EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5478
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Generalized Conformable Hamiltonian Dynamics with Higher-Order Derivatives

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Abstract. In this paper, we investigate higher-order calculus using the conformable derivative and integral. We use a fractional variant of the calculus of variations to obtain the Euler-Lagrange equation. Our route integral quantization approach streamlines the procedure by integrating solely over canonical coordinates q_i, eliminating the requirement to integrate higher-order derivatives ($\overline{q_i} = D_t^{\alpha} q_i$). In addition, we employ the conformable derivative to develop canonical conserved energy-momentum and Ostrogradsky's Hamiltonian. Furthermore, we generalized the Hamilton formulation for higher order derivatives and applied this new formulation to obtained equations of motion for a one dimensional point particle.

2020 Mathematics Subject Classifications: 37J05, 70H20, 49S05, 26A33, 70G45

Key Words and Phrases: Left and right conformable fractional, Ostrogradsky's Hamiltonian, Energy and momentum conservation, Euler-Lagrange, Conformable Higher-Order Derivatives, Lagrange equation

1. Introduction

Higher-Order Derivatives is proving to be invaluable across a spectrum of disciplines including chemistry, biology, and electronics [6, 20]. Recent research has showcased its utility in scaling phenomena [19, 23], classical mechanics, and mathematical modeling [25, 31]. Scholars have ventured into higher-order dynamical systems using Dirac's restricted dynamics [1, 2], examining the harmony between constraints and equations of motion [8], and advancing our understanding of systems with higher-order derivatives [9, 17]. Simplified

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5478

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quantization techniques have been devised for both conservative and non-conservative systems [16], with discrete systems receiving particular attention [4]. Recent inquiries have extended fractional equations to encompass fields like the Lee-Wick field [22], complex scalar fields [5], and the Dirac field [7]. Moreover, Noether's theorem has been invoked to determine conserved quantities [3], while the application of Hamilton formalism with higher-order derivatives has facilitated the derivation of equations of motion, maintaining consistency with conventional mechanics [10]. The Riemann-Liouville derivative has emerged as a powerful tool for problem-solving [12], exhibiting convergence in the time domain as segment size increases. Many studies have advanced Hamilton-Ostrogradskii principle formulations for higher-order dynamical systems, including developments in systems with higher-order derivatives and fractional derivatives, as well as innovations in path integral quantization [13, 15, 24, 29]. On the other hand, the Riemann-Liouville derivative and Caputo derivative do not obey the Leibniz rule and chain rule, which prevent us from applying these derivatives to the ordinary physical system with high-order derivatives. The conformable derivative was introduced in [18, 21]. This derivative obeys the Leibniz rule and the chain rule. The innovative approach, based on Leibniz and chain principles, gives identical derivatives of all orders under one, creating a flexible framework for studying higher-order derivatives. With increasing adoption by researchers, this definition broadens its applicability to include Hamiltonian formalism with high-order identical derivatives, ensuring independence from higher-order coordinate derivatives. This novel mathematical framework enables a more detailed depiction of dynamical systems characterized by complex behaviors such as memory effects, non-local interactions, and anomalous diffusion. Efforts focusing on fractional calculus within classical domains, mechanical systems, and variation problems aim to deepen our understanding of anomalous dynamics and non-local influences in physics and engineering. This paper aims to address the identified limitations by introducing a novel Hamiltonian formulation for systems involving higher-order derivatives. Our approach leverages a new definition of the conformable derivative that complies with both the Leibniz and chain rules, thereby enhancing its applicability. We anticipate that this proposed method will provide a wide range of accurate solutions to generalized differential equations with higher derivatives, effectively overcoming existing limitations. By integrating higher-order derivatives and fractional operators into our framework, we expect to create more adaptable models compared to those derived from traditional calculus. Furthermore, we intend to demonstrate that our Hamiltonian formulation can be developed independently of higher-order derivatives of the coordinates, which will simplify the analysis of complex dynamical systems and improve our understanding of their dynamics.

