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The Non-Perturbative Approach in Examining the Motion of a Simple Pendulum Associated with a Rolling Wheel with a Time-Delay

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Abstract. The present study aims to examine the movement of a simple pendulum that is connected by a lightweight spring and connected with a rotating wheel. The motivation behind this topic is to gain a comprehensive understanding of intricate dynamic systems that involve consistent mechanical components and response with time delay. This system is not only theoretically attractive but also practically appropriate in domains such as robotics, engineering, and control systems. As well-known, all classical perturbation methods exploit Taylor expansion to simplify the practicality of restoring forces. In contrast, the non-perturbative approach, as a novel methodology, transforms any nonlinear ordinary differential equation into a linear one. It scrutinizes the restoring forces, away from employing Taylor expansion; hence it eliminates the previous weakness. The concept of the non-perturbative approach is based mainly on the He's frequency formula. The confidence of the non-perturbative approach comes from the numerical compatibility between the nonlinear and linear ordinary differential equation via the Mathematica Software. Therefore, instead of handling the nonlinear ordinary differential equation, we investigate the linear one. The achieved response is plotted over time to show the impact of the acted parameters during a specified time interval. Moreover, the phase plane curves that correspond to the plotted solution are presented and examined. The stability criteria of the analogous linear ordinary differential equation are provided and drawn to explore the stability/instability zones. The performance is applicable in engineering and other fields due to its ease of adaptation to different nonlinear systems. Therefore, the non-perturbative approach can be regarded as substantial, successful, and interesting and can extended to be applied in further categories within the field of couples dynamical systems.

2020 Mathematics Subject Classifications: 34C15, 34D20, 70K20, 37C75, 93C10

Key Words and Phrases: Simple pendulum, Rolling wheel, Non-perturbative approach, He's frequency formula, Nonlinear stability analysis, Time history

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Abbreviations

Symbol	Meaning	Symbol	Meaning
HFF	He's frequency formula	ODE	Ordinary differential equation
NPA	Non-perturbative analysis	DO	Duffing oscillator
MS	Mathematica Software	HPM	Homotopy perturbation technique
I.C.	Initial conditions	SP	Simple pendulum

1. Introduction

A basic SP, which is only a mass (which is defined as a bob) hanging from a fixed location by a thread or rod that influences in response to gravity, can perform a great deal of the current work. Among SPs, pendulum clocks are the most well-known problem. The hands of the clock are twisting to keep precise time by means of a pendulum that swings back and forth at regular intervals. A SP is a perfect watch because its period is insensitive to changes in amplitude. With the use of a newly proposed formula, one can approximately determine the period of a SP [2]. It was considered both small and large amplitudes of oscillation, while deriving the nonlinear ODE, which represents and explains the swing of a simple SP [9]. The fundamental element of a SP experiment is a pendulum bob, which is an item with a slight mass suspended from a light thread cite3. To solve the nonlinear problems of a mass spring oscillator, which is the same as the DO [44], the vibrational iterative approach and using the Laplace transform was used. Discovering the Lagrange multiplier is a crucial part of the variational iterative method, and thus is a common application of variational theory. This research looked at the motion of a classical point particle in a revolving frame with a central restoring force [8]. These responses were applied to the problems of surface-based pendulums, and most especially to the SP. It is possible for a cubic-quintic DO to arise, for instance, in the case of a magnetic spherical pendulum's governing equation of motion. The current endeavor aims to solve this problem analytically within certain bounds. The problem was resolved by integrating methods such as the HPM, nonlinear expanded frequency, and Laplace transforms [21]. Keep in mind that this issue covers the same ground as the previous one [20], but with certain limitations:

1. A crucial component of this inquiry is the time-delay.
2. The current approach uses the so-called NPA rather than the HPM.
3. Neglecting the disadvantages of increasing the restoring forces through Taylor expansion, which is an essential part of the equation of motion, is considered.

The ODEs played a vital role in many branches of applications, including applied mathematics, physics, chemistry, and biology. The physical significance of the scenario is a qualitative factor that governs dynamic behaviors. Although establishing a significant analysis of a nonlinear DO can often be challenging, the solution to a linear ODE was straightforward. Evaluating minuscule parameters posed a challenging endeavor, demanding the utilization of distinct methodologies. The semi-analytical HPM can be a helpful approach to comprehending common nonlinear DO. The first Chinese mathematician Prof. He suggested this approach to analyze the nonlinear DO [31]. Except for the requirement for a modest parametric assumption, the HPM has all the compensations of all perturbation methodologies. In comparison to earlier approaches, the HPM was overcome by the complexity of calculation, so using less computer memory, and calculating more quickly. Alluhydan et al. [40] studied how a location and velocity time-delay could reduce the excited nonlinear vibration of DO. A modified was employed to attain an approximate uniform solution to the issue being addressed. The linear time-delay system stability problem was created [54]. It began by describing a comprehensive vector of multiple integral inequalities, which can understand many results as unusual situations. Second, these multiples were used to build a delay-dependent stability standard for time-delay systems. For a time-varying delay linear system, the delay-dependent stability problem was suggested [55]. They provided evidence that their approach was more useful for handling time-varying delay systems. Rahman [5] reviewed many models that incorporate time-delays, including discrete, distributed, and hybrid approaches that combine the two, highlighting the rationale behind doing so. A time-delayed nonlinear vibration absorption system's dynamical responses to harmonic excitation were documented in a previous article [49]. A complicated averaging method was used to study the forced system's slow and fast dynamics. The system under consideration was a one-equation of motion model with nonlinear restoration and damping functions [30]. Two species of zooplankton and one phytoplankton species coexisted under certain conditions [3].

The exact solutions of the complicated nonlinear ODEs that govern the dynamical regulations generally seen in engineering and physics are often unknown. Traditional approaches, such as numerical methods and perturbation techniques, are utilized to establish the precise frequency-amplitude connection and predict the dynamic reactions. The complex dynamics of systems can be better understood with the use of these methods, which allow for a thorough examination of quantitative and qualitative aspects of system behavior. An excessive arrangement of studies has been conducted on the efficacy and broad applicability of this methodology [28, 41, 43, 52]. As mentioned earlier, the HFF was played a crucial role in obtaining closed-form analytical solutions for oscillators, especially those using the DO [45]. The HFF formula has evolved into a powerful mathematical instrument in studying nonlinear oscillators with periodic solutions. Prof. He gave an innovative review paper and was the first to present it [32]. The clarity and empirical confirmation of this frequency formula made it a fast favorite among engineers [29, 33, 34, 42, 50, 51, 56, 57]. HFF has been fine-tuned through the years, leading to even greater accuracy, as was previously seen [12, 35, 38]. Furthermore, as mentioned earlier, the scope of this frequency method's utility has expanded to include fractal oscillators [53]. In the study of nonlinear oscillatory systems, the NPA stands in unambiguous contrast to traditional perturbative approaches. This method is a strong mathematical instrument that can deal with many parameter regimes, particularly those with strong nonlinearity. An important method for obtaining analytical approximations in research on nonlinear oscillators was employed. The primary goal of the NPA is to reduce the complexity of the nonlinear model to a more manageable level so that the solutions can be clearly specified. This will allow for a more accurate approximation of the original system's behavior [10, 11, 13–19, 22–27] . The goal of this simplification is to reduce the average difference between the two systems by making the transformation from a nonlinear to a linear form of the equation. Accordingly, departing from perturbation techniques' repetitive refining, the NPA offered a fresh viewpoint. The goal of the NPA is to understand the details of nonlinear systems on their own, without relying on small changes from a known solution. This method can reveal insights into a broader range of system behavior, unconstrained by the constraints of minor disturbances. The ability to probe dynamics that may be inaccessible or concealed by more traditional perturbation approaches makes this breadth all the more important. A two-dimensional asymmetric system was examined [6]. The system's equations of motion were generated and solved analytically. A nonlinear oscillator equation featuring two dominant linear terms was analyzed [1]. An approximate solution was derived with the power series method. Additionally, by including a parameter into the original equation, we identify the fixed points of the altered nonlinear oscillator equation and conduct a stability analysis of these fixed points. The periodic motion of the micro-electro-mechanical system (MEMS) was analyzed [37]. A novel approach using polyvinylidene fluoride unsmooth nanofibers to transfer electronic current was provided [39]. The unique unsmooth surface of these nanofibers provides a high surface energy (geometrical potential), making them highly sensitive to microorganisms (e.g., viruses) absorbed on their surface.

