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A Generalization for Somewhere Dense Sets with Some Applications

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Abstract. In this paper, we present a new generalization for somewhere dense of a topological space $(\mathcal{Z}, \tau)$, namely $\omega\mathcal{SWD}$-open subsets. We introduce the concept of this family and discuss some of their properties with the help of illustrative example. Moreover, we will show if the space $(\mathcal{Z}, \tau)$ is anti-locally countable and τ is finer than the cocountable topology then the class of $\omega\mathcal{SWD}$-open and somewhere dense subsets of $(\mathcal{Z}, \tau)$ will be equivalent. Moreover, we present more properties for the class of somewhere dense subsets of $(\mathcal{Z}, \tau)$, the most important of which is a generalization for a theorem in [1]. Furthermore, we finish this work by shedding light on one type of covering properties where we study the notion of almost $\omega\mathcal{SWD}$-compact spaces with some of their properties.

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1. Introduction

In recent decades, a major area of study for general topology researchers has been the study of various kinds of generalized open sets. Mathematicians examine various topological notions, such as continuity, compactness, etc. In 1937, stone [18] introduced the concept of regular open sets. In 1963, Levine [13] presented the notion of semi-open sets. In 1965, Njasted [16] introduced α-open sets. In 1982, Mashhour et al [15] introduced the concepts of pre-open and studied their topological properties. In 1983, Abd El-Monsef et al [9] studied the notion of β-open sets. In 1996, Andrijevic [6] defined and explored the idea of b-open sets. A subset H of a space $(\mathcal{Z}, \tau)$ is called a regular open (semi-open, α-open, pre-open, β-open , b-open) sets if $H = \mathcal{I}nt(\mathcal{C}l(H))$(resp., $H \subseteq \mathcal{C}l(\mathcal{I}nt(H))$, $H \subseteq \mathcal{I}nt(\mathcal{C}l(\mathcal{I}nt(H)))$, $H \subseteq \mathcal{I}nt(\mathcal{C}l(H))$, $H \subseteq \mathcal{C}l(\mathcal{I}nt(\mathcal{C}l(H)))$, $H \subseteq \mathcal{I}nt(\mathcal{C}l(H)) \cup \mathcal{C}l(\mathcal{I}nt(H))$).

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Another form of generalization of open sets that we need in this work is $\omega-$open sets. A subset H of a space $(\mathcal{Z},\tau)$ is called an $\omega-$closed [11] if it contains all its condensation points, where a point $x\in\mathcal{Z}$ is called a condensation point of H [10] if for each $G\in\tau$ with $x\in G$, the set $G\cap H$ is uncountable. The complement of an $\omega-$closed is called $\omega-$open. Moreover, in [5] the authors introduced an equivalent definition of $\omega-$open subsets, where $H\subseteq\mathcal{Z}$ is an $\omega-$open subsets of $(\mathcal{Z},\tau)$ if for each $x\in H$ there is $G\in\tau$ with $x\in G$ such that $G-H$ is countable. The study and exploitation of these generalizations have become very widespread and many works have been presented based on these sets, for example in [12] the authors presented some applications of pre-open sets, where they introduced and studied topological properties of pre-limit points, pre-interior and pre-closure and other topological notions [see [8],[14]].

In 2017, Al-Shami [1] examined and studied some main properties of somewhere dense sets on topological spaces where a subset $H\subseteq\mathcal{Z}$ is a somewhere dense set of $(\mathcal{Z},\tau)$ if there is a non-empty open set G with $G\subseteq\mathcal{C}l(H)$ which is equivalent to say $\mathcal{I}nt(\mathcal{C}l(H))$ is a non-empty set. Moreover, he showed that, with the expectation of the empty set, all semi-open, α-open sets, pre-open, β-open, and b-open sets are contained in the class of somewhere dense sets. Then, Al-Shami and Noiri [3] used the class of somewhere dense sets to define the concept of $\mathcal{SWD}$-continuous and $\mathcal{SWD}$-homeomorphism functions. Then in [4], they introduced and investigated the notions of almost $\mathcal{SWD}$-compact, almost $\mathcal{SWD}$-lindelöf spaces, nearly $\mathcal{SWD}$-compact, nearly $\mathcal{SWD}$-lindelöf, mildly $\mathcal{SWD}$-compact and mildly $\mathcal{SWD}$-lindelöf spaces and they studied the relationships between them. Moreover, in [2] the author contributed to this area and used the notion of somewhere dense sets to improve the approximations and accuracy measure in rough set theory.

In this work, we study and present more properties of somewhere dense of $(\mathcal{Z},\tau)$. One of the most important of these properties is a generalization for a theorem that was introduced in [1]. Then, based on the class of all somewhere dense and $\omega-$open subsets of $(\mathcal{Z},\tau)$, we introduce and study the class of $\omega\mathcal{SWD}$-open subsets, which is a new generalization for somewhere dense of a topological space $(\mathcal{Z},\tau)$ and hence it is a new kind of generalized open sets. We organize this work as follows: In Sec. 2, we give more properties of somewhere dense sets of $(\mathcal{Z},\tau)$. In Sec. 3, we introduce the notion of $\omega\mathcal{SWD}$-open subsets and we verify some of basic properties of this class with the help of illustrative examples. Also, we investigate what are the conditions to become the class of $\omega\mathcal{SWD}$-open and somewhere dense subsets of $(\mathcal{Z},\tau)$ are equivalent. Then we show that this family is not a topology through an example that shows this family is not closed under finite intersection. Moreover, we use $\omega\mathcal{SWD}$-open subsets to generalize the notions of interior and closure and define $\omega\mathcal{SWD}$-continuous and $\omega\mathcal{SWD}$-irresolute. In Sec.4, we use $\omega\mathcal{SWD}$-open subsets and the closure operator which are discussed in Sec.3 to study one type of covering properties, namely almost $\omega\mathcal{SWD}$-compact spaces and study some of its properties.

Throughout this work the family of all somewhere dense sets of $(\mathcal{Z},\tau)$($resp.$, the family of all closed somewhere dense which is equivalent to the family of the complement of all somewhere dense of $(\mathcal{Z},\tau)$) is denoted by $\mathcal{SWD}(\mathcal{Z},\tau)$ $(resp., \mathcal{SWDC}(\mathcal{Z},\tau))$. Moreover, $\mathcal{SWD}$ interior of H (is denoted by, $\mathcal{I}nt_{\mathcal{SWD}}(H)$) is given by $\mathcal{I}nt_{\mathcal{SWD}}(H)=\cup\{G:G\subseteq H$

and $G \in \mathcal{SWD}(\mathcal{Z}, \tau)$ } and the $\mathcal{SWD}$ closure of H (is denoted by, $\mathcal{C}l_{\mathcal{SWD}}(H)$) is given by $\mathcal{C}l_{\mathcal{SWD}}(H) = \cap\{F : H \subseteq F$ and $F \in \mathcal{SWDC}(\mathcal{Z}, \tau)$ }. Also, the family of all $\omega-$open subsets of a space $(\mathcal{Z}, \tau)$ forms a topology on $\mathcal{Z}$ finer than τ and denoted by τ_ω[5]. The $\omega-$interior(resp., $\omega-$closure) of a subset H of a space $(\mathcal{Z}, \tau)$ is the interior (resp., closure) of H in the space $(\mathcal{Z}, \tau_\omega)$ and it is denoted by $\mathcal{I}nt_\omega(H)$(resp., $\mathcal{C}l_\omega(H)$).

