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Bernstein Polynomials for Solving Fractional Differential Equations with Two Parameters

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Abstract. This work presents a general framework for solving generalized fractional differential equations based on operational matrices of the generalized Bernstein polynomials. This method effectively obtains approximate analytical solutions of many fractional differential equations. The generalized fractional derivative of the Caputo type with two parameters and its properties are studied. Using orthonormal Bernstein polynomials has led to the development of fractional polynomials, which offer an approximate solution for ordinary fractional differential equations. The approach employs the generalized orthogonal Bernstein polynomials (FOBPs) and constructs their operational matrices for fractional integration and derivative in the generalized Caputo sense to achieve this objective. Operational matrices convert ordinary differential equations into a system of algebraic equations, which can be solved using Newton's method. The convergence analysis and error estimate associated with the proposed problem have been investigated using the approximation of generalized orthogonal Bernstein polynomials (FOBPs). The effect of the new parameters of the fractional derivative is presented in several examples. Finally, several examples are included to clarify the proposed technique's validity, efficiency, and applicability via generalized orthogonal Bernstein polynomials (FOBPs) approximation.

2020 Mathematics Subject Classifications: 26A33, 11Cxx, 34A08

Key Words and Phrases: Generalized Fractional Derivative, Bernstein Polynomials, Operational Matrices, Riccati Equation

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1. Introduction

Differential equations are an essential tool for modeling several phenomena in sciences, engineers, and other fields [16]. Studying this topic will help the researchers to understand those models. Generalizing this topic from natural to fractional derivatives will help the

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researcher fit their actual data within the expected results using a nonstandard derivative [4].

In 1695, a new concept of differential equations appeared, fractional differential equations, and it became the subject of interest to many scientists. Several fractional derivative models have been introduced, such as Leibnitz, Euler, Fourier, Abel, Liouville, Riemann, and Hadamard [12, 13, 18, 19, 42]. The difference between them is the kernel that used [1, 2, 6, 11, 14]. The researchers have studied these new fractional operators by introducing new definitions and discussing their important properties. These definitions have applications in various fields, such as physics [3], dynamical systems, engineering [29], mechanics, signal processing, images, and control theory [34].

Definition 1.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$I^{\alpha}_{+a} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \qquad t > a \tag{1}$$

Using this definition of fractional integration, the Riemann-Liouville derivative and the Caputo fractional derivative of order $\alpha > 0$ are defined by:

$${}^{R}D^{\alpha}_{a} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \left(\int_a^t (t-s)^{m-\alpha-1} f(s)ds\right) \tag{2}$$

$${}^{C}D^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds \tag{3}$$

respectively, where $m-1 < \alpha \leq m$, and $m \in \mathbb{N}$. The Caputo definition is one of the most important definitions used to treat many physical problems related to fractional calculus, due to its properties similar to those of ordinary derivatives. The fractional integral of the fractional derivative is given by

$$I^{\alpha}_{a} D^{\alpha}_{a+} f(t) = f(t) - \sum_{n=0}^{m-1} \frac{f^{(k)}(a)}{k!} (t-a)^k, \qquad t > 0. \tag{4}$$

A generalized form of the fractional derivative is with two parameters recently presented by Udita N. Katugampola [23], and Odibat and Baleanu [41]. The new version of the derivative contains two parameters; thus, the solution of the fractional differential equation depends on those parameters. In this paper, we will use Odibat and Baleanu definitions.

Definition 1.2. *The generalized fractional derivative of a continues function f is $D^{\alpha,\rho}_{a+} f$ of order $\alpha > 0$, and $\rho > 0$ can be written as*

$$D^{\alpha,\rho}_{a+} f(t) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_a^t s^{\rho-1} (t^{\rho} - s^{\rho})^{m-\alpha-1} (s^{1-\rho} \frac{d}{ds})^m f(s)ds, \tag{5}$$

where $a > 0$, $\rho > 0$, $m-1 < \alpha \leq m$, and $m \in \mathbb{N}$.

This generalization has investigated several applications. For example, Kumar et al. [28] introduced the solution of the coronavirus disease model in Brazil via the new definition. The solution of influences infection dynamics for a butterfly pathogen and COVID-19 model restively are presented by Kumar and Erturk [20, 27] respectively. There is now no research paper handling the approximations of the solution using analytical methods.

One of the essential main branches of numerical analysis is approximation. Polynomials are useful mathematical tools that are easily defined, characterized, combined, derived, and grouped to form curves that can approximate any function to any required precision. Also, different bases can characterize the polynomials. Each basis type possesses unique strengths, including the monomial power, Jacobi, Bernstein, and Hermite forms. A wide range of problems can be effectively addressed by selecting the appropriate basis, and various complexities can be mitigated or eliminated. One of those polynomials is Bernstein polynomials, which are widely used to approximate the solutions of ordinary and fractional differential equations [32, 35, 39, 40]. The method assumes the solution can be approximated via a linear combination of Bernstein polynomials. Those polynomials have been used to solve several linear and nonlinear fractional differential equations such as Khan et al. [25] solved Brusselator system, Mirzaee, and Alipour [30] solved Volterra integro-differential equations and others [7–10, 17, 21, 22, 24, 33, 36–38, 43]. Up to now, Bernstein polynomials do not use to approximate the generalized Caputo-type fractional derivative.This work will build this algorithm. Unlike other methods, the algorithm does not need to select an initial guess like other methods or use help from other techniques. It can be used directly to solve the problem.