In light of the significance of the aforementioned aspects, the present work emphasizes analyzing the movement of an SP that is connected to a rolling wheel and linked by a light spring. This topic is motivated due to the prospective implementations of the SP in physics, manufacturing, and practical mechanisms. A classic experiment involving a light spring attached to a rolling wheel, followed by a SP, can be used to investigate various physical phenomena. Here are some specific examples:

1. This model is useful for studying oscillations that involve both a spring's restoring force and the rotational dynamics introduced by a rolling wheel. The SP adds an additional degree of freedom, making the system more complex and enhancing its usefulness for studying multi-mode oscillations.

2. By carefully examining the system, investigators can investigate into the preservation of energy in a system that involves both rotational and translational motion. The SP and wheel enable the exploration of how potential and kinetic energy are exchanged within the system.

3. This circumstance can be expanded to encompass damping forces or external driving forces to examine more intricate phenomena such as resonance, damping, and stability in oscillatory systems.

4. When the oscillations reach a large magnitude, the system may exhibit nonlinear

behavior, which can be determined using this prototype. Nonlinear dynamics is an attractive area of focus in physics, particularly when studying chaotic systems.

5. This prototype serves as an educational tool for teaching fundamental concepts in classical mechanics, including rotational dynamics, harmonic motion, and the interaction between different types of motion.

6. In manufacturing, prototypes have the ability to mimic real-world systems found in automobiles, such as suspension systems that involve springs, wheels, and rotating components working together. Gaining insight into these interactions is crucial for developing highly efficient mechanical systems.

This model offers a diverse framework for exploring different mechanical principles, incorporating rotational motion, spring dynamics, and SP oscillation. Several quires are answered at the end of the current work such as:

i. What about the governing equation of motion of the model?

ii. How does the NPA convert the original nonlinear ODE into a linear one?

iii. What is the influence of the time-delay parameter?

iv. What about the time history as well as stability analysis of the system?

To crystallize the presentation of the problem, the subsequent sections of the article will be organized as follows: A brief explanation of the NPA is presented in **§ 2**. The methodology of the problem is presented in **§ 3**. Two scenarios involving the presence/absence of the time-delay factor are discussed. Furthermore, the estimated solution is contrasted with numerical calculations to authenticate the prototype link. The results of the main outcomes are presented in **§ 4**. A summary of the principal conclusions is provided in which the solution attained is graphed over time to illustrate the effects of the operated parameters. Additionally, the phase plane curves corresponding to the depicted solution are showcased and scrutinized. Stability criteria are furnished and depicted to delve into regions of both stability and instability. Lastly, **§ 5** contains the concluding remakes of the obtained results are provided.

2. A Brief Clarification of the NPA

The aim of this Section is to create specific structures in converting the nonlinear configuration to a linear one that has documented solutions and can be rationally approached by the inventive organization [48]. In a few opinions, it is possible to interpret a nonlinear ODE into a linear one in a way that reduces the consistency of the two systems' alterations. It is exemplified that this additional can be simply skillful. The basic idea of HFF is now utilized to linearize a nonlinear oscillator, resulting in a linear oscillator generating a solution that covers the complete time period of the oscillation history [47]. The reality and uniqueness of a comprehensive corresponding linear system have previously been thoroughly examined [36]. The NPA can now designated as follows:

Given a nonlinear ODE, the nonlinear forces may be abstracted as three different features as follows: quadratic nonlinear forces (do not produce secular terms), odd nonlinear damping forces (yield secular terms), and lastly the restoring nonlinear odd force (yield secular terms) [4]. This means that any nonlinear ODE may be reorganized using these modules, which leads to the subsequent example:

$$\ddot{\xi} + F_1(\zeta,\,\dot{\zeta},\,\ddot{\zeta}) + F_2(\zeta,\,\dot{\zeta},\,\ddot{\zeta}) + F_3(\zeta,\,\dot{\zeta},\,\ddot{\zeta}) = 0, \tag{1}$$

where $F_1(\zeta,\,\dot{\zeta},\,\ddot{\zeta})$, $F_2(\zeta,\,\dot{\zeta},\,\ddot{\zeta})$ and $F_3(\zeta,\,\dot{\zeta},\,\ddot{\zeta})$ are the odd damping secular terms, even nonsecular terms, and odd secular terms respectively, in which they are defined as:

$$\left.\begin{array}{r} F_1(\zeta,\,\dot{\zeta},\,\ddot{\zeta}) = a_1\dot{\zeta} + b_1\zeta^2\dot{\zeta} + c_1\zeta\dot{\zeta}^2 + d_1\dot{\zeta}^3 + e_1\ddot{\zeta}\dot{\zeta}^2 \\ F_2(\zeta,\,\dot{\zeta},\,\ddot{\zeta}) = a_2\dot{\zeta}\zeta + b_2\dot{\zeta}^2 + c_2\zeta^2 + d_2\dot{\zeta}\ddot{\zeta} \\ F_3(\zeta,\,\dot{\zeta},\,\ddot{\zeta}) = \omega^2\zeta + b_3\zeta\dot{\zeta}\ddot{\zeta} + c_3\zeta^2\ddot{\zeta} + d_3\zeta^3 + e_3\ddot{\zeta}\zeta^2 \end{array}\right\}, \tag{2}$$

where ω is the natural frequency, $a_j,\, b_j,\, c_j,\, d_j,$ and e_j $(j = 1,\, 2, 3)$ are the constant physical coefficients.

The HFF aims to convert the nonlinear ODE as given in Eq. (1) into the following linear differential equation, following Moatimid et al. [10, 11, 13–19, 22–27] , one gets

$$\ddot{v} + \sigma_{eqv}\dot{v} + \omega_{eqv}^2 v = \Lambda. \tag{3}$$

Eq. (3) is a linear equation that can be determined using conventional techniques. The aim is to calculate the three coefficients that contained in Eq. (3). This equation contains the damping constant σ_{eqv} (equivalent damping), ω_{eqv}^2 (equivalent frequency) and the non-homogeneous part Λ. The total frequency is shortened to Δ^2, which will be determined later. This frequency comes simply from the standard normal form approach. This concept may simply be introduced as: $v(t) = \psi(t)\,Exp(-\sigma_{eqv}t/2)$, where $\psi(t)$ is an unknown function to be determined. Therefore, Eq. (3) is then used to produce the harmonic equation as shown below.

$$\ddot{\psi} + \Delta^2\psi = \Lambda Exp(\sigma_{eqv}t/2), \tag{4}$$

where $\Delta^2 = \omega_{eqv}^2 - \frac{1}{4}\sigma_{eqv}^2$ represents the total frequency of the system.