The structure of this paper is as follows: Section 2 briefly discusses the definitions of conformable derivatives. Section 3 presents the Generalized Ostrogradsky's Construction Form. Section 4 provides illustrative examples, and Section 5 introduces applications of conformable calculus along with suggestions for further research. Finally, the paper concludes with closing remarks.

2. Calculus of variations

This section presents two distinct definitions of derivatives: left- and right-conformable derivatives (denoted CFDs). These definitions are integral to the Hamiltonian formulation and are employed in solving cases that result in equations of motion of various orders, such as $(1/2, 1, 3/2, 2...)$. The left conformable derivative and the right-conformable derivative are defined and explored, along with their corresponding integrals, the left conformable integral (CFI) and the right conformable integral (CFI) [11, 18]. These concepts are fundamental to understanding and applying higher-order calculus in various physical contexts. The right conformable derivative is defined as follows [11, 18]:

$$D^{\alpha}_{s|x}f(t) = \lim_{\epsilon\to\infty} \frac{(t+\epsilon(t-s)^{1-\alpha}) - f(t)}{\epsilon}$$

With the condition that $s < t$ for the derivative to be valid for all $t \geq s$. The left-conformable derivative is defined as follows [11, 18]:

$$D^{\alpha}_{t|s'}f(t) = -\lim_{\epsilon\to\infty} \frac{(t+\epsilon(s^{'}-t)^{1-\alpha}) - f(t)}{\epsilon}$$

Noting that $t \geq s$ to avoid undefined expressions, as $(t-s)^{\alpha}$ becomes negative in such cases. In the case where $0 < \alpha \leq 1$, the left conformable integral (CFI) is defined as follows:

$$I^{\alpha}_{s|t}f(t) = \int_{s}^{t} (\xi - s)^{1-\alpha} f(\xi) d\xi$$

Ensuring the integral limits are correctly defined from s to t. The right conformable integral (CFI) is defined as follows:

$$I^{\alpha}_{t|s'}f(t) = \int_{t}^{s^{'}} (s^{'} - \xi)^{1-\alpha} f(\xi) d\xi$$

The relationships between the conformable derivatives (CFD) and integrals (CFI) are expressed as follows:

$$D^{\alpha}_{s|t} I^{\alpha}_{s|t} f(t) = f(t),$$

$$D^{\alpha}_{t|s'} I^{\alpha}_{t|s'} f(t) = f(t) - f(a),$$

Next, we generalize to the case when $s = 0$, and use the notation $D^{\alpha}_{s|t}$ to denote ($D^{\alpha}_{s|t} = D^{\alpha}_{t}$) as follows :

$$D^{\alpha}_{s|x}f(t) = \lim_{\epsilon\to\infty} \frac{(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

Where $0 < \alpha \leq 1$ and D^{α}_{x} represents a right CFD. The definition of CFD can be reformulated using l'Hôpital's rule as follows [30]:

$$D^{\alpha}_{s|t} f(t) = f^{'}(t) t^{1-t},$$

Assuming f is differentiable, which is necessary for equation 2 to hold true.

3. Generalized Ostrogradsky's Construction Form

This section explores the Lagrangian and Hamiltonian formulations of mechanical systems characterized by higher-order partial derivatives. It defines the configuration space using generalized coordinates and derives equations of motion through the Euler-Lagrange equations [14, 28].

Lagrangian Formulation

We begin with a configuration space defined by n generalized coordinates $q(t)$, $D^{\alpha}_{s|t}q(t)$ and $D^{2\alpha}_{s|t}q(t)$, and $D^{2\alpha}_{s|t}\alpha q(t)$. The equations of motion arise from the Euler-Lagrange equations, represented as :

$$J^{\alpha}[t] = I^{\alpha}_{0|t_0}\mathcal{L}(q_i(t,\epsilon^{\alpha}), D^{\alpha}_{s|t}q_i(t,\epsilon^{\alpha}), D^{2\alpha}_{s|t}q_i(t,\epsilon^{\alpha}))$$

