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Modifications to Mixed $\theta(\nu_1, \nu_2)$-open Sets in Generalized Topological Spaces

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Abstract. Á. Császár and Makai Jr. [5] introduced the concepts of the mixed operation $\gamma_{\theta(\nu_1,\nu_2)}$ and mixed $\theta(\nu_1, \nu_2)$-open sets in generalized topological spaces. In this paper, we extend this framework by introducing the concepts of mixed operation $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$ and mixed $\tilde{\theta}(\nu_1, \nu_2)$-open sets (briefly, $\tilde{\theta}(\nu_1, \nu_2)$-open sets) and investigate their fundamental properties in generalized topological spaces. We explore the relationships among $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$, $\gamma_{\theta(\nu_1,\nu_2)}$, and $\gamma_\theta(\nu)$, as well as the relationships among $\tilde{\theta}(\nu_1, \nu_2)$-open sets, $\theta(\nu_1, \nu_2)$-open sets, and μ-open sets. Additionally, we introduce the notion of $G(\nu_1, \nu_2)$-regularity in generalized topological spaces. Finally, we provide characterizations of $\tilde{\theta}(\nu_1, \nu_2)$-open sets using mixed $G(\nu_1, \nu)$-regular concept.

2020 Mathematics Subject Classifications: 54A05, 54C08

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1. Introduction

Á. Császár [1] introduced the concepts of generalized topology and generalized open sets, as well as the interior and closure operators within generalized topological spaces.. For further details, see [1]. In the same work, he also introduced the notion of $\theta(\nu)$-open sets and investigated their properties. Similarly, in [7], the author defined a weaker form of $\theta(\nu)$-open sets called $\tilde{\theta}(\nu)$-open sets in generalized topological spaces. For additional details, see [6, 12].

Furthermore, in [5], Á. Császár and Makai Jr. modified the concept of $\theta(\nu)$-open sets by considering two generalized topologies ν_1 and ν_2 on a nonempty set X, introducing the notion of mixed $\theta(\nu_1, \nu_2)$-open sets (briefly, $\theta(\nu_1, \nu_2)$-open).

In our research, inspired by the approach in [4, 5], we extend the definitions of $\tilde{\theta}(\nu)$-open sets and the operation $\gamma_{\tilde{\theta}}(\nu)$ by considering a mixture of two generalized topologies ν_1 and ν_2. In Section 3, we introduce the mixed operation $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$ (briefly, $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$) and explore the relationships between this new operation and the operation $\gamma_{\theta(\nu_1,\nu_2)}$. Additionally,

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we establish sufficient conditions for equivalence between the operation $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$ and the previous operation $\gamma_{\theta(\nu_1,\nu_2)}$.

In Section 4, we define the class of mixed $\tilde{\theta}(\nu_1,\nu_2)$-open sets (briefly, $\tilde{\theta}(\nu_1,\nu_2)$-open sets) as a new category lying strictly between the class of ν_1-open sets and the class of $\theta(\nu_1,\nu_2)$-open sets. As the main results of this section, we introduce the concept of relative mixed $G(\nu_1,\nu_2)$-regular (briefly, $G(\nu_1,\nu_2)$-regularity) as a novel separation axiom in generalized topological spaces. Moreover, we provide a characterization of $G(\nu_1,\nu_2)$-regular spaces.

2. Preliminaries

Let X be a nonempty set and ν a collection of subsets of X. ν is defined as a Generalized Topology (GT) on X if it satisfies the following conditions:

(i) $\emptyset \in \nu$.

(ii) Any union of elements within ν is also an element of ν.

This concept was introduced by Á. Császár in [1]. We denote the pair (X,ν) as a Generalized Topological Space (GTS) on X. The subsets in ν are termed ν-open sets, and their complements are ν-closed sets, as defined in [2]. The union of all elements of ν is denoted by $\mathcal{M}_\nu$.

Additionally, a GTS (X,ν) is called strong [11] if $X \in \nu$.

For a subset A of a GTS (X,ν), the ν-closure of A, denoted $c_\nu(A)$, is defined as the intersection of all ν-closed sets containing A. The ν-interior of A, denoted $i_\nu(A)$, is defined as the union of all ν-open sets contained in A (see [1, 2]).

