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A Novel Bipolar Valued Fuzzy Group based on Dib's approach

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Abstract. The numerous extensions of fuzzy groups (FG) and fuzzy subgroups are complicated. Several results depending on different approaches of FG theory were introduced. This study introduces a novel extension, named bipolar-valued fuzzy groups (BVF-groups), which are based on bipolar-valued fuzzy space (BVF-space) and are created using Dib's methodology. The BVF-space replaces the universal set in conventional set theory. The BVF-space generalizes the notion of fuzzy space (F-space) from $[0,\ 1]$ to $[-1,\ 0] \times [0,\ 1]$ for the range of membership function. The novel theory of BVF-group is achieved through the BVF-space and bipolar valued binary operation (BVFBO) to build a new algebraic structure in a natural way, which satisfies four axioms as in classical group and FG theory. The challenges associated with the lack of a bipolar valued fuzzy universal set may also be resolved using this approach. This generalization highlights how to present and explore the BVF-groupoid, BVF-monoid, and BVF-group based on BVF-space. Also, as a connection result, we proved that every intuitionistic fuzzy groupoid (group) is a bipolar valued fuzzy groupoid (group), but the inverse is not true. Some theorems support the relations between BVF-group as a generalization of the classical (fuzzy) group are illustrated in detail.

2020 Mathematics Subject Classifications: 08A72, 06D72

Key Words and Phrases: Fuzzy space, bipolar valued fuzzy space, fuzzy binary operation, bipolar valued fuzzy binary operation, fuzzy function, bipolar valued fuzzy function, fuzzy group, and bipolar valued fuzzy group subgroup

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1. Introduction

The fuzzy sets (FSs) were created by Zadeh [37]. Fuzzy mathematics has been applied and manipulated in lots of fields and discovered applications in an expansive diversity of disciplines [8, 11, 14, 32, 34, 35]. The primary challenge in fuzzy mathematics is translating conventional notions into a fuzzy form. The obstacle lies in choosing a reasonable generalization from many available approaches. In the literature, we found several generalizations of FSs. The notion of the Bipolar-valued fuzzy set (BVFS) [23, 24] is picked to be studied in Algebra. Also, BVFS has been applied in several fields such as [19, 21, 36].

The pioneers of fuzzy mathematics, Rosenfeld [29] in algebra, made a great effort to overcome the difficulties arising caused by the lack of a fuzzy universal set. Rosenfeld discovered a suitable outlet to partially defeat this obstacle on fuzzy groupoid. He defined fuzzy subgroupoids of $\mho$, by using the ordinary binary operation of the assumed (ordinary) groupoid structure on the classical set $\mho$. In the generalization case, Dib [16] developed the concept of F-space to replace the concept of a universal set. Using this concept, the fuzzy group (FG) is constructed naturally. The fuzzy group [16, 17] is defined as a fuzzy binary operation on a F-space, satisfying the usual conditions of the group. Having defined the FG on the F-space $(\mho,\ [0,\ 1])$, the conditions on fuzzy subset A of $\mho$ to be a fuzzy subgroup were naturally deduced. Our objectives can be summarized by introducing and studying properties and algebraic structure on the concept of BVF-space and BVFBO to present a novel approach to study BVF-group theory comparability and parallel to the Dip approach. The method used in this research is to incorporate BVFS and FG by using the Dip approach. Our methodology started by collecting data and related works of BVFS [23, 24] and Dip's works [16, 17]. Then, we analyze these data by applying the concept of BVFS to the structure of FG. Lastly, our expected results are comprehensively compared with ordinary group and FG works.

The BVFS is considered as a generalization of FS. In 2000, Lee [23, 24] created the concept of bipolar valued fuzzy sets. BVFSs are an expansion of FSs whose membership degree range is expanded from the $[0,\ 1]$ to $[-1,\ 0] \times [0,\ 1]$. After that, Anitha et.al [12] created the bipolar valued fuzzy subgroups of a known group. bipolar valued fuzzy BCK/BCI-algebras was presented by Arsham [30]. Also, bipolar interval-valued fuzzy subgroups of a group were defined by Balasubramanian et.al [15]. In 2009, Lee [25] proposed and identified the bipolar-valued fuzzy subalgebras and bipolar-valued fuzzy ideals of BCK/BCI-algebras. Some properties of fuzzy groups were given by Mustafa [2]. Sahaya et.al., [31] initiated the bipolar valued Q-fuzzy subgroups of a group. Shanmugapriya & Arjunan [33] introduced some converters in bipolar valued fuzzy subsemirings of a semiring. Young and Song [22] presented ideas stuck on bipolar-valued fuzzy sets concerning the subalgebras and closed ideals of BCH-algebras. Recently, some researchers followed Rosenfeld's approach to introduce bipolar-valued fuzzy subgroups [12, 13]. Also, Al-Sharo [8] introduced $(\alpha_{1,2},\ \beta_{1,2})$-complex intuitionistic fuzzy subgroups and their algebraic structure. Manivannan et al. [14] introduced a new approach to complex fuzzy ideals in BCK/BCI-Algebras. In 2021, Abu-Hijleh et al. [1] introduced complex fuzzy groups based on Rosenfeld's approach.

Nevertheless, all mentioned scholars did not identify the notion of bipolar-valued fuzzy groupoid as Dib's approach, which is passed on BVF-space. Recently, the bipolar-valued fuzzy function [9] was prepared in terms of two special concepts, bipolar-valued fuzzy Cartesian product, and bipolar-valued fuzzy relation which were established depending on Dib's approach [17]. These results [9] build the basic structure to start our results and create the BVF-group theory. Also, some researchers followed the Dib approach and published several results in Algebra. In 2009, Fathi and Salleh [18] introduced an intuitionistic fuzzy group (IF-group) based on intuitionistic fuzzy space. The difference between the presented work and Fathi and Salleh's approach lies in the negative membership component. BVF-space has two components for each element, negative and positive membership values with range lies in the interval [-1, 0] and [0, 1], respectively. The IF-space has two components for each element, both membership and non-membership values with range lie in the unit interval [0, 1] with intuitionistic fuzzy restriction. In 2016, Alhusban and Salleh and Alhusban et al [3] generalized the fuzzy group to be under the complex numbers realm. They [3, 20] introduced the notion of complex fuzzy group and complex intuitionistic fuzzy group based on complex fuzzy space and complex intuitionistic fuzzy space, respectively. see [4, 5].