Where

$$q_i(t,\epsilon^{\alpha}) = q(t) + \epsilon\eta(t)$$

$$D^{\alpha}_{t}q_i(t,\epsilon^{\alpha}) = D^{\alpha}_{t}q(t) + \epsilon D^{\alpha}_{t}\eta(t) \quad \text{and} \quad \eta(0) = \eta(t_0) = 0$$

Subsequently, the need for an external value can be expressed as follows:

$$[\partial_{\epsilon}S]_{\epsilon=0} = I^{\alpha}_{0|t_0}\left[\partial_{\epsilon}\mathcal{L}(q_i(t,\epsilon), D^{\alpha}_{t}q_i(t,\epsilon), D^{\alpha}_{t}D^{\alpha}_{t}q_i(t,\epsilon))\right]$$

$$= \int_0^t t^{\alpha-1}\left[\frac{\partial\mathcal{L}}{\partial q_i}\frac{\partial q_i}{\partial\epsilon} + \frac{\partial\mathcal{L}}{\partial D^{\alpha}_{t}q_i}\frac{\partial D^{\alpha}_{t}q_i}{\partial\epsilon}\frac{\partial\mathcal{L}}{\partial D^{\alpha}_{t}D^{\alpha}_{t}q_i} + \frac{\partial D^{\alpha}_{t}D^{\alpha}_{t}q_i}{\partial\epsilon}\right]dt$$

$$= \int_0^t t^{\alpha-1}\left[\frac{\partial\mathcal{L}}{\partial q_i}\eta + \frac{\partial\mathcal{L}}{\partial D^{\alpha}_{t}q_i}D^{\alpha}_{t}\eta + \frac{\partial\mathcal{L}}{\partial D^{\alpha}_{t}D^{\alpha}_{t}q_i}D^{\alpha}_{t}D^{\alpha}_{t}\eta\right]dt$$

By employing integration by parts, we obtain the following expression:

$$0 = \int_0^t t^{\alpha-1}\left[\frac{\partial\mathcal{L}}{\partial q_i} - D^{\alpha}_{t}\left(\frac{\partial\mathcal{L}}{\partial D^{\alpha}_{t}q_i}\right) + \left(\frac{\partial\mathcal{L}}{\partial D^{\alpha}_{t}D^{\alpha}_{t}q_i}\right)\right]\eta(t)dt$$

This results in the fractional Euler-Lagrange equation.

$$\frac{\partial\mathcal{L}}{\partial q_i} - D^{\alpha}_{t}\left(\frac{\partial\mathcal{L}}{\partial D^{\alpha}_{t}q_i}\right) + \left(\frac{\partial\mathcal{L}}{\partial D^{\alpha}_{t}D^{\alpha}_{t}q_i}\right) = 0 \tag{1}$$

By comparing Equations $(A.3)$ and $(A.4)$, we observe that the differential representation of the Hamiltonian requires identifying the partial derivatives of H with respect to q_i , P_1 , P_2 ,and t. Using this comparison, we derive the Hamiltonian equations of motion as follows (see appendix A):

- **First equation of motion:** Since the term $D^{\alpha}_{s|t}q_i d(P_1)$ in equation $(A.3)$ should correspond to the term $\frac{\partial H}{\partial P_1}dP_1$ in equation (A.4), we obtain :

$$D^{\alpha}_{s|t}q_i = \frac{\partial H}{\partial P_1},$$

- **Second equation of motion:** Similarly, the term $D^{2\alpha}_{s|t}q_i d(P_2)$ should match the term $\frac{\partial H}{\partial P_2}dP_2$ in Equation (A.4), leading to:

$$D^{2\alpha}_{s|t}q_i = \frac{\partial H}{\partial P_2},$$

- **Third equation of motion:** Moving to the term involving dq_i in Equation $(A.3)$, we observe that it should be equivalent to the term $(D^{\alpha}_{s|t}P_1 + D^{2\alpha}_{s|t}P_2)dq_i$ in Equation (4). Thus, we find:

$$\frac{\partial H}{\partial q_i} = -D^{\alpha}_{s|t}P_1 + D^{2\alpha}_{s|t}P_2,$$