The linear version of the modest harmonic oscillator is represented by Eq. (3). Prof. He recently investigated this topic by leveraging the peculiarities of specific functions [36]. It was provided the subsequent guessing solution:

$$u = A\cos\Delta t, \quad u(0) = A,\, \dot{u}(0) = 0. \tag{5}$$

According to Moatimid et al. [10, 11, 13–19, 22–27] , the three factors that have emerged from Eq. (3) can be written as follows:

(1) **Frequency Formula**

Employing the HFF to show a benefit in scheming frequencies for the developed generalized $F_3(\zeta,\ \dot{\zeta},\ \ddot{\zeta})$. According to Moatimid et al. [10, 11, 13–19, 22–27] , Elías-Zúñiga et al. [46], He, and Liu [7], the corresponding parameter may be determined as:

$$\omega_{eqv}^2 = \int_0^{2\pi/\Delta} \zeta\, F_3(\zeta,\ \dot{\zeta},\ \ddot{\zeta})dt \Big/ \int_0^{2\pi/\Delta} \zeta^2 dt. \tag{6}$$

(2) **Damping Formula**

One may estimate the frequency of a particular function $F_1(\zeta,\ \dot{\zeta},\ \ddot{\zeta})$ using the HFF. As well as the procedure of Moatimid et al. [10, 11, 13–19, 22–27] to achieve the corresponding damping term:

$$\sigma_{eqv} = \int_0^{2\pi/\Omega} \dot{\zeta} F_1(\zeta,\ \dot{\zeta},\ \ddot{\zeta})dt \Big/ \int_0^{2\pi/\Omega} \dot{\zeta}^2 dt. \tag{7}$$

(3) **Non-Secular Part**

It must be noted that the quadratic formula applies to the non-secular component. Accordingly, the inhomogeneity will be calculated by substituting: $u \to kA$, $\dot{u} \to kA\Omega$, and $\ddot{u} \to kA\Omega^2$ in the even non-secular function $g(u,\ \dot{u},\ \ddot{u})$. As shown by Moatimid et al. [10, 11, 13–19, 22–27] , the factor k is defined as $k = 1/2\sqrt{n-r}$, where n indicates the order of the system and r signifies the degree of freedom of the system. Therefore, in the present case: $n = 2$ and $r = 1$, then the significance of k becomes $k = 1/2$. To do this, the nonlinear Eq. (1) is converted into the linear Eq. (3). The standard normal form of Eq. (3) may be used to guess the stability requirements in a humbler procedure, where the total frequency is defined by the formulation: $\Delta^2 = \omega_{eqv}^2 - \sigma_{eqv}^2/4$.
The stability requirements need: $\Delta^2 > 0$ and $\sigma_{eqv} > 0$.

3. Structure of the Model

In our earlier work [20], we deduced the controlling governing equation of the SP linked with a rotating wheel and constrained by a light spring. The fundamental controlling equation for motion is given as:

$$\left(r^2 + 1 + 2r\cos\theta\right)\ddot{\theta} - r\dot{\theta}^2 \sin\theta + kr^2\theta = 0, \tag{8}$$

where r is the radius of the rolling wheel, and k is the stiffness of the light spring.
It should be noted that the stiffness of the light spring is a critical factor in determining the dynamic properties of the system, such as stability, frequency, and energy characteristics. Moreover, the radius of a rolling wheel significantly influences its efficiency, performance, and applicability for particular applications. The sketch of the physical model is shown in Fig. (1).
As previously stated, the current calculations are a complete departure from the earlier investigation [54].

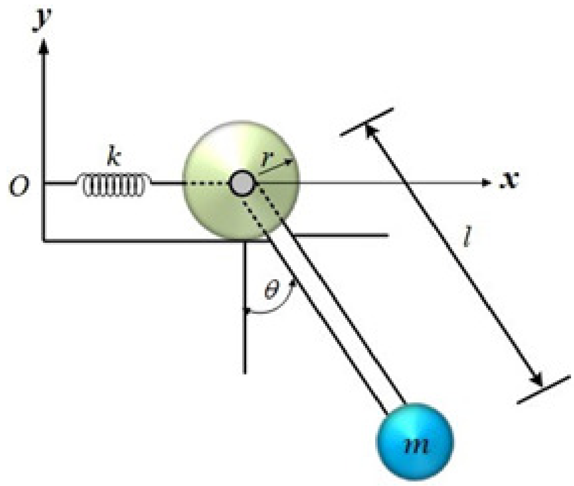

Figure 1: Displays the physical configuration of the prototype.

To repair the weakness in utilizing the Taylor expansion in expanding the restorative forces, the NPA as a novel approach is employed.
Eq. (1) may be formulated as follows:

$$\ddot{\theta} + \frac{2r}{1+r^2}\cos\theta\,\ddot{\theta} - \frac{1}{1+r^2}\sin\theta\,\dot{\theta}^2 + \frac{kr^2}{1+r^2}\theta = 0. \tag{9}$$

The following analysis tackles both the presence and absence of the time delay.

3.1. The Presence of a Time-Delay

As shown in the introduction, the time delay has much significance in diverse fields. Time-delay in a dynamical system is the period of time that elapses between the input and a modification in the system and the corresponding reaction.

1. **Stability:** The presence of a time-delay can have an impact on the stability of a system. Introducing a delay to a system that is initially stable can cause it to become unstable, as the system's reaction may no longer be synchronized with the inputs.

2. **Oscillations:** Time-delays have the ability to cause systems that would otherwise have a stable state to exhibit oscillating behavior. This phenomenon might result in prolonged or even escalating oscillations, which may be undesirable in some applications.

2. **Control Challenges:** When dealing with time-delay, as it adds complexity to the controller design and makes it harder to forecast the future state of the system. It is necessary to implement more advanced control mechanisms in order to guarantee the appropriate level of performance.

4. **Precision:** Delays can lessen the accuracy of the system's reaction, principally in systems that necessitate instantaneous processing, such as communication systems or robots.

In the design and analysis of dynamical systems, it is crucial to properly take into account time-delays to guarantee appropriate functioning and stability.
Therefore, the upcoming analysis is based on the time-delay of the position, i.e., in the last term in Eq. (9). Accordingly, one gets:

$$\ddot{\theta} + \frac{2r}{1+r^2}\cos\theta\,\ddot{\theta} - \frac{1}{1+r^2}\sin\theta\,\dot{\theta}^2 + \frac{kr^2}{1+r^2}\theta(t-\tau) = 0. \tag{10}$$

The time-delay in position denotes the interval between the implementation of a modification in a system and the subsequent observation of its influence on the system's position. This postponement may arise from various factors:

1. Massive objects oppose alterations in their motion because of inertia. Upon the application of a force, there is a time-delay before the item accelerates and its position alters, correspondingly.

2. In several systems, the command or signal for positional alteration requires time to propagate through the system. In mechanical systems, signals must traverse gears or linkages, but in electronic systems, signals propagate at finite velocities.

3. Frictional forces and damping mechanisms, such as air resistance or internal system damping, can induce a delay in response, impeding positional changes.

4. Frictional forces and damping components, such as air resistance or internal system damping, can induce a delay in response, impeding positional changes.

5. In systems regulated by control loops (e.g., robotic arms, motors), intrinsic time-delays arise from feedback processing, decision-making within control algorithms, and actuator reaction times.

6. In structures, material elasticity can induce a lagged response, since applied forces lead to progressive deformations that influence positioning.

These variables accumulatively result in a delayed response of an object's or system's position to an input change, referred to as "time delay in position".
The I.C. should ideally be visualized as in the following manner

$$\theta(0) = A, \;\; \dot{\theta}(0) = 0. \tag{11}$$

where A is the initial oscillation amplitude.
Now, returning again to the fundamental time delay as in Eq. (10), the NPA enables us to transform the nonlinear ODE as given in Eq. (10) into an equivalent linear one like

the simple harmonic motion under the same I.C. as given in Eq. (11). Therefore, the formulation of the guessing solution may proceed as follows:

$$u = A\cos\Omega t, \quad \dot{u} = -A\Omega\sin\Omega t, \quad and \quad \ddot{u} = -\Omega^2 u, \tag{12}$$

where Ω is known as the total frequency that depends on all parameters of the original system. It will be determined later.
Appropriately, the shift of the independent time delay may be expressed as:

$$\begin{aligned} u(t-\tau) &= A\cos\Omega(t-\tau) \\ &= A(\cos\Omega t\cos\Omega\tau + \sin\Omega t\sin\Omega\tau) \\ &= u(t)\cos\Omega\tau - \tfrac{1}{\Omega}\dot{u}(t)\sin\Omega\tau. \end{aligned} \tag{13}$$