Mahmood et. al. [26] have collaborated on numerous publications regarding advanced fuzzy systems. A study from 2023 on bipolar complex fuzzy soft sets, which was applied to pattern recognition and medical diagnosis, was a prominent work. This article explored how trigonometric similarity measures can be utilized in practical situations with fuzzy systems, which corresponds to the decision-making components of our study. Furthermore, the scope of fuzzy algebra has been further broadened by studies like the Analysis of Γ-Semigroups Based on Bipolar Complex Fuzzy Sets [27]. Γ-semigroups, which extended semigroups by permitting the binary operation to be defined on a broader domain, have been examined within the framework of bipolar complex fuzzy sets, offering fresh perspectives on algebraic structures that involve both positive and negative interactions concurrently. This research and other studies are crucial for examining algebraic systems in complicated decision-making settings [6, 7, 10, 26–28].

The BVF-groups introduction holds wide-reaching consequences for multiple practical areas. When making decisions, systems frequently need to consider both favorable and unfavorable assessments, like trust as opposed to distrust, or approval as opposed to disapproval. BVF-groups offer a straightforward method to represent those situations by incorporating dual membership values in algebraic computations. Moreover, in areas like control systems and robotics, where uncertainties can have both positive and negative effects on system stability, BVF-groups can provide a more advanced approach to analysis and design. Moreover, in medical diagnosis, patient symptoms can positively or negatively impact health outcomes, and BVF-algebraic structures provide a more precise portrayal of diagnostic procedures. The BVF-group framework improves the theoretical understanding of fuzzy algebraic structures. Also, BVF-group enhances their practical use in solving complex, real-world problems.

The problem in bipolar-valued fuzzy algebra is essentially different. On the base set $\mho$, one should suppose a priori that the structure is an ordinary groupoid. There is no notion of a bipolar-valued fuzzy subgroupoid of $\mho$ rather than a concept of a bipolar-valued fuzzy groupoid on $\mho$. While the structure of the notion bipolar-valued fuzzy subgroupoid decreases to the ordinary structure in the classical case, the progress of bipolar valued fuzzy algebra is effectively slower than that of bipolar valued fuzzy topology due to the lack of an inherent definition of bipolar-valued fuzzy groupoid. Not because bipolar-valued fuzzy algebra is an impossible or difficult undertaking, but rather due to the lack of sufficient bipolar-valued fuzzy algebraic tools, many significant discoveries in (ordinary) algebra are not yet adapted to bipolar-valued fuzzy algebra. Therefore, Defining the bipolar-valued fuzzy group is not intuitive and not clear in the lack of a notion of a bipolar-valued fuzzy binary operation.

In this study, Dib's technique [16, 17] is utilized to verify a fundamental approach that does not consider any groupoid form on the basis set $\mho$. Dib's solution hinges on the notion of fuzzy binary operations, which we utilize here. The motivations of the presented approach are: (1) A more thorough examination of the characteristics and behaviors of classical (fuzzy) groups is made possible by the BVF-group, which offers a natural generalization of them, (2) It offers a way to measure this uncertainty (BVFS) in the wider context of the BVF group, (3) Because they offer a formalism for FS reasoning, approximation reasoning, and uncertainty management inside the BVFS framework, BVF-groups are essential to these applications. Therefore, upon establishing a bipolar-valued fuzzy binary operation on $\mho$, the notations of bipolar-valued fuzzy groupoid, subgroupoid, monoid, and other findings arise naturally and logically. Also, the definition of bipolar-valued fuzzy group is constructed and formalized. Some relations and results are explored and proved about BVF-group and intuitionistic fuzzy groups. Some theorems are introduced to support and prove that our results of BVF-groups are a generalization of classical (fuzzy) groups.

2. Preliminaries

In this part, we review key theorems and concepts connected to the current findings.

Definition 1. *[37] A fuzzy set A in universe $\mho$ is defined by a membership function* μ_A:$\mho\rightarrow[0,1]$, *indicating the degree of membership of x in A.*

Definition 2. *[24] An intuitionistic fuzzy set A in $\mho$ is defined by a membership function* $\mu_A:\mho\rightarrow[0,1]$ *and a non-membership function* ν_A:$\mho\rightarrow[0,1]$ *such that* $0\leq\ \mu_A\,(\mho)+\nu_A(\mho)\ \leq 1$ *for all $x\in\mho$.*

Definition 3. *[23] A bipolar-valued fuzzy set A in $\mho$ is defined by a membership function* μ_A:$\mho\rightarrow[-1,1]$, *which can represent positive and negative membership degrees:*

(i) $\mu_A\,(x)>0$*: The extent to which x is positively part of A.*

(ii) $\mu_A\,(x)<0$ *: The extent to which x is negatively part to A.*

Definition 4. *[33] A bipolar-valued fuzzy set A is deemed a bipolar-valued fuzzy subsemigroup in a semigroup S if* $\mu_A(x{\cdot}y) \geq \min(\mu_A(x), \mu_A(y))$ *for all* $x, y \in S$.
This guarantees that the characteristics of the subsemigroup are maintained within the fuzzy framework.

Definition 5. *[29] Rosenfeld expanded the idea of fuzzy sets to group theory through the introduction of fuzzy subgroups. A fuzzy subset A in group G is termed a fuzzy subgroup when:*

(i) $\mu_A(x{\cdot}y) \geq \min(\mu_A(x), \mu_A(y))$ *for all* $x, y \in G$,

(ii) $\mu_A(e) = 1$ *where e is the identity element of G,*

(iii) $\mu_A(x^{-1}) = \mu_A(x)$ *for all* $x \in G$.

These conditions ensure that the fuzziness respects the group structure.

Definition 6. *[17] A fuzzy relation R between sets* $\mho$ *and* Y *is a fuzzy set in the Cartesian product* $X \times Y$ *with a membership function* $\mu_R : \mho \times Y \to [0, 1]$.

Definition 7. *[17] A fuzzy function from a fuzzy set A in* $\mho$ *to a fuzzy set B in Y is a function* $f : \mho \to \mathrm{Y}$ *such that the membership value of* $f(x)$ *in B is related to the membership value of x in A.*

Definition 8. *[16] A F-space* $(\mho,\ I = [0, 1])$ *is the set of all ordered pairs* $(x,\ I),\ x \in \mho$ $(\mho,\ I) = \{(x,\ I) :\ x \in \mho\}$ *where* $(x,\ I) = \{(x, r) :\ r \in I\}$. *The ordered pair* $(x,\ I)$ *is called a fuzzy element in the F-space* $(\mho,\ I)$.