Finding the approximate analytic solution based on the orthogonal polynomials for FDE of generalized form will help the scientist fit their real data of the model with several parameters that make it easy to differentiate, integrate, and analyze the models. The main objective of this work is to build a convergent numerical algorithm for solving the fractional differential equation (FDE) with two parameters in terms of an analytic approach. In this paper, we utilized the operational matrix of FDEs to solve the fractional model of differential equations. By employing an operational matrix, we transform the fractional differential equation into a set of linear or nonlinear algebraic equations. Solving this system yields an approximate solution for the original equation.The algorithm can give the best approximation solution to linear and nonlinear FDE in two parameters, as increases the degree of the polynomial the solution converges to the exact one.

2. Bernstein Polynomials

Bernstein polynomials are one of the most important groups of polynomials because they have many properties such as continuity and base group unit of B polynomials over the period $[0, R]$. At $x = 0$ the bases of the Bernstein polynomial disappear except for the first polynomial, which is equal to 1, and at $x = R$ these polynomials disappear except for the last polynomial, which also equals 1 during the interval $[0, R]$. This property is important for increasing the flexibility to enforce boundary conditions at interval endpoints. It also ensures that the sum at any point x of all Bernstein polynomials is unity.

Definition 2.1. *The B-polynomial of n-th degree are defined on the interval* $[0,1]$ *as [15]*

$$B_{i,n}(t)=\binom{n}{i}(1-t)^{n-i}t^{i}, \tag{6}$$

Bernstein polynomials are considered positive, forming the unit for every real x that belongs to the period $[0,1]$, which can be proved easily. These polynomials can be written in terms of the linear combination of the basic functions by using the binomial expansion of $(1-t)^{n-i}$, as:

$$\begin{aligned} B_{i,n}(t) &= \tbinom{n}{i}(1-t)^{n-i}t^{i}, \\ &= \tbinom{n}{i}t^{i}\left(\sum_{k=0}^{n-i}(-1)^{k}\binom{n-i}{k}t^{k}\right), \\ &= \sum_{k=0}^{n-i}(-1)^{k}\binom{n}{i}\binom{n-i}{k}t^{k+i}, \qquad i=1,2,..,n. \end{aligned}$$

Definition 2.2. *The Orthonormal Bernstein polynomial (OBPs) on the interval* $[0,1]$ *of degree* n *in terms of original orthonormal Bernstein basis functions can be defined as [31]*

$$\phi_{i,n}(s)=\sqrt{2(n-i)+1}\sum_{k=0}^{i}(-1)^{k}\frac{\binom{2n+1-k}{i-k}\binom{i}{k}}{\binom{n-k}{i-k}}B_{i-k,n-k}(s), \tag{7}$$

where

$$B_{i,n}(s)=\sum_{k=0}^{n-i}(-1)^{k}\binom{n}{i}\binom{n-i}{k}s^{i+k}. \tag{8}$$

Definition 2.3. *Fractional Orthonormal Bernstein polynomials (FOBPs) are derived by transforming* s *to* t^{v}*, where* $(v>0)$*, using the foundation of OBPs. These FOBPs are represented as* $\phi_{i,n}^{v}(t)$ *for* $i=0,\ldots,n$*. The analytical expression of FOBPs can be obtained through the following formula [31]:*

$$\phi_{i,n}^{v}(t)=\sqrt{2(n-i)+1}(1-t^{v})^{n-i}\sum_{k=0}^{i}(-1)^{k}\binom{2n+1-k}{i-k}\binom{i}{k}t^{v(i-k)}, \tag{9}$$

for $i=0,1,\cdots,n$.

Now we define $\phi^{v}(t)$ be a set of FOBPs of degree n as:

$$\phi^{v}(t)=[\phi_{0,n}^{v}(t),\phi_{1,n}^{v}(t),\cdots,\phi_{n,n}^{v}(t)]^{T}, \tag{10}$$

where $\phi_{i,n}^{v}(t)$ for $i=0,\ldots,n$ are FOBPs defined in Equation (9), and its matrix form as:

$$\phi^{v}(t)=AT_{n}^{v}(t), \tag{11}$$

where

$$T_{n}^{v}(t)=[1,t^{v},t^{2v},\cdots,t^{nv}], \tag{12}$$

And $A = [a_{i,j}]$ is $(n+1) \times (n+1)$ matrix where its elements are

$$a_{i.j} = \sqrt{2(n-i)+1} \sum_{k=max\{0,j-n+i\}}^{min\{i,j\}} \mathfrak{m}_{i,j-k} \mathfrak{n}_{i,k}, \qquad i,j = 0,1,2,\cdots,n \tag{13}$$

Where $\mathfrak{m}_{i,j}$ and $\mathfrak{n}_{i,j}$ are defined as follows:

$$\begin{aligned} \mathfrak{m}_{i,j} &= (-1)^j \tbinom{n-i}{j}, & j = 0,1,\cdots,n-i, \\ \mathfrak{n}_{i,j} &= (-1)^{i-j} \tbinom{2n+1-i+j}{j} \tbinom{i}{i-j}, & j = 0,1,\cdots,i. \end{aligned} .$$