At this stage, Eq. (3) can be expressed as follows:

$$\ddot{\theta} + f_1(\dot{\theta}) + f_2(\theta,\, \dot{\theta},\, \ddot{\theta}) = 0, \tag{14}$$

where

$$\left.\begin{aligned} f_1(\dot{\theta}) &= -\tfrac{kr^2}{\Omega(1+r^2)}\dot{\theta}\sin\Omega\tau \\ f_2(\theta,\, \dot{\theta},\, \ddot{\theta}) &= \tfrac{2r}{1+r^2}\cos\theta\;\ddot{\theta} - \tfrac{r}{1+r^2}\dot{\theta}^2\sin\theta + \tfrac{kr^2}{1+r^2}\theta\cos\Omega\tau \end{aligned}\right\}. \tag{15}$$

Currently, an equivalent frequency ϖ^2 can be evaluated as shown previously by Moatimid et al. [10, 11, 13–19, 22–27] in the following manner:

$$\varpi_{eqv}^2 = \int_0^{2\pi/\Omega} u f_2(u,\, \dot{u},\, \ddot{u})dt \Big/ \int_0^{2\pi/\Omega} u^2 dt = \frac{r}{1+r^2}\left(-2\Omega^2 J_0(A) + kr\cos\Omega\tau\right), \tag{16}$$

where $J_0(A)$ is the Bessel function of the first kind of argument A of order zero.
Additionally, following Moatimid et al. [10, 11, 13–19, 22–27] , Elías-Zúñiga et al. [46], He, and Liu [7], the assessment of the equivalent damping term can be carried out as follows:

$$\Gamma_{eqv} = \int_0^{2\pi/\Omega} \dot{u} f_1(\dot{u})dt \Big/ \int_0^{2\pi/\Omega} \dot{u}^2 dt = -\frac{kr^2}{\Omega(1+r^2)}\sin\Omega\tau. \tag{17}$$

The equivalent linear ODE, as given in the simple harmonic motion, can now be constructed as follows:

$$\ddot{u} + \Gamma_{eqv}\,\dot{u} + \varpi_{eqv}^2 u = 0. \tag{18}$$

Furthermore, the standard normal form can be attained along with the transformation $u(t) = f(t)Exp(-\Gamma_{eqv}\,t/2)$. Elementary, the unknown function satisfies the following simple harmonic differential equation:

$$\ddot{f} + \Omega^2 f = 0, \tag{19}$$

where $\Omega^2 = \varpi_{eqv}^2 - \Gamma_{eqv}^2/4$.

In other words, the total frequency can be obtained by combining the results in Eqs. (10) and (11) with the previous relation to produce:

$$\Omega^2 = \frac{r}{1+r^2}\left(-2\Omega^2 J_0(A) + kr\cos\Omega\tau\right) + \lambda\cos\Omega\tau - \frac{1}{4}\left(-\frac{kr^2}{1+r^2}\sin\Omega\tau\right)^2. \tag{20}$$

As seen, Eq. (20) is a transcendental equation in Ω. For simplicity, Taylor expansion may be employed to approximate the values of the trigonometric functions in Ω as $\sin\varepsilon \cong \varepsilon$, and $\cos\varepsilon \cong 1$. In this simplification, the total frequency can be written as follows:

$$\Omega = \frac{r\sqrt{4k + 4kr^2 - k^2r^2\tau^2}}{2\sqrt{(1+r^2)\left(1+r^2+2rJ_0(A)\right)}}. \tag{21}$$

To determine the value of the equivalent frequency, consider the following dataset:

$$r = 3.0,\ k = 0.5,\ \tau = 0.001 \ \text{ and } \ A = 0.5.$$

The stability standard requires

$$\Omega^2 > 0, \quad \text{and} \quad \Gamma_{eqv} > 0. \tag{22}$$

For more convenience, along with the mathematic Software with the commend NDsolve, the graph of the nonlinear ODE as in Eq. (10) is graphed with the linear ODE as given in Eq. (18).

Returning to the foundational time delay as given in Eq. (10), the NPA enables us to create a comparable linear ODE. The I.C. are given in Eq. (11). They are identical to those used by the resulting linear equation as displayed in Eq. (18) to obtain the non-perturbative solution (NPS). As stated earlier, the corresponding frequency is influenced by each of the original characteristics (15). It is also useful to contrast the equivalent linear ODE solution with the numerical solution (NS) of Eq. (10). The equivalent linear damping parameter is provided in Eq. (19). The two responses are shown in this comparison, as shown in Fig. (2). The two curves are plotted with the previous data. As observed, the results are quite consistent with each other. Moreover, the mathematical software indicates that within a time span of 100 units, the absolute error between the theoretical and computed solutions is 0.00284. When the linear and nonlinear solutions are coinciding, it indicates that, despite differing definitions, they share the same spatial positions. This indicates that the two curves embody the same geometric form or trajectory. It frequently implies that the fundamental physical processes, restrictions, or relationships governing the curves are analogous, or that one curve represents a transformation or particular instance of the other.

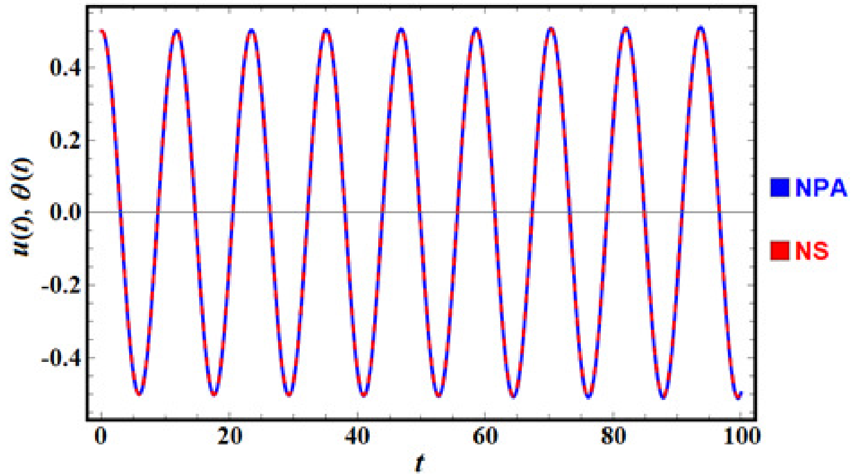

Figure 2: Demonstrates a strong correspondence between the NS of Eq. (10) and the NPS of Eq. (19).

For more convenience, with the aid of MS, Table (1) is presented to indicate the relation between the numerical as well as the actual values.

Table (1): points out the rapprochement between the numerical NS and NPA outcomes

t	Numerical	Equivalent	Absolute error
0	0.5	0.5	0.
10	0.304706739041348	0.30224373475524535	0.0024630042861026324
20	−0.13107368871285635	−0.1291550605636387	0.0019186281492176438
30	−0.4657496564370769	−0.4635706663352717	0.002178990101805167
40	−0.4356924771688938	−0.4334637584494958	0.002228718719398004
50	−0.06289263814986465	−0.06340077978309462	0.0005081416332299721
60	0.3610442523582828	0.35565425149696445	0.005390000861318356
70	0.5029857461263576	0.500386739396239	0.0025990067301185915
80	0.2500442070772532	0.2492719030498598	0.0007723040273934223
90	−0.20066057864782982	−0.19307043080206004	0.007590147845769779
100	−0.4956085135621511	−0.48979923333890363	0.005809280223247448

For more clarification, the stability diagrams are plotted to display the stability configuration due to the influence of the variation of the various physical parameters. Therefore, the stability configurations are plotted to show the impacts of the parameters; r, k, and τ in Figs. (3), (4), and (5). In this regard, Fig. (3) is sketched to illustrate the influence of the radius of the rolling wheel. As seen this parameter has a destabilizing influence. The destabilizing impact of a rotating wheel, especially in the context of automobiles, is caused by a phenomenon known as gyroscopic effect or gyroscopic precession. When a wheel is in motion, it produces angular momentum, which enhances stability within the rotational plane. However, if not adequately balanced, this can also increase the likelihood of tipping over for systems such as bicycles or motorcycles. The gyroscopic effect serves to stabilise the wheel against minor disturbances, but it can also present difficulties in maintaining balance when steering or when external forces, such as bumps or side winds, impact the wheel. The interplay between the steering forces and the wheel's angular momentum can result in a phenomenon called "speed vibrate," characterized by wobbling or oscillation. If left uncontrolled, this can have a destabilizing effect. Furthermore, the area of contact between the wheel and the ground might move as the wheel is rolling, causing alterations in the direction of the frictional force. If these variations are not accounted for by the rider or the design of the vehicle, they can potentially disrupt the motion and make it less stable.