In this paper, we will write $\mathcal{TS}$ instead of topological space. The sets $\mathbb{R}$ and $\mathbb{Q}$, respectively the set of real numbers and rational numbers. The cofinite topology, the cocountable topology, the indiscrete topology, and the usual topology are denoted by τ_{cof}, τ_{coc}, τ_{ind} and τ_u respectively. Also, if H is a subset of a space $(\mathcal{Z}, \tau)$, then the relative topology on H in $(\mathcal{Z}, \tau)$ will be denoted by τ_H.

Definition 1. *[10] A filter on $\mathcal{Z}$ is a family $\mathcal{F} \subseteq \mathcal{P}(\mathcal{Z})$ which satisfies the following:*
(i) $\phi \notin \mathcal{F}$.
(ii) If $H, G \in \mathcal{F}$, then $H \cap G \in \mathcal{F}$.
(iii) If $H \in \mathcal{F}$ and $H \subseteq G \subseteq \mathcal{Z}$, then $G \in \mathcal{F}$.

Moreover, A filter $\mathcal{F}$ on $\mathcal{Z}$ is said to be a maximal filter on $\mathcal{Z}$ if each filter $\mathcal{H}$ on $\mathcal{Z}$ that contains $\mathcal{F}$ we have $\mathcal{F} = \mathcal{H}$. Also, a family $\mathcal{F} \subseteq \mathcal{P}(\mathcal{Z})$ is said to be a filter base on $\mathcal{Z}$ if it is a non-empty such that $\phi \notin \mathcal{F}$ and if $H, G \in \mathcal{F}$ then there is $V \in \mathcal{F}$ with $V \subseteq H \cap G$.

Definition 2. *Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$. Then $(\mathcal{Z}, \tau)$ is said to be:*
(i) Hyperconnected [17] if no mutually disjoint non-empty open sets.
(ii) Strongly hyperconnected [1] if a subset of $\mathcal{Z}$ is dense iff it is non-empty and open.

Theorem 1. *[1] Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$ and $H, G \subseteq \mathcal{Z}$. If $H \in \mathcal{SWD}(\mathcal{Z}, \tau)$ and $(\mathcal{Z}, \tau)$ is:*
(i) hyperconnected, then $H \cap G \in \mathcal{SWD}(\mathcal{Z}, \tau)$ whenever $G \in \tau$.
(ii) strongly hyperconnected, then $H \cap G \in \mathcal{SWD}(\mathcal{Z}, \tau)$ whenever $G \in \mathcal{SWD}(\mathcal{Z}, \tau)$.

Definition 3. *[10] Let $\{(\mathcal{Z}_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a family of topological spaces with $\mathcal{Z}_\alpha \cap \mathcal{Z}_\beta = \phi$ for each $\alpha \neq \beta$. Let $\mathcal{Z} = \underset{\alpha\in\Delta}{\cup} \mathcal{Z}_\alpha$ with the topology $\tau_s = \{G \subseteq \mathcal{Z} : G \cap \mathcal{Z}_\alpha \in \tau_\alpha$ for each $\alpha \in \Delta\}$. Then $(\mathcal{Z}, \tau_s)$ is called the sum of the spaces $\{(\mathcal{Z}_\alpha, \tau_\alpha) : \alpha \in \Delta\}$and denoted by $\mathcal{Z} = \underset{\alpha\in\Delta}{\oplus} \mathcal{Z}_\alpha$.*

Theorem 2. *[1] Let $(\prod\limits_{\alpha=1}^{n} \mathcal{Z}_\alpha, \tau)$ be a finite product $\mathcal{TS}$. Then $H_\alpha \in \mathcal{SWD}(\mathcal{Z}_\alpha, \tau_\alpha)$ for each α = 1, 2, ..., n, iff $\prod\limits_{\alpha=1}^{n} H_\alpha \in \mathcal{SWD}(\prod\limits_{\alpha=1}^{n} \mathcal{Z}_\alpha, \tau)$.*

Theorem 3. *Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$ and $H, G \subseteq \mathcal{Z}$. Then:*
(i) If $H \subseteq G$ and $H \in \mathcal{SWD}(\mathcal{Z}, \tau)$, then $G \in \mathcal{SWD}(\mathcal{Z}, \tau)$ [1].
(ii) If $E \in \tau$ and $H \subseteq E$, then $H \in \mathcal{SWD}(\mathcal{Z}, \tau)$ whenever $H \in \mathcal{SWD}(E, \tau_E)$ [7].

Definition 4. *[5] Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$. Then $(\mathcal{Z}, \tau)$ is said to be anti-locally countable if each non-empty open subset of $(\mathcal{Z}, \tau)$ is uncountable.*

Note that, if $(\mathcal{Z}, \tau)$ is an anti locally countable space, then $(\mathcal{Z}, \tau_\omega)$ is also anti-locally countable.

Definition 5. *[4] Let $(\mathcal{Z},\tau)$ be a $\mathcal{TS}$. Then:*

(i) A family $\mathcal{H}=\{H_\alpha : \alpha\in\Delta\}$ is said to be $\mathcal{SWD}(\mathcal{Z},\tau)$-cover of $\mathcal{Z}$ if $\mathcal{Z}=\bigcup\limits_{\alpha\in\Delta} H_\alpha$ with $H_\alpha\in\mathcal{SWD}(\mathcal{Z},\tau)$.

(ii) $(\mathcal{Z},\tau)$ is said to be almost $\mathcal{SWD}$-compact if for each $\mathcal{SWD}(\mathcal{Z},\tau)$-cover $\mathcal{H}=\{H_\alpha : \alpha\in\Delta\}$ of $\mathcal{Z}$ there is a finite subset $\Delta_\circ\subseteq\Delta$ with $\mathcal{Z}=\bigcup\limits_{\alpha\in\Delta_\circ} Cl_{\mathcal{SWD}}(H_\alpha)$.

2. More Properties of Somewhere Dense sets

In this section, we examine further properties of somewhere dense of a topological space $(\mathcal{Z},\tau)$.

Proposition 1. *Let $(\mathcal{Z},\tau)$ and $(\mathcal{K},\sigma)$ be two $\mathcal{TS}$s and $\Gamma:(\mathcal{Z},\tau)\to(\mathcal{K},\sigma)$ be a continuous, open and surjective function. If $H\subseteq\mathcal{Z}$ and $H\in\mathcal{SWD}(\mathcal{Z},\tau)$, then $\Gamma(H)\in\mathcal{SWD}(\mathcal{K},\sigma)$.*

Proof. Since $H\in\mathcal{SWD}(\mathcal{Z},\tau)$, then there is $G\in\tau$ with $\phi\neq G\subseteq\mathcal{C}l(H)$. Therefore, $\phi\neq\Gamma(\mathcal{I}nt(\mathcal{C}l(G))\subseteq\mathcal{I}nt(\Gamma(\mathcal{C}l(H))\subseteq\mathcal{I}nt(\mathcal{C}l((\Gamma(H))$ and hence $\Gamma(H)\in\mathcal{SWD}(\mathcal{K},\sigma)$.