2.1. Approximation of Function for FOBPs

The set of Bernstein polynomials $\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\}$ in Hilbert space $L^2[0,1]$ is a complete basis [26]. Therefore, The set of FOBPs $\{\phi_{0,m}, \phi_{1,m}, \cdots, \phi_{m,m}\}$ is a complete basis in Hilbert space $L^2[0,1]$. So any function $f(t)$ in Hilbert space $L^2[0,1]$ can be represented by FOBPs as

$$f(t) \simeq \sum_{0}^{m} c_i \phi_{i,m}(t) = C^T \phi(t) \tag{14}$$

where $\phi^T(t) = [\phi_{0,m}, \phi_{1,m}, \cdots, \phi_{m,m}]$ and $C^T = [c_1, c_2, \cdots, c_m]$. The vector C can be obtained by

$$C^T \langle \phi^v(t), \phi^v(t) \rangle_{w(t)} = \langle f(t), \phi^v(t) \rangle_{w(t)}, \tag{15}$$

but $\phi(t)$ are orthogonal with respect to the weight function $w(t)$, so Equation (15) becomes

$$C^T = v \langle f(t), \phi^v(t) \rangle_{w(t)}. \tag{16}$$

3. generalized Caputo-type fractional derivatives

Definition 3.1. *The generalized fractional integral of a continues function f is $I^{\alpha,\rho}_{a+} f$ of order $\alpha > 0$, and $\rho > 0$ can be written as [41].*

$$I^{\alpha,\rho}_{a+} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s) ds. \tag{17}$$

Odibat and Baleanu were able to determine the relationship between the generalized fractional integral given in (17) and the given generalized fractional derivative given in (5) as [41]:

$$I^{\alpha,\rho}_{a^+} D^{\alpha,\rho}_{a+} f(t) = f(t) - \sum_{n=0}^{m-1} \frac{1}{\rho^n n!} (t^\rho - a^\rho)^n [(x^{1-\rho} \frac{d}{dx})^n f(x)]|_{x=a}, \tag{18}$$

where $0 < \alpha \leq m$, $\rho > 0$ and $a \geq 0$.

In the following, we will show some theorems and essential properties of generalized fractional derivatives and integrals with two parameters that we will need in the next section.

(i) $I_0^{\alpha,\rho} C = C \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} t^{\alpha\rho}$.

(ii) $I_0^{\alpha,\rho} t^{\beta} = \frac{\rho^{-\alpha}\Gamma(1+\frac{\beta}{\rho})}{\Gamma(1+\frac{\beta}{\rho}+\alpha)} t^{\beta+\alpha\rho}$.

(iii) $I_a^{\alpha,\rho}[Cf(t) + g(t)] = C I_a^{\alpha,\rho} f(t) + I_a^{\alpha,\rho} g(t)$.

(iv) $D_{a^+}^{\alpha,\rho} C = 0$.

(v) $D_0^{\alpha,\rho} t^{\beta} = \{ \begin{array}{cc} 0 & : \ \beta < m, \ m-1 \leq \alpha < m \\ \frac{\rho^{\alpha-m}\Gamma(\beta+1)\Gamma(\frac{\beta+\rho-m\rho}{\rho})}{\Gamma(\beta-m+1)\Gamma(m-\alpha+\frac{\beta+\rho-m\rho}{\rho})} t^{\beta-\alpha\rho} & : \text{otherwise} \end{array}$.

(vi) $D_a^{\alpha,\rho}[Cf(t) + g(t)] = C D_{a^+}^{\alpha,\rho} f(t) + D_{a^+}^{\alpha,\rho} g(t)$.

4. generalized operational matrix of orthogonal Bernstein polynomials

The main objective of this section is to derive the FOBPs operational matrices for generalized Caputo fractional derivatives. We derive the FOBPs operational matrices of fractional integration and derivative, the identities, and zero matrices of order $(n+1)$ are I and O, respectively.

4.1. Operational Matrix of Fractional Integration Based on FOBPs

The generalized fractional integral of $\phi^v(t)$ is defined as:

$$I^{\alpha,\rho}\phi^v(t) = I^{\alpha,\rho,v}\phi^v(t), \tag{19}$$

where $I^{\alpha,\rho,v}$ is called the operational matrix of generalized fractional integration of order $\alpha > 0, \rho > 0$. From Equation (11) and the properties of the operator $I^{\alpha,\rho}$ we get.

$$\begin{aligned} I^{\alpha,\rho}\phi^v(t) &= I^{\alpha,\rho} A T_n^v(t) = A I^{\alpha,\rho}(T_n^v(t)) = A[I^{\alpha,\rho}1, I^{\alpha,\rho}t^v, \cdots, I^{\alpha,\rho}t^{nv}]^T, \\ &= A[\frac{\rho^{-\alpha}\Gamma[1]}{\Gamma[1+\alpha]} t^{\alpha\rho}, \rho^{-\alpha}\frac{\Gamma[1+\frac{v}{\rho}]}{\Gamma[1+\alpha+\frac{v}{\rho}]} t^{v+\alpha\rho}, \cdots, \rho^{-\alpha}\frac{\Gamma[1+\frac{nv}{\rho}]}{\Gamma[1+\alpha+\frac{nv}{\rho}]} t^{nv+\alpha\rho}]^T, \\ &= AB\bar{T}_n^v(t). \end{aligned} \tag{20}$$

where $B = [b_{i,j}]$ is an $(n+1) \times (n+1)$ matrix, and for $i, j = 0, \ldots, n$, the $b_{i,j}$ is given by:

$$b_{ij} = \left\{ \begin{array}{ll} \rho^{-\alpha}\frac{\Gamma[1+\frac{iv}{\rho}]}{\Gamma[1+\alpha+\frac{iv}{\rho}]} & : \ i = j \\ 0 & : \text{otherwise} \end{array} \right. \tag{21}$$