Similarly, the stiffness of the light spring has a destabilizing impact as seen in Fig. (4). The destabilizing effect of the rigidity of a lightweight spring generally pertains to the inclination of a system with a stiff (high spring constant) spring to become less stable under specific circumstances. Within mechanical systems, a spring with higher stiffness applies a larger force when subjected to a specific displacement, resulting in more prominent oscillations or vibrations. If these oscillations are not adequately attenuated, they can amplify in magnitude, potentially resulting in instability. Stiff springs, whether used in control systems or structural dynamics, can enhance the sensitivity of a system to external forces or disturbances. The heightened sensitivity might result in a more intense response of the system to perturbations, so making it more challenging to maintain balance and perhaps leading to instability.

In contrast, the time-delay parameter has a stabilizing influence as seen in Fig. (5). The stabilizing effect of time delay pertains to the occurrence where the introduction of a delay in the feedback loop of a system can result in enhanced stability under specific circumstances. Although delays are commonly linked to destabilizing effects, this may appear counterintuitive. However, in certain systems, especially in control systems or in models of population dynamics, the introduction of a time delay can mitigate oscillations or decrease the magnitude of fluctuations, thus leading to the stabilization of the system. The stabilization is a result of the delay's ability to efficiently mitigate abrupt fluctuations in the system's condition, hence limiting excessive responses to disruptions. By reducing the speed of the feedback, the system is less prone to quick oscillations or chaotic behavior that could arise from immediate feedback, thereby improving overall stability.

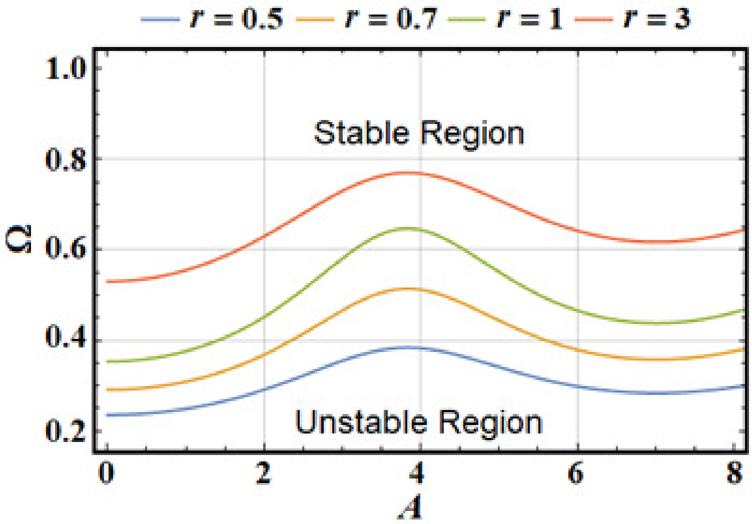

Figure 3: Displays the influence of the radius of the rolling wheel in the stability profile.

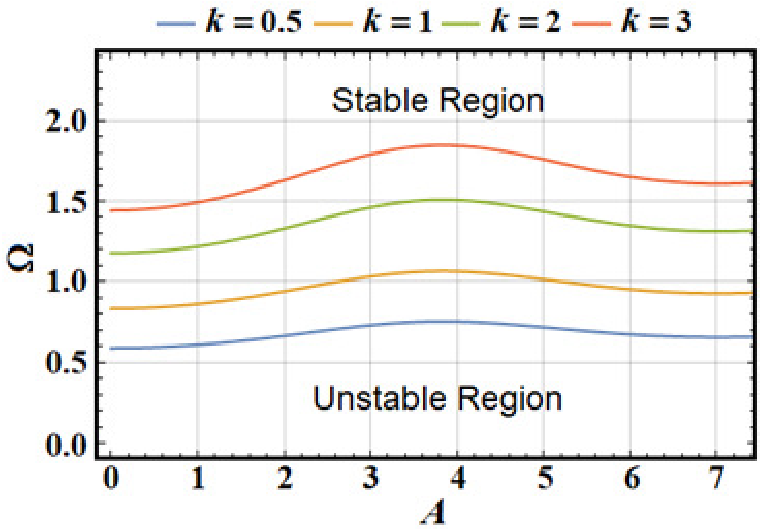

Figure 4: Illustrates how the spring's stiffness affects the stability profile.

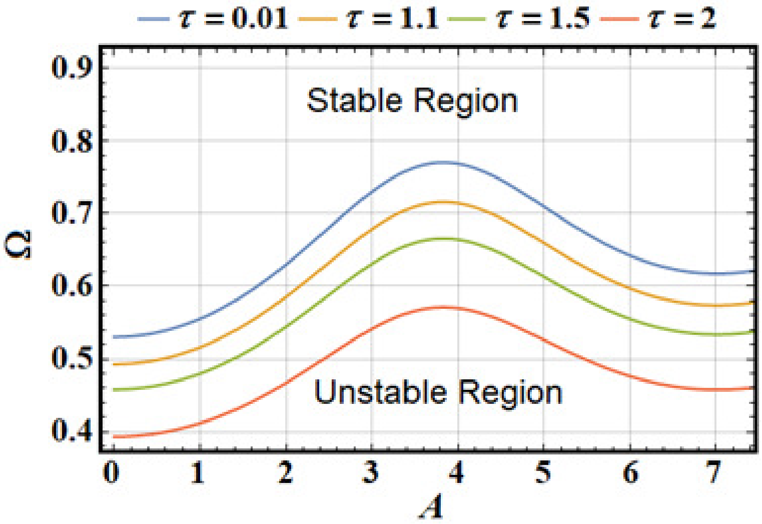

Figure 5: Demonstrates the impact of the time-delay parameter on the stability profile.

3.2. The Non-Existence of a Time-Delay

Simply setting $\tau = 0$ and using the same justifications as in the preceding subsection, one finds that the damping term disappears. Additionally, some modification is found in the equivalent frequency. In this scenario, the total frequency has the same form as the equivalent frequency.

Therefore, it is given as: $\Omega = \frac{r\sqrt{k}}{\sqrt{1+r^2}\sqrt{1+2rJ_0(A)/(1+r^2)}}$. The same information from Fig. (2) is presented in Fig. (6) but without the time-delay aspect. The Mathematica software 12.0.0.0 additionally established that the absolute error concerning the theoretical and computational solutions is 0.00130537 up to a temporal interval of 100 units. The convergence of linear and nonlinear solutions signifies that, despite their distinct definitions, they occupy identical spatial places. This signifies that the two curves represent the same geometric shape or path. It often suggests that the underlying physical processes, constraints, or relationships driving the curves are similar, or that one curve signifies a transition or specific case of the other.