Theorem 4. *Let $\{(\mathcal{Z}_\alpha,\tau_\alpha):\alpha\in\Delta\}$ be a family of topological spaces with $\mathcal{Z}_\alpha\cap\mathcal{Z}_\beta=\phi$ for each $\alpha\neq\beta$. For each $\alpha\in\Delta$, let $\phi\neq H_\alpha\subseteq\mathcal{Z}_\alpha$ and put $H=\bigcup\limits_{\alpha\in\Delta} H_\alpha$. Then:*

(i) $H\in\mathcal{SWD}(\mathcal{Z},\tau_s)$ iff there is $\alpha_\circ\in\Delta$ with $H_{\alpha_\circ}\in\mathcal{SWD}(\mathcal{Z}_{\alpha_\circ},\tau_{\alpha_\circ})$.

(ii) If $H_\alpha\in\mathcal{SWD}(\mathcal{Z}_\alpha,\tau_\alpha)$ for each $\alpha\in\Delta$, then $H\in\mathcal{SWD}(\mathcal{Z},\tau_s)$.

Proof. (i) First, note that for all $\alpha\in\Delta$, $\mathcal{C}l_\alpha(H_\alpha)=\mathcal{C}l(H_\alpha)$ where $\mathcal{C}l_\alpha(H_\alpha)$ is the closure of H_α in $\mathcal{Z}_\alpha$ while $\mathcal{C}l(H_\alpha)$ is the closure of H_α in $\mathcal{Z}$. Now, choose $\alpha_\circ\in\Delta$ with $H_{\alpha_\circ}\in\mathcal{SWD}(\mathcal{Z}_{\alpha_\circ},\tau_{\alpha_\circ})$. Then there is $G_{\alpha_\circ}\in\tau_{\alpha_\circ}$ with $\phi\neq G_{\alpha_\circ}\subseteq\mathcal{C}l_{\alpha_\circ}(H_{\alpha_\circ})=\mathcal{C}l(H_{\alpha_\circ})\subseteq\mathcal{C}l(H)$ and hence $H\in\mathcal{SWD}(\mathcal{Z},\tau_s)$. Conversely, since $H\in\mathcal{SWD}(\mathcal{Z},\tau_s)$ and the family $\{H_\alpha:\alpha\in\Delta\}$ is locally finite in $(\mathcal{Z},\tau_s)$, then there is $G\in\tau_s$ with $\phi\neq G\subseteq\mathcal{C}l(H)=\mathcal{C}l(\bigcup\limits_{\alpha\in\Delta}H_\alpha)=\bigcup\limits_{\alpha\in\Delta}\mathcal{C}l(H_\alpha)=\bigcup\limits_{\alpha\in\Delta}\mathcal{C}l_\alpha(H_\alpha)$. Since $\phi\neq G$, choose $x_{\alpha_\circ}\in G$ for some $\alpha_\circ\in\Delta$. Then $G\cap\mathcal{Z}_{\alpha_\circ}$ is a non-empty set in $(\mathcal{Z}_{\alpha_\circ},\tau_{\alpha_\circ})$ such that $G\cap\mathcal{Z}_{\alpha_\circ}\subseteq\bigcup\limits_{\alpha\in\Delta}\mathcal{C}l_\alpha(H_\alpha)\cap\mathcal{Z}_{\alpha_\circ}=\mathcal{C}l_{\alpha_\circ}(H_{\alpha_\circ})$. Therefore, $H_{\alpha_\circ}\in\mathcal{SWD}(\mathcal{Z}_{\alpha_\circ},\tau_{\alpha_\circ})$.

(ii) Follows from part (i).

The following theorem is one of the most important results that we present in this section. Since Theorem 5 (Part ii) is a generalization of Theorem 2.

Theorem 5. *Let $\mathcal{Z}=\prod\limits_{\alpha\in\Delta}\mathcal{Z}_\alpha$ be the product space of the spaces $(\mathcal{Z}_\alpha,\tau_\alpha),\alpha\in\Delta$ with the Tychonoff topology τ_p. Let $H_\alpha\subseteq Z_\alpha$ for each $\alpha\in\Delta$. Then the following are equivalent:*

(i) $H_\alpha\in\mathcal{SWD}(\mathcal{Z}_\alpha,\tau_\alpha)$ for each $\alpha\in\Delta$.

(ii) For each finite subset $\Delta^*\subseteq\Delta$, the set $H=\prod\limits_{\alpha\in\Delta^*}H_\alpha\times\prod\limits_{\beta\in\Delta-\Delta^*}\mathcal{Z}_\beta\in\mathcal{SWD}(\mathcal{Z},\tau_p)$.

(iii) For each $\alpha\in\Delta$, the set $H=H_\alpha\times\prod\limits_{\substack{\beta\in\Delta\\ \beta\neq\alpha}}\mathcal{Z}_\beta\in\mathcal{SWD}(\mathcal{Z},\tau_p)$.

Proof. The implication $(ii\longrightarrow iii)$ is obvious.

$(i \longrightarrow ii)$ For each $\alpha \in \Delta^*$, there is $G_\alpha \in \tau_\alpha$ with $\phi \neq G_\alpha \subseteq \mathcal{Z}_\alpha$ and $G_\alpha \subseteq \mathcal{C}l(H_\alpha)$. Then $G = \prod_{\alpha\in\Delta^*} G_\alpha \times \prod_{\beta\in\Delta-\Delta^*} \mathcal{Z}_\beta$ is a non-empty open set in $(\mathcal{Z}, \tau_p)$ such that $G \subseteq \prod_{\alpha\in\Delta^*} \mathcal{C}l(H_\alpha) \times \prod_{\beta\in\Delta-\Delta^*} \mathcal{Z}_\beta = \mathcal{C}l(\Pi(\prod_{\alpha\in\Delta^*} H_\alpha) \times \prod_{\beta\in\Delta-\Delta^*} \mathcal{Z}_\beta) = \mathcal{C}l(H)$. Therefore, $H \in \mathcal{SWD}(\mathcal{Z}, \tau_p)$.

$(iii \to i)$ Since for each $\alpha \in \Delta$, the projection function $\pi_\alpha : (\mathcal{Z}, \tau_p) \to (\mathcal{Z}_\alpha, \tau_\alpha)$ is continuous, open and surjective such that $\pi_\alpha(H_\alpha \times \prod_{\substack{\beta\in\Delta \\ \beta\neq\alpha}} \mathcal{Z}_\beta) = H_\alpha$, then by Proposition 1, $H_\alpha \in \mathcal{SWD}(\mathcal{Z}_\alpha, \tau_\alpha)$ for each $\alpha \in \Delta$.