And

$$\bar{T}_n^v(t) = [t^{\alpha\rho}, \cdots, t^{nv+\alpha\rho}]^T. \tag{22}$$

Now, we approximate $t^{iv+\alpha\rho}$ for $i = 0, 1, \ldots, n$, in terms of FOBPs as

$$t^{iv+\alpha\rho} \simeq E_i^T \phi^v(t), \tag{23}$$

where

$$\begin{aligned} E_i &= v \int_0^1 t^{iv+\alpha\rho} \phi^v(t) w(t) dt, \\ &= v \left[\int_0^1 t^{iv+\alpha\rho} \phi_{0,n}^v(t) t^{v-1} dt, \cdots, \int_0^1 t^{iv+\alpha\rho} \phi_{n,n}^v(t) t^{v-1} dt \right]^T. \end{aligned}$$

We will define a new matrix of order $(n+1) \times (n+1)$ denoted $E = [E_{ij}]$ where $E_{i,j}$ is given by:

$$E_{ij} = \sqrt{2(n-j)+1} \sum_{k=0}^{j} (-1)^k \binom{2n+1-k}{j-k} \binom{j}{k} \frac{\Gamma[i+j+\frac{\alpha\rho}{v}-k+1]\Gamma[n-j+1]}{\Gamma[i+n+\frac{\alpha\rho}{v}-k+2]}. \tag{24}$$

We have

$$I^{\alpha,\rho} \phi^v(t) = ABE^T \phi^v(t) \tag{25}$$

So, $I^{\alpha,\rho,v} = ABE^T$.

4.2. Operational Matrix of Fractional Differentiation Based on FOBPs

The generalized fractional differential of $\phi^v(t)$ is defined as:

$$D_t^{\alpha,\rho} \phi^v(t) = D^{\alpha,\rho,v} \phi^v(t). \tag{26}$$

Where $D^{\alpha,\rho,v}$ is called the operational matrix of generalized fractional differential of order $\alpha > 0$, $\rho > 0$. From Equation (11) and the properties of the operator $D_t^{\alpha,\rho}$ we get.

$$\begin{aligned} D_t^{\alpha,\rho} \phi^v(t) &= D_t^{\alpha,\rho} A T_n^v(t) = A D_t^{\alpha,\rho}(T_n^v(t)) = A[D_t^{\alpha,\rho} 1, D_t^{\alpha,\rho} t^v, \cdots, D_t^{\alpha,\rho} t^{nv}]^T, \\ &= A[0, \rho^{-1+\alpha} \tfrac{\Gamma[1+v]\Gamma[\frac{v+\rho-1\rho}{\rho}]}{\Gamma[v]\Gamma[1-\alpha+\frac{v}{\rho}]} t^{v-\alpha\rho}, \cdots, \rho^{-1+\alpha} \tfrac{\Gamma[1+nv]\Gamma[\frac{nv}{\rho}]}{\Gamma[n]\Gamma[1-\alpha+\frac{n}{\rho}]} t^{nv-\alpha\rho}]^T, \qquad (27) \\ &= AB\bar{T}_n^v(t). \end{aligned}$$

where $B = [b_{i,j}]$ is an $(n+1) \times (n+1)$ matrix, and for $i, j = 0, 1, \ldots, n$, the $b_{i,j}$ is given by:

$$b_{ij} = \begin{cases} \rho^{-1+\alpha} \frac{\Gamma[1+iv]\Gamma[\frac{iv}{\rho}]}{\Gamma[iv]\Gamma[1-\alpha+\frac{i}{\rho}]} & : \ i = j \neq 0 \\ 0 & : \text{otherwise} \end{cases} \tag{28}$$

And

$$\bar{T}_n^v(t) = [0, t^{v-\alpha\rho}, \cdots, t^{nv-\alpha\rho}]^T. \tag{29}$$

Now, we approximate $t^{iv-\alpha\rho}$ for $i = 1, \dots, n$, in terms of FOBPs as

$$t^{iv-\alpha\rho} \simeq \bar{E}_i^T \phi^v(t), \tag{30}$$

where

$$\begin{aligned} \bar{E}_i &= v \int_0^1 t^{iv-\alpha\rho} \phi^v(t) w(t) dt, \\ &= v \left[\int_0^1 t^{iv-\alpha\rho} \phi_{0,n}^v(t) t^{v-1} dt, \cdots, \int_0^1 t^{iv-\alpha\rho} \phi_{n,n}^v(t) t^{v-1} dt \right]^T. \end{aligned}$$