4. Discussion of the Results

The current section presents a discussion of the achieved results which are drawn in Figs. (7), (8), and (9). The graphed curves in Fig. (7) are calculated when $k = 0.5, A = 0.5$, and $\tau = 0.01$ at various values of $r (= 0.5, 0.7, 0.9)$, whereas the drawn ones in Fig. (8) are plotted when the values $r = 0.5, A = 0.5$, $\tau = 0.01$, and $k (= 0.5, 0.7, 1)$ are considered. Furthermore, the inspection on the graphed curves in Fig. (9) are explored when $r = A = k = 0.5$ and $\tau (= 0.001, 0.01, 0.05)$. The figures show the time histories of the solution of Eq. (18) and the corresponding phase plane diagrams as indicated, respectively,

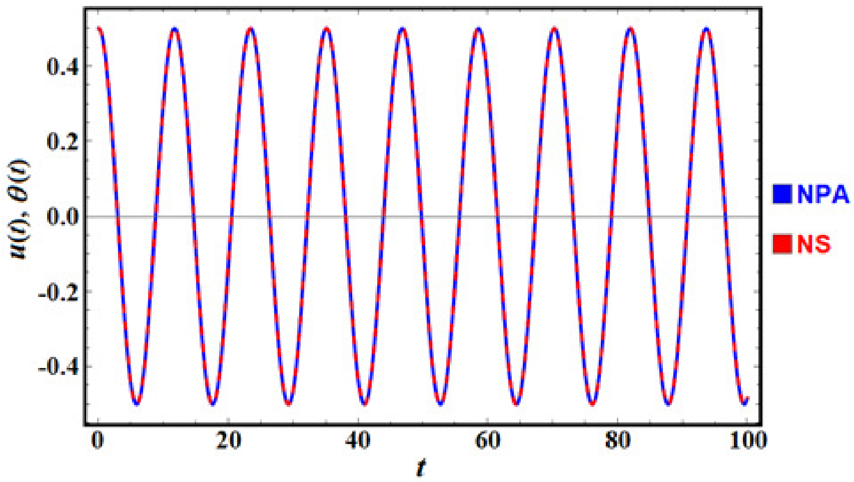

Figure 6: Sketches a comparison between the NS of Eq. (9) in the absence of time delay and the NPS of Eq. (19).

in parts (a) and (b) of these figures. It is observed that the behavior of the waves describing the obtained solution using NPA has periodicity manner, which give an induction about the stationary behavior of the obtained solution. Moreover, the amplitudes of the drawn waves besides the oscillation's number increase, to some extent, with the increase of r and k values, as illustrated in Figs. (7a) and (8a), respectively. The increase of τ values produces standing periodic waves with some distinct nodes, where the amplitudes of the drawn curves are changed slightly with the increase of τ values, as seen Fig. (9a).

Based on the above, one can conclude that the obtained solution using NPA has a stable performance and is free of chaos. This supposition has been asserted through the graphed curves of the corresponding phase plane of the NPA, as seen in Figs. (7b), (8b), and (9b), in which closed curves are plotted for the identical considered values of r, k, and τ, respectively.

Within dynamical systems analysis, the phase plane appears as a surface on which the interaction of variables can be graphically represented. The plotted closed curves provide an interesting story about equilibrium and periodicity inside these systems. These curves, which elegantly repeat themselves along axes of symmetry, capture the essence of stability and cyclic behavior. Their symmetry is evidence of the system's intrinsic balance, where every closed curve draws a line that loops back onto itself with ease.

The notions of stability and instability zones hold significant relevance across diverse disciplines including engineering, physics, and ecology. These zones help to determine the conditions under which a system will remain stable or become unstable. Stability criteria are used to analyze the stability of a system and predict its behavior under different circumstances. To draw the stability and instability areas, conditions (22) are plotted when r and k have the same considered values above to produce the stable regions and the unstable ones, as seen in parts of Fig. (10). It noted that the stable zones diminish as the values of parameters r and k increase, as depicted in sections (a) and (b) of this figure, respectively.

In dynamic systems, especially within control theory or ODEs involving delays, time delay can profoundly influence the stability of the system. Here is a concise elucidation:

1. In instances of brief time delays, the feedback loop promptly adjusts to alterations, enabling the system to respond nearly instantaneously to its prior conditions. This may induce quick oscillations and overreaction, resulting in unstable behavior, as the system lacks sufficient time to stabilize before the subsequent input, influenced by outdates data, impacts it. The system becomes too responsive, exacerbating errors or disturbances rather than mitigating them.

2. A prolonged latency in the feedback loop integrates antiquated data, hence impeding the system's responsiveness. This delay enables the system to respond more gradually, mitigating oscillations and averting excessive activity. The postponed reaction serves as a damping mechanism, successfully stabilizing the system by inhibiting immediate alterations that could otherwise lead to instability.

In conclusion, brief delays may induce instability through excessively quick feedback and adjustments, whereas prolonged delays might enhance system stability by tempering responses to changes, so enabling better absorption of shocks.

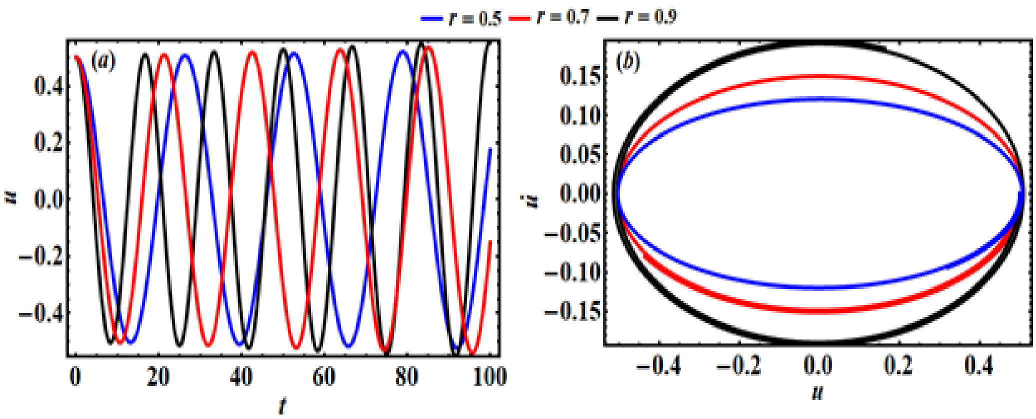

Figure 7: (a) Presents the time history of the obtained solution using NPA at $r(= 0.5, 0.7, 0.9)$, and (b) the corresponding phase plane curves.

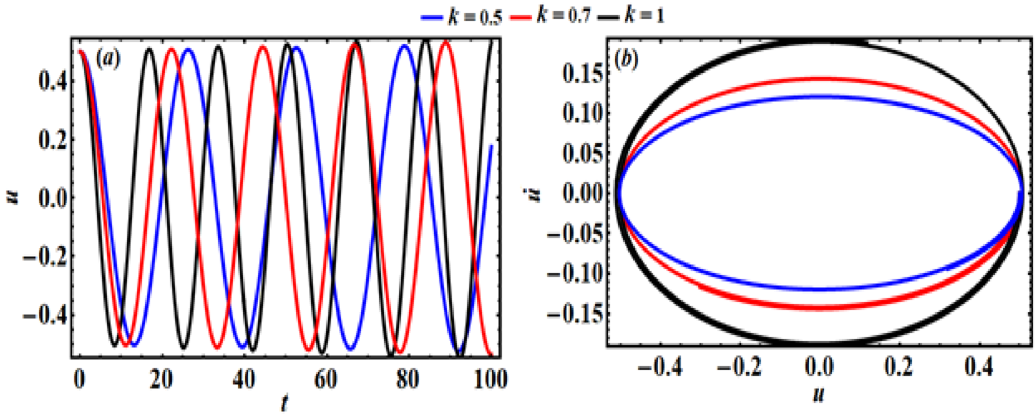

Figure 8: (a) Shows the behavior of $u(t)$ at $k(= 0.5, 0.7, 1)$, and (b) the related phase plane curves.

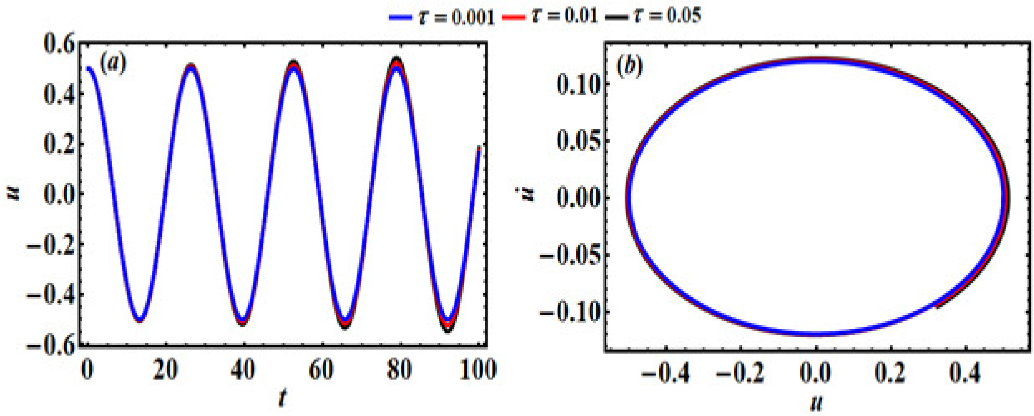

Figure 9: (a) Describes the behavior of $u(t)$ at $\tau(= 0.001, 0.01, 0.05)$, and (b) the related phase plane curves.

