta_9\}, \{\zeta_3, \zeta_7, \zeta_8, \zeta_9\}\}$.

iii) $\tau_u = \{\mathfrak{B}, \phi, \{\zeta_8\}, \{\zeta_2, \zeta_7\}, \{\zeta_3, \zeta_9\}, \{\zeta_2, \zeta_7, \zeta_8\}, \{\zeta_3, \zeta_8, \zeta_9\}, \{\zeta_2, \zeta_3, \zeta_7, \zeta_9\}\}$.

iv) $\tau_i = \tau_{discrete}$, which has a best accuracy in Table 3.

The results of these topologies on G can be investigated as follows:

i) The topologies τ_r are τ_l are independent.

ii) $\tau_u \subseteq \tau_r$ and $\tau_u \subseteq \tau_l$.

iii) τ_i is finer than any topology which reduce from any subgraph of G.

iv) Core minimal intersection topology τ_i is the best topology because it represents all parts of the heart that can be used for the best diagnosis. It is considered the ideal choice from a topological point of view, as topological scientists use it in their studies.

In the application that was presented, we have suggested many different topologies that help experts in diagnosing the heart. Many topological tools can be used, such as separation axioms, connectivity, compactness, and continuity. These tools have a fundamental impact in the medical field.

7. Conclusion and Future Work

In the current paper, we define core minimal neighborhood which is a generalization of rough set theory, and we have studied its properties and reached some results. Also, a comparison between neighborhood, core neighborhood, minimal neighborhood, and core minimal neighborhood are introduced. We investigate four various types of generalizations for RST, which contain four types of dual approximations constructed by core minimal neighborhoods. The characteristics of these approximations are examined. There are several comparisons between our generalizations and others. Further topological developments in RST and its applications are made possible by the approximations operators. Our research established four topologies and studied a comparison between them. The example shows medical applications that are utilized to make decisions in the human heart. Furthermore, this discovery will be beneficial and offer new prospects in the research of topological spaces that approach RST via minimal neighborhoods, and the examination of core minimal neighborhoods as applications of these novel ideas. In future work, there are many studies to combine rough sets with many topological concepts such as neighborhoods and ideals that preserve the diagnosis and cure of dengue. Moreover, we study several relationships between dual approximations, accuracies, and boundaries of neighborhood, core neighborhood, minimal neighborhood, and core minimal neighborhood.

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