Now, $\bar{E}$ can be written as $(n+1) \times (n+1)$ matrix in the form $\bar{E} = [\bar{E}_{ij}]$ where:

$$\bar{E}_{ij} = \begin{cases} 0 & : i = j = 0 \\ \sqrt{2(n-j)+1} \sum_{k=0}^{j} (-1)^k \binom{2n+1-k}{j-k} \binom{j}{k} \frac{\Gamma[i+j-\frac{\alpha\rho}{v}-k+1]\Gamma[n-j+1]}{\Gamma[i+n-\frac{\alpha\rho}{v}-k+2]} & : \text{otherwise} \end{cases} \tag{31}$$

Then,

$$D_t^{\alpha,\rho} \phi^v(t) = AB\bar{E}^T \phi^v(t), \tag{32}$$

and $D^{\alpha,\rho,v} = AB\bar{E}^T$. So, The generation of the above formula can be given by diagram 1

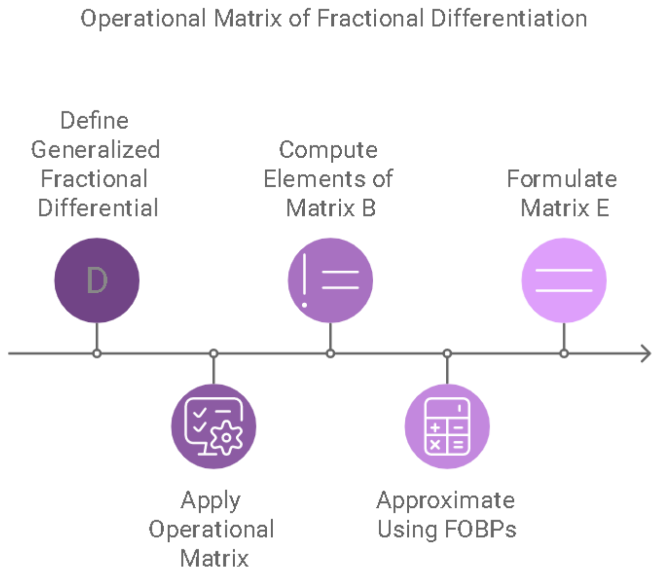

Figure 1: Operational Matrix of Fractional Differentiation.

5. Convergence Analysis and Error Estimate

In this section, we provide the convergence theorem of the method based on some existing results in [5, 30].

Theorem 5.1. *[5] Let $f : [0,1] \to \mathbb{R}$ such that $f \in C^{n+1}[0,1]$, and $s_n = span\{\phi^v_{0,n}, \phi^v_{1,n}, \cdots, \phi^v_{n,n}\}$, now if $K^T\phi^v$ be the best approximation f out of S_n then*

$$||f - K^T\phi^v||_2 \leq \frac{\bar{C}}{(n+1)!^2\sqrt{2n+1}}, \tag{33}$$

where

$$\bar{C} = \max_{t\in[0,1]} |f^{(n+1)}(t)|. \tag{34}$$

6. Examples.

Example 1. *Consider the following linear ordinary differential equation on* $[0,1]$.

$$D^{\alpha,\rho}_t u(t) = u(t) + t, \qquad u(0) = 1, \qquad 0 < \alpha \leq 1, \rho > 0 \tag{35}$$

When $\alpha = \rho = 1$, the equation has exact solution $u(t) = 2e^{2t} - t - 1$.

First suppose that $u(t) = K^T\phi^v(t)$, and $D^{\alpha,\rho}_0 u(t) = K^T D^{\alpha,\rho}\phi^v(t)$, next write t in term of 1D-FOBPs as $t = c^T\phi^v(t)$,where

$$c^T = v\int_0^1 t\phi^v(t)^T dt. \tag{36}$$

We can construct n linear equation using

$$\int_0^1 (K^T D^{\alpha,\rho} - K^T - c^T)\phi^v(t)t^{i+1}dt = 0, \qquad i = 0, 1, \cdots n-1. \tag{37}$$

Apply the initial condition $u(0) = 1$ to get

$$K^T\phi^v(0) = 1. \tag{38}$$

Thus, we have $(n+1)$ equations for $(n+1)$ unknown variables of the vector K. After solving the linear system, we can calculate the approximation solution $u(t)$.

Figure 2. Shows the exact solution and the approximate solutions of Example 1 for $\alpha, \rho = 1$, $v = 1$, and $n = 6$. The absolute error between the exact solution and approximation solutions of Example 1 when $n = 6$, with $\alpha, \rho =$, and $v = 1$ is plotted in Figure 3. Of course, by increasing the value of n of FOBPs, the approximate values of $u(t)$ converge to the exact solutions. Figure 4. Present the solution when $n = 6$, $\rho = 1$, $v = 1$, and different values of α. The effect of ρ is presented in Figure **??**. *The solution does not depend only on α but also on ρ. The solution for different values of α and ρ is presented in Figure*

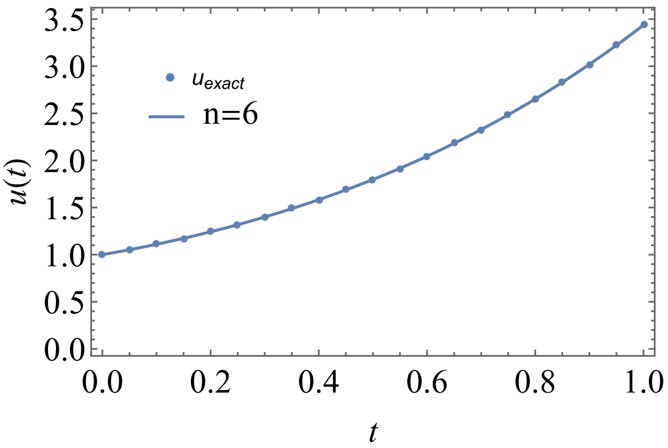

Figure 2: The exact solution and approximation solutions for Example 1 when $n = 6$, with $\alpha = 1$, $\rho = 1$, and $v = 1$.

