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Analyzing Cancer Survival Times Using the Exponentiated Fréchet-Gompertz Distribution

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Abstract. Understanding the survival of cancer patients is essential for determining optimal treatment strategies. This research introduces a robust distribution for analyzing survival data known as the Exponential Fréchet-Gompertz distribution (EFG). The EFG combines the unique characteristics of the Exponential Fréchet and Gompertz distributions, enhancing its efficiency and providing greater flexibility in representing complex datasets. The study specifically investigates the EFG distribution's effectiveness in modeling cancer patients' survival times. A comprehensive analysis of the distribution's properties is presented, including the quantile and quartile functions, shape indices, moments, moment-generating function, characteristic function, mean residual life, mean waiting time, Rényi entropy, and order statistics. The parameters of the distribution are derived using five distinct methods: Maximum Likelihood Estimation (MLE), Ordinary Least Squares (OLS), Weighted Least Squares (WLS), Cramér-von Mises (CVM), and Maximum Product of Spacings (MPS). A Monte Carlo simulation technique is employed to evaluate the performance of these estimation methods. The simulation results indicate that as sample size increases, the mean square error (MSE) values for all estimators decrease. Notably, the MLE exhibits the lowest MSE, while the MPS has the highest MSE, particularly for smaller sample sizes. Furthermore, the study presents a comprehensive comparison of the effectiveness of these estimation methods in analyzing survival times for various cancer types, including bladder, bone, blood and brain cancer. The results indicate that the EFG distribution is an optimal model for representing the survival times of patients across these different cancers data. Furthermore, the W.LS method yielded superior estimations for most datasets concerning cancer patient survival. Overall, the EFG distribution demonstrates exceptional capability in accurately fitting survival times for cancer patients compared to other competing distributions.

2020 Mathematics Subject Classifications: 62E10 , 62N05

Key Words and Phrases: Estimation method, probability distributions, maximum likelihood method, characteristic function, Exponentiated Fréchet distribution, Gompertz distribution

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1. Introduction

Estimating the survival time for cancer patients is a crucial aspect of cancer treatment. It helps doctors and patients make informed decisions about the treatment plan and manage expectations regarding the outcome. Researchers are working diligently to develop accurate models and statistical estimation techniques that can be used to predict the lifetime of cancer patients. [28] analyzed the lifetime of patients suffering from leukemia using a generalized linear exponential distribution. [34] employed McDonald log-logistic distribution with maximum likelihood technique to estimate the lifetime for breast cancer patients. [9] proposed the Alpha Power Weibull–Exponential model and used it to estimate the survival time for head and neck cancer patients. [24] conducted a survival analysis of cancer patients using Alpha power Kumaraswamy Weibull distribution with maximum likelihood estimation technique.

Researchers have put significant effort into developing various methods for creating new distributions that can adapt to represent different types of data, particularly survival data effectively. Techniques such as compounding, adding parameters, composing, and transforming have been advanced to broaden the scope of distributions. In 2013, [3] proposed a general approach known as the T-X transformation, that enables the use of any baseline distribution to generate a new distribution. The Lomax-G family by [14], the Weibull-G family by [10], the Lindley-G family by [11], the power Lindley-G family by [19], the Gompertz-G family by [2] are some generated families of distributions by T-X transformation technique. [21] introduced the new lifetime exponential-X family which used to generate a variety of distributions, such as the exponential Fréchet distribution [4], the exponentiated Weibull [5], the exponential inverted Topp-Leone distribution [30]. Furthermore, [8] provided the Exponentiated Fréchet generator of distribution. This family seems to be a great fit for modeling complex data and is particularly useful in reliability analysis. The cumulative function (CDF) and the probability density (PDF) of this family is given as

$$F(x) = 1 - \left[1 - exp\left\{-\left(\frac{\lambda}{-log[1 - G(x;\Theta)])}\right)^{\beta}\right\}\right]^{\alpha}, x > 0; \lambda, \gamma, \theta, \beta, \alpha > 0. \quad (1)$$

$$f(x) = \alpha\beta\lambda^{\beta}\frac{g(x;\Theta)}{1 - G(x;\Theta)}\left\{-log[1 - G(x;\Theta)]\right\}^{-(\beta+1)} exp\left\{-\left(\frac{\lambda}{-log[1 - G(x;\Theta)]}\right)^{\beta}\right\} \quad (2)$$

where Θ represents the parameter vector of the baseline distribution G.

The Gompertz distribution is a statistical distribution utilized in survival analysis to model lifetime data. This traditional distribution is employed to model the survival function based on mortality laws and is crucial in estimating various life-related events, such as human mortality rates and financial outcomes. This makes it a suitable choice for representing survival time for cancer patients. The Gompertz distribution was first introduced

by [18]. The CDF of Gompertz is

$$F(x) = 1 - e^{-\frac{\lambda}{\beta}\left(e^{\beta x}-1\right)}, x > 0; \lambda, \beta > 0. \tag{3}$$

and its PDF is

$$f(x) = \lambda e^{\beta x} e^{-\frac{\lambda}{\beta}\left(e^{\beta x}-1\right)} \tag{4}$$

While this distribution effectively models the survival function, its capacity to represent various lifetime data is restricted due to its tendency to display an exponentially increasing failure rate over lifetime data. This characteristic makes it unsuitable for certain applications. As a result, scientists developed the Gompertz distribution and explored various generalizations and combinations with other distributions to enhance its performance. Beyond the development of the Gompertz distribution, [2] introduced the Gompertz generalized family utilizing the $T - X$ transformation technique. This approach led to the creation of several derived distributions, including the Gompertz Normal, Gompertz Beta, Gompertz Gamma, Gompertz Log-Logistic, Gompertz Exponentiated Weibull, and Gompertz Lomax [32]. [23] implemented the same transformation to create a new model called the generalized Gompertz-G family. [16] introduced a generalized Gompertz distribution with three parameters utilizing the exponentiated method. Building on this foundation, [15] enhanced [16] model by incorporating two additional shape parameters through the exponentiated generalized technique proposed by [13]. Furthermore, [1] combined the exponentiated approach in [16] with a Gompertz exponential derived from [2], resulting in a new generalized Gompertz distribution. This model offers greater flexibility for analyzing survival data. Another significant advancement is the Kumaraswamy-G generalized Gompertz distribution, proposed by [17]. Recently, [7] introduced a new family of distributions by combining the odd Weibull family with the inverse Gompertz distribution, merging features from both.

This research aims to develop the exponential Frechet-Gompertz (EFG) distribution by amalgamating the distinctive characteristics of both the exponential Frechet and Gompertz distributions into a single model. The research entails an exhaustive examination of the distribution's characteristics, including the estimation of distribution parameters through the utilization of five distinct estimation methods. Moreover, the study includes a meticulous comparison of the efficacy of these estimation methods in analyzing the survival time for several types of cancer.

This article is classified as follows: Section 2 describes the EFG using graphical representations. Section 3 gives useful expansion for the EFG's CDF and PDF. Section 4 derived statistical properties for EFG. In section 5, five estimation methods are employed to estimate the EFG parameters: maximum likelihood (MLH), ordinary least squares (O.LS), weighted least squares (W.LS), Cramér-von Mises (CRM) and maximum product of spacing (MPS). Section 6 presents assessing the performance of the estimation methods using Monte Carlo simulation studies . Section 7 investigates different datasets of cancer

patient data are analyzed to assess the EFG's modeling effectiveness and compare its performance against competing distributions. Section 8 concludes with some final remarks.

2. Exponentiated Fréchet Gompertz distribution

The EFG's CDF and PDF are found by replacing the G(x) and g(x) in (1) and (2) by (3) and (4) as follows:

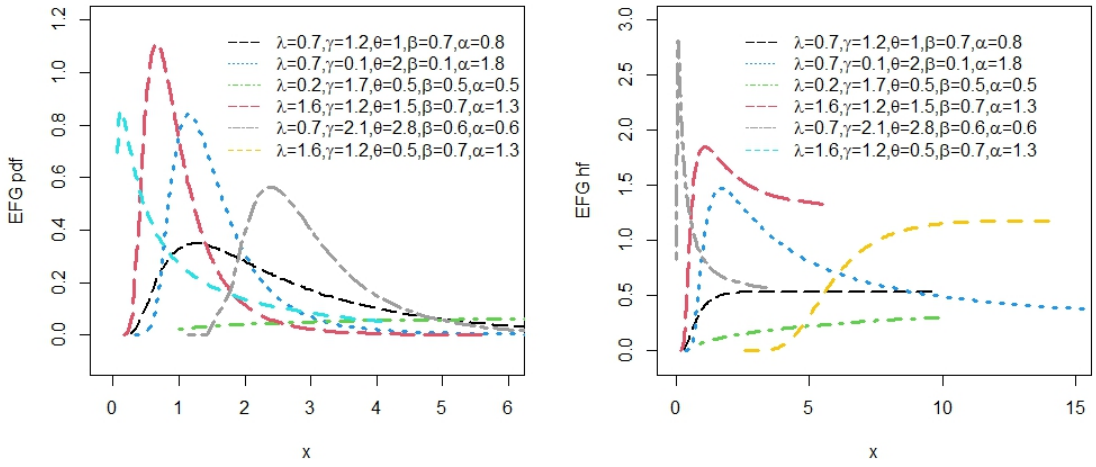
$$F(x) = 1 - \left[1 - exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta}\right\}\right]^{\alpha}, x > 0; \lambda, \gamma, \theta, \beta, \alpha > 0. \tag{5}$$

$$f(x) = \frac{\alpha\beta\theta\lambda^{\beta}e^{x\theta}}{\gamma^{\beta}} \frac{exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta}\right\}}{[(e^{x\theta}-1)]^{(\beta+1)}} \left[1 - exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta}\right\}\right]^{\alpha-1} \tag{6}$$

The survival and the hazard rate functions of EFG are given respectively as follows

$$S(x) = \left[1 - exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta}\right\}\right]^{\alpha} \tag{7}$$

$$H(x) = \frac{\alpha\beta\theta\lambda^{\beta}e^{x\theta}}{\gamma^{\beta}} \frac{exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta}\right\}}{[(e^{x\theta}-1)]^{(\beta+1)}} \left[1 - exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta}\right\}\right]^{-1} \tag{8}$$



(a) $f(x)$ (b) $H(x)$

Figure 1: The density and hazard function plots

Figure 1a shows a variety of shapes of EFG distribution, symmetric and asymmetric with different degrees of skewness. Additionally, Figure 1b represents the hazard function of EFG which exhibits a range of behaviors that demonstrate its capacity to adapt to different data types, indicating a high level of flexibility.

3. Useful Expansion for the EFG's CDF and PDF

This subsection provides the expansion for the EFG's Cumulative and Probability Functions. The binomial series (9), (10) and the exponential function (11) given below are applied to expand the EFG's CDF and PDF in (5) and (6), respectively.

$$(1-r)^n = \sum_{l=0}^{\infty} (-1)^l \binom{n}{l} r^l, \quad |r| < 1, \quad n \text{ is any real number} \tag{9}$$

$$(1-u)^{-n} = \sum_{w=0}^{\infty} \binom{n+w-1}{w} u^w. \tag{10}$$

$$e^{-z} = \sum_{v=0}^{\infty} \frac{(-1)^v (z)^v}{v!}, \tag{11}$$

3.1. CDF Expansion

First, the binomial series (9) is applied to expand (5). The EFG's CDF might then be written as

$$F(x) = \sum_{l_1=0}^{1} \sum_{l_2=0}^{\infty} (-1)^{l_1+l_2} \binom{1}{l_1} \binom{\alpha l_1}{l_2} exp\left\{ - \left(\frac{\lambda}{\gamma \left(e^{\theta x} - 1\right)} \right)^{\beta l_2} \right\}$$

Then, the exponential function (11)is utilized resulting the following form for $F(x)$:

$$F(x) = \sum_{l_1=0}^{1} \sum_{l_2=0}^{\infty} \sum_{v_1=0}^{\infty} \frac{(-1)^{l_1+l_2+v_1-l_2 v_1 \beta}}{v_1!} \binom{1}{l_1} \binom{\alpha l_1}{l_2} \left(\frac{\lambda}{\gamma} \right)^{l_2 v_1 \beta} \left(1 - e^{\theta x}\right)^{-l_2 v_1 \beta}$$

By applying the series in (10), the EFG's CDF can be reduced to

$$F(x) = \eta_1 e^{\theta x w_1} \tag{12}$$

where

$$\eta_1 = \sum_{l_1=0}^{1}\sum_{l_2=0}^{\infty}\sum_{v_1=0}^{\infty}\sum_{w_1=0}^{\infty}\frac{(-1)^{l_1+l_2+v_1-l_2v_1\beta}}{v_1!}\binom{1}{l_1}\binom{\alpha l_1}{l_2}\binom{l_2v_1\beta+w_1-1}{v_1}\left(\frac{\lambda}{\gamma}\right)^{l_2v_1\beta} \tag{13}$$

3.2. PDF Expansion

To find the expansion for the EFG'S PDF, first, the series (9) and (11) are employed to expand (6). Then the $f(x)$ might be written as below.

$$f(x) = \sum_{l_3=0}^{\infty}\sum_{v_2=0}^{\infty}\sum_{v_3=0}^{\infty}\frac{(-1)^{l_3+v_2+v_3-\beta(l_3v_2+v_3+1)-1}}{v_2!v_3!}\binom{\alpha-1}{l_2}\left(\frac{\lambda}{\gamma}\right)^{\beta(l_3v_2+v_3+1)}\frac{\alpha\beta\theta e^{\theta x}}{(1-e^{\theta x})^{\beta(l_3v_2+v_3+1)+1}}$$

Follow that, the series (10) is applied to reduce the EFG's PDF to the following form.

$$f(x) = \eta_2\ \alpha\ \beta\ \theta\ e^{\theta(1+w_2)x}, \tag{14}$$

where

$$\eta_2\ = \sum_{l_3=0}^{\infty}\sum_{v_2=0}^{\infty}\sum_{v_3=0}^{\infty}\sum_{w_2=0}^{\infty}\frac{(-1)^{l_3+v_2+v_3-\beta(l_3v_2+v_3+1)-1}}{v_2!v_3!}\binom{\alpha-1}{l_2}\binom{\beta(l_3v_2+v_3+1)-1}{w_3} \tag{15}$$

4. Statistical Properties of the EFG

This section derives several statistical properties of the EFG distribution, including the quantile function, moments, moment-generating function, characteristic function, Rényi entropy, and order statistics. Statistical properties are vital for comprehensively describing and analyzing data from diverse perspectives.