Definition 9. *[16] A fuzzy group* $((\mho,\ I), \underline{F})$ *is called a commutative or abelian fuzzy group if* $(x,\ I)\,\underline{F}\,(y,\ I) = (y,\ I)\,\underline{F}\,(x,\ I)$, *for all fuzzy elements* $(x,\ I)$ *and* $(y,\ I)$ *of the F-space* $(\mho,\ I)$. *It is clear that* $((\mho,\ I), \underline{F})$ *is a commutative fuzzy group iff* $(\mho, F)$ *is an ordinary commutative group.*

Definition 10. *[18] An intuitionistic fuzzy binary operation (IFBO)* $\boldsymbol{F}$ *on an intuitionistic fuzzy space (IF-space)* $(\mho,\ I,\ I)$ *is an intuitionistic fuzzy function* $\boldsymbol{F} : (\mho,\ I,\ I) \times (\mho,\ I,\ I) \longrightarrow (\mho,\ I,\ I)$ *with comembership functions* $\underline{f}_{xy}$ *and co-nonmembership functions* $\overline{f}_{xy}$ *satisfying:*

$$(1)\underline{f}_{xy}\ (r,\ s) \neq 0\ iff\ r \neq 0,\ s \neq 0\ and\ \overline{f}_{xy}(w,\ z) \neq 1\ iff\ w \neq 1,\ z \neq 1.$$

(2) $\underline{f}_{xy}$, $\overline{f}_{xy}$ *are onto. That is,* $\underline{f}_{xy}\ (I \times I) = I$ *and* $\overline{f}_{xy}(I \times I) = I$,
where $I = [0,\ 1]$,
Thus, the intuitionistic fuzzy binary operation $\boldsymbol{F} = (F,\ \underline{f}_{xy},\ \overline{f}_{xy})$ *over the IF-space X is defined by*
(3) $(x,\ I,\ I)\,\boldsymbol{F}\,(y,\ I,\ I) = \boldsymbol{F}((x,\ I,\ I),\ (y,\ I,\ I))$
$= (F(x,\ y),\ \underline{f}_{xy}\ (I \times I),\ \overline{f}_{xy}(I \times I)) = (F(x,\ y),\ I,\ I)$
where, $(x,\ I,\ I)$, $(y,\ I,\ I)$ *of the IF-space X are intuitionistic fuzzy elements (IF-element), and* $\boldsymbol{F} = (F,\ \underline{f}_{xy},\ \overline{f}_{xy})$ *is any IFBO defined on an IF-space* $\mho$.

An IFBO is identified to be uniform if both $\underline{f}_{xy}$ and $\overline{f}_{xy}$ are identical. That is, $\underline{f}_{xy} = \overline{f}_{xy} = f$ for all $x,\ y \in\ V$. A left uniform (right uniform) IFBO is IFBO having identical comembership functions (co-nonmembership functions).

Definition 11. *[18] The structure of* $((G,\ I,\ I),\ \boldsymbol{F})$*, where IF-space* G *and* $I = [0,\ 1]$*, with IFBO* $\boldsymbol{F}$ *defined on IF-space* G*, is called an intuitionistic fuzzy group (IFG) if the following conditions are fulfilled:*
(1) For any IF-element of $(x,\ I,\ I)$, $(y,\ I,\ I)$, $(z,\ I,\ I) \in ((G,\ I,\ I),\ \boldsymbol{F})$

$$((x,\ I,\ I)\,\boldsymbol{F}\,(y,\ I,\ I))\,\boldsymbol{F}\ \ (z,\ I,\ I) = (x,\ I,\ I)\,\boldsymbol{F}\ \ ((y,\ I,\ I)\,\boldsymbol{F}\,(z,\ I,\ I)).$$

(2) There exists an IF-element $(e,\ I,\ I) \in (G, I,\ I)$ *such that for all* $(x,\ I,\ I)$ *in* $((G,\ I,\ I),\ \boldsymbol{F})$:

$$(e,\ I,\ I)\,\boldsymbol{F}\,(x,\ I,\ I) = (x,\ I,\ I)\,\boldsymbol{F}\,(e,\ I,\ I) = (x,\ I,\ I).$$

(3) For every IF-element $(x,\ I,\ I)$ *in* $((G,\ I,\ I),\ \boldsymbol{F})$ *there exists an IF-element* $(x^{-1},\ I,\ I)$ *in* $((G,\ I,\ I),\ \boldsymbol{F})$ *such that:*

$$(x,\ I,\ I)\,\boldsymbol{F}\left(x^{-1},\ I,\ I\right) = \left(x^{-1},\ I,\ I\right)\boldsymbol{F}\,(x,\ I,\ I) = (e,\ I,\ I).$$

An IFG $((G,\ I,\ I),\ \boldsymbol{F})$ *is called an abelian IFG iff for all*
$(x,\ I,\ I),\ (y,\ I,\ I) \in\ ((G,\ I,\ I),\ \boldsymbol{F})$, *and*
$(x,\ I,\ I)\,\boldsymbol{F}\ \ (y,\ I,\ I) = (y,\ I,\ I)\,\boldsymbol{F}\ \ (x,\ I,\ I)$ *is true.*

Definition 12. *[9] The BVFCP of two ordinary sets* U *and* V*, denoted by* $U\overline{\times}V$*, is the collection of all K-BVF subsets of* $U{\times}V$ *that is* $U\overline{\times}V{=}K^{U\times V}$*, An element of* $U\overline{\times}V$ *is then a function* $M{:}U{\times}V{\to}K$*, or*

$$M{=}\{\ ((u,\ v)\,, [\ (\delta^{-},\ \delta^{+})\,, (\vartheta^{-},\ \vartheta^{+})\ ])\,{:}\,(u,\ v)\,{\in}U{\times}V,\ \ [\ (\delta^{-},\ \delta^{+})\,, (\vartheta^{-},\ \vartheta^{+})\ ]\ {=}M(u,\ v){\to}K\}.$$

The BVFCP of a BVF subset $H{=}\{(u, (\delta^{-},\ \delta^{+})\,)\}$ *of* U *and a BVF subset* $T{=}\ ((v, (\vartheta^{-},\ \vartheta^{+})\,)\}$ *of* V *is the K-BVF subset* $H\underline{\times}T$ *of* $U{\times}V$ *defined by:*
$H\underline{\times}T{=}\{\ ((u,\ v)\,,\ ((H^{-}\,(u)\,, H^{+}\,(u)\,),\ (T^{-}\,(v)\,, T^{+}\,(v)))\ {:}u{\in}U\ \ ,\ v{\in}V\}$
$\equiv\{((u,\ v),\ (\ (\delta^{-},\ \delta^{+})\,, (\vartheta^{-},\ \vartheta^{+})\,))\}$*. Therefore,* $H\underline{\times}T$ *is an element of* $U\overline{\times}V$, $\forall H{\in}W^{U}$ *and* $\forall T{\in}W^{V}$.