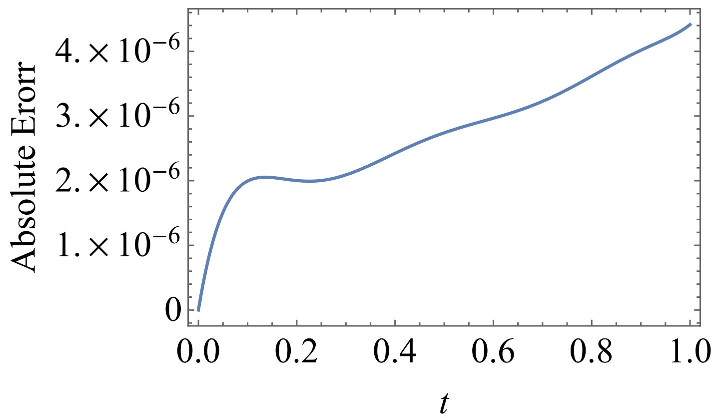

Figure 3: The absolute error between the exact solution and approximation solutions, for example, 1 when $\alpha = \rho = 1$, and $v = 1$, for $n = 6$.

6. It is worth mentioning that the solution is changed for every single value of α and ρ. Table 1 shown approximate solutions of example 1 when $t = 1$, $v = 1$, and different value of α, ρ. Finally, we note that the CPU time for this example is 2.372s for $n = 6$ using Mathematica software.

Example 2. *Consider the following nonlinear Riccati Equation on [0,1], with a given initial condition*

$$D_0^{\alpha,\rho}u(t) = 2u(t) - u^2(t) + 1, \qquad u(0) = 0, \qquad 0 < \alpha \leq 1, \rho > 0. \tag{39}$$

The exact solution when $\alpha, \rho = 1$ is

$$u(t) = \frac{e^{2\sqrt{2}t} - 1}{-e^{2\sqrt{2}t} + \sqrt{2}e^{2\sqrt{2}t} + 1 + \sqrt{2}}. \tag{40}$$

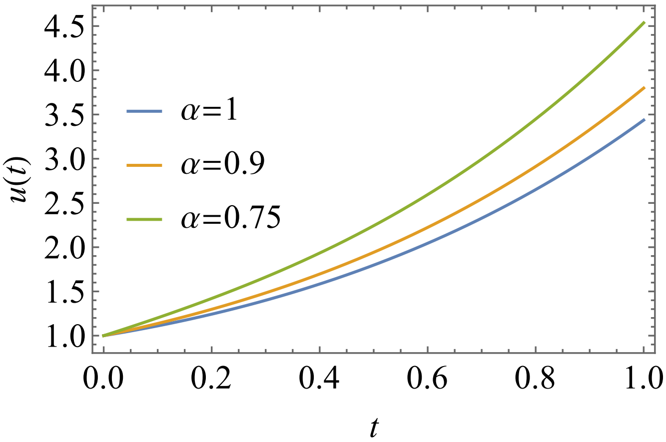

Figure 4: The approximate solution for Example 1 for fixed $n = 6$, $v = 1$, $\rho = 1$, and vary α.

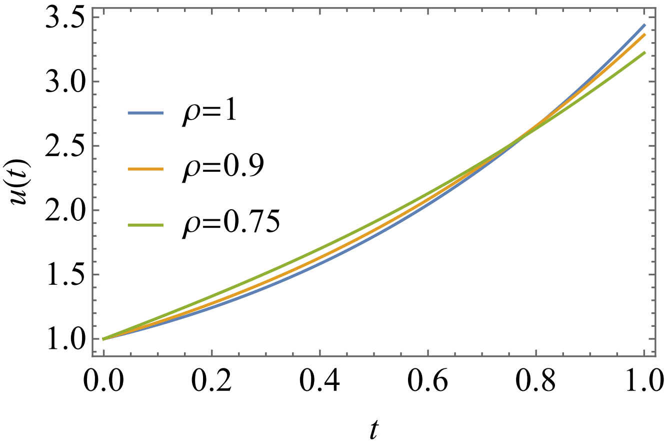

Figure 5: The approximate solutions for Example 1 for fixed $n = 6$, $\alpha = 1$, $v = 1$, and vary ρ.