- **Fourth equation of motion:** The term $\frac{\partial H}{\partial(t)}$ represents the temporal change of the Hamiltonian, which is related to the temporal change of the term as :

$$\frac{\partial H}{\partial t} = -\frac{\partial l}{\partial t'}$$

It should be highlighted that the results deduced from the above equations are closely aligned with what is observed in classical field theory, especially when dealing with integer orders within equations of motion.

$$L = L(q_i, D^{\alpha}_t q_i, D^{2\alpha}_t q_i, D^{3\alpha}_t q_i, \cdots, D^{n\alpha}_t q_i, t)$$

Integrating the Lagrangian values with respect to time yields the action, which is a functional that describes the path followed in configuration space, represented as:

$$S = \int L(q_i, D^{\alpha}_t q_i, D^{2\alpha}_t q_i, D^{3\alpha}_t q_i, \cdots, D^{n\alpha}_t q_i, t)dt,$$

The evolution of the classical system is determined by solving the Euler–Lagrange equations of motion, which are derived as follows :

$$\sum_{i=0}^{n}(-1)^i D^{i\alpha}_t \left(\frac{dL}{dq^i}\right) = 0 \qquad q_i = D^{i\alpha}_t q \qquad i = 1, 2, 3, \cdots$$

The Hamiltonian formulation is defined as follows:

$$\sum_{i=1}^{n} p_i q_i - L, \qquad q_i = D^{i\alpha}_t q, \qquad i = 1, 2, 3, \cdots$$

When calculating the total differential of the Hamiltonian, we obtain the following result.

$$dH = \sum_{i=1}^{n} p_i d(D^{i\alpha}_t q_i) + (D^{i\alpha}_t q_i)dp_i - \frac{\partial L}{\partial(D^{i\alpha}_t q_i)}d(D^{i\alpha}_t q_i) - \frac{\partial L}{\partial q_i}, \quad q_i = D^{i\alpha}_t q, \quad i = 1, 2, 3, \cdots \tag{2}$$

To provide an alternative perspective on the generalized momenta p_i, which is linked to the expression $q^i = D_t^{i\alpha} q$, let's examine the following formulation. We can uniquely present the generalized momenta by utilizing the equation $q_i = D_t^{i\alpha} q$:

$$p_i = \frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial D_t^{i\alpha} q} \tag{3}$$

After substituting the momentum values into (3), we obtain at the following outcome :

$$dH = \sum_{i=1}^{n} D_{s|t}^{i\alpha} q d(p_i) - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt, \tag{4}$$

By employing the Euler-Lagrange (1), we obtain at the following result:

$$-\frac{\partial L}{\partial q} = \sum_{i=1}^{n} (-1)^i D_{s|t}^{i\alpha} (\frac{\partial L}{\partial q^i}) = \sum_{i=1}^{n} (-1)^i (\frac{\partial^i p_i}{\partial t^i}) \quad i = 1, 2, 3, \cdots, n. \tag{5}$$

When we substitute (5) with (4), the outcome is as follows:

$$dH = \sum_{i=1}^{n} p_i d(D_{s|t}^{i\alpha} q d(p_i)) + (-1)^i (\frac{\partial^i p_i}{\partial t^i}) dq - D_{s|t}^{i\alpha} L dt. \tag{6}$$

This indicates that the Hamiltonian can be expressed in the following manner:

$$H = H(q, P_i, t) \qquad i = 1, 2, 3, \cdots, n.$$

The total differential of this function is represented as follows:

$$dH = \frac{\partial H}{\partial q} \partial q + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} d(p_i) + \frac{\partial H}{\partial t} dt, \tag{7}$$

When we compare (6) and (7), we can derive the following set of Hamilton's equations of motion.

$$\frac{\partial H}{\partial q} = \sum_{i=1}^{n} (-1)^i p_i, \qquad D_{s|t}^{i\alpha} = \frac{\partial H}{\partial p_i},$$