Definition 13. *[9] A BVFR* β *maps* U *to* V *is a subset of the BVFCP* $U\overline{\times}V$*. In other words,* β *is a member of K-BVF subsets* $M\ {:}U{\times}V{\to}\ K$*. A BVFR from* U *to* U *is said to be a BVFR in* U*.*

Definition 14. *[9] Let* β_1 *and* β_2*:* $U{\to}V$ *to* V *be two BVFRs. We call that* β_2 *is containing* β_1*, denoted by* $\beta_1{\subset}\beta_2$ *if and only if when* $((u,\ v),\ (\ (\delta^{-},\ \delta^{+})\,, (\vartheta^{-},\ \vartheta^{+})\,)){\in}H{\in}\beta_1$*, there exists* $B{\in}\beta_2$ *such that* $((u,\ v),\ (\ (\delta^{-},\ \delta^{+})\,, (\vartheta^{-},\ \vartheta^{+})\,)){\in}T{\in}\beta_2$*. If* $\beta_1{\subset}\beta_2$ *and* $\beta_2\ {\subset}\beta_1$*, then* β_1 *and* β_2 *are equal, that is* $\beta_1{=}\beta_2$ *.*

Definition 15. *[9] Let* $\beta{:}U{\to}V$ *be a BVFR. The inverse of* $\beta = \beta^{-1}{:}V{\to}U$ *is the BVFR defined by* $\beta^{-1}{=}\{\ M^{-1}\ :\ M\epsilon\beta\}$.

Definition 16. *[9] Let* $\beta{:}U{\to}V$ *and* $\gamma{:}V{\to}Z$ *be two BVFRs. The composition of* β *and* γ*, denoted* $\gamma\circ\beta{:}U{\to}Z$*, is a BVFR defined by*
$\gamma\circ\beta{=}\ \{((u,\ z),\ (\ (\delta^-,\ \delta^+)\ ,(\alpha^-,\ \alpha^+)\ ))\in M\ :M\in U\overline{\times}Z\}$. *Where a K-BVF subsets* $M\in U\overline{\times}Z$ *defined by:*

$((u,\ z),\ (\ (\delta^-,\ \delta^+)\ ,(\alpha^-,\ \alpha^+)\ ))\in M$ *if and only if* $\exists(v,\ (\vartheta^-,\ \vartheta^+)\ )\in V\times W$ *such that* $((u,\ v),\ \ (\ (\delta^-,\ \delta^+)\ ,(\vartheta^-,\ \vartheta^+)\ ))\in A$ *and* $((v,\ z),\ \ (\ (\vartheta^-,\ \vartheta^+)\ ,(\alpha^-,\ \alpha^+)\ ))\in B$ *for some* β *and* $B\in\gamma$.

Definition 17. *[9]Let* β *be a BVFR in* U*, i.e.,* $\beta\subset U\overline{\times}U$*. Then*
1. β *is called reflexive in* U *if and only if* $\forall u\in U$ *and* $\forall\,(\delta^-,\ \delta^+)\in W$*,* $\exists H\in\beta$ *such that* $((u,u),(\ (\delta^-,\ \delta^+)\ ,(\delta^-,\ \delta^+)\ ))\in H\in\beta$*, that is if and only if* $\Delta_U\subset\beta$.
2. β *is called symmetric if and only if whenever* $((u,v),(\ (\delta^-,\ \delta^+)\ ,(n^-,\ n^+)\ ))\in H\in\beta$*,* $\exists H\in\rho$ *such that* $((v,u),(\ (n^-,\ n^+)\ ,(\delta^-,\ \delta^+)\ ))\in T\in\beta$*, that is if and only if* $\beta^{-1}{=}\beta$.
3. β *is called transitive if and only if whenever* $((u,v),(\ (\delta^-,\ \delta^+)\ ,(\vartheta^-,\ \vartheta^+)\ ))\in H\in\beta$ *and* $((v,z)\ ,((\vartheta^-,\ \vartheta^+)\ ,(\alpha^-,\ \alpha^+)))\in T\in\beta$*,* $\exists C\in\beta$ *such that* $((u,z),(\ (\delta^-,\ \delta^+)\ ,(\alpha^-,\ \alpha^+)\ ))\in C\in\beta$*, that is if and only if* $\beta\circ\beta\subset\beta$.

A BVFR in U is called a BVFER in U if and only if it satisfies all three axioms above.

Definition 18. *[9] Let* U *and* V *be nonempty sets. A BVF function from* U *to* V *can be described as a function* $\mathbf{F}$ *from* W^U *to* W^V *characterized by the ordered pair* $\left(F,\left\{\left(f_u(\delta^-),\ \ f_u(\delta^+)\right)\right\}_{\mathrm{u}\in\mathrm{U}}\right)$*, where* $F{:}U{\to}V$ *is a function from* U *to* V *and* $\left\{\left(f_u(\delta^-),\ \ f_u(\delta^+)\right)\right\}_{\mathrm{u}\in\mathrm{U}}$ *is a family of functions* $\left(f_u(\delta^-),\ \ f_u(\delta^+)\right){:}W{\to}W$ *that satisfy the following conditions:*

(i) $f_u(\delta^-),\ \ f_u(\delta^+)$ *are nondecreasing on* W*, and*

(ii) $f_u\left(\delta^-{=}\ 0\right)=0=f_u\left(\delta^+{=}\ 0\right),\ f_u\left(\delta^-{=}\ -1\right)=-1,$ *and* $f_u(\delta^+{=}\ 1)=1.$

3. Bipolar Valued Fuzzy Space

The central result of this part is to generalize F-space to the BVF-space. This generalization can be done by enlarge the codomain of the membership for each element in the F-space from [0, 1] to [-1, 0]×[0, 1] in BVF-space. The concept of BVF-space is a replacement of universal set (F-space) in classical mathematics (fuzzy mathematics). Moreover, BVF-space is considered a cornerstone for the theory of bipolar valued fuzzy algebra.