4.1. Quantile Function and Quartiles

The quantile function of EFG is written as

$$Q_p = ln\left[1-\frac{\beta\lambda}{\gamma}\frac{1}{ln\left[1-(1-p)^{\frac{1}{\alpha}}\right]}\right]/\beta, \quad 0<p<1. \tag{16}$$

The median of the EFG distribution can be obtained as

$$Q_{(0.5)} = ln\left[1-\frac{\beta\lambda}{\gamma}\frac{1}{ln\left[1-(1-0.5)^{\frac{1}{\alpha}}\right]}\right]/\beta,$$

Hence, the 25^{th} percentile and the 75^{th} percentile of the EFG distribution are given as

$$Q_{(0.25)} = ln\left[1 - \frac{\beta\lambda}{\gamma}\frac{1}{ln\left[1-(1-0.75)^{\frac{1}{\alpha}}\right]}\right]/\beta,$$

$$Q_{(0.75)} = ln\left[1 - \frac{\beta\lambda}{\gamma}\frac{1}{ln\left[1-(1-0.25)^{\frac{1}{\alpha}}\right]}\right]/\beta,$$

4.2. Shape Indices

The shape of the EFG can be assessed using Galton's skewness and Moors' kurtosis [31], which can be calculated by utilizing the quantile function (16) and respectively given as follows:

$$Skewness = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}},$$

and

$$Kurtosis = \frac{Q_{0.875} - Q_{0.625} + Q_{0.375} - Q_{0.125}}{Q_{0.75} - Q_{0.25}}.$$

4.3. Moments

The moment of of X for $f(x)$ is written as

$$E\left(x^r\right) = \int_0^\infty x^r f(x)dx$$

If X follows the EFG $(\lambda, \gamma, \theta, \beta, \alpha)$, then the r^{th} moment of X is written as

$$E\left(x^r\right) = \eta_2\ \alpha\ \beta\ \theta \int_0^\infty x^r e^{\theta(1+w_2)x}dx,$$

where η_2 is given by (15).

By using Laplace transformation, $L_t\left[f(t)\right](s) = \int_0^\infty f(t)e^{-st}dt$ where $f(t)$ is defined for $t \geq 0$ [25], with taking $f(t) = t^r$ then $\int_0^\infty f(t)e^{-st}dt = \frac{r!}{s^{r+1}}$.

Therefore, $f(x) = x^r$ then $\int_0^\infty f(x)e^{-sx}dx = \frac{r!}{s^{r+1}}$, where $s = -\theta(1+w_2)$. Then the r^{th} moment is expressed as

$$\mu_r = E(x^r) = \eta_2\ \alpha\ \beta\ \theta\left(\frac{-1}{\theta(1+w_2)}\right)^{r+1}\Gamma(r+1),\ \ r \geq 0 \tag{17}$$

Then, the EFG's mean is written as

$$\mu = E\left(x\right) = \frac{\eta_2\ \alpha\ \beta}{\theta}\left(\frac{1}{1+w_2}\right)^2 \tag{18}$$

The EFG's variance is determined by

$$\sigma^2 = E(x^2) - \mu^2 = \frac{2\,\eta_2\,\alpha\,\beta}{\theta^2}\left(\frac{-1}{1+w_2}\right)^3 - \mu^2,$$

where η_2 is given by (15).

4.4. Moment Generating Function

The moment generating function (MGF) of X for $f(x)$ is written as

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

If X follows the EFG $(\lambda, \gamma, \theta, \beta, \alpha)$, then the EFG's MGF is written as

$$M_x(t) = E(e^{tx}) = \eta_2\,\alpha\,\beta\,\theta \int_0^\infty e^{tx}\, e^{\theta(1+w_2)x} dx,$$

By using Laplace transformation, with taking $f(t) = 1$ then $\int_0^\infty f(t)e^{-st}dt = \frac{1}{s}$.

Therefore, $f(x) = 1$ then $\int_0^\infty f(x)e^{-sx}dx = \frac{1}{s}$, where $s = -(t + \theta(1 + w_2))$. Then the then MGF will be given as

$$M_x(t) = E(e^{tx}) = \eta_2\,\alpha\,\beta\,\theta\left(\frac{-1}{t+\theta(1+w_2)}\right), \tag{19}$$

where η_2 is given by (15).

4.5. Characteristic Function

The characteristic function of EFG is simply constructed as :

$$\phi_x(t) = E(e^{itx}) = \eta_2\,\alpha\,\beta\,\theta\left(\frac{-1}{it+\theta(1+w_2)}\right), \tag{20}$$

where η_2 is given by (15).

4.6. Mean residual life and mean waiting time

If $X \sim EFG(\lambda, \gamma, \theta, \beta, \alpha)$ with $S(t)$ provided in (7) , then the mean residual life, $\mu(t)$, is expressed as

$$\mu(t) = \frac{1}{S(t)}\left(E(t) - \int_0^t xf(x)dx\right) - t. \tag{21}$$

If the incomplete moment, $I_{inc} = \int_0^t xf(x)dx$, then

$$I_{inc} = \eta_2\ \alpha\ \ \beta\ \theta \int_0^t xe^{\theta(1+w_2)x}dx.$$

Setting $y = \theta(1 + x_2)x$, then simplifying will be obtained,

$$I_{inc} = \left(\frac{\eta_2\alpha\beta}{\theta(1+w_2)^2}\right) \int_0^{\theta(1+w_2)t} ye^y dy,$$

Using integration by part by taking $u = y$ and $dv = e^y dy$, the incomplete moment will be obtained as

$$I_{inc} = \left(\frac{\eta_2\alpha\beta}{\theta(1+w_2)^2}\right) \left[\theta(1+w_2)te^{\theta(1+w_2)t} - e^{\theta(1+w_2)t} + 1\right], \tag{22}$$

Substituting (18), and (22) in (21), $\mu(t)$ might be rewritten as

$$\mu(t) = \frac{1}{S(t)} \frac{\eta_2\alpha\beta}{\theta(1+w_2)^2} \left[e^{\theta(1+w_2)t} - \theta(1+w_2)te^{\theta(1+w_2)t}\right] - t,$$

In a similar manner, the mean waiting time, $\bar{\mu}(t)$, might be defined as

$$\bar{\mu}(t) = t - \frac{1}{F(t)} \int_0^t xf(x)dx, \tag{23}$$

where $F(t)$ is provided in (12). Then, $\bar{\mu}(t)$ of the EFG can be found by substituting (12) and (22) in (23) as follows

$$\bar{\mu}(t) = t - \left(\frac{\eta_2\alpha\beta}{\eta_1\theta(1+w_2)^2e^{\theta x w_1}}\right) \left[\theta(1+w_2)te^{\theta(1+w_2)t} - e^{\theta(1+w_2)t} + 1\right]$$

Mean residual life and mean waiting time are important concepts in reliability theory and queuing theory, respectively, as they provide insights into the expected remaining lifespan of a system or the time until an event occurs. Incomplete moments, which summarize essential features of distributions without requiring full data (censored data), complement these concepts by offering partial insights into uncertainty and variability. They are particularly useful in situations where data is limited or when focusing on essential features is necessary, such as in risk assessment and model simplification. Derived results from incomplete moments offer partial insights that can inform decision-making and enable comparative analyses, despite acknowledging the inherent uncertainty in not having a full distribution. Together, Mean residual life and mean waiting time can guide decision-making in fields like finance and operations, while the interpretations of their derived results enable comparisons and assumptions about system performance, even when data is limited.

4.7. Rényi entropy

$RE_X(\zeta)$ is the Rényi entropy function which given as

$$RE_X(\zeta) = \frac{1}{1-\zeta}\log\left(\int_0^\infty f(x)^\zeta dx\right); \quad \zeta > 0, \zeta \neq 1.$$

Then, applying the EFG's PDF in (6)

$$f(x)^\zeta = \left(\frac{\alpha\beta\theta\lambda^\beta}{\gamma^\beta}\right)^\zeta e^{\theta x\zeta}\frac{exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta\zeta}\right\}}{[(e^{x\theta}-1)]^{(\beta+1)\zeta}}\left[1-exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^\beta\right\}\right]^{(\alpha-1)\zeta}$$

Applying the same approach in Subsection 3 and using (9), (10) and (11), then

$$f(x)^\zeta = \eta_2^*\ \alpha^\zeta\ \beta^\zeta\ \theta^\zeta\ e^{\theta(\zeta+w_3)x},$$

where

$$\eta_2^* = \sum_{l_4=0}^{\infty}\sum_{v_4=0}^{\infty}\sum_{v_5=0}^{\infty}\sum_{w_3=0}^{\infty}\frac{(-1)^{l_4+v_4+v_5-\beta(l_4v_4+v_5\zeta+\zeta)-\zeta}}{v_4!v_5!}\binom{(\alpha-1)\zeta}{l_4}\binom{\beta(l_4v_4+v_5\zeta+\zeta)-\zeta+w_3-1}{w_3}$$

Using Laplace transformation, since $f(t) = 1$ then $L_t\left[f(t)\right](s) = \frac{1}{s}$, where $s = -(\zeta+w_3)\beta$, then the Rényi entropy of the EFG, is then will be reduced to

$$RE_x(\zeta) = \frac{1}{1-\zeta}\log\left[\frac{-\eta_2^*\ \alpha^\zeta\ \beta^\zeta\ \theta^{\zeta-1}}{(\zeta+w_3)}\right].$$

4.8. Order statistics

The density function, $f_{j:n}(x)$, of the j^{th} order statistics is given as

$$f_{j:n}(x) = \frac{1}{B(j,n-j+1)}f(x)\left[F(x)\right]^{j-1}\left[1-F(x)\right]^{n-j}.$$

By employing the series formula , $f_{j:n}(x)$ can be expressed as

$$f_{j:n}(x) = \frac{1}{B(j,n-j+1)}\sum_{l_5=0}^{n-j}(-1)^{l_5}\binom{n-j}{l_5}f(x)\left[F(x)\right]^{l_5+j-1}. \tag{24}$$

By substituting the CDF (12) and PDF of EFG (14) into (24), the PDF of $X_{j:n}$ is

$$f_{j:n}(x) = \frac{1}{B(j,n-j+1)}\sum_{l_5=0}^{n-j}(-1)^{l_5}\binom{n-j}{l_5}\left[\eta_2\ \alpha\ \beta\ \theta\ e^{\theta(1+w_2)x}\right]\left[\eta_1 e^{\theta x w_1}\right]^{l_5+j-1} \tag{25}$$

where η_1 and η_2 are given by (13) and (15), respectively.

5. Estimation methods

5.1. Maximam Likelihood Method (MLH)

For a random sample $x_1, x_2, ..., x_n$ from EFG, the log-likelihood function (ℓ), for $\Theta = (\lambda, \gamma, \theta, \beta, \alpha)$, is written as

$$\begin{aligned} L(\Theta) = {} & n\, ln\, \alpha + n\, ln\, \beta + n\, ln\, \theta + n\, \beta\, ln\, \lambda - n\, \beta\, ln\, \gamma \\ & + \beta \sum_{i=1}^{n} x_i - \left(\frac{\lambda}{\gamma}\right)^{\beta} \sum_{i=1}^{n} (e^{\theta x_i} - 1)^{-\beta} - (\beta + 1) \sum_{i=1}^{n} ln\, (e^{\theta x_i} - 1) \\ & + (\alpha - 1) \sum_{i=1}^{n} ln\, \left[1 - exp\left\{ - \left(\frac{\lambda}{\gamma (e^{\theta x_i} - 1)} \right)^{\beta} \right\}\right] \end{aligned} \quad (26)$$

The following equations from (27) to (31) represent the partial derivation from ℓ function, (26), regards to the parameters $(\lambda, \gamma, \theta, \beta, \alpha)$. The ML estimates, $\hat{\lambda}_{MLH}, \hat{\gamma}_{MLH}, \hat{\theta}_{MLH}, \hat{\beta}_{MLH}$ and $\hat{\alpha}_{MLH}$ can be obtained by maximizing the equation (26) or by solving the equations from (27) to (31) using numerical iterative technique.