Theorem 6. *Let $\mathcal{Z} = \prod_{\alpha\in\Delta} \mathcal{Z}_\alpha$ be the product space of the spaces $(\mathcal{Z}_\alpha, \tau_\alpha), \alpha \in \Delta$ with the Tychonoff topology τ_p. Let $H_\alpha \subseteq \mathcal{Z}_\alpha$ for each $\alpha \in \Delta$. Then :*

(i) If $H_\alpha \in \mathcal{SWD}(\mathcal{Z}_\alpha, \tau_\alpha)$ for each $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta}(H_\alpha \times \prod_{\substack{\beta\in\Delta \\ \beta\neq\alpha}} \mathcal{Z}_\beta) \in \mathcal{SWD}(\mathcal{Z}, \tau_p)$.

(ii) If $\prod_{\alpha\in\Delta} H_\alpha \in \mathcal{SWD}(\mathcal{Z}, \tau_p)$, then $H_\alpha \in \mathcal{SWD}(\mathcal{Z}_\alpha, \tau_\alpha)$ for each $\alpha \in \Delta$.

Proof. (i) Follows from Theorem 5 (Part iii) and Theorem 3 (Part i).

(ii) Follows from Proposition 1 (see the proof of the implication $(iii \to i)$ in Theorem 5).

The following example shows that the converses of Theorem 6 need not be true in general:

Example 1. *(i) Let $\mathcal{Z} = \mathbb{R}$ and consider the spaces $(\mathcal{Z}_1, \tau_1) = (\mathbb{R}, \tau_u)$ and $(\mathcal{Z}_2, \tau_2) = (\mathbb{R}, \tau_{ind})$. Then the set $(\{2\} \times \mathcal{Z}_2) \cup (\mathcal{Z}_1 \times \{1\})$ is somewhere dense of $(\mathbb{R}, \tau_u) \times (\mathbb{R}, \tau_{ind})$ while $\{2\} \notin \mathcal{SWD}((\mathbb{R}, \tau_u)$.*

(ii) For each $\alpha \in \Delta$ with Δ is an infinite set, consider $(\mathcal{Z}_\alpha, \tau_\alpha) = (\mathcal{K}_\alpha, \tau_{ind})$ where $\mathcal{K}_\alpha$ any set with $|K_\alpha| > 1$. For each $\alpha \in \Delta$, choose $x_\alpha \in \mathcal{K}_\alpha$, then for each $\alpha \in \Delta$, $H_\alpha = \{x_\alpha\} \in \mathcal{SWD}(\mathcal{K}_\alpha, \tau_{ind})$ while $\prod_{\alpha\in\Delta} H_\alpha \notin \mathcal{SWD}(\mathcal{Z}, \tau_p)$.

Theorem 7. *Let $\mathcal{Z} = \prod_{\alpha\in\Delta} \mathcal{Z}_\alpha$ be the Cartesian product of the spaces $(\mathcal{Z}_\alpha, \tau_\alpha)$ with the topology τ_b which is generated by the base $\{\prod_{\alpha\in\Delta} V_\alpha : V_\alpha \in \tau_\alpha$ for each $\alpha \in \Delta\}$(τ_b is called the box topology). Then $H_\alpha \in \mathcal{SWD}(\mathcal{Z}_\alpha, \tau_\alpha)$ for each $\alpha \in \Delta$ iff $\prod_{\alpha\in\Delta} H_\alpha \in \mathcal{SWD}(\mathcal{Z}, \tau_b)$.*

Proof. For each $\alpha \in \Delta$, there is $G_\alpha \in \tau_\alpha$ with $\phi \neq G_\alpha \subseteq \mathcal{Z}_\alpha$ and $G_\alpha \subseteq \mathcal{C}l(H_\alpha)$. Then $G = \prod_{\alpha\in\Delta} G_\alpha$ is a non-empty open set of $\mathcal{Z}$ such that $G \subseteq \prod_{\alpha\in\Delta} \mathcal{C}l(H_\alpha) = \mathcal{C}l(\Pi(\prod_{\alpha\in\Delta} H_\alpha))$. Conversely, let $\alpha_\circ \in \Delta$. Then $H_{\alpha_\circ} \times \prod_{\substack{\alpha\in\Delta \\ \alpha\neq\alpha_\circ}} H_\alpha \in \mathcal{SWD}(\mathcal{Z}, \tau_b)$ and hence there is $G \in \tau_b$ such that $\phi \neq G \subseteq \mathcal{C}l(H_{\alpha_\circ} \times \prod_{\substack{\alpha\in\Delta \\ \alpha\neq\alpha_\circ}} H_\alpha)$ and so there is a basic open set $V = \prod_{\alpha\in\Delta} V_\alpha$ with $V_{\alpha_\circ} \times \prod_{\substack{\alpha\in\Delta \\ \alpha\neq\alpha_\circ}} V_\alpha \subseteq \mathcal{C}l(H_{\alpha_\circ} \times \prod_{\substack{\alpha\in\Delta \\ \alpha\neq\alpha_\circ}} H_\alpha)$. Therefore, $V_{\alpha_\circ} \subseteq \mathcal{C}l(H_{\alpha_\circ})$ and thus $H_{\alpha_\circ} \in \mathcal{SWD}(\mathcal{Z}_{\alpha_\circ}, \tau_{\alpha_\circ})$.

Corollary 1. *(i) Let* $\mathcal{Z} = \prod_{\alpha\in\Delta} \mathcal{Z}_\alpha$ *be the product space of the spaces* $(\mathcal{Z}_\alpha, \tau_\alpha), \alpha \in \Delta$ *with the topology* τ_p *and* $F_\alpha \subseteq \mathcal{Z}_\alpha$ *for each* $\alpha \in \Delta$*. If* $\bigcup_{\alpha\in\Delta}(F_\alpha \times \prod_{\substack{\beta\in\Delta\\ \beta\neq\alpha}} \mathcal{Z}_\beta) \in \mathcal{SWDC}(\mathcal{Z}, \tau_p)$*, then* $F_\alpha \in \mathcal{SWDC}(\mathcal{Z}_\alpha, \tau_\alpha)$*.*

(ii) Let $\mathcal{Z} = \prod_{\alpha\in\Delta} \mathcal{Z}_\alpha$ *be the product space of the spaces* $(\mathcal{Z}_\alpha, \tau_\alpha), \alpha \in \Delta$ *with the topology* τ_b *and* $F_\alpha \subseteq \mathcal{Z}_\alpha$*. Then* $F_\alpha \in \mathcal{SWDC}(\mathcal{Z}_\alpha, \tau_\alpha)$ *iff* $\bigcup_{\alpha\in\Delta}(F_\alpha \times \prod_{\substack{\beta\in\Delta\\ \beta\neq\alpha}} \mathcal{Z}_\beta) \in \mathcal{SWDC}(\mathcal{Z}, \tau_b)$*.*