Definition 19. *Let* $\mho$ *be a nonempty set. A BVF-space denoted by* $(\mho,\ [-1,0],\ [0,\ 1])$ *is a set of all triple elements on the form* $(x,\ [-1,0],\ [0,\ 1])$*, where* $(x,\ [-1,0],\ [0,\ 1]) = \{(x,\ n,\ m) : n\in[-1,\ 0]\,,\ m\in[0,\ 1]\,,\ \text{and}\ x\in\mho\}$*. The element* $(x,\ -I,\ I)$ *is called a bipolar valued fuzzy element (BVF-element) of the BVF-space* $(\mho,\ [-1,0],\ [0,\ 1])$.

For elements, the first component represents the conventional element, the second component represents the negative and the third component represents the positive membership values.

Support of BVFS B is a crisp set B_0 having elements with negative and positive values less than zero and greater than zero, respectively. that is $B_0 = \{x : \mu^+(x) > 0 \; and \; \mu^-(x) < 0\}$.

Definition 20. *suppose B_0 is the support of a specify bipolar valued fuzzy subset (BVF-subset) B of $\mho$. A bipolar valued fuzzy subspace (BVF-subspace) B of the BVF-space $(\mho, \; [-1, 0], \; [0, \; 1])$ is the collection of all elements $(x, \; b_x^-, \; b_x^+)$, where $x \in B_0$ and $b_x^-, \; b_x^+$ are subset of $[-1, 0]$ and $[0, 1]$ respectively, such that b_x^- and b_x^+ include the zero element with at least one more element. If $x \notin B_0$, then $b_x^- = 0, \; b_x^+ = 0$. The element $(x, \; b_x^-, \; b_x^+)$ is named a BVF-element of the BVF-subspace B. Also, the empty BVF-subspace indicated by ϕ is expressed as $\phi = \{(x, \; \phi_x^-, \; \phi_x^+) : x \notin \mathrm{B}_0 \}$.*

Example 1. *Consider $\mho$ is a BVF-space and suppose $B = (B^-(x), \; B^+(x))$ is a BVF-subset of $\mho$. The BVF-subset B induces some BVF-subspaces as follows:*

(i) BVF-subspace induced by B (Lower form):

$$S_l(B) = \{(x, \; [B^-(x), \; 0], [0, \; B^+(x)]) : x \in B_0 \}$$

(ii) BVF-subspace induced by B (Upper form)

$$S_u(B) = \{(x, \; [-1, \; B^-(x)] \cup \{0\}, \{0\} \cup [\; B^+(x), \; 1]) : x \in B_0 \}$$

(iii) BVF-subspace induced by B (Finite form):

$$S_0(B) = \{(x, \; \{B^-(x), 0\}, \{0, B^+(x)\}) : x \in B_0 \}.$$

Definition 21. *Let $B = \{(x, \; b_x^-, \; b_x^+) : x \in B_0\}$, and $C = \{(x, \; c_x^-, \; c_x^+) : x \in C_0\}$be two BVF-subspace of a BVF-space $\mho$. The union $B \cup C$ and the intersection $B \cap C$ are given respectively by:*

$$B \cup C = \{(x, b_x^- \cap c_x^-, b_x^+ \cup c_x^+ \;) : x \in B_0 \cup C_0\},$$
$$B \cap C = \{(x, b_x^- \cup c_x^-, b_x^+ \cap c_x^+ \;) : x \in B_0 \cap C_0\}.$$

Clearly, the union and intersection of any two BVF-subspaces of BVF-space $\mho$ is indeed a BVF-subspace of the BVF-space $\mho$.

Example 2. *Let $B = \{(x_1, \; -0.3, \; 0.6), (x_2, \; -0.8, \; 0.2) : x \in B_0\}$, and $C = \{(x_1, \; -0.5, \; 0.9), (x_2, \; -0.1, \; 0.5) : x \in C_0\}$ be two BVF-subspace of a BVF-space $\mho$. The union $B \cup C$ and the intersection $B \cap C$ are calculated, respectively as:*

$$B \cup C = \{(x_1, \; -0.5, \; 0.9), (x_2, \; -0.8, \; 0.5) : x \in B_0 \cup C_0\},$$

$$B \cap C = \{(x_1, \; -0.3, \; 0.6), (x_2, \; -0.1, \; 0.2) : x \in B_0 \cap C_0\}.$$

4. Discussion Bipolar Valued Fuzzy Group

The concept of BVFBO is established. This definition adds the negative comembership function to the structure of fuzzy function in Dib approach.

Definition 22. *An bipolar valued fuzzy binary operation* $\boldsymbol{F}$ *on an BVF-space* $(\mho,\ [-1,0],\ [0,\ 1])$ *is an bipolar valued fuzzy function* $\boldsymbol{F}\ :\ (\mho,[-1,0],\ [0,\ 1])\ \times\ (\mho,\ [-1,0],\ [0,\ 1])\ \longrightarrow\ (\mho,\ [-1,0],\ [0,\ 1])$ *with negative comembership functions* f_{xy}^{-} *and positive comembership functions* f_{xy}^{+} *satisfying:*

(i) $f_{xy}^{-}\ (n^{-},\ m^{-}) \neq 0\ iff\ n^{-} \neq 0,\ m^{-} \neq 0,\ f_{xy}^{-}\ (w^{-},\ z^{-}) \neq -1\ iff\ w^{-} \neq -1,\ z^{-} \neq -1$ *and* $f_{xy}^{+}\ (n^{+},\ m^{+}) \neq 0\quad iff\ n^{+} \neq 0,\ m^{+} \neq 0$ *, and*

$$f_{xy}^{+}\ \left(w^{+},\ z^{+}\right) \neq 1 \quad iff\ w^{+} \neq 1,\ z^{+} \neq 1.$$

(ii) f_{xy}^{-}*,* f_{xy}^{+} *are onto. That is,* $f_{xy}^{-}\ ([-1,0] \times [-1,0]) = [-1,0]$ *and* $f_{xy}^{+}\ ([0,1] \times [0,1]) = [0,1]$.