$$\frac{\partial \ell}{\partial \lambda} = (\alpha - 1)\, \beta \cdot \sum_{i=1}^{n} \left\{ \frac{\left(\frac{\lambda}{(\mathrm{e}^{\theta x_i} - 1)\gamma}\right)^{\beta} \mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x_i} - 1)\gamma}\right)^{\beta}}}{\lambda \cdot \left(1 - \mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x} - 1)\gamma}\right)^{\beta}}\right)} \right\} - \frac{\beta}{\lambda} \cdot \left(\frac{\lambda}{\gamma}\right)^{\beta} \sum_{i=1}^{n} \left(\mathrm{e}^{\theta x_{\mathrm{i}}} - 1\right)^{-\beta} + \frac{\beta n}{\lambda} \quad (27)$$

$$\frac{\partial \ell}{\partial \gamma} = -\frac{(\alpha - 1)\, \beta}{\gamma} \cdot \sum_{i=1}^{n} \left\{ \frac{\left(\frac{\lambda}{(\mathrm{e}^{\theta x_i} - 1)\gamma}\right)^{\beta} \mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x_i} - 1)\gamma}\right)^{\beta}}}{\left(1 - \mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x_i} - 1)\gamma}\right)^{\beta}}\right)} \right\} + \beta\gamma \cdot \left(\frac{\lambda}{\gamma}\right)^{\beta} \sum_{i=1}^{n} \left(\mathrm{e}^{\theta x_{\mathrm{i}}} - 1\right)^{-\beta} - \frac{\beta n}{\gamma} \quad (28)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + - (\beta + 1) \cdot \sum_{i=1}^{n} \left\{ \frac{x_{\mathrm{i}} \mathrm{e}^{x_{\mathrm{i}} \theta}}{\mathrm{e}^{x_{\mathrm{i}} \theta} - 1} \right\} + \beta \cdot \left(\frac{\lambda}{\gamma}\right)^{\beta} \sum_{i=1}^{n} x_{\mathrm{i}} \mathrm{e}^{\theta x_{\mathrm{i}}} \left(\mathrm{e}^{\theta x_{\mathrm{i}}} - 1\right)^{-\beta - 1}$$

$$-\left(\frac{(\alpha-1)\cdot\beta\cdot\lambda^{\beta}}{\gamma^{\beta}}\right)\cdot\sum_{i=1}^{n}\left\{\frac{x_i\cdot\left(\mathrm{e}^{\theta x_i}-1\right)^{-(\beta+1)}\,\mathrm{e}^{\theta x_i-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_i}-1\right)}\right)^{\beta}}}{\left(1-\mathrm{e}^{-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_i}-1\right)}\right)^{\beta}}\right)}\right\} \quad (29)$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}+n\,ln\,\lambda-n\,ln\gamma+\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}ln(e^{\theta x_i}-1)-\sum_{i=1}^{n}\left\{\left(\frac{\lambda}{\gamma\cdot(\mathrm{e}^{\theta x_i}-1)}\right)^{\beta}\cdot ln\left(\frac{\lambda}{\gamma\,(\mathrm{e}^{\theta x_i}-1)}\right)\right\}$$
$$+(\alpha-1)\cdot\sum_{i=1}^{n}\left\{\frac{\left(\frac{\lambda}{(\mathrm{e}^{\theta x_i}-1)\gamma}\right)^{\beta}\ln\left(\frac{\lambda}{(\mathrm{e}^{\theta x_i}-1)\gamma}\right)}{\mathrm{e}^{\left(\frac{\lambda}{(\mathrm{e}^{\theta x_i}-1)\gamma}\right)^{\beta}}-1}\right\} \quad (30)$$

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(1-\mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x}-1)\gamma}\right)^{\beta}}\right) \quad (31)$$

5.2. Ordinary Least Square Method (O.LS)

O.LS method is proposed by [33], which is based on the difference between the empirical and theoretical cdf. Suppose a random sample from EFG distribution with size n and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are its order statistics. The sum of squares for the difference between the empirical and theoretical cdf of EFG distribution is formulated in equation (32).

$$U_1(\Theta)=\sum_{i=1}^{n}\left[1-\left[1-exp\left\{-\left(\frac{\lambda}{\gamma(e^{\theta x_{(i)}}-1)}\right)^{\beta}\right\}\right]^{\alpha}-\left(\frac{i}{n+1}\right)\right]^2 \quad (32)$$

where $\left(\frac{i}{n+1}\right)$ is the empirical cdf and $i=1,2,...,n$.
The partial derivation from the equation (32) in regard to $\Theta=(\lambda,\gamma,\theta,\beta,\alpha)$ can be written as follows:

$$\frac{\partial\ell}{\partial\lambda}=-\frac{2\alpha\beta\lambda^{(\beta-1)}}{\gamma^{\beta}}\cdot\sum_{i=1}^{n}\left(\frac{1}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}$$

$$\cdot \left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right]\cdot \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}} \tag{33}$$

$$\frac{\partial \ell}{\partial \gamma}=\frac{2\alpha\beta\lambda^{\beta}}{\gamma^{(\beta+1)}}\cdot\sum_{i=1}^{n}\left(\frac{1}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}$$
$$\cdot\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right]\cdot \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}} \tag{34}$$

$$\frac{\partial \ell}{\partial \theta}=2\alpha\beta\left(\frac{\lambda}{\gamma}\right)^{\beta}\cdot\sum_{i=1}^{n}x_{(i)}\cdot\left(\frac{1}{\mathrm{e}^{\theta x_{(i)}}-1}\right)^{\beta+1}\mathrm{e}^{\theta x_{(i)}-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}\cdot$$
$$\cdot\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right]\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}\right]^{\alpha-1} \tag{35}$$

$$\frac{\partial \ell}{\partial \beta}=-2\alpha\cdot\sum_{i=1}^{n}\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}\ln\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)\cdot\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}$$
$$\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}\cdot\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right] \tag{36}$$

$$\frac{\partial \ell}{\partial \alpha} = -2\sum_{i=1}^{n}\left[1-\mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x}-1)\gamma}\right)^{\beta}}\right]^{\alpha}\ln\left[1-\mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x}-1)\gamma}\right)^{\beta}}\right]\cdot\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x}-1)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right] \tag{37}$$

The O.LS estimates for the parameters Θ can be obtained by minimizing the equation (32) concerning the $\Theta = (\lambda, \gamma, \theta, \beta, \alpha)$ or by solving the equations from (33) to (37) using numerical techniques available in statistical software.

5.3. Weighted Least Square Method (W.LS)

The W.LS method is similar to the O.LS method, which depends on the differences between the empirical and theoretical of the cdf, in addition to the variance of the order statistic as a wight [33]. Therefore, the W.LS function for EFG is written as

$$U_2(\Theta) = \sum_{i=1}^{n}\omega_i\left[1-\left[1-exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta}\right\}\right]^{\alpha}-\left(\frac{i}{n+1}\right)\right]^{2} \tag{38}$$

where $\omega_i = \frac{(n+1)^2(n+2)}{i(n+1-i)}$

The equation (38) is derived for each parameter in the EFG distribution and the derivation equations are given as follows. The W.LS estimation is obtained by minimizing the equation (38) or by solving the nonlinear equations from (39) to (43) using numerical iterative technique available in any statistical software.

$$\frac{\partial \ell}{\partial \lambda} = -\frac{2\alpha\beta\lambda^{(\beta-1)}}{\gamma^{\beta}}\cdot\sum_{i=1}^{n}\omega_{\mathrm{i}}\cdot\left(\frac{1}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}$$
$$\cdot\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right]\cdot\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}} \tag{39}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{2\alpha\beta\lambda^{\beta}}{\gamma^{(\beta+1)}}\cdot\sum_{i=1}^{n}\omega_{\mathrm{i}}\cdot\left(\frac{1}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}$$

$$\cdot \left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right]\cdot \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}} \tag{40}$$

$$\frac{\partial \ell}{\partial \theta}=2\alpha\beta\left(\frac{\lambda}{\gamma}\right)^{\beta}\cdot\sum_{i=1}^{n}\omega_{\mathrm{i}}\cdot x_{(i)}\cdot\left(\frac{1}{\mathrm{e}^{\theta x_{(i)}}-1}\right)^{\beta+1}\mathrm{e}^{\theta x_{(i)}-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}\cdot$$
$$\cdot\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right]\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}\right]^{\alpha-1} \tag{41}$$

$$\frac{\partial \ell}{\partial \beta}=-2\alpha\cdot\sum_{i=1}^{n}\omega_{\mathrm{i}}\cdot\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}\ln\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)\cdot\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}$$
$$\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right]\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1} \tag{42}$$

$$\frac{\partial \ell}{\partial \alpha}=-2\sum_{i=1}^{n}\omega_{\mathrm{i}}\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}\ln\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]$$
$$\cdot\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{\mathrm{i}}{n+1}\right] \tag{43}$$

5.4. Cramér-von Mises Method (CRM)

[27] introduced the CRM method for estimation parameters. The function of the CRM method for EFG distribution is presented in equation (44).

$$CRM(\Theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[1 - \left[1 - exp\left\{-\left(\frac{\lambda}{\gamma(e^{x\theta}-1)}\right)^{\beta}\right\}\right]^{\alpha} - \left(\frac{2i-1}{2n}\right)\right]^2 \tag{44}$$

The equations below show the partial derivatives of the parameters $\Theta = (\lambda, \gamma, \theta, \beta, \alpha)$ from (44). The CRM estimates for EFG parameters are derived by solving the equations from (45) to (49) using numerical technique or by minimizing (44) using optimization technique available in R Package.

$$\frac{\partial \ell}{\partial \lambda} = -\frac{2\alpha\beta\lambda^{(\beta-1)}}{\gamma^{\beta}} \cdot \sum_{i=1}^{n}\left(\frac{1}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)^{\gamma}}\right)^{\beta}}\right]^{\alpha-1} \left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)^{\gamma}}\right)^{\beta}}\right]^{\alpha} - \frac{2i-1}{2n}\right]\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)^{\gamma}}\right)^{\beta}} \tag{45}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{2\alpha\beta\lambda^{\beta}}{\gamma^{(\beta+1)}} \cdot \sum_{i=1}^{n}\left(\frac{1}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta} \cdot \left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)^{\gamma}}\right)^{\beta}}\right]^{\alpha-1} \cdot \left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)^{\gamma}}\right)^{\beta}}\right]^{\alpha} - \frac{2i-1}{2n}\right]\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)^{\gamma}}\right)^{\beta}} \tag{46}$$

$$\frac{\partial \ell}{\partial \theta} = 2\alpha\beta\left(\frac{\lambda}{\gamma}\right)^{\beta} \cdot \sum_{i=1}^{n} x_{(i)} \cdot \left(\frac{1}{\mathrm{e}^{\theta x_{(i)}}-1}\right)^{\beta+1} \mathrm{e}^{\theta x_{(i)} - \left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}$$

$$\cdot \left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}\right]^{\alpha}-\frac{2i-1}{2n}\right]\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\gamma\cdot\left(\mathrm{e}^{\theta x_{(i)}}-1\right)}\right)^{\beta}}\right]^{\alpha-1} \tag{47}$$

$$\frac{\partial \ell}{\partial \beta}=-2\alpha\cdot\sum_{i=1}^{n}\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}\ln\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}$$
$$\cdot\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{2i-1}{2n}\right]\cdot\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1} \tag{48}$$

$$\frac{\partial \ell}{\partial \alpha}=-2\sum_{i=1}^{n}\left[1-\mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x}-1)\gamma}\right)^{\beta}}\right]^{\alpha}\ln\left[1-\mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x}-1)\gamma}\right)^{\beta}}\right]\left[1-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{(\mathrm{e}^{\theta x}-1)\gamma}\right)^{\beta}}\right]^{\alpha}-\frac{2i-1}{2n}\right] \tag{49}$$

5.5. Maximum product of spacing Method (MPS)

[12] proposed the MPS method in order to improve the performance of the MLH estimator. Let a random sample from EFG distribution with size n and $x_{(1)}, x_{(2)}, ..., x_{(n)}$ is the corresponding ordered sample. The idea of the MPS method is to optimize the geometric mean of spacings, which refers to the variations between the CDF values of adjacent data points. The spacings between neighboring ordered values can be defined as $D_i(\Theta) = F(x_{(i)}; \Theta) - F(x_{(i-1)}; \Theta), \quad \Theta = (\lambda, \gamma, \theta, \beta, \alpha), \quad i = 1, \ldots, n+1.$
Therefore, the MPS function for EFG is given as

$$M(\Theta)=\frac{1}{n+1}\sum_{i=1}^{n+1} logD_i(\Theta)$$

$$M(\Theta)=\frac{1}{n+1}\sum_{i=1}^{n+1} log\left[\left[1-exp\left\{-\left(\frac{\lambda}{\gamma(e^{\theta x_{(i-1)}}-1)}\right)^{\beta}\right\}\right]^{\alpha}-\left[1-exp\left\{-\left(\frac{\lambda}{\gamma(e^{\theta x_{(i)}}-1)}\right)^{\beta}\right\}\right]^{\alpha}\right] \tag{50}$$

The first partial derivative of (50) with respect to $\Theta = (\lambda, \gamma, \theta, \beta, \alpha)$, are given as follows. The MPS estimates for EFG parameters are derived by solving the equations

from (51) to (55) using numerical technique or via maximizing (50) using optimization technique available in R Package.

$$\frac{\partial \ell}{\partial \lambda} = \frac{\alpha\beta\lambda^{(\beta-1)}}{\gamma^{\beta}\cdot(n+1)}\cdot\sum_{i=1}^{n+1}\left[\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}\right]^{-1}\cdot$$

$$\cdot\left\{\left(\frac{1}{\mathrm{e}^{\theta x_{(i)}^{-1}}-1}\right)^{\beta}\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}^{-1}}-1\right)\gamma}\right)^{\beta}}\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}^{-1}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}\right\}$$

$$-\left\{\left(\frac{1}{\mathrm{e}^{\theta x_{(i-1)}}-1}\right)^{\beta}\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}}-1\right)\gamma}\right)^{\beta}}\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}\right\} \quad (51)$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{\alpha\beta\lambda^{\beta}}{\gamma^{(\beta+1)}\cdot(n+1)}\cdot\sum_{i=1}^{n+1}\left[\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}-\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha}\right]^{-1}$$

$$\cdot\left\{\left(\frac{1}{\mathrm{e}^{\theta x_{(i)}}-1}\right)^{\beta}\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}\right\}$$

$$-\left\{\left(\frac{1}{\mathrm{e}^{\theta x_{(i-1)}}-1}\right)^{\beta}\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}}-1\right)\gamma}\right)^{\beta}}\left[1-\mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}}-1\right)\gamma}\right)^{\beta}}\right]^{\alpha-1}\right\} \quad (52)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{\alpha\beta\lambda^{\beta}}{(n+1)\cdot\gamma^{\beta}} \cdot \sum_{i=1}^{n+1} \left[\left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta}} \right]^{\alpha} - \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta}} \right]^{\alpha} \right]^{-1}$$

$$\cdot \left\{ x_{(i-1)} \cdot \left(\frac{1}{\mathrm{e}^{\theta x_{(i-1)}} - 1} \right)^{\beta+1} \cdot \mathrm{e}^{\theta x_{(i-1)} - \left(\frac{\lambda}{\gamma \cdot \left(\mathrm{e}^{\theta x_{(i-1)}} - 1 \right)} \right)^{\beta}} \cdot \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\gamma \cdot \left(\mathrm{e}^{\theta x_{(i-1)}} - 1 \right)} \right)^{\beta}} \right]^{\alpha-1} \right\}$$

$$- \left\{ x_{(i)} \cdot \left(\frac{1}{\mathrm{e}^{\theta x_{(i)}} - 1} \right)^{\beta+1} \cdot \mathrm{e}^{\theta x_{(i)} - \left(\frac{\lambda}{\gamma \cdot \left(\mathrm{e}^{\theta x_{(i)}} - 1 \right)} \right)^{\beta}} \cdot \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\gamma \cdot \left(\mathrm{e}^{\theta x_{(i)}} - 1 \right)} \right)^{\beta}} \right]^{\alpha-1} \right\} \quad (53)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{\alpha}{(n+1)} \cdot \sum_{i=1}^{n+1} \left[\left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta}} \right]^{\alpha} - \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta}} \right]^{\alpha} \right]^{-1}$$

$$\cdot \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta}} \right]^{\alpha-1} \cdot \left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta} \cdot \ln \left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)\gamma} \right) \cdot \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta}}$$

$$- \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta}} \right]^{\alpha-1} \cdot \left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta} \cdot \ln \left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)\gamma} \right) \cdot \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)\gamma} \right)^{\beta}} \quad (54)$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{1}{(n+1)} \cdot \sum_{i=1}^{n+1} \left[\left[\left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)^{\gamma}} \right)^{\beta}} \right]^{\alpha} - \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)^{\gamma}} \right)^{\beta}} \right]^{\alpha} \right]^{-1} \right.$$
$$\cdot \left[\ln \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)^{\gamma}} \right)^{\beta}} \right] \cdot \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i)}{}^{-1}} - 1 \right)^{\gamma}} \right)^{\beta}} \right] \right]$$
$$\left. - \left[\ln \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)^{\gamma}} \right)^{\beta}} \right] \cdot \left[1 - \mathrm{e}^{-\left(\frac{\lambda}{\left(\mathrm{e}^{\theta x_{(i-1)}{}^{-1}} - 1 \right)^{\gamma}} \right)^{\beta}} \right] \right] \right] \quad (55)$$