(iii) Let $\mathcal{Z} = \prod_{\alpha=1}^{n} \mathcal{Z}_\alpha$ *be the finite product space of the spaces* $(\mathcal{Z}_\alpha, \tau_\alpha)$ *and* $F_\alpha \subseteq \mathcal{Z}_\alpha$ *for each* $\alpha \in \{1, 2.., n\}$*. Then for each* $\alpha \in \{1, 2.., n\}, F_\alpha \in \mathcal{SWDC}(\mathcal{Z}_\alpha, \tau_\alpha)$ *iff* $\bigcup_{\alpha=1}^{n}(F_\alpha \times \prod_{\substack{\beta\in\{1,2...,n\}\\ \beta\neq\alpha}} \mathcal{Z}_\beta)$ *is closed somewhere dense of* $\mathcal{Z}$*.*

Note that, in Example 1 (Part (ii)), $F_\alpha = \{x_\alpha\} \in \mathcal{SWDC}(\mathcal{Z}_\alpha, \tau_\alpha)$ while $\bigcup_{\alpha\in\Delta}(F_\alpha \times \prod_{\substack{\beta\in\Delta\\ \beta\neq\alpha}} \mathcal{Z}_\beta) \notin \mathcal{SWDC}(\mathcal{Z}, \tau_p)$, since $\mathcal{Z} - \bigcup_{\alpha\in\Delta}(F_\alpha \times \prod_{\substack{\beta\in\Delta\\ \beta\neq\alpha}} \mathcal{Z}_\beta) = \prod_{\alpha\in\Delta}(\mathcal{Z}_\alpha - F_\alpha) \notin \mathcal{SWD}(\mathcal{Z}, \tau)$ and so the converse of Corollary 1 (Part i) is not true in general.

3. $\omega-$Somewhere Dense Open Sets With Some Applications

In this section we introduce the notion of $\omega\mathcal{SWD}$-open subsets, denoted by $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$, as a new generalization for somewhere dense of a topological space $(\mathcal{Z}, \tau)$. Then we discuss the sufficient conditions for the equivalence between the classes $\mathcal{SWD}(\mathcal{Z}, \tau)$, $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$ and $\omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega)$. Also, we study some applications by using $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$.

Definition 6. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$ *with* $H \subseteq \mathcal{Z}$*. A point* $x \in \mathcal{Z}$ *is an* $\omega\mathcal{SWD}-$*condensation point of* H *if for each* $S \in \mathcal{SWD}(\mathcal{Z}, \tau)$ *with* $x \in S$*, the set* $S \cap H$ *is uncountable. If* H *contains all its* $\omega\mathcal{SWD}-$*condensation points, then* H *is said to be* $\omega\mathcal{SWD}$*-closed subset of* $(\mathcal{Z}, \tau)$ *and its complement is an* $\omega\mathcal{SWD}$*-open subset of* $(\mathcal{Z}, \tau)$*. The collection of all* $\omega\mathcal{SWD}$*-closed (resp.,* $\omega\mathcal{SWD}$*-open) subsets of* $(\mathcal{Z}, \tau)$ *will be denoted by* $\omega\mathcal{SWDC}(\mathcal{Z}, \tau)$ *(resp.,* $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$*).*

The proofs of the following results are straightforward and thus are omitted.

Proposition 2. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$ *with* $H \subseteq \mathcal{Z}$*. Then* $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ *iff for each* $x \in H$ *there is* $S \in \mathcal{SWD}(\mathcal{Z}, \tau)$ *with* $x \in S$ *and* $S - H$ *is countable.*

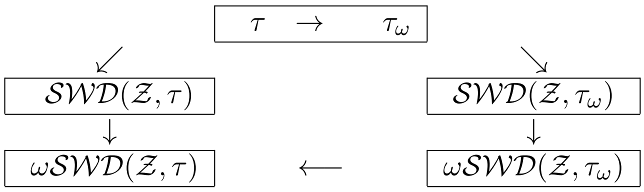
Corollary 2. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$ *with* $H \subseteq \mathcal{Z}$*. Then* $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ *iff for each* $x \in H$ *there is* $S \in \mathcal{SWD}(\mathcal{Z}, \tau)$ *and a countable set* G *with* $x \in S - G \subseteq H$*.*

Theorem 8. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$*.Then* $\omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega) \subseteq \omega\mathcal{SWD}(\mathcal{Z}, \tau)$*.*

Proof. Let $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega)$ and $x \in H$. Then there is $S \in \mathcal{SWD}(\mathcal{Z}, \tau_\omega)$ with $x \in S$ and $C = S - H$ is countable. Then there is $G \in \tau_\omega$ with $\phi \neq G \subseteq Cl_\omega(S)$.

Choose $t \in G$. *Then, there is* $V \in \tau$ *with* $t \in V$ *and* $C_1 = V - G$ *is countable. Thus,* $V \subseteq G \cup C_1 \subseteq \mathcal{C}l_\omega(S) \cup \mathcal{C}l_\omega(C_1) = \mathcal{C}l_\omega(S \cup C_1) \subseteq \mathcal{C}l(S \cup C_1)$. *Therefore,* $S \cup C_1 \in \mathcal{SWD}(\mathcal{Z}, \tau)$ *with* $x \in S \cup C_1$ *and* $(S \cup C_1) - H = (S - H) \cup (C_1 - H)$ *is countable and hence* $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$.

By using Definition 6 and Theorem 8 we generate the following diagram where none of these implications being reversible.



Example 2. *(i) Consider* $\mathcal{Z} = \{1, 2, 3\}$ *with* $\tau = \{\phi, \mathcal{Z}, \{1, 2\}\}$. *Then* $\{3\} \in \tau_\omega - \{\phi\} \subseteq \mathcal{SWD}(\mathcal{Z}, \tau_\omega)$ *while* $\{3\} \notin \mathcal{SWD}(\mathcal{Z}, \tau)$.

(ii) Consider $(\mathbb{R}, \tau_u)$ *with* $H = \mathbb{Q}$. *Then* $H \in \mathcal{SWD}(\mathbb{R}, \tau_u)$ *while* $H \notin \mathcal{SWD}(\mathbb{R}, (\tau_u)_\omega)$ *since* $\mathcal{I}nt_\omega\mathcal{C}l_\omega(H) = \mathcal{I}nt_\omega(H) = \mathcal{I}nt(H) = \phi$.

(iii) Consider $(\mathbb{R}, \tau_{cof})$ *with* $H = \{1\}$. *Then* $H \in \omega\mathcal{SWD}(\mathbb{R}, \tau_{cof})$ *while* $H \notin \mathcal{SWD}(\mathbb{R}, \tau_{cof})$.

(iv) Consider $\mathcal{Z} = \mathbb{R}$ *with* $\tau = \{\mathbb{R}\} \cup \{G \subseteq \mathbb{R} : G \subseteq \mathbb{R} - \mathbb{Q}\}$ *and* $H = \mathbb{Q}$. *Then* $H \in \omega\mathcal{SWD}(\mathbb{R}, \tau_\omega)$ *while* $H \notin \mathcal{SWD}(\mathbb{R}, \tau_\omega)$.*To show that,* $H \in \omega\mathcal{SWD}(\mathbb{R}, \tau_\omega)$, *let* $x \in H$. *Choose* $r \in \mathbb{R} - \mathbb{Q}$. *Then* $G = \{r\} \in \tau$ *such that* $\phi \neq G \subseteq H \cup \{r\} \subseteq \mathcal{C}l_\omega(H \cup \{r\})$. *So* $H \cup \{r\} \in \mathcal{SWD}(\mathbb{R}, \tau_\omega)$ *with* $x \in H \cup \{r\}$ *and* $(H \cup \{r\}) - H$ *is countable. Therefore,* $H \in \omega\mathcal{SWD}(\mathbb{R}, \tau_\omega)$.