Recalling from [3], let ν be a GT on the nonempty set X, and $\mathcal{P}(X)$ denote the power set of X. Define $\theta(\nu) \subseteq \mathcal{P}(X)$ such that $A \in \theta(\nu)$ if for each $x \in A$, there exists $M \in \nu$ containing x with $M \subseteq c_\nu(M) \subseteq A$. Then $\theta(\nu)$ forms a GT on X, included in ν. The sets in $\theta(\nu)$ are known as $\theta(\nu)$-open sets, and their complements are referred to as $\theta(\nu)$-closed sets. The operation $\gamma_\theta : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined for $A \subseteq X$ by

$$\gamma_\theta(A) = \{x \in X : c_\nu(M) \cap A \neq \emptyset, \forall M \in \nu, x \in M\}.$$

In [7], Min extended this by defining $\tilde{\theta}(\nu) \subseteq \mathcal{P}(X)$ such that $A \in \tilde{\theta}(\nu)$ if for each $x \in A$, there exists $M \in \nu$ containing x with $M \subseteq c_\nu(M) \cap \mathcal{M}_\nu \subseteq A$. $\tilde{\theta}(\nu)$ is a GT on X, contained in ν, and $\theta(\nu) \subseteq \tilde{\theta}(\nu)$. The elements of $\tilde{\theta}(\nu)$ are referred to as $\tilde{\theta}(\nu)$-open sets, while their complements are known as $\tilde{\theta}(\nu)$-closed sets. The operation $\gamma_{\tilde{\theta}} : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined for $A \subseteq X$ by

$$\gamma_{\tilde{\theta}}(A) = \{x \in X : (c_\nu(M) \cap \mathcal{M}_\nu) \cap A \neq \emptyset, \forall M \in \nu, x \in M\}.$$

Furthermore, in [5], Á. Császár and Makai Jr. introduced $\theta(\nu_1,\nu_2)$ for combining two GTs ν_1 and ν_2 on X. A set $A \subseteq X$ belongs to $\theta(\nu_1,\nu_2)$ if $x \in A$ implies the existence of $M \in \nu_1$ with $x \in M \subseteq c_{\nu_2}(M) \subseteq A$. $\theta(\nu_1,\nu_2)$ is also a GT contained in ν_1 on X. The elements of $\theta(\nu_1, \nu_2)$ are called $\theta(\nu_1, \nu_2)$-open sets, and their complements are $\theta(\nu_1, \nu_2)$-closed sets. The operation $\gamma_{\theta(\nu_1,\nu_2)} : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined for $A \subseteq X$ by

$$\gamma_{\theta(\nu_1,\nu_2)}(A) = \{x \in X : c_{\nu_2}(M) \cap A \neq \emptyset, \forall M \in \nu_1, x \in M\}.$$

In conclusion, we revisit the following definitions and facts due to their significance in our paper's content.

Lemma 1. *[9] Let ν_1 and ν_2 be two GTs on a nonempty set X, and let $A \subseteq X$. If $A \in \nu_2$, then $\gamma_{\theta(\nu_1,\nu_2)}(A) = c_{\nu_1}(A)$.*

Definition 1. *[5] Let ν_1 and ν_2 be two GTs on a nonempty set X. A subset A of X is called (ν_1, ν_2)-regular-open if $A = i_{\nu_1}\big(c_{\nu_2}(A)\big)$.*

Theorem 1. *[5] Let ν_1 and ν_2 be two GTs on a nonempty set X, and let $A \subseteq X$. Then A is $\theta(\nu_1, \nu_2)$-closed if and only if $\gamma_{\theta(\nu_1,\nu_2)}(A) = A$.*

Definition 2. *[8] Let (X, ν) be a GTS. We say that X is G-regular with respect to $\mathcal{M}_\nu$ if, for every point $x \in \mathcal{M}_\nu$ and every ν-closed set F such that $x \notin F$, there exist sets U and V in ν satisfying the following conditions: $x \in U$, $F \cap \mathcal{M}_\nu \subseteq V$, and $U \cap V = \emptyset$.*

Definition 3. *[9] Let ν_1 and ν_2 be two GTs defined on a nonempty set X. We say that X is (ν_1, ν_2)-regular if, for every point $x \in X$ and every ν_1-closed set F with $x \notin F$, there exist open sets $U \in \nu_1$ and $V \in \nu_2$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$.*

3. Properties of the mixed operation $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$

We begin this section by introducing our primary Definition of the mixed operation $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$ and presenting intriguing results associated with it.

Definition 4. *Let ν_1 and ν_2 be two GTs defined on a nonempty set X, and let $A \subseteq X$. Define $\gamma_{\tilde{\theta}(\nu_1,\nu_2)} : \mathcal{P}(X) \to \mathcal{P}(X)$ as a mixed operation by:*

$$\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \{x \in X : (c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}) \cap A \neq \emptyset, \text{ for all } M \in \nu_1, x \in M\}.$$

If $x \in X - \mathcal{M}_{\nu_1}$, then by definition, $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

According to this definition, $x \notin \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ if and only if there exists $M \in \nu_1$ such that $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A = \emptyset$.