Thus for any two BVF-elements $(x,\ [-1,0],\ [0,\ 1])$, $(y,[-1,0],\ [0,\ 1])$ of the BVF-space $\mho$ and any BVFBO $\boldsymbol{F} = (F,\ f_{xy}^{-},\ f_{xy}^{+})$ defined on a BVF-space $\mho$, the action of the BVFBO $\boldsymbol{F} = (F,\ f_{xy}^{-},\ f_{xy}^{+})$ over the BVF-space $\mho$ is given by

$$\begin{aligned}(x,\ -I,\ I)\,\boldsymbol{F}\,(y,-\ I,\ I) &= \boldsymbol{F}\left((x,\ [-1,0],\ [0,\ 1])\,,\ (y,\ [-1,0],\ [0,\ 1])\right)\\ &= \left(F\left(x,\ y\right),\ f_{xy}^{-}\left([-1,0] \times [-1,0]\right),\ f_{xy}^{+}\left([0,1] \times [0,1]\right)\right)\\ &= \left(F\left(x,\ y\right),\ [-1,0],\ [0,\ 1]\right).\end{aligned}$$

Example 3. *Let* $(Q^{+},\ [-1,0],\ [0,\ 1])$ *be BVF-space with BVFBO* $\boldsymbol{F}$ *defined by* $F(x,\ y) = x/y,\ f_{xy}^{-}(n^{-},\ m^{-}) = \min\{n^{-},m^{-}\}$*, and* $f_{xy}^{+}(n^{+},\ m^{+}) = \max\{n^{+},m^{+}\}$. *Therefore* F *is a BVFBO on a BVF-space* $(Q^{+},\ [-1,0],\ [0,\ 1])$.

Example 4. *Let* $(\mho = \{h,\ k,\ l\},\ [-1,0],\ [0,\ 1])$ *be BVF-space with BVFBO* $\boldsymbol{F}$ *defined by Table 1.*

Table 1: BVFBO **F** defined on BVF-space $(\mho = \{h,\ k,\ l\},\ [-1,0],\ [0,\ 1])$.

$\boldsymbol{F}((x,\ -I,\ I),(y,\ -I,\ I))$	$(h,\ -0.3,\ 0.5)$	$(k,\ -1,\ 0.2)$	$(l,\ -0.8,\ 1)$
$(h,\ -0.3,\ 0.5)$	$(k,\ -0.3,\ 0.5)$	$(l,\ -1,\ 0.5)$	$(k,\ -0.8,\ 1)$
$(k,\ -1,\ 0.2)$	$(h,\ -1,\ 0.5)$	$(l,\ -1,\ 0.2)$	$(k,\ -1,\ 1)$
$(l,\ -0.8,\ 1)$	$(l,\ -0.8,\ 1)$	$(k,\ -1,\ 1)$	$(h,\ -0.8,\ 1)$

A BVFBO is called a uniform if the f_{xy}^{-}*, and* f_{xy}^{+} *are identical. That is,* $\left|f_{xy}^{-}\right| = f_{xy}^{+} = f$ *for all* $x,\ y \in \mho$*. A left uniform (right uniform) BVFBO is a BVFBO having identical negative comembership functions (positive comembership functions).*

Definition 23. *A bipolar valued fuzzy groupoid, denoted by* $((\mho, [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$*, is a BVF-space* $(\mho, [-1,0],\ [0,\ 1])$ *together with a BVFBO* $\boldsymbol{F}$ *defined over it. A uniform (left uniform, right uniform) bipolar valued fuzzy groupoid is a bipolar valued fuzzy groupoid with uniform (left uniform, right uniform) bipolar valued fuzzy binary operation.*

The following theorem establishes a relationship between bipolar valued fuzzy groupoids and ordinary (fuzzy) groupoids.

Theorem 1. *(1) Associated to each bipolar valued fuzzy groupoid* $(\mho,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *where* $\boldsymbol{F} = (F,\ f_{xy}^{-},\ f_{xy}^{+})$ *a fuzzy groupoid*$((\mho,\ [0,1])\,,\ \underline{F})$ *where* $\underline{F} = \left(F,\ f_{xy}^{+}\right)$ *which is isomorphic to the bipolar valued fuzzy groupoid* $(\mho,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *by the correspondence* $(x, -[-1,0],\ [0,\ 1])\ \leftrightarrow\ (x,\ [0,1])$.

(2) There is an associated (ordinary) groupoid $(\mho, F)$ *to any BVF-groupoid* $((\mho, [-1,0],\ [0,\ 1]), F)$ *that is isomorphic to the bipolar valued fuzzy groupoid via the corresponding* $(x, [-1,0],\ [0,\ 1]) \leftrightarrow x$.

Proof. (1) Let $((\mho, [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ be a given bipolar valued fuzzy groupoid. Now redefine $\boldsymbol{F} = \left(F,\ f_{xy}^{-},\ f_{xy}^{+}\right)$to be $\boldsymbol{F} = (F,\ f_{xy}^{+})$. Since f_{xy}^{+} meets the conditions of fuzzy Comembership function, $\boldsymbol{F} = (F,\ f_{xy}^{+})$ is a fuzzy binary operation. That is $((\mho,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F} = (F,\ f_{xy}^{+}))$ is a fuzzy groupoid.
(2) Again, consider the bipolar valued fuzzy groupoid $(\mho, [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$, Now using the isomorphism $(x,\ [-1,0],\ [0,\ 1]) \leftrightarrow x$ we can redefine the bipolar valued fuzzy function $\boldsymbol{F} = (F,\ f_{xy}^{-},\ f_{xy}^{+})$ to be $\boldsymbol{F} = F : \mho \times \mho \to \mho$. That is F defines an ordinary binary operation over $\mho$. Thus, $(\mho,\ F)$ is the associated ordinary groupoid.