(v) Consider $(\mathbb{R}, \tau_{ind})$ *with* $H = \{1\}$. *Since* $\omega\mathcal{SWD}(\mathbb{R}, \tau_{ind}) = \mathcal{P}(\mathbb{R})$, *then* $\{1\} \in \omega\mathcal{SWD}(\mathbb{R}, \tau_{ind})$ *while* $\{1\} \notin \omega\mathcal{SWD}(\mathbb{R}, (\tau_{ind})_\omega)$. *Since if there is* $\mathcal{S} \in \mathcal{SWD}(\mathbb{R}, (\tau_{ind})_\omega)$ *with* $1 \in S$ *and* $S - \{1\}$ *is countable, then* S *is countable and hence* $Int_\omega\mathcal{C}l_\omega(\mathcal{S}) = Int_\omega(\mathcal{S}) = \phi$. *Therefore,* $\{1\} \notin \omega\mathcal{SWD}(\mathbb{R}, (\tau_{ind})_\omega)$.

Theorem 9. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$. *If* $(\mathcal{Z}, \tau)$ *is an anti-locally countable and* τ *is finer than the cocountable topology, then the following families are equivalent:*

(i) $\mathcal{SWD}(\mathcal{Z}, \tau)$.

(ii) $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$.

(iii) $\omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega)$.

Proof. $(i \to ii)$ *Trivial.*

$(ii \to iii)$ *Let* $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ *and* $x \in H$. *Then there is* $S \in \mathcal{SWD}(\mathcal{Z}, \tau)$ *with* $x \in S$ *and* $C = S - H$ *is countable. Hence* $S \subseteq H \cup C$. *Now, choose* $G \in \tau$ *with* $\phi \neq G \subseteq \mathcal{C}l(S) \subseteq \mathcal{C}l(H \cup C) \subseteq \mathcal{C}l(H) \cup \mathcal{C}l(C) = \mathcal{C}l(H) \cup C$ *(since* $\tau_{coc} \subseteq \tau$*). Since* $(\mathcal{Z}, \tau)$ *is an anti-locally countable and* $\phi \neq G \in \tau$, *then* $G - C \neq \phi$. *Now, we claim* $G - C \subseteq \mathcal{C}l_\omega(S)$. *Suppose not, then there is* $t \in G - C$ *and* $t \notin \mathcal{C}l_\omega(S)$ *and so there is* $V \in \tau_\omega$ *with* $t \in V$ *and* $V \cap S = \phi$. *Now, choose,* $O \in \tau$ *with* $t \in O$ *and* $C_1 = O - V$ *is countable. Then we have* $t \in O - C_1 \in \tau$ *with* $O - C_1 \subseteq V$. *Finally, put* $W = (G - C) \cap (O - C_1)$. *Then* $W \in \tau$

with $t \in W$ and hence $W \cap S \neq \phi$. On the other hand, $W \cap S \subseteq (O - C_1) \cap S \subseteq V \cap S = \phi$, which is a contradiction. Therefore, $S \in \mathcal{SWD}(\mathcal{Z}, \tau_\omega)$ and hence $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega)$.

$(iii) \to (i)$ Let $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega)$ and $x \in H$. Then there is $S \in \mathcal{SWD}(\mathcal{Z}, \tau_\omega)$ with $x \in S$ and $C = S - H$ is countable. Choose $G \in \tau_\omega$ with $t \in G \subseteq \mathcal{C}l_\omega(S) \subseteq \mathcal{C}l_\omega(H \cup C) \subseteq \mathcal{C}l_\omega(H) \cup \mathcal{C}l_\omega(C) = \mathcal{C}l_\omega(H) \cup C$. Now, choose $O \in \tau$ with $t \in O$ and $O - G = C_1$ is countable. Since, $O - C_1 \subseteq G \subseteq \mathcal{C}l_\omega(H) \cup C$, then $\phi \neq (O - C_1) - C \subseteq \mathcal{C}l_\omega(H) \subseteq \mathcal{C}l(H)$. Therefore, $H \in \mathcal{SWD}(\mathcal{Z}, \tau)$.

Corollary 3. *Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$. If $(\mathcal{Z}, \tau)$ is an anti-locally countable, then:*
(i) $\mathcal{SWD}(\mathcal{Z}, \tau_\omega) = \omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega)$.
(ii) $\mathcal{SWD}(\mathcal{Z}, \tau) = \mathcal{SWD}(\mathcal{Z}, \tau_\omega)$ whenever $\tau_{coc} \subseteq \tau$.

Proof. (i) Since $(\mathcal{Z}, \tau_\omega)$ is anti locally countable and $\tau_{coc} \subseteq \tau_\omega$, then by Theorem 9, $\mathcal{SWD}(\mathcal{Z}, \tau_\omega) = \omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega)$.
(ii) From part(i) and Theorem 9.

Note that from Example 2 (Part iii) imposing the condition of anti locally countable on $(\mathcal{Z}, \tau)$ alone in Theorem 9 is not enough and hence we looked for another condition on $(\mathcal{Z}, \tau)$.

Theorem 10. *Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$. Then $\underset{\alpha \in \Delta}{\cup} H_\alpha \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ whenever $H_\alpha \subseteq \mathcal{Z}$ and $H_\alpha \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ for each $\alpha \in \Delta$.*

Proof. If $\underset{\alpha \in \Delta}{\cup} H_\alpha = \phi$, then $\underset{\alpha \in \Delta}{\cup} H_\alpha \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$. Now, let $x \in \underset{\alpha \in \Delta}{\cup} H_\alpha$. Then there is $\alpha(x) \in \Delta$ such that $x \in H_{\alpha(x)}$ and hence there is $S \in \mathcal{SWD}(\mathcal{Z}, \tau)$ with $x \in S$ and $S - H_{\alpha(x)}$ is countable. Since $S - \underset{\alpha \in \Delta}{\cup} H_\alpha \subseteq S - H_{\alpha(x)}$, then $S - \underset{\alpha \in \Delta}{\cup} H_\alpha$ is countable. Therefore, $\underset{\alpha \in \Delta}{\cup} H_\alpha \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$.

Corollary 4. *Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$.Then $\underset{\alpha \in \Delta}{\cap} H_\alpha \in \omega\mathcal{SWDC}(\mathcal{Z}, \tau)$ whenever $H_\alpha \subseteq \mathcal{Z}$ and $H_\alpha \in \omega\mathcal{SWDC}(\mathcal{Z}, \tau)$ for each $\alpha \in \Delta$.*

The next example shows that the intersection of two $\omega\mathcal{SWD}$-open subsets of $(\mathcal{Z}, \tau)$ is not an $\omega\mathcal{SWD}$-open in general and hence we can conclude for any topology τ on $\mathcal{Z}$, $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$ may not be a topology on $\mathcal{Z}$.