To further establish the validity of our method for handling high-order derivatives, we will provide two examples of generalized conformable derivative concepts. These examples will serve to introduce and clarify the application of these concepts in supporting our approach. In this context, the path integral quantization is given as follows :

$$K = \int dp dq e^{i \sum_{k=1}^{n} (P_k q^k - H(q, q^k, t)} dt$$

For quadratic $H(q, q^k, t)$ in terms of P_k, and after integration over the momenta P_k we obtain :

$$K = \int dq e^{i\int L} dt$$

One should notice that the path integral quantization is obtained as an integration over the canonical coordinates q_i without any need to integrate over higher order derivatives $(q_i, \overline{q_l})$ as given by Ostrogradskii formulation.

4. Generalized Conformable Fractional Higher-Order Derivatives

Classical mechanics describes the motion of particles and systems under forces, often described by a Lagrangian function. Recent interest in incorporating fractional derivatives has led to the development of fractional mechanics, using conformable derivatives. This work builds upon fractional mechanics and conformable derivatives to derive a generalized form of Euler-Lagrange equations for higher order conformable derivatives, establishing energy and momentum conservation in the presence of time independence and translational invariance of the Lagrangian [26].

Let's begin by revisiting some fundamental concepts regarding theories involving particles in one dimensional motion. These theories are described by a Lagrangian function denoted as "L" which depends on variables like position (x), velocity $(D^{\alpha}_{s|t}x)$, and acceleration $(D^{2\alpha}_{s|t}x))$ at a given time (t).

$$\delta L - \delta t \frac{\partial L}{\partial t} = \delta q_i \frac{\partial L}{\partial q_i} + \delta D^{\alpha}_t q_i \frac{\partial L}{\partial D^{\alpha}_t q_i} + \delta D^{2\alpha}_t q_i \frac{\partial L}{\partial D^{2\alpha}_t q_i} =$$
$$\delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial D^{\alpha}_t q_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial D^{2\alpha}_t q_i} \right) +$$
$$\frac{d}{dt} \left(\delta q_i \frac{\partial L}{\partial D^{\alpha}_t q_i} + \delta D^{\alpha}_t q_i \frac{\partial L}{\partial D^{2\alpha}_t q_i} - \delta q_i \frac{d}{dt} \frac{\partial L}{\partial D^{2\alpha}_t q_i} \right).$$

Therefore, the equation of motion is as follows:

$$\frac{\partial L}{\partial q_i} - D^{\alpha}_t \frac{\partial L}{\partial D^{\alpha}_t q_i} + D^{2\alpha}_t \frac{\partial L}{\partial D^{2\alpha}_t q_i} = 0 \tag{8}$$

This aligns with the Euler-Lagrange outcome (refer to (1)). Through the utilization of fractional calculus in the preservation principles of classical mechanics, with a particular emphasis on fractional energy and momentum, we establish what we term fractional mechanics. Within this framework, we reassess the equation of motion.

$$D^{i\alpha}_{s|t} \left(L - D^{\alpha}_t q_i \frac{\partial L}{\partial D^{\alpha}_t q_i} - D^{2\alpha}_t q_i \frac{\partial L}{\partial D^{2\alpha}_t q_i} + D^{\alpha}_t q_i \frac{d}{dt} \frac{\partial L}{\partial D^{\alpha}_t q_i} \right) = \frac{\partial L}{\partial t}$$

When the Lagrangian remains constant over time, indicating its time independence, it implies the existence of a conserved quantity known as energy. Consequently, the total energy remains constant as time progresses.

$$E = D^{\alpha}_{s|t}q_i \frac{\partial L}{\partial D^{\alpha}_{s|t}q_i} + D^{2\alpha}_{s|t}q_i \frac{\partial L}{\partial D^{2\alpha}_{s|t}q_i} - D^{\alpha}_{s|t}q_i \frac{d}{dt}\frac{\partial L}{\partial D^{\alpha}_{s|t}q_i} - L$$