Remark 1. *Let ν be a GT on a nonempty set X. For any subset $A \subseteq X$, it holds that $\gamma_{\tilde{\theta}(\nu,\nu)}(A) = \gamma_{\tilde{\theta}(\nu)}(A)$.*

In Remark 1 above, for a strong GTS (X, ν), the following equality holds:

$$\gamma_{\tilde{\theta}(\nu,\nu)}(A) = \gamma_{\tilde{\theta}(\nu)}(A) = \gamma_{\theta(\nu)}(A).$$

Theorem 2. *Let ν_1 and ν_2 be two GT's on a nonempty set X. Then $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq \gamma_{\theta(\nu_1,\nu_2)}(A)$ for any $A \subseteq X$.*

Proof. Let $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ and $M \in \nu_1$ such that $x \in M$. Then $(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}) \cap A \neq \emptyset$. Since $(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}) \cap A \subseteq c_{\nu_2}(M) \cap A$, it follows that $c_{\nu_2}(M) \cap A \neq \emptyset$. Therefore, $x \in \gamma_{\theta(\nu_1,\nu_2)}(A)$.

The following example shows that generally $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \neq \gamma_{\theta(\nu_1,\nu_2)}(A)$.

Example 1. *Consider the set $X = \{a, b, c, d\}$ equipped with two generalized topologies: $\nu_1 = \{\emptyset, \{b, d\}\}$ and $\nu_2 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{a, c\}$.*

Observe the following:

$$c_{\nu_2}(\{b, d\}) = X \quad \textit{and} \quad \mathcal{M}_{\nu_1} = \{b, d\}.$$

Additionally,

$$c_{\nu_2}(\{b, d\}) \cap A \neq \emptyset \quad \textit{and} \quad \big(c_{\nu_2}(\{b, d\}) \cap \mathcal{M}_{\nu_1}\big) \cap A = \emptyset.$$

Therefore,

$$b, d \in \gamma_\theta(\nu_1, \nu_2)(A) \quad \textit{and} \quad b, d \notin \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A).$$

Thus,

$$\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \{a, c\} \quad \textit{and} \quad \gamma_\theta(\nu_1, \nu_2)(A) = X.$$

Consequently,

$$\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subset \gamma_\theta(\nu_1, \nu_2)(A).$$

Corollary 1. *Let ν_1 and ν_2 be two GT's on a nonempty set X and let $A \subseteq X$. If (X, ν_1) is a strong GTS, then $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \gamma_{\theta(\nu_1,\nu_2)}(A)$.*

Theorem 3. *Let ν_1 and ν_2 be two GT's on a nonempty set X and $A, B \subseteq X$. Then the operation $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$ has the following properties.*

(i) if $A \subseteq B$, then $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(B)$.

(ii) $A \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

(iii) if $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq A$, then $A = \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

Proof. (i) Let $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ and $M \in \nu_1$ such that $x \in M$. Then, $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A \neq \emptyset$. Since $A \subseteq B$, it follows that $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap B \neq \emptyset$, and hence $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(B)$.

(ii) Case 1: If $x \in A$ and $x \in \mathcal{M}_{\nu_1}$, then for each ν_1-open set M containing x, $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A \neq \emptyset$, so $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

Case 2: If $x \in A$ and $x \notin \mathcal{M}_{\nu_1}$, then by Definition 4, $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. Therefore, $A \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. From cases 1 and 2, we derive that $A \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

(iii) Let $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq A$. Then by (ii), $A \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. Hence, $A = \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

Theorem 4. *Let ν_1 and ν_2 be two GT's on a nonempty set X and let $A \subseteq X$. Then the following hold.*

(i) If $A \subseteq X - \mathcal{M}_{\nu_1}$, then $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = X - \mathcal{M}_{\nu_1}$.

(ii) $X - \mathcal{M}_{\nu_1} \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

Proof. (i) Let $A \subseteq X - \mathcal{M}_{\nu_1}$ and $x \in X - \mathcal{M}_{\nu_1}$. By Definition 4, $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ implies $X - \mathcal{M}_{\nu_1} \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

Conversely, if $x \in \mathcal{M}_{\nu_1}$, then for any $M \in \nu_1$ containing x, $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A = \emptyset$, hence $x \notin \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. This implies $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = X \setminus \mathcal{M}_{\nu_1}$.

(ii) This follows directly from the definition of the operation $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$.

Theorem 5. *Let ν_1 and ν_2 be two GT's on a nonempty set X and $A \subseteq X$. Then the following hold.*

(i) If $A \in \nu_1$, then $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \gamma_{\theta(\nu_1,\nu_2)}(A)$.

(ii) If $A \in \nu_2$, then $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \gamma_{\theta(\nu_1,\nu_2)}(A)$.

Proof. (i) This follows directly from Definition 4.