Example 3. *Consider the space $(\mathbb{R}, \tau_{coc})$. Take $H = [0, 1]$ and $G = [1, 2]$. Then $H, G \in \mathcal{SWD}(\mathbb{R}, \tau_{coc}) \subseteq \omega\mathcal{SWD}(\mathbb{R}, \tau_{coc})$, while $H \cap G = \{1\} \notin \omega\mathcal{SWD}(\mathbb{R}, \tau_{coc})$, since there is no $S \in \mathcal{SWD}(\mathbb{R}, \tau_{coc})$ with $1 \in S$ and $S - \{1\}$ is countable.*

Proposition 3. *Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$. Then the family $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$ is a topology on $\mathcal{Z}$ if one of the following hold:*
(i) $(\mathcal{Z}, \tau)$ is strongly hyperconnected.
(ii) $\mathcal{Z}$ is countable or τ is the indiscrete topology.

Proof. Straightforward.

Theorem 11. *Let $(\mathcal{Z},\tau)$ be a hyperconnected $\mathcal{TS}$ and $H,G \subseteq \mathcal{Z}$. If $H \in \tau_\omega$ and $G \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$, then $H \cap G \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$.*

Proof. Let $x \in H \cap G$. Then there are $V \in \tau$ with $x \in V$ and $S \in \mathcal{SWD}(\mathcal{Z},\tau)$ with $x \in S$ such that $V - H$ and $S - G$ are countable sets. By Theorem1, $V \cap S \in \mathcal{SWD}(\mathcal{Z},\tau)$ with $x \in V \cap S$ and since $(V \cap S) - (H \cap G) \subseteq (V - H) \cup (S - G)$, then $(V \cap S) - (H \cap G)$ is countable. Therefore, $H \cap G \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$.

Corollary 5. *Let $(\mathcal{Z},\tau)$ be hyperconnected $\mathcal{TS}$ and $H,G \subseteq \mathcal{Z}$. If $\mathcal{Z} - H \in \tau_\omega$ and $G \in \omega\mathcal{SWDC}(\mathcal{Z},\tau)$, then $H \cup G \in \omega\mathcal{SWDC}(\mathcal{Z},\tau)$.*

From Example 3 we note that in Theorem 11 is not enough to be $(\mathcal{Z},\ \tau)$ is hyperconnected and hence we looked for another condition on the sets. Also, note that this example (Example 3), $\mathbb{Q} \cap (\mathbb{R} - Q) \in \omega\mathcal{SWD}(\mathbb{R},\tau_{coc})$ and $(\mathbb{R} - Q) \in \omega\mathcal{SWD}(\mathbb{R},\tau_{coc})$ but $\mathbb{Q} \notin \tau_\omega$ and hence the converse of Theorem 11 is not true in general.

Theorem 12. *Let $(\mathcal{Z},\tau)$ be a $\mathcal{TS}$ and $H,E \subseteq \mathcal{Z}$. If $E \in \tau$ and $H \subseteq E$, then:*
(i) If $H \in \omega\mathcal{SWD}(E,\tau_E)$, then $H \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$.
(ii) If $H \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$, then $H \in \omega\mathcal{SWD}(E,\tau_E)$ provided that E is dense.

Proof. (i) Obvious by using Theorem 3 (Part (ii)).
(ii) Let $x \in H$. Then there is $S \in \mathcal{SWD}(\mathcal{Z},\tau)$ with $x \in S$ and $S - H$ is countable. Since E is an open dense subset, then for some $G \in \tau$ we can have $\phi \neq G \cap E \subseteq \mathcal{C}l(S) \cap E \subseteq \mathcal{C}l(S \cap E)$ and hence $G \cap E \subseteq \mathcal{C}l(S \cap E) \cap E = \mathcal{C}l_E(S \cap E)$(The closure of $S \cap E$ in (E,τ_E)). Therefore, $S \cap E \in \mathcal{SWD}(E,\tau_E)$ and hence $H \in \omega\mathcal{SWD}(E,\tau_E)$.

Remark 1. *Let $(\mathcal{Z},\tau)$ and $(\mathcal{K},\sigma)$ be two $\mathcal{TS}$s. Then:*
(i) If $\omega\mathcal{SWD}(\mathcal{Z},\tau) \subseteq \omega\mathcal{SWD}(\mathcal{K},\sigma)$, then it is not true in general $\tau \subseteq \sigma$.
(ii) If $\tau \subseteq \sigma$, then it is not true in general $\omega\mathcal{SWD}(\mathcal{Z},\tau) \subseteq \omega\mathcal{SWD}(\mathcal{K},\sigma)$.

The following example illustrates Remark 1.

Example 4. *(i) Note that, $\omega\mathcal{SWD}(\mathbb{R},\tau_{coc}) \subseteq \omega\mathcal{SWD}(\mathbb{R},\tau_{ind})$ while $\tau_{coc} \nsubseteq \tau_{ind}$.*
(ii) Note that, $\tau_{ind} \subseteq \tau_{coc}$ while $\omega\mathcal{SWD}(\mathbb{R},\tau_{ind}) \nsubseteq \omega\mathcal{SWD}(\mathbb{R},\tau_{coc})$.

Theorem 13. *Let $(\mathcal{Z},\ \tau)$ be a $\mathcal{TS}$. Then:*
(i) $\tau = \mathcal{I}nt(\omega\mathcal{SWD}(\mathcal{Z},\tau)) = \{\mathcal{I}nt(H) : H \in \omega\mathcal{SWD}(\mathcal{Z},\tau)\}$.
(ii) $\tau_\omega = \mathcal{I}nt_\omega(\omega\mathcal{SWD}(\mathcal{Z},\tau)) = \{\mathcal{I}nt_\omega(H) : H \in \omega\mathcal{SWD}(\mathcal{Z},\tau)\}$.

Proof. Let $G \in \tau$. Then $G \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$ and hence $\mathcal{I}nt(G) = G \in \mathcal{I}nt(\omega\mathcal{SWD}(\mathcal{Z},\tau))$. Conversely, is obvious since $\{\mathcal{I}nt(H) : H \in \omega\mathcal{SWD}(\mathcal{Z},\tau)\} \subseteq \tau$.
(ii)The proof is similar technique in part (i).

Definition 7. *Let $(\mathcal{Z},\ \tau)$ be a $\mathcal{TS}$ and $H \subseteq \mathcal{Z}$. Then:*
(i) $\mathcal{I}nt_{\omega\mathcal{SWD}}(H) = \cup\{G : G \subseteq H \text{ and } G \in \omega\mathcal{SWD}(\mathcal{Z},\tau)\ \}$.
(ii) $\mathcal{C}l_{\omega\mathcal{SWD}}(H) = \cap\{F : H \subseteq F \text{ and } F \in \omega\mathcal{SWDC}(\mathcal{Z},\tau)\ \}$.