6. Numerical Study

The Monte Carlo simulation technique is carried out to assess and compare the effectiveness of the various estimation methods that are used for estimating the parameters of EFG. The simulation involved using gradient samples of varying sizes, beginning with a small size of 15 and gradually increasing to larger sizes of 30, 50, 100, and 200. Furthermore, two sets of parameters have been implemented:
Set I: (λ=2.4, γ=0.5, θ=0.3, β=0.7, α=1.2).
Set II: (λ=0.5, γ=3, θ=0.7, β=1.5, α=2.4).
The bias and mean square error (MSE) of the parameter estimation are calculated for each simulation scenario.
Tables 1 and 2 show the simulation results. The table displays the parameter estimation values and corresponding bias and MSE for each method of estimation. It appears that with an increase in the sample size, the MSE values for all the estimators decrease. Out of all the estimators, the MLH estimator has the lowest MSE value.
Based on the Figures 2 and 5, it seems that MLH estimator is the closest estimate of the parameter at different sample sizes, while the other estimators get closer to the correct parameter values as the sample size increases. Additionally, the figures show that the MPS estimate is the farthest from the actual value for certain parameters at smaller sample sizes. However, all estimators are equally accurate at a large sample size of 200.
Based on Figures 3, 4, 6 and 7. It seems that as the sample size increases, the MSE values for all estimators decrease. The MLH estimator appears to have the least MSE compared to the other estimators. For smaller sample sizes, the MSE values for the estimators O.LS, W.LS and CRM are quite similar. However, the MPS estimator has the highest MSE values compared to the other estimators, particularly for smaller sample sizes.

Table 1: Parameter estimation from five different methods (Set I)

		Set I				
		(λ=2.4, γ=0.5, θ=0.3, β=0.7, α=1.2)				
n	Par.	MLH	O.LS	W.LS	CRM	PMS
15	$\hat{\lambda}$	2.3512	2.3514	2.2721	2.2945	2.7462
	Bias	0.0880	1.3073	1.3129	1.4186	2.2728
	MSE	0.1449	2.4218	2.4173	6.1409	5.2112
	$\hat{\gamma}$	0.5853	0.7343	0.7397	0.5872	0.8009
	Bias	0.3438	0.6234	0.6168	0.5049	0.8180
	MSE	0.4272	0.9822	1.0735	0.7979	1.9747
	$\hat{\theta}$	0.4146	0.4041	0.4078	0.4029	0.5314
	Bias	0.1279	0.2022	0.2041	0.1879	0.3605
	MSE	0.1874	0.2770	0.2781	0.2570	0.4948
	$\hat{\beta}$	1.2062	1.3493	1.3998	1.4720	1.4974
	Bias	0.5773	0.8197	0.8667	0.9135	1.0597
	MSE	0.7961	1.3511	1.4236	1.4539	1.9745
	$\hat{\alpha}$	1.0688	1.5481	1.5404	1.8106	2.2968
	Bias	0.4159	0.9719	1.0091	1.2483	1.8040
	MSE	0.4933	1.6519	1.6428	5.8150	6.9145
30	$\hat{\lambda}$	2.3543	2.1697	2.2125	2.1767	2.2907
	Bias	0.0796	0.8750	1.0115	0.9276	1.4084
	MSE	0.1150	1.4818	2.8512	1.5135	2.2168
	$\hat{\gamma}$	0.6052	0.6801	0.6010	0.5952	0.6313
	Bias	0.3305	0.5076	0.4423	0.4615	0.5785
	MSE	0.3994	0.8231	0.6472	0.7740	1.1232
	$\hat{\theta}$	0.3975	0.3724	0.3965	0.3889	0.5049
	Bias	0.1080	0.1486	0.1725	0.1577	0.3068
	MSE	0.1655	0.1973	0.2327	0.2131	0.4302
	$\hat{\beta}$	1.1226	1.2623	1.2033	1.2796	1.1178
	Bias	0.4793	0.6863	0.6331	0.7080	0.6009
	MSE	0.6604	1.1075	0.9828	1.1464	1.0554
	$\hat{\alpha}$	1.0755	1.4750	1.4975	1.6354	1.5666
	Bias	0.3603	0.8252	0.8498	0.9924	0.9858
	MSE	0.4352	1.3811	1.6379	1.7390	1.6794
50	$\hat{\lambda}$	2.3654	2.1632	2.1517	2.1655	2.3983
	Bias	0.0629	0.6827	0.6992	0.6994	1.2418

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Table1 :Parameter estimation...(Set I) – (continue)

		Set I				
		(λ=0.5, γ=3, θ=0.7, β=1.5, α=2.4)				
n	Par.	MLH	O.LS	W.LS	CRM	PMS
	MSE	0.0858	1.2851	1.1158	1.4107	2.1186
	$\hat{\gamma}$	0.5890	0.6261	0.5891	0.6034	0.6500
	Bias	0.2698	0.4180	0.3784	0.3881	0.5074
	MSE	0.3298	0.5994	0.5783	0.6171	0.9859
	$\hat{\theta}$	0.3767	0.3706	0.3849	0.3767	0.4654
	Bias	0.0824	0.1319	0.1398	0.1322	0.2635
	MSE	0.1319	0.1778	0.1966	0.1807	0.3756
	$\hat{\beta}$	1.0309	1.1495	1.1011	1.1891	1.0577
	Bias	0.3635	0.5474	0.4859	0.5680	0.4699
	MSE	0.4649	0.8643	0.7305	0.8702	0.7161
	$\hat{\alpha}$	1.1120	1.3688	1.3172	1.3952	1.4319
	Bias	0.2647	0.6773	0.6025	0.7374	0.8438
	MSE	0.3332	1.0368	0.8795	1.1953	1.4303
100	$\hat{\lambda}$	2.3723	2.1838	2.2314	2.1722	2.3261
	Bias	0.0497	0.4811	0.4522	0.4706	0.8865
	MSE	0.0661	0.7516	0.7180	0.7981	1.5979
	$\hat{\gamma}$	0.5763	0.6084	0.5951	0.5827	0.5667
	Bias	0.2128	0.3248	0.2888	0.3086	0.3828
	MSE	0.2574	0.5113	0.4116	0.4396	0.8550
	$\hat{\theta}$	0.3623	0.3611	0.3640	0.3662	0.4438
	Bias	0.0653	0.1030	0.0999	0.1063	0.2198
	MSE	0.0985	0.1401	0.1406	0.1460	0.3195
	$\hat{\beta}$	0.9861	1.0927	1.0592	1.1041	0.9392
	Bias	0.2997	0.4475	0.3985	0.4560	0.3131
	MSE	0.3687	0.6403	0.5507	0.6640	0.4185
	$\hat{\alpha}$	1.1205	1.2691	1.2199	1.2531	1.3295
	Bias	0.1763	0.5499	0.4654	0.5245	0.6425
	MSE	0.2299	0.8246	0.6782	0.7788	0.9742
200	$\hat{\lambda}$	2.3693	2.2044	2.2811	2.1933	2.399
	Bias	0.0470	0.3583	0.3040	0.3442	0.7237
	MSE	0.0602	0.5660	0.4806	0.5366	1.4105
	$\hat{\gamma}$	0.6100	0.6133	0.6261	0.6138	0.5544
	Bias	0.2043	0.2594	0.2363	0.2608	0.2879
	MSE	0.2459	0.3524	0.3196	0.3639	0.4193
	$\hat{\theta}$	0.3482	0.3466	0.3416	0.3446	0.4114
	Bias	0.0508	0.0814	0.0686	0.0795	0.1728
	MSE	0.0771	0.1127	0.0979	0.1092	0.2581

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Table1 :Parameter estimation...(Set I) – (continue)

		Set I				
		(λ=0.5, γ=3, θ=0.7, β=1.5, α=2.4)				
n	Par.	MLH	O.LS	W.LS	CRM	PMS
	$\hat{\beta}$	0.9965	1.0396	1.0239	1.0433	0.9337
	Bias	0.3015	0.3651	0.3361	0.3671	0.2667
	MSE	0.3472	0.4695	0.4147	0.4762	0.3266
	$\hat{\alpha}$	1.1079	1.1799	1.1661	1.2015	1.2353
	Bias	0.1660	0.3560	0.3165	0.3699	0.4783
	MSE	0.2057	0.4830	0.4182	0.5231	0.6832

Table 2: Parameter estimation from five different methods (Set II)

		Set II				
		(λ=2.4, γ=0.5, θ=0.3, β=0.7, α=1.2)				
n	Par.	MLH	O.LS	W.LS	CRM	PMS
15	$\hat{\lambda}$	1.3173	1.1633	1.2466	1.3151	1.3479
	Bias	0.9575	1.1856	1.2501	1.2147	1.6603
	MSE	1.3253	2.6838	2.6338	2.4629	4.3668
	$\hat{\gamma}$	2.5823	3.3157	3.1059	2.9964	3.1173
	Bias	0.4383	2.0975	1.9570	2.0707	2.5715
	MSE	0.7287	3.6010	3.1051	3.8074	5.3895
	$\hat{\theta}$	1.4061	0.9920	1.0960	1.1645	1.2406
	Bias	0.8966	0.9272	0.9912	0.9403	1.2764
	MSE	1.2245	1.7337	1.7678	1.5917	2.0885
	$\hat{\beta}$	1.3022	1.4707	1.4469	1.5112	1.6331
	Bias	0.5074	0.9303	0.9227	0.9219	1.1114
	MSE	0.6403	1.3643	1.3735	1.3535	1.9297
	$\hat{\alpha}$	2.2546	3.7934	3.9230	4.2419	5.9237
	Bias	1.0808	2.7599	2.8551	3.1269	5.0923
	MSE	1.2991	6.1538	6.5440	6.8010	18.1692
30	$\hat{\lambda}$	1.1721	1.1353	1.1595	1.2092	0.9323
	Bias	0.7999	0.9932	1.0928	1.0549	1.0962
	MSE	1.0942	2.1260	2.2174	2.1770	2.4700
	$\hat{\gamma}$	2.6875	3.0399	3.1581	2.9533	3.2234
	Bias	0.3306	1.5615	1.6747	1.6252	2.1068
	MSE	0.5654	2.2631	2.6847	3.7378	4.2600
	$\hat{\theta}$	1.1944	0.9915	0.9529	1.0952	0.9429

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Table2 :Parameter estimation...(Set II) – (continue)

		Set II				
		(λ=2.4, γ=0.5, θ=0.3, β=0.7, α=1.2)				
n	Par.	MLH	O.LS	W.LS	CRM	PMS
	Bias	0.7035	0.7708	0.8299	0.8488	0.9901
	MSE	0.9602	1.4288	1.3899	1.4607	1.6179
	$\hat{\beta}$	1.1308	1.2555	1.2452	1.2808	1.3091
	Bias	0.4412	0.7083	0.7196	0.7061	0.7921
	MSE	0.5113	0.9514	0.9675	0.8969	1.2750
	$\hat{\alpha}$	2.2898	3.5279	3.8083	3.5672	4.9085
	Bias	0.8181	2.3179	2.5358	2.3424	3.7576
	MSE	1.0171	5.2966	6.4424	4.4101	13.8399
50	$\hat{\lambda}$	1.1081	1.0230	0.9116	1.0299	0.7250
	Bias	0.7285	0.8321	0.8173	0.8375	0.7688
	MSE	1.0041	1.6203	1.6048	1.6681	1.8030
	$\hat{\gamma}$	2.7400	3.1485	3.0437	2.9673	3.1842
	Bias	0.2777	1.3605	1.3479	1.3011	1.8411
	MSE	0.4723	1.9355	2.0288	1.7906	3.1472
	$\hat{\theta}$	1.0677	0.8891	0.8341	0.9384	0.7971
	Bias	0.5804	0.6812	0.6886	0.6898	0.8131
	MSE	0.7856	1.1375	1.0790	1.1663	1.6433
	$\hat{\beta}$	1.0720	1.1172	1.0975	1.1837	1.1255
	Bias	0.4519	0.5965	0.5688	0.5880	0.6143
	MSE	0.5018	0.6921	0.6572	0.7098	0.7777
	$\hat{\alpha}$	2.3206	3.2388	3.2671	3.2239	4.0809
	Bias	0.6817	1.8042	1.7741	1.8734	2.7098
	MSE	0.8571	3.0798	3.1481	3.7484	7.1000
100	$\hat{\lambda}$	0.9495	0.8179	0.7823	0.8777	0.6716
	Bias	0.5370	0.5686	0.5446	0.6090	0.6284
	MSE	0.7206	1.0441	0.9622	1.1496	1.1765
	$\hat{\gamma}$	2.8457	3.0526	3.0523	3.0116	3.3954
	Bias	0.1683	1.0339	0.9871	1.0464	1.6882
	MSE	0.2711	1.3585	1.3000	1.3737	2.8454
	$\hat{\theta}$	0.9178	0.7481	0.7597	0.8205	0.6769
	Bias	0.4089	0.4939	0.5212	0.5410	0.6537
	MSE	0.5590	0.7567	0.7646	0.8684	1.0054
	$\hat{\beta}$	1.0485	1.0705	1.0508	1.0858	1.0268
	Bias	0.4562	0.5283	0.05091	0.5250	0.5384
	MSE	0.4870	0.5948	0.5585	0.5901	0.6105
	$\hat{\alpha}$	2.3061	3.0350	2.8741	2.9541	3.4503
	Bias	0.5229	1.4461	1.2220	1.3575	1.8577