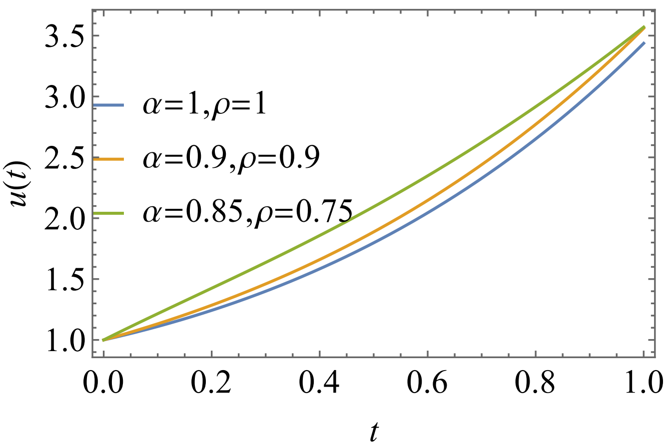

Figure 6: The approximation solutions for Example 1 for fixed $n = 6$, $v = 1$, and vary α, ρ.

Table 1: Approximate solutions of Example 1 when $t = 1$, $v = 1$, and different value of α, ρ

n	$\alpha = 1,\rho = 1$	$\alpha = 1,\rho = 0.9$	$\alpha = 0.95,\rho = 0.75$	$\alpha = 0.9,\rho = 1.2$
5	3.43666	3.35715	3.31114	4.02278
6	3.43656	3.36259	3.32984	4.01898
7	3.43656	3.36669	3.34406	4.01691
8	3.43656	3.36986	3.35519	4.01535

Now, we will Suppose that $u(t) = K^T\phi^v(t)$, and $D_0^{\alpha,\rho}u(t) = K^TD^{\alpha,\rho,v}\phi^v(t)$, next we write 1 in term of FOBPs as $1 = c^T\phi^v(t)$, where

$$c^T = v\int_0^1 \phi^v(t)dt. \tag{41}$$

Substitute these assumptions into the Equation 39. Thus we get

$$K^TD^{\alpha,\rho,v}\phi^v(t) - 2K^T\phi^v(t) + (K^T\phi^V(t))^2 - C^T\phi^v(t) = 0. \tag{42}$$

Construct the n equation as

$$\int_0^1 [K^TD^{\alpha,\rho,v}\phi^v(t) - 2K^T\phi^v(t) + (K^T\phi^V(t))^2 - C^T\phi^v(t)] \sqrt[i+1]{t} = 0, \qquad i = 1, 2, \cdots, n, \tag{43}$$

the initial condition gives

$$K^T\phi^v(0) = 0. \tag{44}$$

Thus, we have $(n+1)$ nonlinear equation for $(n+1)$ unknown variable of vector K. After solving the nonlinear system, we can calculate the approximation solution.

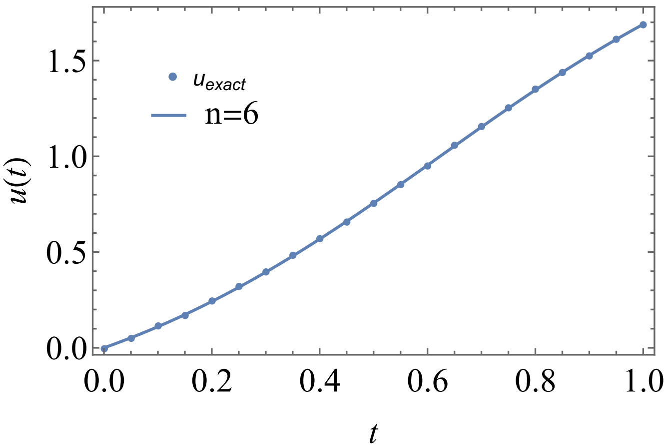

Figure 7: The exact solution and approximation solutions for Example 2 when $n = 6$, with $\alpha = 1$, $\rho = 1$, and $v = 1$.

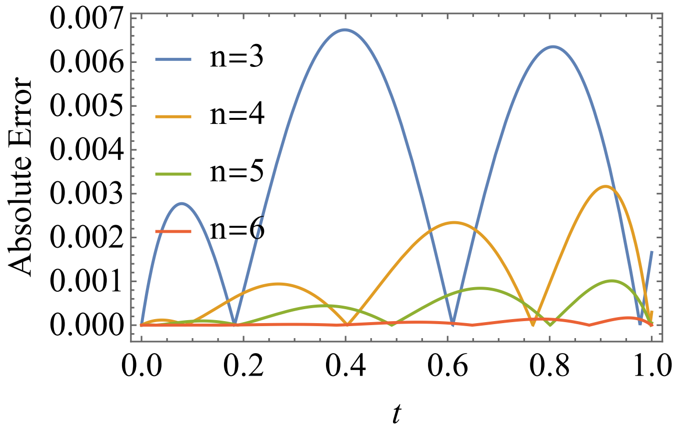

Figure 8: The absolute error between the exact solution and approximate solution, for example, 2 when $\alpha = 1$, $\rho = 1$, $v = 1$ and vary n.