Theorem 2. *Every intuitionistic fuzzy groupoid is a bipolar valued fuzzy groupoid, the inverse is not true.*

Proof.
Let $((\mho, [0,1],\ [0,\ 1]),\ \boldsymbol{F})$ be a given intuitionistic fuzzy groupoid. Now redefine $\boldsymbol{F} = \left(F,\ \underline{f_{xy}},\ \overline{f_{xy}}\right)$to be $\boldsymbol{F} = \left(F,\ f_{xy}^{-} = -(\overline{f_{xy}}),\ f_{xy}^{+} = \underline{f_{xy}}\right)$. Since f_{xy}^{-}, and f_{xy}^{+} satisfy the axioms of negative and positive bipolar valued fuzzy Comembership function, $\boldsymbol{F} = \left(F,\ f_{xy}^{-},\ f_{xy}^{+}\right)$ will be a BVFBO. That is $((\mho,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F} = (F,\ f_{xy}^{-}, f_{xy}^{+}))$ will define a bipolar valued fuzzy groupoid.
The invers is proved by counter example as follows:

Example 5 (Counter example)**.** *Let* $(6,\ -0.7,\ 0.5)$ *and* $(2,\ -0.8,\ 0.9)$ *be two BVF-elements in BVF-space*$(Q^{+},\ -I,\ I)$ *with BVFBO* $\boldsymbol{F}$ *defined as Example 3.*
Then $(6,\ -0.7,\ 0.5)\,\boldsymbol{F}(2,\ -0.8,\ 0.9) = (4,\ -0.8,\ 0.9)$. *Now, If we redefine Now redefine* $\boldsymbol{F} = \left(F,\ f_{xy}^{-}, f_{xy}^{+}\right)$*to be* $\boldsymbol{F} = \left(F,\ \overline{f_{xy}} = \left|f_{xy}^{-}\right|,\ \underline{f_{xy}} = f_{xy}^{+}\right)$, *this implies the element* $(6,\ 0.7,\ 0.5)\,\boldsymbol{F}(2,\ 0.8,\ 0.9) = (4,\ 0.8,\ 0.9)$ *which are not IF-element and not defined on IFBO and its not in IF-space because the sum of "0.7 and 0.5", "0.8 and 0.9", and "0.8 and 0.9" is not less than 1.*

Definition 24. *A bipolar valued fuzzy subgroupoid,* $(U;\ \boldsymbol{F})$*, of the BVF-groupoid* $((\mho,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *iff U is closed under the BVFBO* $\boldsymbol{F}$ *and U is a BVF-subspace of the BVF-space* $\mho$*.*

We may now develop the concept of bipolar valued fuzzy groupoid to bipolar valued fuzzy semi-groups and bipolar valued fuzzy monoids, just as we do with ordinary (fuzzy) groupoid.

Definition 25. *A bipolar valued fuzzy groupoid that is associative is called a bipolar-valued fuzzy semi-group. A bipolar valued fuzzy monoid is a bipolar valued fuzzy semi-group with existence of an identity.*

Now, we are able to introduce the concept of bipolar-valued fuzzy group by using the definitions 23, 24, and 25.

Definition 26. *For all BVF-elements have an inverse of a bipolar valued fuzzy monoid is called a bipolar valued fuzzy group. Equivalently, a bipolar valued fuzzy groupoid* $((G,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *is a BVF-group iff the following restrictions are hold:*
(l) For any BVF-elements,
$(x,[-1,0],\ [0,\ 1]),\ (y,[-1,0],\ [0,\ 1]),\ (z,\ [-1,0],\ [0,\ 1]) \in (G,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$*:*

$$((x,[-1,0]\,,\ [0,\ 1])\,\boldsymbol{F}\,(y,\ [-1,0]\,,\ [0,\ 1]))\,\boldsymbol{F}\ \ (z,\ [-1,0]\,,\ [0,\ 1])$$

$$= (x,\ [-1,0],\ [0,\ 1])\,\boldsymbol{F}\ \ ((y,[-1,0],\ [0,\ 1])\,\boldsymbol{F}\,(z,[-1,0],\ [0,\ 1]))\,.$$

(2) There exists a BVF-element $(e,[-1,0],\ [0,\ 1]) \in (G,\ [-1,0],\ [0,\ 1])$ *such that for all* $(x,\ [-1,0],\ [0,\ 1])$ *in* $((G,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *:*

$$(e,\ [-1,0]\,,\ [0,\ 1])\,\boldsymbol{F}\,(x,[-1,0]\,,\ [0,\ 1]) = (x,[-1,0]\,,\ [0,\ 1])\,\boldsymbol{F}\,(e,[-1,0]\,,\ [0,\ 1])$$

$$= (x,\ [-1,0],\ [0,\ 1])\,.$$

(3) For every BVF-element $(x,\ [-1,0]\,,\ [0,\ 1])$ *in* $((G,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *there exists a BVF-element* $(x^{-1},[-1,0],\ [0,\ 1])$ *in* $(G,[-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *such that:*

$$(x,[-1,0]\,,\ [0,\ 1])\,\boldsymbol{F}\left(x^{-1},[-1,0]\,,\ [0,\ 1]\right) = \left(x^{-1},\ [-1,0]\,,\ [0,\ 1]\right)\boldsymbol{F}\,(x,[-1,0]\,,\ [0,\ 1])$$

$$= (e,[-1,0],\ [0,\ 1])\,.$$

A BVF-group $((G,[-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ is named an abelian BVF-group if and only if for all $(x,[-1,0],\ [0,\ 1])\,,\ (y,\ [-1,0],\ [0,\ 1]) \in\ ((G,[-1,0],\ [0,\ 1])\,,\ \boldsymbol{F})\,.$

$$(x,[-1,0],\ [0,\ 1])\,\boldsymbol{F}\ \ (y,[-1,0],\ [0,\ 1]) = (y,\ [-1,0],\ [0,\ 1])\,\boldsymbol{F}\ \ (x,\ [-1,0],\ [0,\ 1])\,.$$

Identical to the bipolar valued fuzzy groupoid, the following theorem establishes a relationship between BVF-groups and both ordinary and fuzzy groups.

Theorem 3. *(1) Associated to each bipolar valued fuzzy group* $((G,[-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *where* $\boldsymbol{F} = (F,\ \ f_{xy}^{-},\ f_{xy}^{+})$ *a fuzzy group* $((G,\ [0,1]),\ \underline{F})$ *where* $\underline{F} = (F,\ f_{xy}^{+})$ *which is isomorphic to the bipolar valued fuzzy group* $(G,[-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *by the correspondence* $(x,\ [-1,0],\ [0,\ 1]) \leftrightarrow (x,\ [0,1])$.

(2) There is an associated (ordinary) group (G,F) *to any bipolar valued fuzzy group* $((G,[-1,0],\ [0,\ 1]),\boldsymbol{F})$ *that is isomorphic to the bipolar valued fuzzy group via the corresponding* $(x,\ [-1,0],\ [0,\ 1]) \leftrightarrow\ x$.

Proof. The proof is like Theorem 1

As a result of the prior theorems, the following corollary supplies an adequate and mandatory condition for a bipolar valued fuzzy group.