(ii) By Theorem 2, $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq \gamma_{\theta(\nu_1,\nu_2)}(A)$.

For the converse inclusion, let $x \in \gamma_{\theta(\nu_1,\nu_2)}(A)$ and $M \in \nu_1$ such that $x \in M$. Then $c_{\nu_2}(M) \cap A \neq \emptyset$. Hence, there exists $z \in c_{\nu_2}(M) \cap A$. Since A is a ν_2-open set containing z, it follows that $M \cap A \neq \emptyset$. As $M \cap A = (M \cap \mathcal{M}_{\nu_1}) \cap A$, we have $(M \cap \mathcal{M}_{\nu_1}) \cap A \neq \emptyset$. Thus, $(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}) \cap A \neq \emptyset$. This implies $\gamma_{\theta(\nu_1,\nu_2)}(A) \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. Finally, we conclude $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \gamma_{\theta(\nu_1,\nu_2)}(A)$.

Based on Lemma 1 and the implication (ii) from Theorem 5 above, we derive the following corollary.

Corollary 2. *Let ν_1 and ν_2 be two GT's on a nonempty set. If $A \in \nu_2$, then $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \gamma_{\theta(\nu_1,\nu_2)}(A) = c_{\nu_1}(A)$.*

Theorem 6. *Let ν_1 and ν_2 be two GT's on a nonempty set X and $A \subseteq X$. Then $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1})$*

Proof. Since $A \cap \mathcal{M}_{\nu_1} \subseteq A$, by Theorem 3(i), we have $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1}) \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

For the converse inclusion, suppose $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ and $M \in \nu_1$ contains x. Then $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A \neq \emptyset$. By the equality:

$$\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A = \big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap \big[(A \cap \mathcal{M}_{\nu_1}) \cup (A \cap (X - \mathcal{M}_{\nu_1}))\big],$$

it follows that $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap (A \cap \mathcal{M}_{\nu_1}) \neq \emptyset$. Hence, $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1})$, implying $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1})$.

Therefore, $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1})$, completing the proof.

Definition 5. *Let ν_1 and ν_2 be two GTs defined on a nonempty set X, and let $A \subseteq \mathcal{M}_{\nu_1}$. We define the restriction operation with respect to $\mathcal{M}_{\nu_1}$ as follows:*

$$(\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \{x \in \mathcal{M}_{\nu_1} : c_{\nu_2}(M) \cap A \neq \emptyset, \forall M \in \nu_1, x \in M\}.$$

The following lemma is crucial for proving the next theorem.

Lemma 2. *Let ν_1 and ν_2 be two GT's on a nonempty set X and $A \subseteq X$. Then*

$$\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = (X - \mathcal{M}_{\nu_1}) \cup (\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1}).$$

Proof. Let $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ and $M \in \nu_1$ such that $x \in M$. By the definition of $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$,

$$\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A \neq \emptyset$$

and

$$\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A = c_{\nu_2}(M) \cap \big(\mathcal{M}_{\nu_1} \cap A\big).$$

Since $\mathcal{M}_{\nu_1} \cap A \subseteq \mathcal{M}_{\nu_1}$, by Definition 5, $x \in (\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1})$, hence

$$\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq (\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1}).$$

Obviously,

$$\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq (X - \mathcal{M}_{\nu_1}) \cup (\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1}). \tag{1}$$

For the other inclusion, from Theorem 4(ii), $X - \mathcal{M}_{\nu_1} \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. Let $x \in (\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1})$. Then for each ν_1-open set M containing x,

$$c_{\nu_2}(M) \cap (\mathcal{M}_{\nu_1} \cap A) \neq \emptyset.$$

Since

$$c_{\nu_2}(M) \cap (\mathcal{M}_{\nu_1} \cap A) = \big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A,$$

it follows that $x \in \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ and thus

$$(\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1}) \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A).$$

Thus,

$$(X - \mathcal{M}_{\nu_1}) \cup (\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1}) \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A). \tag{2}$$

From equalities (1) and (2), we conclude that

$$\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = (X - \mathcal{M}_{\nu_1}) \cup (\gamma \mid_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1}).$$

Theorem 7. *Let ν_1 and ν_2 be two GT's on a nonempty set X, and let $A \subseteq X$. The following properties then hold:*

(i) $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(X) = X$.

(ii) $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(X - \mathcal{M}_{\nu_1}) = X - \mathcal{M}_{\nu_1}$.

(iii) $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(\emptyset) = X - \mathcal{M}_{\nu_1}$.

(iv) If $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = A$*, then* $X - \mathcal{M}_{\nu_1} \subseteq A$.

Proof. (i) By Theorem 3(ii), $X \subseteq \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(X)$, implying $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(X) = X$.