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Table2 :Parameter estimation...(Set II) – (continue)

		Set II				
		(λ=2.4, γ=0.5, θ=0.3, β=0.7, α=1.2)				
n	Par.	MLH	O.LS	W.LS	CRM	PMS
	MSE	0.6534	2.5415	1.9669	2.2529	2.8001
200	$\hat{\lambda}$	0.9068	0.7664	0.7823	0.7629	0.6124
	Bias	0.4635	0.4518	0.5446	0.4391	0.4917
	MSE	0.6317	0.6975	0.9622	0.6648	0.8302
	$\hat{\gamma}$	2.8778	3.0805	3.0523	3.0105	3.4361
	Bias	0.1324	0.8169	0.9871	0.7978	1.3674
	MSE	0.2184	1.0906	1.3000	1.0826	2.0207
	$\hat{\theta}$	0.8507	0.7006	0.7597	0.7193	0.5915
	Bias	0.3253	0.4022	0.5212	0.3869	0.5609
	MSE	0.4267	0.5675	0.7646	0.5341	0.8099
	$\hat{\beta}$	1.0180	1.0246	1.0508	1.0351	0.9873
	Bias	0.4821	0.5069	0.5091	0.4959	0.5298
	MSE	0.4991	0.5511	0.5585	0.5407	0.5694
	$\hat{\alpha}$	2.3671	2.8549	2.8741	2.8335	3.3185
	Bias	0.4047	1.0719	1.2220	1.0796	1.5162
	MSE	0.5222	1.5927	1.9669	1.6631	2.1942

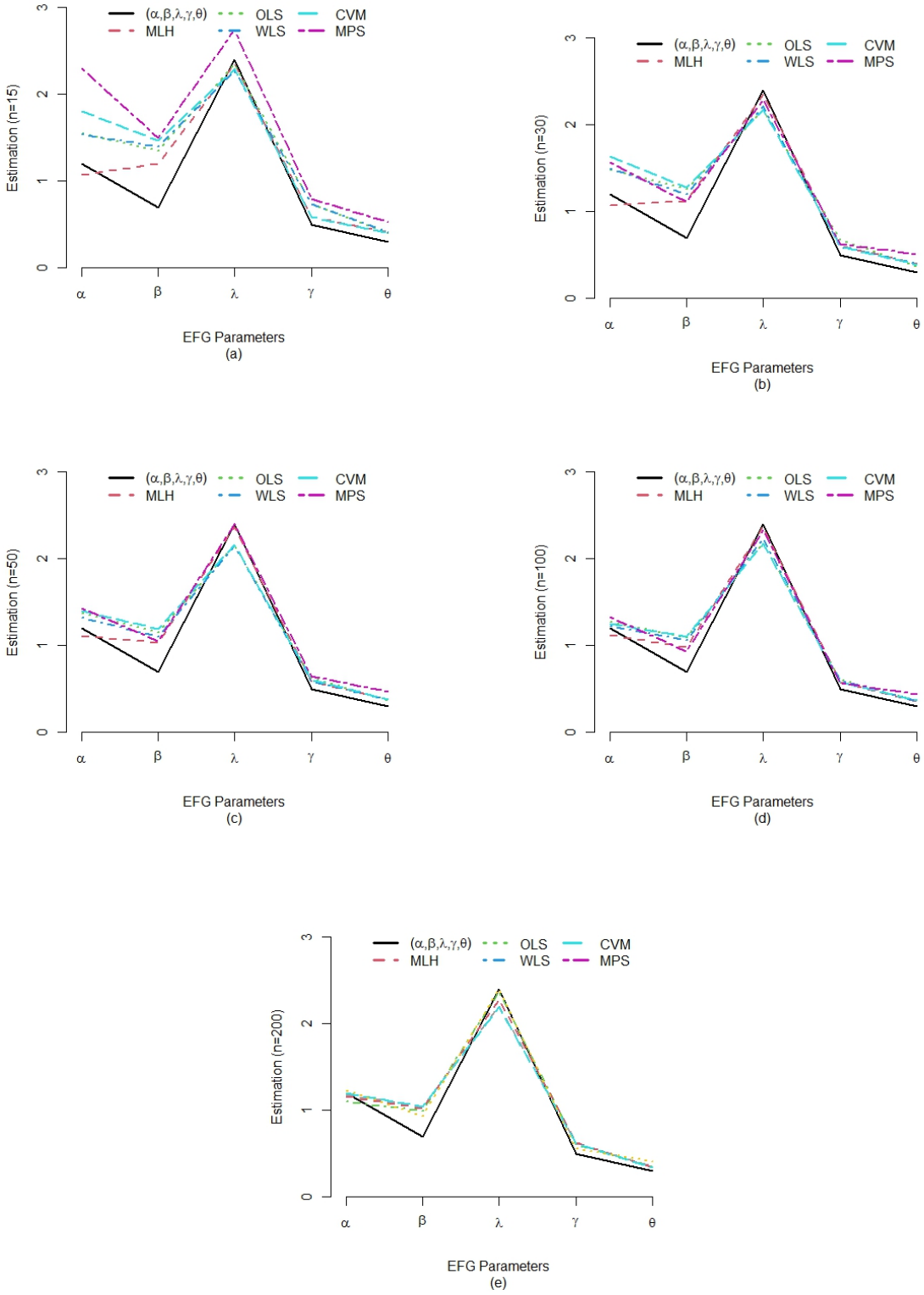

Figure 2: Compare parameter estimation from different methods at each sample size set I