In the presence of translational invariance within the Lagrangian $\frac{\partial L}{\partial q_i} = 0$, we anticipate the conservation of momentum. Considering the equation for the equation of motion, we introduce the momentum p_x with the following expression:

$$p_{q_i} = \frac{L}{\partial D^{\alpha}_{s|t}q_i} - \frac{d}{dt}\frac{\partial L}{\partial D^{2\alpha}_{s|t}q_i}$$

This results in equation (11) being expressed as follows:

$$D^{2\alpha}_{s|t}p_{q_i} = \frac{\partial L}{\partial q_i}$$

Therefore, when the Lagrangian exhibits translational invariance, the conservation of momentum in the x-direction is expressed as p_x follows. Consequently, the energy can be defined as:

$$E = D^{\alpha}_{s|t}q_i p_{q_i} + D^{2\alpha}_{s|t}q_i \frac{\partial L}{\partial D^{2\alpha}_{s|t}q_i} - L$$

Example 1. *let us consider the Lagrangian [22]*

$$L = \frac{1}{2}ax(D^{2\alpha}_{t}x)^2 - \frac{1}{2}bx(D^{\alpha}_{t}x)^2$$

By applying the Euler-Lagrange equation 1 with respect to the independent variable q_i, we obtain:

$$\frac{3}{2}\alpha(D^{2}_{s|t}q_i)^2 + bq_i D^{2}_{s|t}q_i + 2\alpha(D^{s|t}_{1}q_i)q_i^{(3)} + \frac{1}{2}b(D^{s|t}_{1}q_i)^2 + aq_i q_i^{(4)} = 0. \quad (9)$$

The momenta p_1 and p_2 are given as:

$$p_1 = \frac{\partial L}{\partial D^{\alpha}_{s|t}x} = -bxD^{\alpha}_{t}x.$$

$$p_2 = \frac{\partial L}{\partial D^{2\alpha}_{s|t}x} = axD^{2\alpha}_{s|t}x.$$

The Hamiltonian density can be written as:

$$H = p_1 D^{\alpha}_{s|t}q + p_2 D^{2\alpha}_{s|t}q - L = \frac{p_1^2}{2bx} + \frac{p_2^2}{2ax} \quad (10)$$

Additionally, the equations of motion according to Hamilton's formalism are:

$$D^{\alpha}_{s|t}x = \frac{-p_1}{bx}.$$

$$D^{2\alpha}_{s|t}x = \frac{p_2}{ax}.$$

$$\frac{H}{\partial x} = -D^{\alpha}_{s|t}p_1 + D^{\alpha}_{s|t}p_2.$$

Using the equation above, we obtain the following outcome:

$$\frac{3}{2}a(D^{2\alpha}_{s|t}q_i)^2 + bq_iD^{2\alpha}_{s|t}q_i + 2a(D^{\alpha}_{s|t}q_i)q_i^{(3)} + \frac{1}{2}b(D^{\alpha}_{s|t}q_i)^2 + aq_iq_i^{(4)} = 0 \tag{11}$$

The above equation is exactly the same as the equation that has been derived by Euler-Lagrange 9 in fractional form. For $\alpha = 1$, we get:

$$\frac{3}{2}a(D^{2}_{s|t}q_i)^2 + bq_iD^{2}_{s|t}q_i + 2a(D^{1}_{t}q_i)q_i^{(3)} + \frac{1}{2}b(D^{1}_{s|t}q_i)^2 + aq_iq_i^{(4)} = 0 \tag{12}$$

Using the conformable derivative [29] and assuming $D^2_{s|t}q_i = \ddot{q}_l$, $D^1_{s|t} = \dot{q}_l$

$$\frac{3}{2}a(\ddot{q}_l)^2 + bq_i\ddot{q}_l + 2a(\dot{q}_l)q_i^{(3)} + \frac{1}{2}b(\dot{q}_l)^2 + aq_iq_i^{(4)} = 0 \tag{13}$$