(ii) From Lemma 2, we obtain

$$\begin{aligned}\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(X - \mathcal{M}_{\nu_1}) &= (X - \mathcal{M}_{\nu_1}) \cup (\gamma\,|_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}\big((X - \mathcal{M}_{\nu_1}) \cap \mathcal{M}_{\nu_1}\big)\\ &= (X - \mathcal{M}_{\nu_1}) \cup (\gamma\,|_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(\emptyset)\\ &= (X - \mathcal{M}_{\nu_1}) \cup \emptyset\\ &= X - \mathcal{M}_{\nu_1}.\end{aligned}$$

(iii) This follows directly from Lemma 2 and Definition 5.

(iv) Let $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = A$. Then by Lemma 2,

$$\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = (X - \mathcal{M}_{\nu_1}) \cup (\gamma\,|_{\mathcal{M}_{\nu_1}})_{\tilde{\theta}(\nu_1,\nu_2)}(A \cap \mathcal{M}_{\nu_1}) = A,$$

which implies $X - \mathcal{M}_{\nu_1} \subseteq A$.

4. Mixed $\tilde{\theta}(\nu_1, \nu_2)$-Open Sets

Definition 6. *Let* ν_1 *and* ν_2 *be two GTs defined on a nonempty set* X*. A subset* A *of* X *is mixed* $\tilde{\theta}(\nu_1, \nu_2)$*-open (briefly,* $\tilde{\theta}(\nu_1, \nu_2)$*-open) if for every* $x \in A$*, there exists* $M \in \nu_1$ *such that* $x \in M$ *and* $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$*. The complement of a* $\tilde{\theta}(\nu_1, \nu_2)$*-open set is called a* $\tilde{\theta}(\nu_1, \nu_2)$*-closed set. The family of all* $\tilde{\theta}(\nu_1, \nu_2)$*-open sets in* X *is denoted by* $\tilde{\theta}(\nu_1, \nu_2)$.

Remark 2. *Let* ν *be a GT on a nonempty set* X*. Then every* $\tilde{\theta}(\nu, \nu)$*-open set in* X *is* $\tilde{\theta}(\nu)$*-open.*

Theorem 8. *Let* ν_1 *and* ν_2 *be two GT's on a nonempty set* X*. Then*

$$\theta(\nu_1, \nu_2) \subseteq \tilde{\theta}(\nu_1, \nu_2) \subseteq \nu_1.$$

Proof. To show $\theta(\nu_1, \nu_2) \subseteq \tilde{\theta}(\nu_1, \nu_2)$, take $A \in \theta(\nu_1, \nu_2)$ and $x \in A$. There exists $M \in \nu_1$ such that $M \subseteq c_{\nu_2}(M) \subseteq A$. Since $c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq c_{\nu_2}(M)$, we have $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$. Hence, A is $\tilde{\theta}(\nu_1, \nu_2)$-open, implying $A \in \tilde{\theta}(\nu_1, \nu_2)$.

Next, to show $\tilde{\theta}(\nu_1, \nu_2) \subseteq \nu_1$, suppose $A \in \tilde{\theta}(\nu_1, \nu_2)$ and $x \in A$. There exists $M \in \nu_1$ such that $x \in M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$. Therefore, $A = \bigcup_{x \in A} M_x \in \nu_1$.

Remark 3. *According to Theorem 8, the diagram below illustrates the relationship.*

$$\theta(\nu_1, \nu_2)\text{-open} \Longrightarrow \tilde{\theta}(\nu_1, \nu_2)\text{-open} \Longrightarrow \nu_1\text{-open}$$

The implications stated above do not work in reverse, as illustrated by the following example.

Example 2. *Let $X = \{a, b, c, d\}$. Consider two generalized topologies $\nu_1 = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\nu_2 = \{\emptyset, \{b, d\}\}$ on X. It can be verified that:*
The set $\{a, b, c\}$ is $\tilde{\theta}(\nu_1, \nu_2)$-open but not $\theta(\nu_1, \nu_2)$-open.
The set $\{a, b\}$ is ν_1-open but not $\tilde{\theta}(\nu_1, \nu_2)$-open.

Remark 4. *Let ν_1 and ν_2 be two GT's on a nonempty set X. If the GTS (X, ν_1) is strong, then $\tilde{\theta}(\nu_1, \nu_2) = \theta(\nu_1, \nu_2)$.*

Theorem 9. *Let ν_1 and ν_2 be two GT's on a nonempty set X. Then $\tilde{\theta}(\nu_1, \nu_2)$ is also a generalized topology on X contained in ν_1.*

Proof. It is evident that $\emptyset \in \tilde{\theta}(\nu_1, \nu_2)$. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of $\tilde{\theta}(\nu_1, \nu_2)$-open sets in X, and let $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$. There exists $\alpha_0 \in \Lambda$ such that $x \in A_{\alpha_0}$. Since A_{α_0} is $\tilde{\theta}(\nu_1, \nu_2)$-open, there exists $M \in \nu_1$ such that $x \in M$ and $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A_{\alpha_0} \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$. Therefore, $\bigcup_{\alpha\in\Lambda} A_\alpha$ is $\tilde{\theta}(\nu_1, \nu_2)$-open.