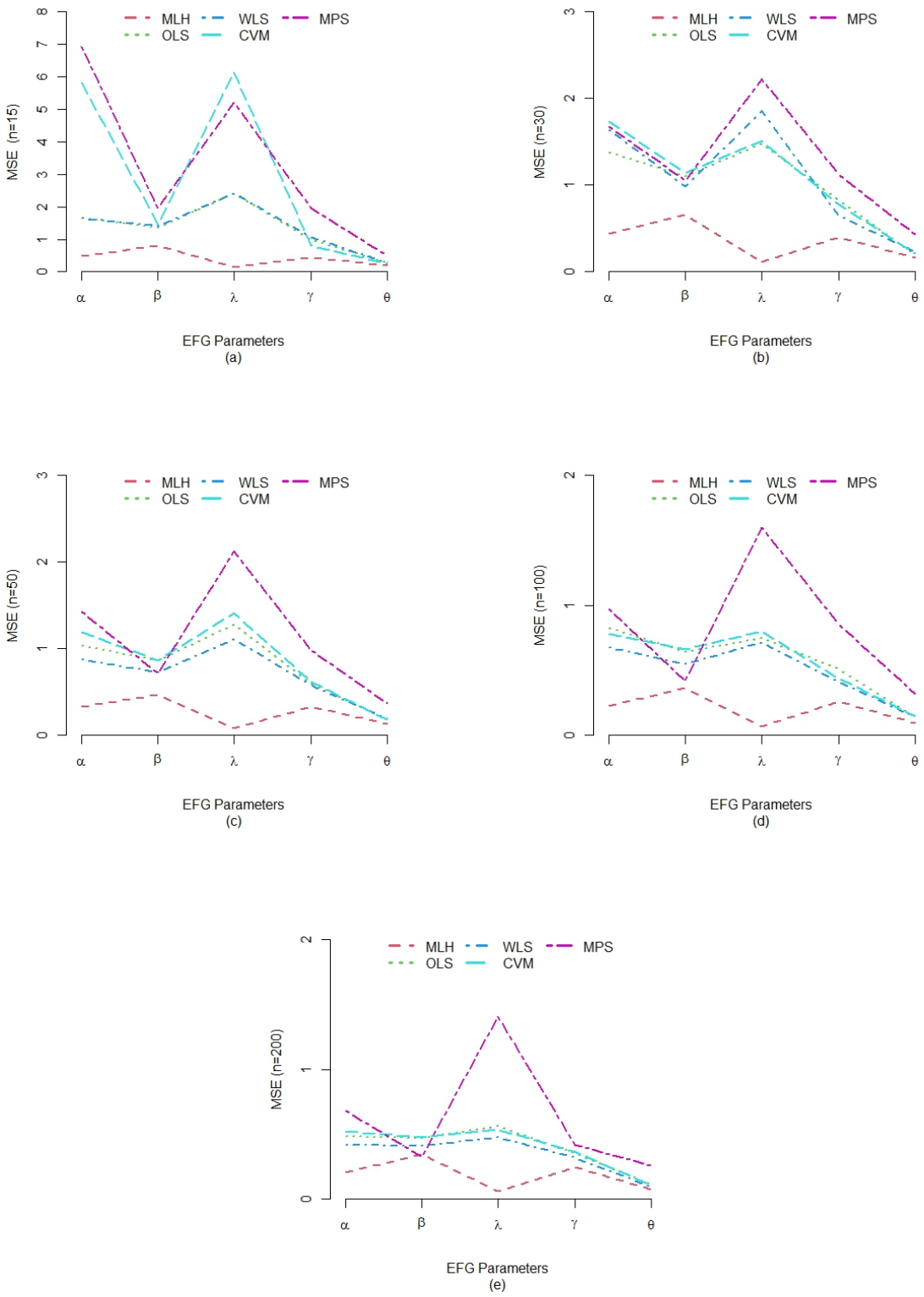

Figure 3: Compare MSE for different estimation methods at each sample size set I

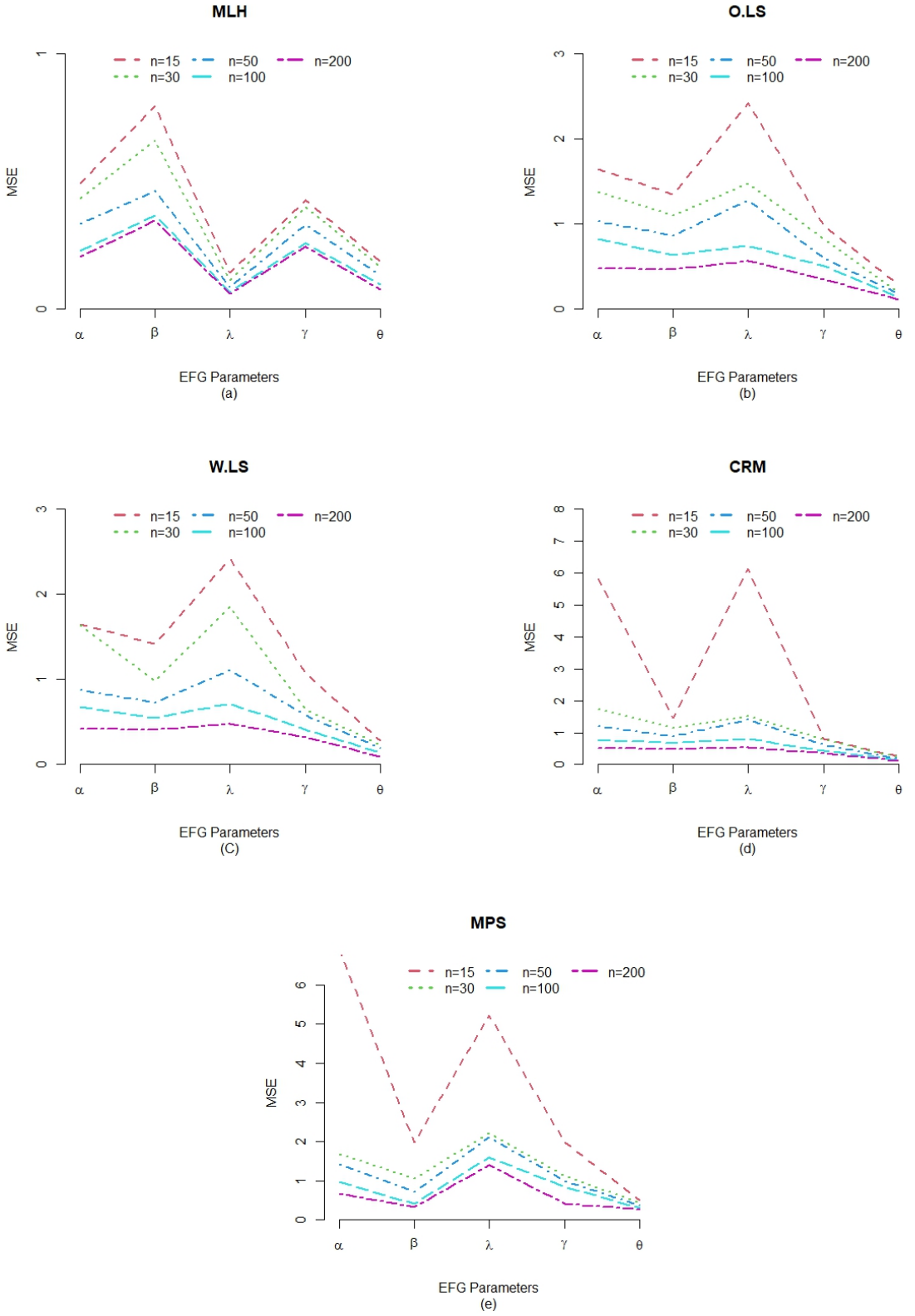

Figure 4: MSE at different n set I

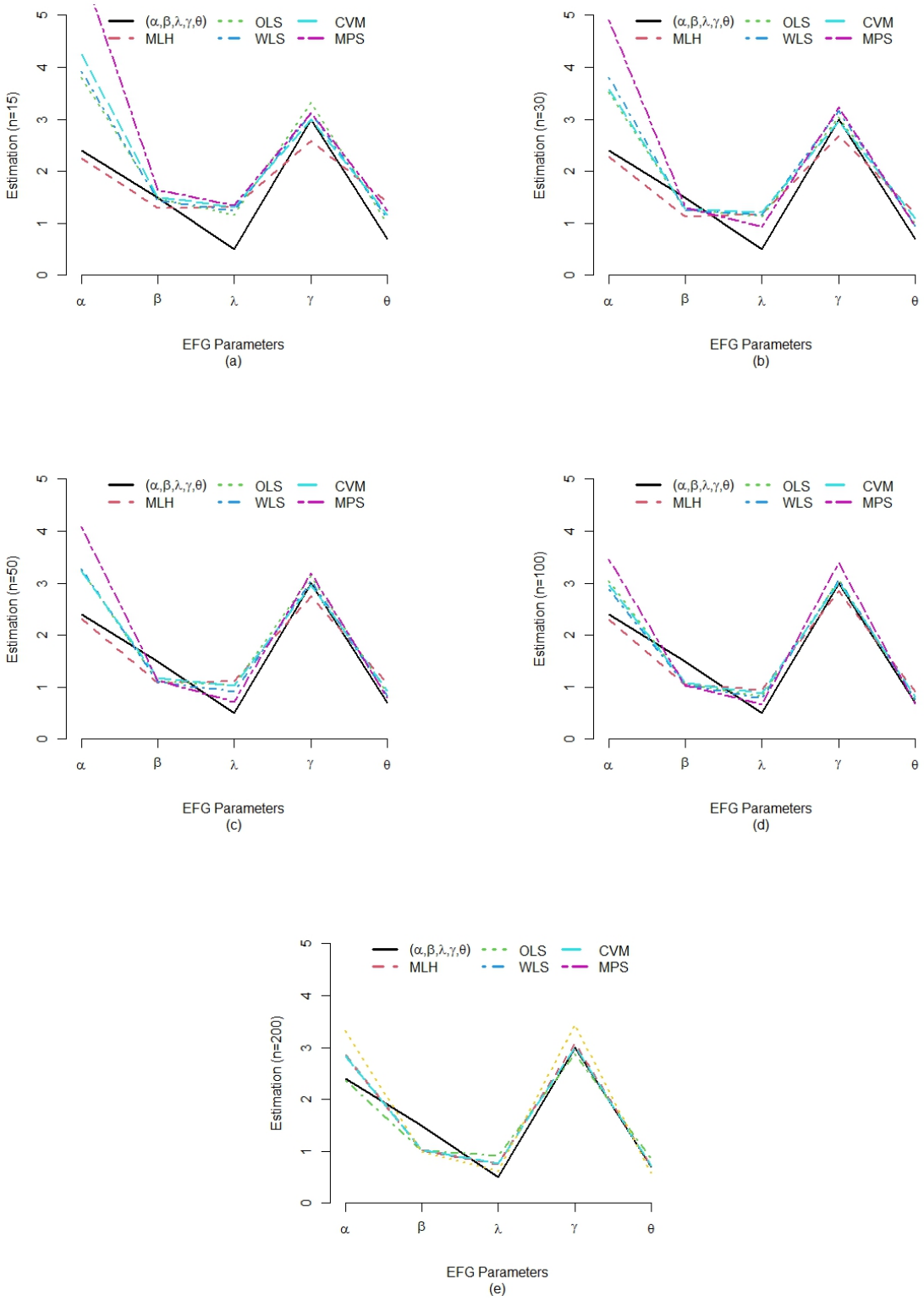

Figure 5: Compare parameter estimation from different methods at each sample size set II

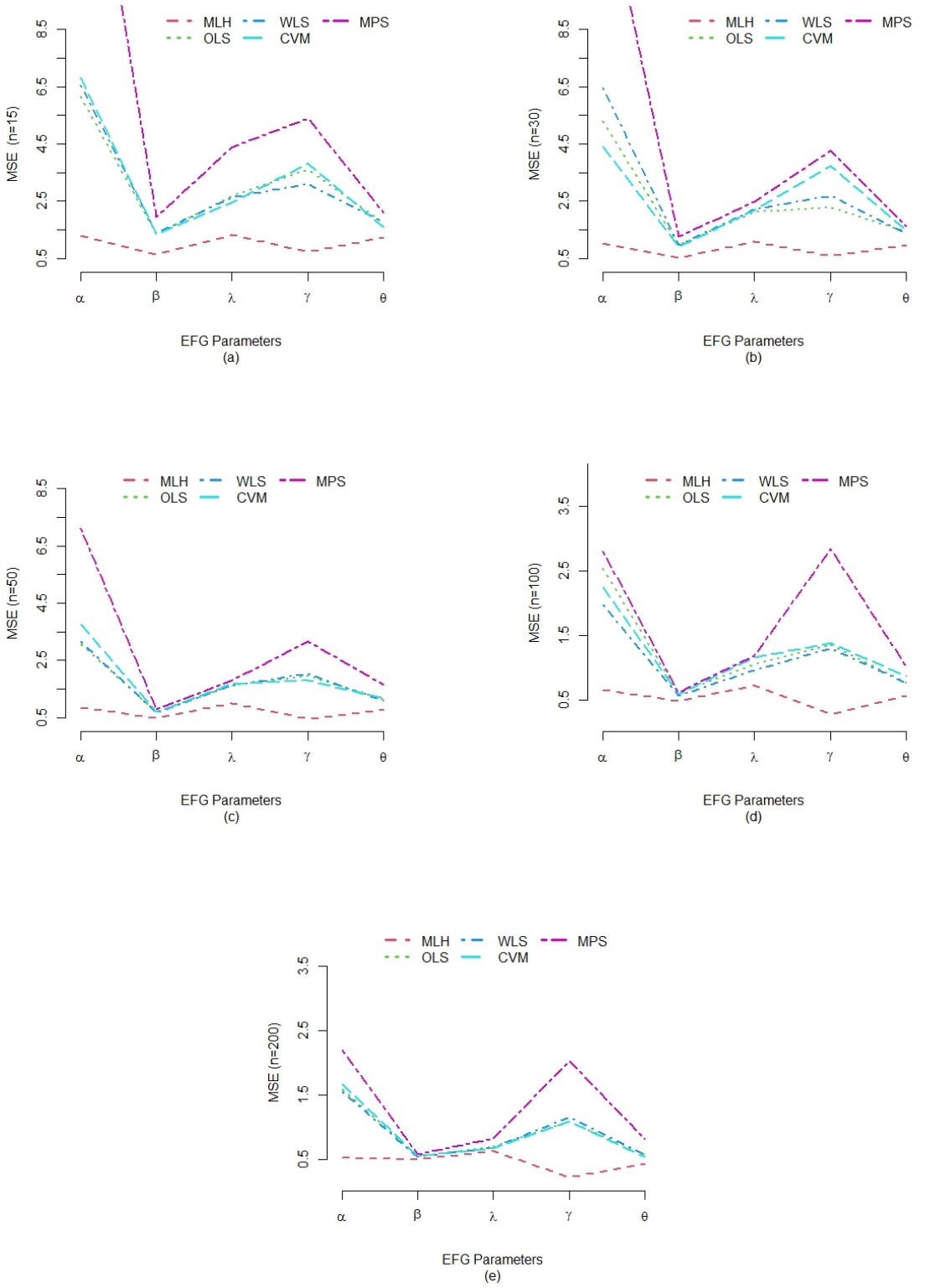

Figure 6: Compare MSE for different estimation methods at each sample size set II

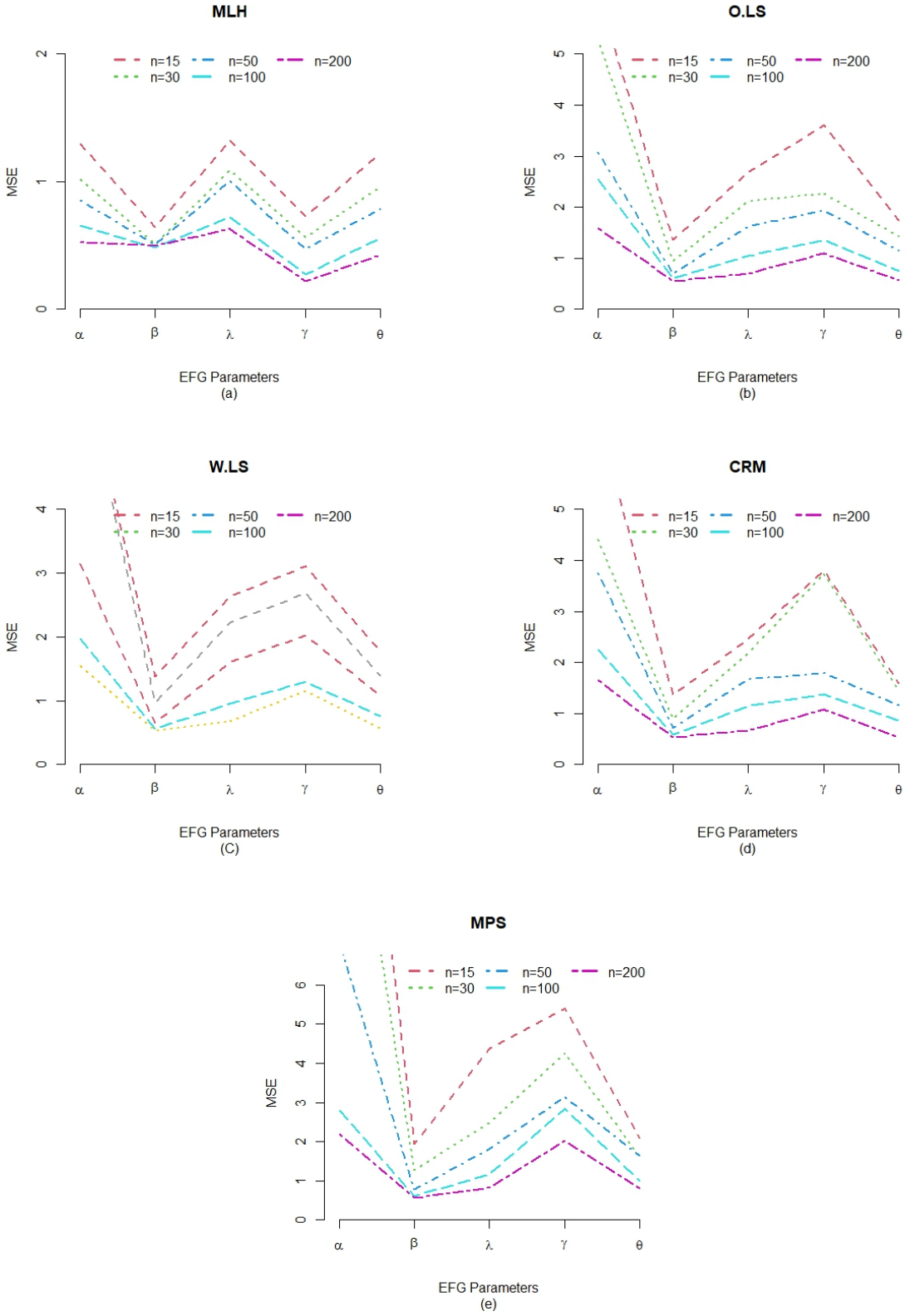

Figure 7: MSE at different n set II

7. Application

In this section, the newly developed distribution is applied to real-world data from patients with various types of cancer, including brain, bladder, Bone and blood cancers. The analysis involves comparing different estimation methods, as well as evaluating the performance of the new distribution against other competing distributions.

7.1. Compare between estimation methods

In this section, different datasets of cancer patient data are analyzed, each group suffering from a different type of cancer such as bladder cancer, brain cancer, bone cancer and blood cancer . The data are modeled using the EFG model and the model parameters are estimated using the five different estimation methods. The datasets are provided below.

Dataset I: Bladder Cancer

The remission times by months of bladder cancer patients were recorded in a dataset that included 36 individuals [20].

Dataset II: Brain Cancer

This data represents the lifetime by day for 87 patients with brain cancer diseases collected from the Atomic Medicine and Radiance Hospital in Baghdad [22].

Dataset III: Bone Cancer

Simulated data has been considered by [29] to represent the survival times (in days) of 73 patients who have been diagnosed with acute bone cancer.

Dataset IV: Blood Cancer

This data consists of the lifetime (in years) of 40 blood cancer (leukemia) patients from one of the Ministry of Health hospitals in Saudi Arabia reported in [6]

Dataset V: Bladder Cancer

This data represents the duration of remission (in months) for a group of 128 patients who were diagnosed with bladder cancer [26].

Table 3: Descriptive statistics for the datasets

Data	Min	Q_1	$Median$	$Mean$	Q_3	Max	Skw	Kur
Dataset I	0.08	1.16	2.08	1.94	2.71	3.36	-0.29	1.86
Dataset II	11	37.5	67.5	78.26	92	274	1.47	5.37
Dataset III	0.09	0.92	1.57	3.76	2.75	86.01	6.79	51.78
Dataset IV	0.32	2.19	3.35	3.14	4.26	5.38	-0.42	2.27
Dataset V	0.08	3.35	6.39	9.37	11.84	79.05	3.29	18.48

Table 3 displays a summary for the datasets. The table presents a range of skewness and kurtosis values that reflect different shapes of the data distribution. The skewness values vary in terms of direction and severity, with some being positive and others negative,

and some being large while others are small. Similarly, the kurtosis values also differ, with some being small and others large, indicating variations in the width of the distribution's tail.