Corollary 1. *Let* $(\mho,\ [-1,0],\ [0,\ 1])$ *be an BVF-space and let* $\boldsymbol{F} = (F,\ \ f_{xy}^{-},\ f_{xy}^{+})$ *be an bipolar valued fuzzy binary operation defined over* $(\mho,\ [-1,0],\ [0,\ 1])$. *The algebraic structure* $((\mho,[-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *defines an BVF-group* iff $((\mho,\ [0,1]),\ \underline{F})$ *and* $((\mho,\ [0,1]),\ \overline{F})$ *are both fuzzy groups, where* $\underline{F} = \left(F,\ f_{xy}^{+}\right)$ *and* $\overline{F} = \left(F,\ \left|f_{xy}^{-}\right|\right)$.

Example 6. *Let* $G = \{b\}$ *be a singleton set. Define the BVFBO* $\boldsymbol{F} = (F,\ \ f_{xy}^{-},\ f_{xy}^{+})$ *over the BVF-space* $(G,[-1,0],\ [0,\ 1])$ *such that:*
$F(b,\ b)\ =\ b$ *and* $f_{bb}^{+}(n^{+},\ m^{+}) = n^{+} \wedge m^{+},\ \ f_{bb}^{-}(n^{-},\ m^{-}) = n^{-} \vee m^{-}$.
Thus, the BVF-space $(G,[-1,0],\ [0,\ 1])$ *together with* $\boldsymbol{F}$ *identify a trivial BVF-group* $(G,[-1,0],\ [0,\ 1]),\ \boldsymbol{F})$.

Example 7. *Let* $Z_5\ =\ \{0,\ 1,\ 2,\ 3,4\}$ *be a set. Define the BVFBO* $\boldsymbol{F} = (F,\ \ f_{xy}^{-},\ f_{xy}^{+})$ *over the BVF-space* $(Z_5,[-1,0],\ [0,\ 1])$ *as follows:*
$F(x,y) =\ x +_5\ y$, *where* $+_5$ *refers to addition modulo 5, and* $f_{xy}^{+}(n^{+},\ m^{+}) = n^{+}.m^{+}$, $f_{xy}^{-}(n^{-},\ m^{-}) = -(n^{-}.\ m^{-})$. *Thus* $((Z_5,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *is a BVF-group.*

The following theorem derives directly from Theorem 1 and the notion of a BVF-group.

Theorem 4. *For any BVF-group* $((G,[-1,0],\ [0,\ 1]),\ \boldsymbol{F})$*, the next statements are true:*

(i) The identity of element of BVF-group is unique.

(ii) The inverse of each BVF-element $(x,[-1,0],\ [0,\ 1])\ \in ((G,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$ *is unique.*

(iii) $\left((x^{-1})^{-1},\ [-1,0],\ [0,\ 1]\right) = (x,\ [-1,0],\ [0,\ 1])$

(iv) For all $(x,\ [-1,0],\ [0,\ 1]),\ (y,\ [-1,0],\ [0,\ 1]) \in ((G,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F}):$

$$((x,\ [-1,0],\ [0,\ 1])\,\boldsymbol{F}\,(y,\ [-1,0],\ [0,\ 1]))^{-1} = \left(y^{-1},[-1,0],\ [0,\ 1]\right)\boldsymbol{F}\left(x^{-1},[-1,0],\ [0,\ 1]\right).$$

(v) for all $(x,\ [-1,0],\ [0,\ 1]),(y,\ [-1,0],\ [0,\ 1]),(z,\ [-1,0],\ [0,\ 1]) \in\ ((G,\ [-1,0],\ [0,\ 1]),\ \boldsymbol{F})$. $If\ (x,\ [-1,0],\ [0,\ 1])\,\boldsymbol{F}\ (y,\ [-1,0],\ [0,\ 1])$, *and* $(z,\ [-1,0],\ [0,\ 1])\,\boldsymbol{F}\ (y,\ [-1,0],\ [0,\ 1])$, *then*

$$(x,\ [-1,0],\ [0,\ 1]) =\ (z,\ [-1,0],\ [0,\ 1]).$$

$If\ (y,\ [-1,0]\,,\ [0,\ 1])\,\boldsymbol{F}\ (x,\ [-1,0]\,,\ [0,\ 1]) = (y,\ [-1,0]\,,\ [0,\ 1])\,\boldsymbol{F}\,(z,\ [-1,0]\,,\ [0,\ 1])\,,$
then

$$(x,\ [-1,0],\ [0,\ 1])\ =\ (z,\ [-1,0],\ [0,\ 1]).$$

Proof. The proof is straightforward.

5. Conclusions

In this research, we investigated the expansions of fuzzy groups to a novel framework for BVF-groups. We extended traditional fuzzy set theory by introducing the BVF-space, which permits membership values to vary between $[-1,0] \times [0,1]$ instead of just $[0,1]$. The BVF-space provides a more thorough depiction of BVF-groups, acting as a substitute for the universal set in classical set theory. The incorporation of the BVFBO into this BVF-space allowed for the creation of BVF-groupoids, following the core ideas of classical groupoid and fuzzy groupoid theories, as outlined in Dib's method. These principles ensure that the fundamental algebraic structure of BVF-groupoids is preserved, despite the additional complexity of bipolar-valued membership. Our model tackles the difficulties brought about by the lack of a bipolar-valued fuzzy universal set, providing a strong base for additional algebraic structures like BVF-groupoids, BVF-monoids, and BVF-subgroups. This generalization expands the theoretical scope and improves the practical usefulness of fuzzy groups by incorporating both positive and negative membership values in real-world problems. The theoretical foundation of BVF-groups is well established, yet its practical implementation is still in its initial phases. Additional empirical research is required to assess the suitability and efficacy of the method in real-world scenarios like decision-making, control systems, and social networks. The complete effectiveness of the method will remain theoretical until additional real-world testing is carried out. As future research, the BVF-space and BVFBO offer a practical and efficient approach to defining and examining bipolar valued fuzzy subgroups, bipolar valued fuzzy normal subgroups and homeomorphism between two bipolar valued fuzzy algebraic structures. This development in BVF-group theory provides new possibilities for study and application in different fields that demand a thorough depiction of uncertainty and duality.

Author contributions

Fadi Al-Zu'bi Validated the research outputs and wrote the original draft of this research; Abd Ulazeez Alkouri and Fadi Prepared and created the published work by those from the original research group and helped to create the final form of this research. Abdul Ghaffur Ahmad: Scrutinized the formal analysis, methodology, ideas and formulated the overarching research goals and aims and supervised the presented research. Maslina Darua: Played the role of project administration and investigation in this research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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