Theorem 10. *Let ν_1 and ν_2 be two GT's on a nonempty set X, and let $A \subseteq X$. Then A is $\tilde{\theta}(\nu_1, \nu_2)$-closed if and only if $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = A$.*

Proof. Let A be a $\tilde{\theta}(\nu_1, \nu_2)$-closed set. Assume $x \in X - A$. Then $X - A$ is $\tilde{\theta}(\nu_1, \nu_2)$-open. According to Definition 6, there exists $M \in \nu_1$ such that $x \in M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq X - A$. Hence, $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A = \emptyset$, implying $x \notin \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. Therefore, $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq A$. By Theorem 3 (ii), we conclude $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = A$.

Conversely, suppose $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = A$. If $x \in X - A$, then there exists $M \in \nu_1$ containing x such that $\big(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\big) \cap A = \emptyset$, implying $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq X - A$. Hence, $X - A$ is $\tilde{\theta}(\nu_1, \nu_2)$-open, showing A is $\tilde{\theta}(\nu_1, \nu_2)$-closed.

Based on (ii) of Theorems 5 and 10, the following corollary follows.

Corollary 3. *Let ν_1 and ν_2 be two topologies on a nonempty set X, and let $A \subseteq X$. If $A \in \nu_2$ and $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = A$, then A is $\theta(\nu_1, \nu_2)$-closed set.*

Theorem 11. *Let ν_1 and ν_2 be two GT's on a nonempty set X, and let $A \subseteq X$. If A is $\tilde{\theta}(\nu_1, \nu_2)$-open and $x \in A$, then there exists a (ν_1, ν_2)-regular-open set U containing x such that $U \subseteq c_{\nu_2}(U) \cap \mathcal{M}_{\nu_1} \subseteq A$.*

Proof. Let A be $\tilde{\theta}(\nu_1, \nu_2)$-open in X, and suppose $x \in A$. Thus, there exists a ν_1-open set M such that $x \in M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$. Define $U = i_{\nu_1}\big(c_{\nu_2}(M)\big)$. Then U is (ν_1, ν_2)-regular-open, with $M \subseteq U \subseteq c_{\nu_2}(U) = c_{\nu_2}\big(i_{\nu_1}\big(c_{\nu_2}(M)\big)\big) \subseteq c_{\nu_2}(M)$.

This implies $x \in M \subseteq U \subseteq c_{\nu_2}(U) \cap \mathcal{M}_{\nu_1} \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$. Therefore, $x \in U \subseteq c_{\nu_2}(U) \cap \mathcal{M}_{\nu_1} \subseteq A$ for some (ν_1, ν_2)-regular-open set U.

Since every set that is (ν_1, ν_2)-regular-open is ν_1-open in X, the following corollary is clearly derived.

Corollary 4. *Let ν_1 and ν_2 be two GT's on a nonempty set X, and let $A \subseteq X$. Then A is $\tilde{\theta}(\nu_1, \nu_2)$-open if and only if there exists a (ν_1, ν_2)-regular-open set U containing x such that $U \subseteq c_{\nu_2}(U) \cap \mathcal{M}_{\nu_1} \subseteq A$.*

Definition 7. *Let ν_1 and ν_2 be two GTs defined on a nonempty set X. We say that X is $G(\nu_1, \nu_2)$-regular with respect to $\mathcal{M}_{\nu_1}$ (or simply $G(\nu_1, \nu_2)$-regular) if, for every $x \in \mathcal{M}_{\nu_1}$ and every ν_1-closed set F with $x \notin F$, there exist open sets $U \in \nu_1$ and $V \in \nu_2$ such that:*

$$x \in U, \quad F \cap \mathcal{M}_{\nu_1} \subseteq V, \quad \text{and} \quad U \cap V = \emptyset.$$

Proposition 1. *Let ν_1 and ν_2 be two GT's on a nonempty set X such that $\nu_1 = \nu_2$. If X is (ν_1, ν_2)-regular, then X is either ν_1-regular or ν_2-regular.*

Proposition 2. *Let ν_1 and ν_2 be two GT's on a nonempty set X. If X is a (ν_1, ν_2)-regular, then X is $G(\nu_1, \nu_2)$-regular.*

Proof. Let X be (ν_1, ν_2)-regular. Take $x \in \mathcal{M}_{\nu_1}$ and consider any ν_1-closed set F such that $x \notin F$. By the definition of (ν_1, ν_2)-regularity, there exist $U \in \nu_1$, $V \in \nu_2$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$. Since $F \cap \mathcal{M}_{\nu_1} \subseteq F \subseteq V$, we conclude that X is $G(\nu_1, \nu_2)$-regular.