Table 4: Estimation Survival time for bladder cancer patients (Dataset I)

Est.	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	KS	P.value
MLH	7.2725	0.2072	9.8959	0.0040	1.9577	0.1467	0.4208
	(5.7753)	(0.0587)	(26.4236)	(0.0031)	(0.7532)		
O.LS	6.3994	0.2570	2.1266	0.0068	1.3603	0.1487	0.4036
	(29.7235)	(0.3706)	(20.2968)	(0.0188)	(3.6681)		
W.LS	3.0007	0.2090	5.7650	0.0048	2.6659	0.1442	0.4422
	(0.6459)	(0.0142)	(2.4854)	(0.0012)	(0.3396)		
MPS	1.7373	0.2571	2.1588	0.0152	2.5544	0.1554	0.3493
	(0.9347)	(0.0862)	(9.3294)	(0.0699)	(0.9404)		
CVM	1.0113	0.5146	0.0609	0.0050	1.7158	0.1670	0.2681
	(3.2626)	(1.9698)	(0.7743)	(0.0114)	(6.1166)		

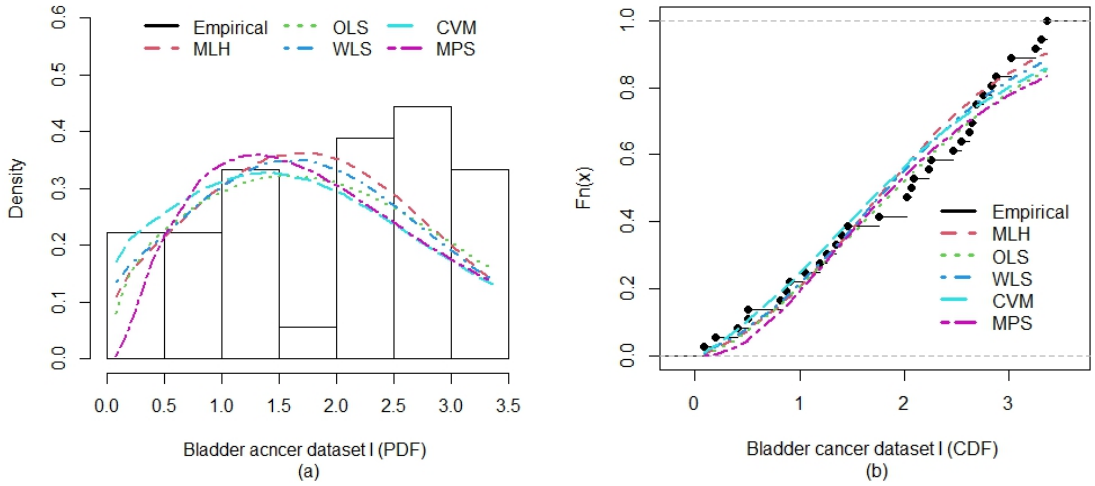

Figure 8: Estimation Survival time for Bladder cancer patients (Dataset I)

Table 5: Estimation Survival time for Brain cancer patients

Est.	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	KS	P.value
MLH	0.6469	0.5550	0.3064	0.0390	0.0560	0.1244	0.4211
	(0.6738)	(0.2586)	(2.0015)	(0.2472)	(0.0528)		
O.LS	0.0685	0.1171	1.3718	2.4640	1.3474	0.1454	0.2410
	(0.2126)	(0.3623)	(0.5018)	(0.5015)	(0.1552)		
W.LS	0.0832	0.0502	2.3820	1.1014	2.5989	0.1459	0.2373
	(0.0081)	(0.0050)	(0.1833)	(0.3195)	(0.0910)		
MPS	0.4982	0.4982	0.3169	0.0325	0.0729	0.1213	0.4542
	(0.4306)	(0.2841)	(2.1811)	(0.2197)	(0.0659)		
CVM	0.0460	0.0776	1.5981	0.6614	3.0128	0.1497	0.2122
	(0.1406)	(0.2408)	(0.0884)	(0.0885)	(0.1441)		

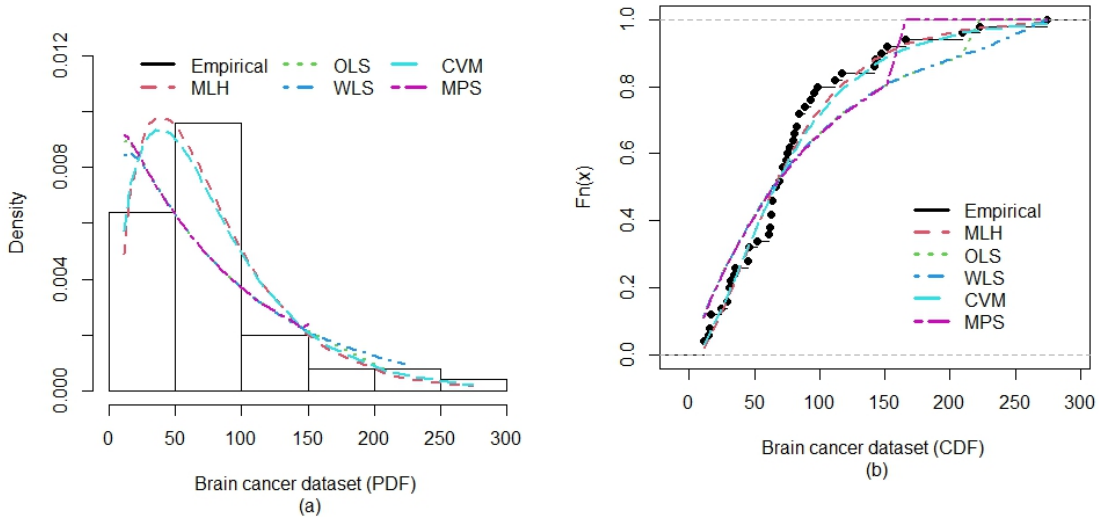

Figure 9: Estimation survival time for brain cancer patients

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Table 6: Estimation survival time for acute bone cancer patients

Est.	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	KS	P.value
MLH	3.7435	0.5731	0.0898	2.5659	0.0080	0.0908	0.5843
	(2.0732)	(0.1339)	(0.7592)	(20.2149)	(0.0042)		
O.LS	1.2000	0.7829	0.2822	0.2372	0.5734	0.0683	0.8852
	(28.0826)	(5.7028)	(14.3646)	(10.5554)	(10.3191)		
W.LS	1.8908	0.7619	0.0752	0.1117	0.2702	0.0714	0.8503
	(0.4005)	(0.0941)	(0.0767)	(0.0640)	(0.1330)		
MPS	1.1891	0.8510	0.0123	1.1132	0.0092	0.1330	0.1510
	(0.3298)	(0.1210)	(0.0131)	(1.3190)	(0.0051)		
CVM	1.2552	0.8531	0.0054	0.0065	0.4297	0.0702	0.8649
	(7.5602)	(1.7563)	(0.0052)	(0.0087)	(1.9309)		

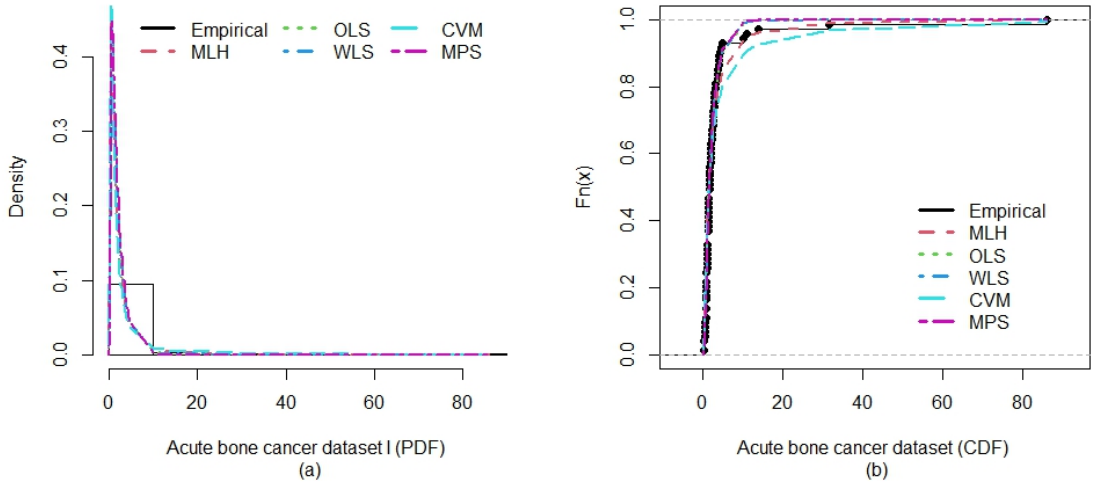

Figure 10: Estimation Survival time for acute bone cancer patients

Table 7: Estimation survival time for blood cancer patients

Est.	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	KS	P.value
MLH	12.5138	0.2339	11.5206	0.0044	1.0836	0.1104	0.7143
	(10.2835)	(0.0876)	(39.7697)	(0.0026)	(0.4653)		
O.LS	0.3832	0.7476	0.4354	0.0084	2.01428	0.1403	0.4103
	(1.7182)	(3.4603)	(4.1605)	(0.0570)	(6.5835)		
W.LS	1.6424	0.2544	1.1654	0.0005	2.4439	0.1012	0.8070
	(0.3505)	(0.0263)	(0.4141)	(0.0003)	(0.0797)		
MPS	1.5480	0.3466	1.0598	0.0139	1.4797	0.1399	0.4139
	(0.8931)	(0.1544)	(3.1758)	(0.0613)	(0.6700)		
CVM	2.0440	0.2554	1.8712	0.0009	2.1143	0.1010	0.8092
	(10.1339)	(0.4582)	(16.1734)	(0.0042)	(0.9868)		

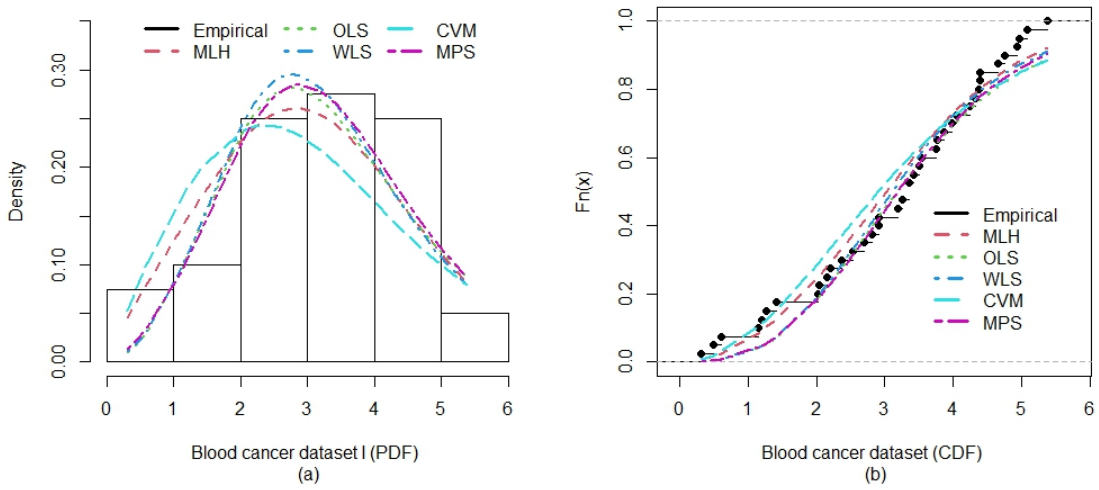

Figure 11: Estimation Survival time for blood cancer patients

Table 8: Estimation survival time for bladder cancer patients (Dataset V)

Est.	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	KS	P.value
MLH	0.2559	0.4308	0.1958	0.0647	1.0659	0.0708	0.5419
	(0.0229)	(0.0036)	(0.0036)	(0.0036)	(0.0036)		
O.LS	0.1478	0.6019	1.5195	0.3035	1.4830	0.0321	0.9994
	(0.0747)	(0.1446)	(4.3697)	(0.1478)	(0.1464)		
W.LS	2.2327	0.5686	0.0349	0.0313	0.0842	0.0349	0.9977
	(0.2789)	(0.0255)	(0.0055)	(0.0050)	(0.0120)		
MPS	0.2632	0.4361	0.6715	0.2136	1.0025	0.0831	0.3400
	(0.0287)	(0.0136)	(0.0150)	(0.0148)	(0.0646)		
CVM	0.9312	0.7291	0.1447	0.1240	0.1832	0.0370	0.9948
	(4.8391)	(1.7438)	(3.2033)	(2.4158)	(0.8814)		

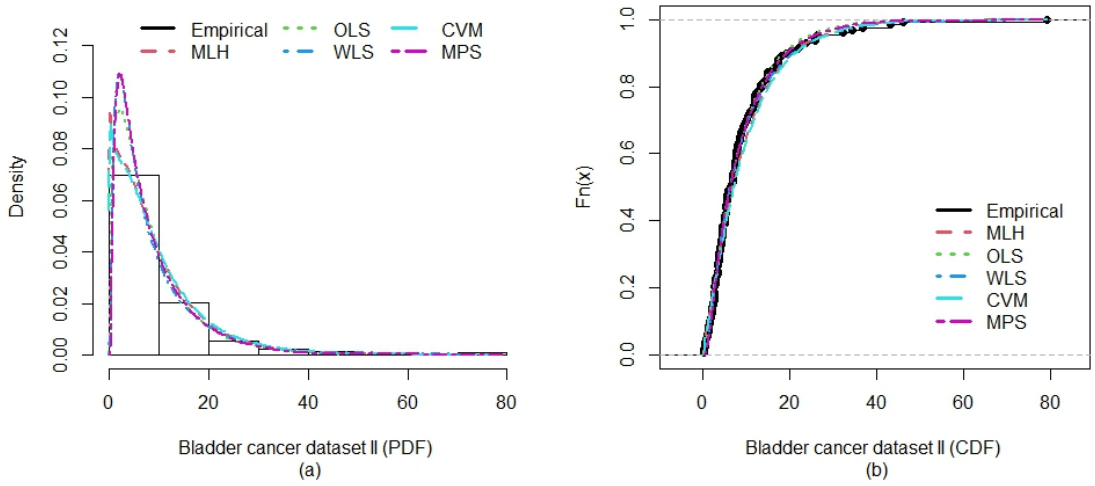

Figure 12: Estimation Survival time for bladder cancer patients (Dataset V)

The Tables from 4 to 8 display the results of parameter estimation and its standard error (SE) (in parentheses). The tables also include the test statistic of the Kolmogorov Smirnov (KS) goodness of fit test along with its P.value which is used to check the fitting of data with the EFG distribution. The estimated distribution with the lowest KS and highest p.value is considered as the best fit for data. Additionally, to compare the fitting of the data with the estimated EFG distribution by the estimation methods, the figures from 8 to 12 display the empirical distribution of each dataset concurrently with the estimated pdf and cdf of EFG distribution by the five estimation methods.