It is worth noting that the results in Equation 13 are consistent with those found in Muslih et al [27]. Besides, the path integral is given by

$$K = \int dq e^{i(\frac{1}{2}ax(D^{2\alpha}_{s|t}x)^2 - \frac{1}{2}bx(D^{\alpha}_{s|t}x)^2)dt} \tag{14}$$

The path integral (14)*, is an integration over the canonical coordinate x, without any need to any integration over the velocity $\overline{x} = D^{\alpha}_{s|t}x$ as given by Ostrogradskii formulation.*

5. Higher-Order Calculus: Applications and Recommendations for Future Work

Understanding conformable calculus is crucial due to its high potential value, especially in higher order derivatives, Hamiltonian systems, nonlinear motion, and differential equations. The Hamiltonian formalism for fractional differential equations is essential for comprehending the dynamics of intricate physical systems, particularly at the nanoscale. This approach enables the exploration of fractional derivatives of point particles, offering valuable insights into the movements of nanoscale materials and particles. These practical implications highlight the importance of derivatives in understanding complex systems in engineering and applied physics. This method helps researchers delve deeper into the stability, oscillation, and periodicity of systems exhibiting memory effects or long-range interactions. By incorporating the Hamiltonian method into the study of conformable derivatives, we can achieve a more comprehensive understanding of how systems respond to external influences and disturbances. Further research on conformable derivatives and generalized point particles is vital for advancing our understanding of these complex systems. The insights gained from such studies have the potential to revolutionize fields like nanotechnology, biomechanics, and advanced engineering, offering innovative solutions and a deeper comprehension of the fundamental principles governing these intricate phenomena.

6. Conclusion

In this research, we investigated two systems using the conformable version of the calculus of variations to derive the fractional Euler-Lagrange equation. It applies this calculus to the conformable version of classical mechanics, introducing the conformable Lagrangian and deriving the equation of motion. The path integral quantization was obtained directly as an integration over the canonical coordinate q_i without the need to integrate over the variable $D_t^{\alpha} q_i$. The classical equations of motion obtained in this work perfectly matched those obtained through the Lagrangian formulation. The Hamiltonian Method for Generalized Conformable Differential Equations with Higher-Order Derivatives offers a structured framework for studying nonlinear dynamics and invariance principles in complex systems involving controlled Lagrangians with higher-order derivatives. This research enhances theoretical applications in this domain, demonstrating invariance results concerning state variables and exploring a natural Hamiltonian formulation for composite higher derivative theories involving time derivatives.
The approach, along with a numerical example, enables us to identify both the static and dynamic parameters of particles or structures in motion. This forms the foundation for a new research project we are currently developing.

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Appendix

In this appendix, we shift from the Lagrangian approach to the Hamiltonian approach, we introduce paired generalized momenta (p_i and Φ) associated with and associated in conjunction with their respective generalized coordinates.

$$p_i = D^{\alpha}_{s|t}\frac{\partial L}{\partial D^{\alpha}_{s|t}q_i} + D^{2\alpha}_{s|t}\frac{\partial L}{\partial (D^{2\alpha}_{s|t}q_i)}$$

$$\pi_i = \frac{\partial L}{\partial (D^{\alpha}_{s|t}q_i)}$$

We expand the phase space to include the canonical variables (q_i, p_i) and their associated counterparts $(\overline{q_l}, \pi_i)$, where $(\overline{q_l} = D^{\alpha}_{s|t})$. Within this augmented phase space, the Hamiltonian is expressed as follows:

$$H = p_i\overline{q_i} + \pi_i D^{2\alpha}_{s|t}q_i - L$$

This implies that the Hamiltonian can be expressed as a function, as follows:

$$H = H(q_i, p_i, \overline{q_i}, \pi_i, t)$$

In the generalized case:

$$L = L(q_i, D^{\alpha}_{s|t}q_i, D^{2\alpha}_{s|t}q_i, \cdots, D^{n\alpha}_{s|t}q_i, t)$$