Theorem 12. *Let X be a nonempty set and ν_1, ν_2 be two GT's on X. The following statements are equivalent:*

(i) X is $G(\nu_1, \nu_2)$-regular.

(ii) For every $x \in X$ and every ν_1-open set U containing x, there exists a ν_1-open set V containing x such that $V \subseteq c_{\nu_2}(V) \cap \mathcal{M}_{\nu_1} \subseteq U$.

Proof. $(i) \Rightarrow (ii)$: Assume X is $G(\nu_1, \nu_2)$-regular. For $x \in \mathcal{M}_{\nu_1}$ and a ν_1-open set U containing x, there exist $V \in \nu_1$ and $W \in \nu_2$ such that $x \in V$, $(X - U) \cap \mathcal{M}_{\nu_1} \subseteq W$, and $V \subseteq X - W$. Since $X - W$ is ν_2-closed, $c_{\nu_2}(V) \subseteq X - W$. Hence, $c_{\nu_2}(V) \cap \big((X-U) \cap \mathcal{M}_{\nu_1}\big) \subseteq c_{\nu_2}(V) \cap W = \emptyset$, implying $V \subseteq c_{\nu_2}(V) \cap \mathcal{M}_{\nu_1} \subseteq U$.

$(ii) \Rightarrow (i)$: Let F be a ν_1-closed set and $x \in \mathcal{M}_{\nu_1}$ with $x \notin F$. Since $X - F$ is a ν_1-open set containing x, by hypothesis, there exists a ν_1-open set V containing x such that $x \in V \subseteq c_{\nu_2}(V) \cap \mathcal{M}_{\nu_1} \subseteq X - F$. This implies $c_{\nu_2}(V) \cap \mathcal{M}_{\nu_1} \cap F = \emptyset$. Hence, $F \cap \mathcal{M}_{\nu_1} \subseteq X - c_{\nu_2}(V)$, and since $X - c_{\nu_2}(V) \in \nu_2$ and $V \cap \big(X - c_{\nu_2}(V)\big) = \emptyset$, we conclude X is $G(\nu_1, \nu_2)$-regular.

Theorem 13. *Let ν_1 and ν_2 be two GT's on a nonempty set X. If X is $G(\nu_1, \nu_2)$-regular, then every ν_1-open set is $\tilde{\theta}(\nu_1, \nu_2)$-open.*

Proof. Let X be $G(\nu_1,\nu_2)$-regular, and consider any ν_1-open set A in X. For each $x \in A$, by Theorem 12, there exists a ν_1-open set V such that $x \in V \subseteq c_{\nu_2}(V) \cap \mathcal{M}_{\nu_1} \subseteq A$. Hence, A is $\tilde{\theta}(\nu_1,\nu_2)$-open.

Corollary 5. *Let ν_1 and ν_2 be twoGT's on a nonempty set X. If X is $G(\nu_1,\nu_2)$-regular, then $\nu_1 = \tilde{\theta}(\nu_1,\nu_2)$.*

Proof. It can be deduced from Theorem 8 and Theorem 13.

Definition 8. *Let ν_1 and ν_2 be two GTs defined on a nonempty set X, and let $A \subseteq X$. Define the following notions:*

$$c_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \bigcap \left\{ F \subseteq X \mid A \subseteq F \textit{ for } \tilde{\theta}(\nu_1,\nu_2)\textit{-closed set } F \textit{ in } X \right\};$$

$$i_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \bigcup \left\{ V \subseteq X \mid V \subseteq A \textit{ for } \tilde{\theta}(\nu_1,\nu_2)\textit{-open set } V \textit{ in } X \right\};$$

$$l_{\tilde{\theta}(\nu_1,\nu_2)}(A) = \{ x \in X \mid c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A \textit{ for some } \nu_1\textit{-open set } M \textit{ containing } x \}.$$

Note that $x \in c_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ if and only if $\forall U \in \tilde{\theta}(\nu_1,\nu_2)$, $(x \in U \Rightarrow U \cap A \neq \emptyset)$.

Theorem 14. *Let ν_1 and ν_2 be two GT's on a nonempty set X, and let $A \subseteq X$. Then the following hold:*

(i) $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq c_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq c_{\theta(\nu_1,\nu_2)}(A)$.