Based on the results presented in the tables from 4 to 8, it seems that the EFG distribution could effectively represent the survival time data for cancer patients since the

p.values are greater than 0.05 in all of the six datasets. Furthermore, the W.LS method provides the best estimation for EFG parameters fitting with the bladder cancer datasets (Table 4 and Table 8). Even though, the CVM method produces the highest p.value for representing the bone cancer and blood cancer datasets as seen in Tables 6 and 7, the W.LS still provides a close fit to the data with a smaller SE comparing to CVM. According to brain cancer and breast cancer datasets, the MPS method gives the best fitting to these datasets.

7.2. Compare EFG against other distributions

This section provides a comparison of the performance of the EFG model against four competing distributions, each with a different number of parameters. The CDFs of these distributions are outlined below.

- Exponentiated Gompertz Exponential (EGoG) [1]
  $$F(x) = \left\{1 - e^{\frac{\gamma}{\theta}[1-e^{\lambda}x\theta]}\right\}^{\alpha}, x > 0; \theta, \gamma, \lambda, \alpha > 0.$$
- The Fréchet
  $$F(x) = exp\left\{-\left(\frac{x-\lambda}{\theta}\right)^{-\alpha}\right\}, x > 0; \alpha, \theta > 0, -\infty < \lambda < \infty.$$
- Gompertz given in (3).
- The exponential(E)
  $$F(x) = \theta\ e^{-\theta x},.$$

The EFG model is evaluated using several goodness-of-fit indices (GoF). These include the negative log-likelihood value ($-\ell$), Akaike Information Criterion (AIC), corrected AIC (CAIC) and Bayesian Information Criterion (BIC). Kramér-von Mises (KVM) test statistic, Kolmogorov-Smirnov (KS) test statistic, and their corresponding *p-values* are calculated as well. A model is deemed the best fit for the data when its values for these statistics are lower than those of the competing models.

The tables from (9) to (12) present the GoF measurements for the EFG model and the competing distributions. The results indicate that the exponential and Gompertz distributions did not adequately fit all the data, as they yielded significant *p-values* less than 0.05. In contrast, other distributions, such as EGoG and Fréchet, demonstrated *p-values* greater than 0.05 for most datasets, suggesting they can effectively represent the data. While these models fit the majority of datasets (*p-value*$>$ 0.05), the EFG model exhibited the highest *p-value* among all fitted models. Additionally, the tables show that the NEG distribution achieved the lowest values across all GoF indices, indicating that it fits all five datasets better than the other competing distributions.

Table 9: The Performance of EFG for Bladder cancer data.

Distributions	EFG	EGoE	Fréchet	Gompertz	E
$-\ell$	49.481	72.603	53.814	66.089	59.857
AIC	108.962	153.206	113.628	136.179	121.714
CAIC	113.529	156.374	116.004	137.762	122.505
BIC	116.879	159.541	118.379	139.346	123.297
KVM	0.126	0.437	0.159	1.089	0.656
p-value	0.473	0.057	0.137	0.001	0.016
KS	0.146	0.190	0.175	0.333	0.230
p-value	0.428	0.147	0.222	0.001	0.044

Table 10: The Performance of EFG for Brain cancer data.

Distributions	EFG	EGoE	Fréchet	gompertz	E
$-\ell$	260.789	302.709	270.203	266.799	268.003
AIC	531.578	613.417	546.405	537.599	538.006
CAIC	536.358	617.241	549.273	539.511	549.273
BIC	541.138	621.065	552.141	541.423	539.918
KVM	0.089	6.290	0.858	0.556	0.522
p-value	0.646	$2.2\text{x}10^{-16}$	0.005	0.028	0.035
KS	0.124	0.558	0.284	0.226	0.186
p-value	0.421	$5.9\text{x}10^{-14}$	0.001	0.012	0.062

Table 11: The Performance of EFG for Bone cancer data.

Distributions	EFG	EGoE	Fréchet	gompertz	E
$-\ell$	142.975	146.544	155.419	203.825	169.589
AIC	295.951	301.088	316.839	411.651	341.179
CAIC	301.677	305.669	320.275	413.941	320.275
BIC	307.403	310.249	323.710	416.232	343.4693
KVM	0.139	0.172	0.222	0.423	1.546
p-value	0.426	0.329	0.229	0.062	$1.2\text{x}10^{-4}$
KS	0.103	0.091	0.1219	0.182	0.251
p-value	0.584	0.379	0.2281	0.016	$1.9\ \text{x}10^{-4}$

Table 12: The Performance of EFG for Blood cancer data.

Distributions	EFG	EGoE	Fréchet	gompertz	E
$-\ell$	68.592	75.771	74.386	83.953	85.778
AIC	147.185	159.542	154.772	171.9054	173.556
CAIC	151.407	162.920	157.306	173.594	174,401
BIC	155.629	166.298	159.839	175.283	175.245
KVM	0.094	0.221	0.201	1.077	1.084
p-value	0.616	0.230	0.267	0.001	0.001
KS	0.110	0.139	0.136	0.299	0.300
p-value	0.714	0.421	0.451	0.002	0.001

8. Conclusion

Obtaining accurate estimates regarding patients' condition is extremely important as this helps the doctor determine the appropriate treatment. This study focused on investigating the EFG distribution and its effectiveness in representing the survival time of cancer patients. It also provides a comprehensive overview of the statistical properties of the distribution such as the quantile function,moments, moment-generating function, characteristic function, R´enyi entropy, and order statistics More addition, the study also examined the performance of five different types of estimates, MLH, O.LS, W.LS, CRM and MPS including simulation and application to real data for cancer patients. The results revealed that the EFG distribution is an ideal representation of the survival time for patients with various types of cancer. Additionally, the W.LS method provided the best estimate for survival time with minimal SEs. Moreover, the performance of the EFG model is assessed in comparison to various competing distributions. Notably, EFG proves to be a more suitable choice than several of these models, as it consistently achieves the lowest values across multiple goodness-of-fit criteria. This underscores EFG's superiority in both effectiveness and flexibility when analyzing survival times for cancer patients. Based on these findings, the EFG model is a promising option for modeling survival time data, especially within the medical field.

References

[1] Adewara Ademola, John Adeyeye, Mundher Khaleel, and Aako Olubisi. Exponentiated gompertz exponential (egoe) distribution: Derivation, properties and applications. *Istatistik Journal of the Turkish Statistical Association*, 13(1):12–28, 2021.

[2] Morad Alizadeh, Gauss M Cordeiro, Luis Gustavo Bastos Pinho, and Indranil Ghosh. The gompertz-g family of distributions. *Journal of statistical theory and practice*, 11:179–207, 2017.

[3] Ayman Alzaatreh, Carl Lee, and Felix Famoye. A new method for generating families of continuous distributions. *Metron*, 71(1):63–79, 2013.

[4] Omar Alzeley, Ehab M Almetwally, Ahmed M Gemeay, Huda M Alshanbari, Eslam H Hafez, and MH Abu-Moussa. Statistical inference under censored data for the new exponential-x fréchet distribution: Simulation and application to leukemia data. *Computational Intelligence and Neuroscience*, 2021.

[5] Muhammad Arif, Dost Muhammad Khan, Muhammad Aamir, Mahmoud El-Morshedy, Zubair Ahmad, and Zardad Khan. A new flexible exponentiated-x family of distributions: Characterizations and applications to lifetime data. *IETE Journal of Research*, pages 1–13, 2022.

[6] MA Atallah, MAW Mahmoud, and BM Al-Zahrani. A new test for exponentiality versus nbumgf life distributions based on laplace transform. *Quality and Reliability Engineering International*, 30(8):1353–1359, 2014.

[7] Lamya A Baharith. New generalized weibull inverse gompertz distribution: Properties and applications. *Symmetry*, 16(2):197, 2024.

[8] Lamya A Baharith and Hanan H Alamoudi. The exponentiated fréchet generator of distributions with applications. *Symmetry*, 13(4):572, 2021.

[9] Lamya A Baharith and Wedad H Aljuhani. New method for generating new families of distributions. *Symmetry*, 13(4):726, 2021.

[10] Marcelo Bourguignon, Rodrigo B Silva, and Gauss M Cordeiro. The weibull-g family of probability distributions. *Journal of data science*, 12(1):53–68, 2014.

[11] Selen Cakmakyapan and Gamze Ozel. The lindley family of distributions: properties and applications. *Hacettepe Journal of Mathematics and Statistics*, 46(6):1113–1137, 2016.

[12] RCH Cheng and NAK Amin. Maximum product-of-spacings estimation with applications to the lognormal distribution. *Math report*, 791, 1979.

[13] Gauss M Cordeiro, Edwin MM Ortega, and Daniel CC da Cunha. The exponentiated generalized class of distributions. *Journal of data science*, 11(1):1–27, 2013.

[14] Gauss M Cordeiro, Edwin MM Ortega, Božidar V Popović, and Rodrigo R Pescim. The lomax generator of distributions: Properties, minification process and regression model. *Applied Mathematics and Computation*, 247:465–486, 2014.

[15] Thiago AN De Andrade, Subrata Chakraborty, Laba Handique, and Frank Gomes-Silva. The exponentiated generalized extended gompertz distribution. *Journal of Data Science*, 17(2):299–330, 2019.

[16] Awad El-Gohary, Ahmad Alshamrani, and Adel Naif Al-Otaibi. The generalized gompertz distribution. *Applied mathematical modelling*, 37(1-2):13–24, 2013.

[17] Zahra Karimi Ezmareh and Gholamhossein Yari. Kumaraswamy-g generalized gompertz distribution with application to lifetime data. *International Journal of Industrial Engineering*, 33(4):1–22, 2022.

[18] Benjamin Gompertz. Xxiv. on the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. in a letter to francis baily, esq. frs &c. *Philosophical transactions of the Royal Society of London*, (115):513–583, 1825.

[19] Amal S Hassan and Said G Nassr. Power lindley-g family of distributions. *Annals of Data Science*, 6:189–210, 2019.

[20] Rafif Hibatullah, Yekti Widyaningsih, and Sarini Abdullah. Marshall-olkin extended power lindley distribution with application. *J. Ris. and Ap. Mat*, 2(2):84–92, 2018.

[21] Xiaoyan Huo, Saima K Khosa, Zubair Ahmad, Zahra Almaspoor, Muhammad Ilyas, and Muhammad Aamir. A new lifetime exponential-x family of distributions with applications to reliability data. *Mathematical Problems in Engineering*, 2020(1):1316345, 2020.

[22] SA Jasim. *Estimate Survival Function for the Brain Cancer Disease by using three Parameters Weibull Distribution*. PhD thesis, MSc Thesis, Baghdad University, College of Education (Ibn Al-Haitham), 2010.

[23] Jibril Yahaya Kajuru, Hussaini Dikko Garba, Aminu Mohammed Suleiman, and Aliyu Fulatan Ibrahim. The generalized gompertz-g family of distributions: Statistical properties and applications. *UMYU Scientifica*, 3(1):120–128, Mar. 2024.

[24] Hadeel S Klakattawi. Survival analysis of cancer patients using a new extended weibull distribution. *Plos one*, 17(2):e0264229, 2022.

[25] Granino Arthur Korn and Theresa M Korn. *Mathematical handbook for scientists and engineers: definitions, theorems, and formulas for reference and review*. Courier Corporation, 2000.

[26] Elisa T Lee and John Wang. *Statistical methods for survival data analysis*, volume 476. John Wiley & Sons, 2003.

[27] PD Macdonald. Estimation procedure for mixtures of distributions. *Journal of the Royal Statistical Society Series B - Statistical Methodology*, 33(2):326–+, 1971.

[28] Mohamed AW Mahmoud and Farouq Mohammad A Alam. The generalized linear exponential distribution. *Statistics & probability letters*, 80(11-12):1005–1014, 2010.

[29] M Mansour, Haitham M Yousof, WA Shehata, and Mohamed Ibrahim. A new two parameter burr xii distribution: properties, copula, different estimation methods and modeling acute bone cancer data. *Journal of Nonlinear Science and Applications*, 13(5):223–238, 2020.

[30] Ahmed Sayed M Metwally, Amal S Hassan, Ehab M Almetwally, BM Golam Kibria, and Hisham M Almongy. Reliability analysis of the new exponential inverted Topp–Leone distribution with applications. *Entropy*, 23(12), 2021.

[31] JJA Moors. A quantile alternative for kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 37(1):25–32, 1988.

[32] Pelumi E Oguntunde, Mundher A Khaleel, Mohammed T Ahmed, Adebowale O Adejumo, Oluwole A Odetunmibi, et al. A new generalization of the lomax distribution with increasing, decreasing, and constant failure rate. *Modelling and Simulation in Engineering*, 2017, 2017.

[33] James J Swain, Sekhar Venkatraman, and James R Wilson. Least-squares estimation of distribution functions in johnson's translation system. *Journal of Statistical Computation and Simulation*, 29(4):271–297, 1988.

[34] Muhammad Hussain Tahir, Muhammad Mansoor, Muhammad Zubair, and Gholamhossein Hamedani. Mcdonald log-logistic distribution with an application to breast cancer data. *Journal of Statistical Theory and Applications*, 2014.