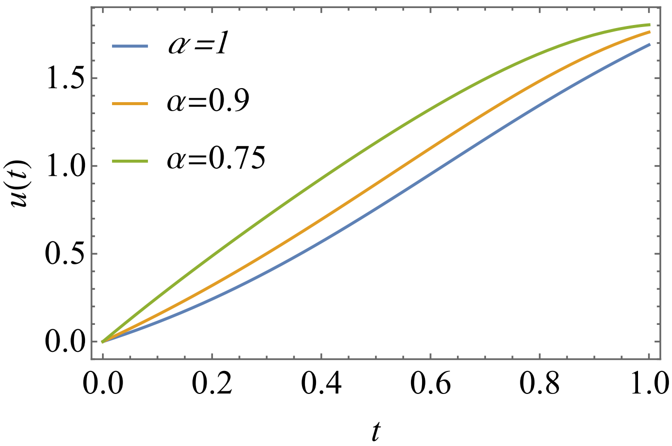

Figure 9: The approximation solutions for Example 2 for fixed $n = 4$, $\rho = 1$, and $v = 1$ and vary α.

Figures 7 show the exact solution together with the approximate solution, for example, 2 when $n = 6$, and the absolute error between them for $\alpha, \rho = 1$, $v = 1$, and different values of n are shown in Figures 8, We note, by increasing the value of n of FOBPs, the approximate solution of $u(t)$ converges to the exact solutions.

In Figure 9, we plot the approximate solution, for example, 2 when $n = 4$, $\rho = 1$, $v = 1$, and different values of α. As α approaches 1, we note that the approximate solution converges to the exact solution. The approximate solution for example 2 when $n = 4$, $\alpha = 1$, $v = 1$, and different value of ρ are shown in Figure 10. The solution does not depend only on α but also on ρ. The effect of changing two values of α, ρ for changing an approximate solution for example 2 is presented in figure 11, and we note that as α, ρ approaches 1, the numerical solution converges to the exact solution. Table 2 shows Approximate solutions of fractional Riccati equation when $t = 1$, $v = 1$, and different values of α, ρ. It is clear that the solution behaviors depend on the two fractional parameters which give the scientists a benefit in choosing which one can fit the real data

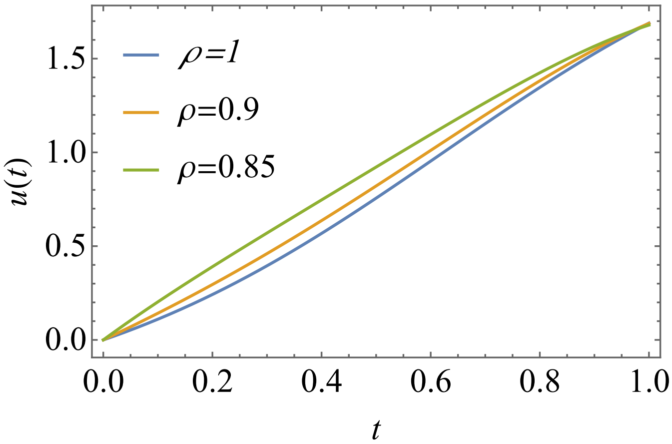

Figure 10: The approximation solutions for Example 2 for fixed $n = 4$, $\alpha = 1$, and $v = 1$, and vary ρ.

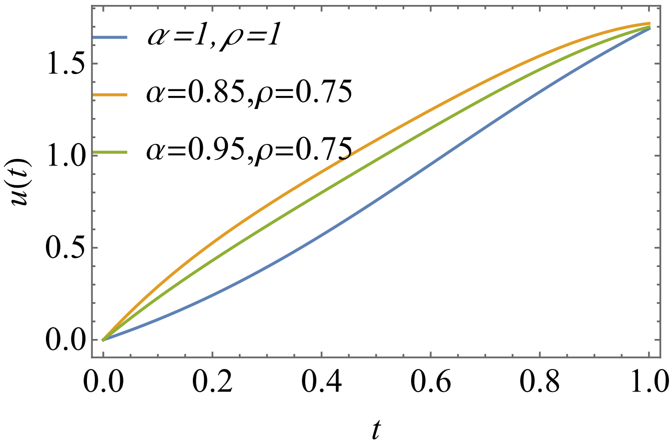

Figure 11: The approximation solutions for Example 2 for fixed $n = 4$, and $v = 1$ and vary α, ρ.

more accurately.

7. Conclusion

In this paper, we apply the method of operational matrices for Bernstein polynomials to solve the generalized fractional differential of the Caputo type with two parameters. Operational matrices convert differential equations into algebraic equations to calculate approximate solutions to linear and nonlinear fractional differential equations. The accuracy of the approximate solutions was verified by comparing the approximate solutions when $\alpha, \rho = 1$, and $v = 1$ with the exact solutions in the case of linear and nonlinear equations. The method used to analyze and solve fractional differential equations is implemented. The operational matrices method is effective and reveals the existence of the approximate solution. From the solved linear and nonlinear problems, it was found that the approximate solutions obtained using the presented algorithm are very close to the

Table 2: Approximate solutions of fractional Riccati equation when $t = 1$, and different values of α, ρ

n	$\alpha = 1,\rho = 1$	$\alpha = 1,\rho = 0.9$	$\alpha = 0.95,\rho = 0.75$	$\alpha = 0.9,\rho = 1.2$
3	1.69116	1.68978	1.70788	1.79401
4	1.6892	1.68927	1.70997	1.78634
5	1.68945	1.68947	1.7106	1.78903
6	1.68937	1.68891	1.71064	1.79657

exact solutions. Therefore, this study will begin further implementing and investigating generalized partial Caputo systems. Physical interpretation of the generalized fractional differential equations can also be investigated in the future.

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