(ii) For any $x \in X$, $x \in l_{\tilde{\theta}(\nu_1,\nu_2)}(A)$ if and only if there exists a ν_1-open set M such that $x \in M$ and $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$.

Proof. (i) Let $x \notin c_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. This implies there exists a $\tilde{\theta}(\nu_1,\nu_2)$-open set V such that $x \in V$ and $V \cap A = \emptyset$. Since V is $\tilde{\theta}(\nu_1,\nu_2)$-open, there exists $M \in \nu_1$ such that $x \in M$ and $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq V \subseteq X - A$. This implies $\left(c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1}\right) \cap A = \emptyset$, hence $x \notin \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. Thus $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq c_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. Since every $\theta(\nu_1,\nu_2)$-open set in X is $\tilde{\theta}(\nu_1,\nu_2)$-open, it follows that $c_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq c_{\theta(\nu_1,\nu_2)}(A)$.

(ii) The proof is clear from the definition.

Corollary 6. *Let ν_1 and ν_2 be two generalized topologies on a nonempty set X, and let $A \subseteq X$. Then $i_{\tilde{\theta}(\nu_1,\nu_2)}(A) \subseteq l_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.*

Proof. Let $x \in i_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. This means there exists a $\tilde{\theta}(\nu_1,\nu_2)$-open set V in X containing x, such that $x \in V \subseteq A$. Since V is $\tilde{\theta}(\nu_1,\nu_2)$-open, there exists $M \in \nu_1$ such that $x \in M$ and $M \subseteq c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq V \subseteq A$. Therefore, $x \in l_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

Let ν_1 and ν_2 be two GT's on a nonempty set X. The notations are defined as follows: $l_{\theta(\nu_1,\nu_2)}(A) = \{x \in X : c_{\nu_2}(M) \subseteq A$ for some ν_1-open set M containing $x\}$; $l_{\tilde{\theta}(\nu_1)}(A) = \{x \in X : c_{\nu_1}(M) \cap \mathcal{M}_{\nu_1} \subseteq A$ for some ν_1-open set M containing $x\}$ [10].

Corollary 7. *Let ν_1 and ν_2 be two GT's on a nonempty set X. Then for any subset $A \subseteq X$,*

$$l_{\theta(\nu_1,\nu_2)}(A) \subseteq l_{\tilde{\theta}(\nu_1,\nu_2)}(A).$$

Proof. Let $x \in l_{\theta(\nu_1,\nu_2)}(A)$. This means there exists a ν_1-open set M containing x such that $c_{\nu_2}(M) \subseteq A$. Since $c_{\nu_2}(M) \cap \mathcal{M}_{\nu_1} \subseteq c_{\nu_2}(M) \subseteq A$, it follows that $x \in l_{\tilde{\theta}(\nu_1,\nu_2)}(A)$. Therefore, $l_{\theta(\nu_1,\nu_2)}(A) \subseteq l_{\tilde{\theta}(\nu_1,\nu_2)}(A)$.

Remark 5. *Let ν be a GT on a nonempty set X, and let $A \subseteq X$. Then $l_{\tilde{\theta}(\nu,\nu)}(A) = l_{\tilde{\theta}(\nu)}(A)$.*

Theorem 15. *Let ν_1 and ν_2 be two GTs on a nonempty set X and let $A \subseteq X$. Then the following properties hold:*

(i) $i_{\tilde{\theta}(\nu_1,\nu_2)}(A) = X - c_{\tilde{\theta}(\nu_1,\nu_2)}(X - A)$ *and* $c_{\tilde{\theta}(\nu_1,\nu_2)}(A) = X - i_{\tilde{\theta}(\nu_1,\nu_2)}(X - A)$.

(ii) $l_{\tilde{\theta}(\nu_1,\nu_2)}(A) = X - \gamma_{\tilde{\theta}(\nu_1,\nu_2)}(X - A)$ *and* $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}(A) = X - l_{\tilde{\theta}(\nu_1,\nu_2)}(X - A)$.

Proof. The proof is straightforward and hence omitted.

Conclusion

In this work, we have introduced and studied the operation $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$ and $\tilde{\theta}(\nu_1,\nu_2)$-open sets in generalized topological spaces. We have established several significant results concerning these concepts. The relationships among $\gamma_{\tilde{\theta}(\nu_1,\nu_2)}$, $\gamma_{\theta(\nu_1,\nu_2)}$, and $\gamma_\theta(\nu)$, as well as the relationships among $\tilde{\theta}(\nu_1,\nu_2)$-open sets, $\theta(\nu_1,\nu_2)$-open sets, and μ-open sets have been thoroughly investigated. Finally, we have derived various properties and characterizations in terms of the concept of $G(\nu_1,\nu_2)$-regularity.

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