Considering the variation principle, it is evident that the Euler–Lagrange equations governing the motion of the system are expressed as follows :

$$\sum_{i=0}^{n}(-1)^i D_{s|t}^{i\alpha} q_i(\frac{dL}{dq^i}) = 0 \quad q^i = D_{s|t}^{i\alpha} q^i \quad i = 0, 1, 2, \cdots$$

To streamline the representation of these equations through Ostrogradskii's method and the introduction of paired generalized momenta (p_i, π_i) linked to the generalized coordinates $(q_i, D_{s|t}^{\alpha} q_i)$ the equations can be reformulated as follows :

$$p_i^{k-1} = \frac{\partial L}{\partial(D_{x_\lambda}^{\alpha} D_{s|x_\lambda}^{\alpha} q_i)} - \frac{d}{dt}(\frac{\partial L}{\partial(\partial D q_i^{k+1})}), \quad k = 0, 1, 2, \cdots, n-1, \quad \pi_i = \frac{dL}{dq^n}.$$

The phase space, defined by the canonical variables (q_i and p_i^{k-1})and their associated counterparts ($\overline{q_i}^k$ and π_i) encompasses the generalized coordinates ($\overline{q_i}^k$ and q_i^{k+1}). The formulation of the Hamiltonian is as follows :

$$H = \sum_{k=1}^{n-1} p_k q^k + \pi_i q_i^n - L$$

This implies that the Hamiltonian can be represented as follows:

$$H = H(q_i, p_i^{k-1}, \overline{q}_i^{k-1}, \pi_i, t),$$

Now, the Hamiltonian formalism, denoted as H, is obtained by employing the Legendre transformation in the following manner:

$$H = P_1 D_{s|t}^{\alpha} q_i + P_1 D_{s|t}^{2\alpha} q_i - L,$$

Determining the total differential of this Hamiltonian leads us to the following outcome.

$$dH = P_1 d(D_{s|t}^{\alpha} q_i) + D_{s|t}^{\alpha} q_i d(P_1) + P_2 d(D_{s|t}^{\alpha} D_{s|t}^{\alpha} q_i) + D_{s|t}^{2\alpha} q_i d(P_2) - \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial D_{s|t}^{\alpha} q_i} D_{s|t}^{\alpha} q_i - \frac{\partial L}{\partial D_{s|t}^{2\alpha} q_i} D_{s|t}^{2\alpha} q_i, \quad (A.2)$$

The generalized momenta P_1 and P_2 corresponding to $D_{s|t}^{\alpha} q_i$ and $D_{s|t}^{2\alpha} q_i$ can be defined as follows:

$$P_1 = \frac{\partial L}{\partial(D_{s|t}^{\alpha} q_i)},$$

$$P_2 = \frac{\partial L}{\partial(D_{s|t}^{2\alpha} q_i)},$$

Substituting the values of momenta P_1 and P_2 into (2), we get :

$$dH = P_1 d(D_{s|t}^{\alpha} q_i) + D_{s|t}^{\alpha} q_i d(P_1) + P_2 d(D_{s|t}^{\alpha} D_{s|t}^{\alpha} q_i) + D_{s|t}^{2\alpha} q_i d(P_2) - \frac{\partial L}{\partial q_i},$$

By applying (1) of the Euler-Lagrange equation, we arrive at the following conclusion:

$$dH = P_1 d(D_{s|t}^{\alpha} q_i) + D_{s|t}^{\alpha} q_i d(P_1) + P_2 d(D_{s|t}^{2\alpha} q_i) + D_{s|t}^{2\alpha} q_i d(P_2) + \frac{\partial L}{\partial t} - (D_{s|t}^{\alpha} P_1 - D_{s|t}^{2\alpha} P_2) dq_i, \quad (A.3)$$

This indicates that the Hamiltonian can be represented in the following as follows:

$$H = H(q_i, P_1, P_2, t)$$

The total differential for this function can be expressed as:

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial P_1} dP_1 + \frac{\partial H}{\partial P_2} dP_2 + \frac{\partial H}{\partial t} dt \qquad (A.4)$$