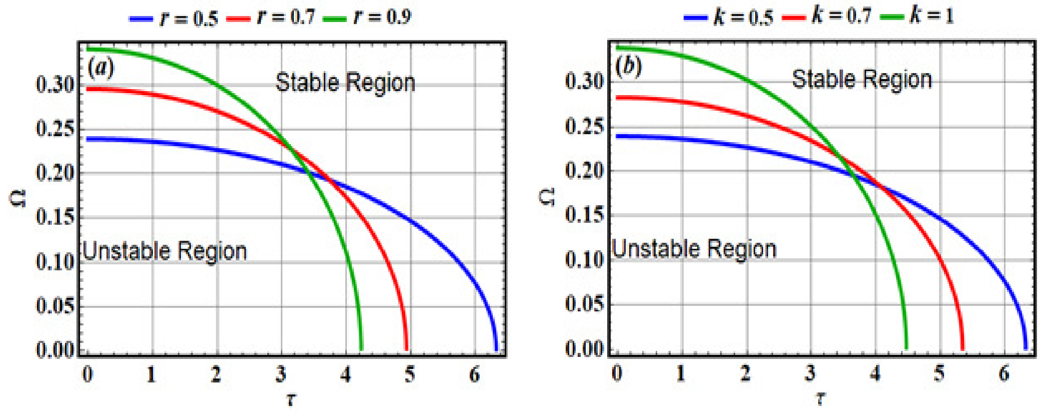

Figure 10: Shows stable and unstable regions: (a) at $r(= 0.5, 0.7, 0.9)$ and (b) at $k(= 0.5, 0.7, 1)$.

5. Conclusions

The current investigation centers on the analysis of the motion of a SP that is linked to a rolling wheel and attached to a spring that has a slight weight. The rationale behind the current subject is seen in the potential applications of the SP in the fields of physics, manufacturing, and practical mechanisms. As well-known classical perturbation methods are often employ Taylor expansion to simplify the current situation by increasing the existing restoring forces. Conversely, the NPA as a novel methodology converts the nonlinear ODE into a linear one. The NPA concept primarily relies on the HFF. It also allows for the assessment of the stability of the SP. Therefore, while analyzing approximations for highly nonlinear oscillators in the MS, the NPA proves to be a more helpful instrument for guaranteeing precision and dependability. To guarantee the precision of the outcomes, a quantitative approach utilizing the MS is utilized to attain a strong level of concurrence between the two systems. The derived analytical solution serves as a foundation for comprehending the interrelated nonlinear dynamics of the oscillators. Numerical computations are visually performed to validate the new approach and assess the effectiveness and applicability of the strategy. The results were compared to the precise numerical responses and demonstrated complete accuracy. The technique is widely relevant in engineering and other fields since it can be easily adapted to many nonlinear systems. The empirical approximation inquiry allows for a qualitative analysis of the findings. The temporal results are examined for different values of the physical frequency and time-delay parameters. The results are determined based on the visible curves. Graphical representations are employed to illustrate the influence of parameters on the behavior of motion. These representations consist of solutions that have been developed over a period of time, together with their related phase plane graphs. Furthermore, stability and instability zones have been drawn and examined. Furthermore, the NPA can be expanded to encompass further categories within the realm of couples of dynamical systems and is considered significant, productive, and captivating. The main outcomes derived from this study can be summarized as follows:

i. The Taylor expansion is used by all conventional techniques to make the provided problem simpler when restoring forces are involved. This drawback is no longer present with the adopted scheme.

ii. This approach, in contrast to earlier conventional methods, enables us to examine the instability analysis of the problem.

iii. In conclusion, it appears that the novel technique is a straightforward, valuable, and efficient tool. It has the potential to be employed in the analysis of various classifications of nonlinear oscillations.

iv. Based on the numerical calculations and the stability requirements, it is discovered that the parameters $r$and k have a dual role in the stability configuration.

In light of the great significance in the coupled system in various classes of the dynamical system, in the future work, the NPA will be developed to analyze such problems. Fractal oscillators are technical analysis tools used in financial markets to identify turning points or reversals in price movements by examining patterns that repeat at different scales. Therefore, another aim of the progress works is to inspect the fractal oscillators.

Authors Statements

Khalid Alluhydan: Methodology, Resources, Validation, Formal Analysis, Visualization and Reviewing.

Galal M. Moatimid: Resources, Conceptualization, Formal Analysis, Methodology, Validation, Writing- Original draft preparation, Visualization and Reviewing.

T.S. Amer: Methodology, Investigation, Conceptualization, Data curation, Validation, Reviewing and Editing.

Conflict of Interest

The authors have disclosed no competing interests.

Data Availability

This work contains all of the data that were created or examined during this study.

References

[1] Abu-As'ad A. and Asad J. Power series approach to nonlinear oscillators. *Journal of Low Frequency Noise, Vibration and Active Control*, 43(1):220–238, 2024.

[2] Big-Alabo A. Approximate period for large-amplitude oscillations of a simple pendulum based on quintication of the restoring force. *European Journal of Physics*, 41(1):015001, 2020.

[3] Chatterjee A. and Meng W. Nonlinear dynamics of interacting population in a marine ecosystem with a delay effect. *Nonlinear Dynamics*, 112:16639–16656, 2024.

[4] Nayfeh A.H. *Perturbation methods.* Wiley, New York, 1973.

[5] Rahman B. Time-delay systems: An overview. *Nonlinear Phenomena in Complex Systems*, 23(2):192–195, 2020.

[6] Rath B., Erturk V.S., Asad J., Mallick P., and Jarrar R. An asymmetric model two-dimensional oscillator. *Journal of Low Frequency Noise, Vibration and Active Control*, 43(2):744–754, 2024.

[7] He C-H. and Liu C. A modified frequency-amplitude formulation for fractal vibration systems. *Fractals*, 30(03):2250046, 2022.

[8] Guiot E. Trajectories of two-dimensional harmonic oscillators in a rotating frame: application to foucault pendulum problem. *Meccanica*, 59:491–501, 2024.

[9] De Oliveira E.F. and Rodrigues C.G. Simple pendulum: Period dependent on amplitude of oscillation. *Cuadernos de Educación y Desarrollo*, 15(12):17662–17685, 2023.

[10] Ismail G.M., Moatimid G.M., and Yamani M.I. Periodic solutions of strongly nonlinear oscillators using he's frequency formulation. *European Journal of Pure and Applied Mathematics*, 17(3):2154–2171, 2024.

[11] Moatimid G.M. and Sayed A. Nonlinear ehd stability of a cylindrical interface separating two rivlin-ericksen fluids: A novel analysis. *Chinese Journal of Physics*, 87:379–397, 2024.

[12] Moatimid G.M., El-Sayed A.T., and Salman H.F. Dynamical analysis of an inverted pendulum with positive position feedback controller approximate uniform solution. *Scientific Reports*, 13:8849, 2023.

[13] Moatimid G.M., El-Sayed A.T., and Salman H.F. Different controllers for suppressing oscillations of a hybrid oscillator via non-perturbative analysis. *Scientific Reports*, 14:307, 2024.

[14] Moatimid G.M. and Mostafa D.M. Nonlinear stability of two superimposed electrified dusty fluids of type rivlin-ericksen: Non-perturbative approach. *Partial Differential Equations in Applied Mathematics*, 10:100745, 2024.

[15] Moatimid G.M., Mostafa D.M., and Zekry M.H. A new methodology in evaluating nonlinear electrohydrodynamic azimuthal stability between two dusty viscous fluids. *Chinese Journal of Physics*, 90:134–154, 2024.

[16] Moatimid G.M., Mohamed M.A.A., and Elagamy Kh. Nonlinear kelvin-helmholtz instability of a horizontal interface separating two electrified walters' b liquids: A new approach. *Chinese Journal of Physics*, 85:629–648, 2023.

[17] Moatimid G.M., Mohamed M.A.A., and Elagamy Kh. An innovative approach in inspecting a damped mathieu cubic-quintic duffing oscillator. *Journal of Vibration Engineering & Technologies*, 2024. Published online: 23 July.

[18] Moatimid G.M., Mohamed M.A.A., and Elagamy Kh. Insightful inspection of the nonlinear instability of an azimuthal disturbance separating two rotating magnetic liquid columns. *The European Physical Journal Plus*, 139:590, 2024.

[19] Moatimid G.M., Mohamed M.A.A., and Elagamy Kh. Inspection of the nonlinear instability of electrified casson fluids: a novel approach. *Waves in Random and Complex Media*, 2024. Published online: 02 August.

[20] Moatimid G.M. and Amer T.S. Analytical solution for the motion of a pendulum with rolling wheel: stability analysis. *Scientific Reports*, 12:12628, 2022.

[21] Moatimid G.M. and Amer T.S. Analytical approximate solutions of a magnetic spherical pendulum: Stability analysis. *Journal of Vibration Engineering & Technologies*, 11:2155–2165, 2023.

[22] Moatimid G.M. and Amer T.S. Dynamical system of a time-delayed ϕ^6-van der pole oscillator: A non-perturbative approach. *Scientific Reports*, 13:11942, 2023.

[23] Moatimid G.M., Amer T.S., and Galal A.A. Studying highly nonlinear oscillators using the non-perturbative methodology. *Scientific Reports*, 13:20288, 2023.

[24] Moatimid G.M., Amer T.S., and Galal A.A. Inspection of some extremely nonlinear oscillators using an inventive approach. *Journal of Vibration Engineering & Technologies*, 2024. Published online: 06 July.

[25] Moatimid G.M., Amer T.S., and Ellabban Y.Y. A novel methodology for a time-delayed controller to prevent nonlinear system oscillations. *Journal of Low Frequency Noise, Vibration and Active Control*, 43(1):525–542, 2024.

[26] Moatimid G.M. and Mohamed Y.M. A novel methodology in analyzing nonlinear stability of two electrified viscoelastic liquids. *Chinese Journal of Physics*, 89:679–706, 2023.

[27] Moatimid G.M. and Mohamed Y.M. Nonlinear electro-rheological instability of two moving cylindrical fluids: An innovative approach. *Physics of Fluids*, 36(2):024110, 2024.

[28] Ahmad H., Khan T.A., Stanimirović P. S., Chu Y-M., and Ahmad I. Modified variational iteration algorithm-ii: Convergence and applications to diffusion models. *Complexity*, 2020:Article ID 8841718, 14 Pages, 2020.

[29] Fan J. He's frequency–amplitude formulation for the duffing harmonic oscillator. *Computers & Mathematics with Applications*, 58(11-12):2473–2476, 2009.

[30] Shaik J., Uchida T.K., and Vyasarayani C.P. Nonlinear dynamics near a double hopf bifurcation for a ship model with time-delay control. *Nonlinear Dynamics*, 111:21441–21460, 2023.

[31] He J-H. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, 178:257–262, 1999.

[32] He J-H. Some asymptotic methods for strongly nonlinear equations. *International Journal of Modern Physics B*, 20(10):1141–1199, 2006.

[33] He J-H. Comment on 'he's frequency formulation for nonlinear oscillators'. *European Journal of Physics*, 29(4):L19, 2008.

[34] He J-H. An improved amplitude-frequency formulation for nonlinear oscillators. *International Journal of Nonlinear Sciences and Numerical Simulation*, 9(2):211–212, 2008.

[35] He J-H. Amplitude-frequency relationship for conservative nonlinear oscillators with odd nonlinearities. *International Journal of Applied and Computational Mathematics*, 3:1557–1560, 2017.

[36] He J-H. Special functions for solving nonlinear differential equations. *International Journal of Applied and Computational Mathematics*, 7:84, 2021. 6 Pages.

[37] He J-H. Periodic solution of a micro-electromechanical system. *Facta Universities Series: Mechanical Engineering*, 22(2):187–198, 2024.

[38] He J-H. and García A. The simplest amplitude-period formula for non-conservative oscillators. *Reports in Mechanical Engineering*, 2(1):143–148, 2021.

[39] He J-H., He C-H., Qian M-Y., and Alsolami A.A. Piezoelectric biosensor based on ultrasensitive mems system. *Sensors and Actuators A: Physical*, 376:115664, 2024.

[40] Alluhydan Kh., Moatimid G.M., Amer T.S., and Galal A.A. Inspection of a time-delayed excited damping duffing oscillator. *Axioms*, 13(6):416, 2024.

[41] Cvetićanin L. Oscillator with strong quadratic damping force. *Publications de L'Institut Mathématique*, 85(99):119–130, 2009.

[42] Geng L. and Cai X-C. He's frequency formulation for nonlinear oscillators. *European Journal of Physics*, 28(5):923, 2007.

[43] Dehghan M. and Ghesmati A. Application of the dual reciprocity boundary integral equation technique to solve the nonlinear klein–gordon equation. *Computer Physics Communications*, 181(8):1410–1418, 2010.

[44] Khan M.N., Haider J.A., Wang Z., Lone S.A., Almutlak S.A., and Elseesy I.E. Application of laplace-based variational iteration method to analyze generalized nonlinear oscillations in physical systems. *Modern Physics Letters B*, 37(34):2350169, 2023.

[45] Elías-Zú niga A. Exact solution of the cubic-quintic duffing oscillator. *Applied Mathematical Modelling*, 37(4):2574–2579, 2013.

[46] Elías-Zú niga A., Palacios-Pineda L.M., Jiménez-Cede no I.H., Martínez-Romero O., and Olvera-Trejo D. Enhanced he's frequency-amplitude formulation for nonlinear oscillators. *Results in Physics*, 19:103626, 2020.

[47] Spanos P.T.D. and Iwan W.D. On the existence and uniqueness of solutions generated by equivalent linearization. *International Journal of Non-Linear Mechanics*, 13(2):71–78, 1979.

[48] Caughey T.K. Equivalent linearization techniques. *The Journal of the Acoustical Society of America*, 35(11):1706–1711, 1963.

[49] Mao X. and Ding W. Dynamics of a nonlinear vibration absorption system with time delay. *Nonlinear Dynamics*, 112:5177–5193, 2024.

[50] Cai X-C. and Liu J-F. Application of the modified frequency formulation to a non-linear oscillator. *Computers and Mathematics with Applications*, 61:2237–2240, 2011.

[51] Cai X-C. and Wu W-Y. He's frequency formulation for the relativistic harmonic oscillator. *Computers & Mathematics with Applications*, 58(11-12):2358–2359, 2009.

[52] Geng Y. Exact solutions for the quadratic mixed-parity helmholtz–duffing oscillator by bifurcation theory of dynamical systems. *Chaos, Solitons & Fractals*, 81(A):68–77, 2015.

[53] Tian Y. Frequency formula for a class of fractal vibration system. *Reports in Mechanical Engineering*, 3(1):55–61, 2022.

[54] Tian Y. and Wang Z. A new multiple integral inequality and its application to stability analysis of time-delay systems. *Applied Mathematics Letters*, 105:106325, 2020.

[55] Tian Y. and Wang Z. Composite slack-matrix-based integral inequality and its application to stability analysis of time-delay systems. *Applied Mathematics Letters*, 120:107252, 2021.

[56] Ren Z. Theoretical basis of he's frequency-amplitude formulation for nonlinear oscillators. *Nonlinear Science Letters A*, 9(1):86–90, 2018.

[57] Ren Z-F., Liu G-Q., Kang Y-X., Fan H-Y., Li H-M., Ren X-D., and Gui W-K. Application of he's amplitude-frequency formulation to nonlinear oscillators with discontinuities. *Physica Scripta*, 80(4):045003, 2009.