Theorem 14. *Let (Z,τ) be a $\mathcal{TS}$ and $H,G \subseteq Z$. Then:*
(i) $\mathcal{I}nt_{\mathcal{SWD}}(H) \subseteq \mathcal{I}nt_{\omega\mathcal{SWD}}(H)$ *and* $\mathcal{C}l_{\omega\mathcal{SWD}}(H) \subseteq \mathcal{C}l_{\mathcal{SWD}}(H)$.
(ii) $H \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$ *iff* $H = \mathcal{I}nt_{\omega\mathcal{SWD}}(H)$.
(iii) $\mathcal{I}nt_{\omega\mathcal{SWD}}(\mathcal{I}nt_{\omega\mathcal{SWD}}(H)) = \mathcal{I}nt_{\omega\mathcal{SWD}}(H)$.
(iv) $H \in \omega\mathcal{SWDC}(\mathcal{Z},\tau)$ *iff* $H = \mathcal{C}l_{\omega\mathcal{SWD}}(H)$.
(v) $x \in \mathcal{C}l_{\omega\mathcal{SWD}}(H)$ *iff for each* $G \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$ *with* $x \in G$ *we have* $G \cap H \neq \phi$.
(vi) $\mathcal{I}nt_{\omega\mathcal{SWD}}(\mathcal{Z} - H) = \mathcal{Z} - \mathcal{C}l_{\omega\mathcal{SWD}}(H)$.
(vii) $\mathcal{C}l_{\omega\mathcal{SWD}}(\mathcal{Z} - H) = \mathcal{Z} - \mathcal{I}nt_{\omega\mathcal{SWD}}(H)$.

Proof. Straightforward.

In general, in Theorem 14 the reverse inclusion of (Part i) does not hold, since in Example 2 (Part iii), $\mathcal{I}nt_{\mathcal{SWD}}(\{1\}) = \phi$ while $\mathcal{I}nt_{\omega\mathcal{SWD}}(\{1\}) =\{1\}$. Also, if $\mathcal{Z} = \{1,2,3\}$ with the topology $\sigma = \{\phi, \mathcal{Z}, \{1\}\}$. Then $\mathcal{C}l_{\mathcal{SWD}}(\{1\}) = \mathcal{Z}$ while $\mathcal{C}l_{\omega\mathcal{SWD}}(\{1\}) = \{1\}$.

Definition 8. *Let $(\mathcal{Z},\tau)$ and $(\mathcal{K},\sigma)$ be two $\mathcal{TS}$s. Then a function $\Gamma : (\mathcal{Z},\tau) \to (\mathcal{K},\sigma)$ is said to be:*
(i) An $\omega\mathcal{SWD}$-continuous iff $\Gamma^{-1}(G) \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$ *for each* $G \in \sigma$.
(ii) An $\omega\mathcal{SWD}$-irresolute iff $\Gamma^{-1}(G) \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$ *for each* $G \in \omega\mathcal{SWD}(\mathcal{K},\sigma)$.

Proposition 4. *Each $\omega\mathcal{SWD}$-irresolute function is $\omega\mathcal{SWD}$-continuous.*

The following example will show the converse of Proposition 4 is not true in general.

Example 5. *Consider the identity function $\Gamma : (\mathbb{R}, \tau_{coc}) \to (\mathbb{R}, \tau_{cof})$. Then Γ is an $\omega\mathcal{SWD}$-continuous while it is not $\omega\mathcal{SWD}$-irresolute since $\Gamma^{-1}(\{1\}) \notin \omega\mathcal{SWD}(\mathcal{Z}, \tau_{coc})$.*

In the following theorem, we use the family $\omega\mathcal{SWD}(\mathcal{Z},\tau)$ to present a theorem similar to Theorem 4.

Theorem 15. *Let $\{(\mathcal{Z}_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a family of topological spaces with $\mathcal{Z}_\alpha \cap \mathcal{Z}_\beta = \phi$ for each $\alpha \neq \beta$. For each $\alpha \in \Delta$, let $\phi \neq H_\alpha \subseteq \mathcal{Z}_\alpha$ and put $H = \underset{\alpha\in\Delta}{\cup} H_\alpha$. Then:*
(i) $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau_s)$ *iff there is* $\alpha_\circ \in \Delta$ *with* $H_{\alpha_\circ} \in \omega\mathcal{SWD}(\mathcal{Z}_{\alpha_\circ}, \tau_{\alpha_\circ})$.
(ii) If $H_\alpha \in \omega\mathcal{SWD}(\mathcal{Z}_\alpha, \tau_\alpha)$*, then* $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau_s)$.

Proof. (i) Let $x \in H$. Then there is $S \in \mathcal{SWD}(\mathcal{Z}, \tau_s)$ with $x \in S$ and $S - H$ is countable. Write $S = \underset{\alpha\in\Delta}{\cup} S_\alpha$. Then by Theorem 4 (Part (i)) there is $\alpha_\circ \in \Delta$ with $S_{\alpha_\circ} \in \mathcal{SWD}(\mathcal{Z}_{\alpha_\circ}, \tau_{\alpha_\circ})$ such that $S_{\alpha_\circ} - H_{\alpha_\circ} \subseteq S_{\alpha_\circ} - \underset{\alpha\in\Delta}{\cup} H_\alpha \subseteq S - H$. Therefore, $S_{\alpha_\circ} - H_{\alpha_\circ}$ is countable. Now, let $x_\circ \in H_{\alpha_\circ}$. Then $S^*_{\alpha_\circ} = S_{\alpha_\circ} \cup \{x_\circ\} \in \mathcal{SWD}(\mathcal{Z}_{\alpha_\circ}, \tau_{\alpha_\circ})$ with $x_\circ \in S^*_{\alpha_\circ}$ and $S^*_{\alpha_\circ} - H_{\alpha_\circ}$ is countable. Therefore, $H_{\alpha_\circ} \in \omega\mathcal{SWD}(\mathcal{Z}_{\alpha_\circ}, \tau_{\alpha_\circ})$. Conversely, choose $x_\circ \in H_{\alpha_\circ}$. Then there is $S_{\alpha_\circ} \in \mathcal{SWD}(\mathcal{Z}_{\alpha_\circ}, \tau_{\alpha_\circ})$ with $x_\circ \in S_{\alpha_\circ}$ and $S_{\alpha_\circ} - H_{\alpha_\circ}$ is countable. Now, let $x \in H$ and put $S = S_{\alpha_\circ} \cup \{x\}$. Then by Theorem 4 (Part (i)), $S \in \mathcal{SWD}(\mathcal{Z}, \tau_s)$ with $x \in S$ and $S - H \subseteq S - H_{\alpha_\circ} \subseteq (S_{\alpha_\circ} - H_{\alpha_\circ}) \cup \{x\}$. Since $(S_{\alpha_\circ} - H_{\alpha_\circ}) \cup \{x\}$ is countable, then $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau_s)$.
(ii) Follows from part (i).

In the next example we will show that the converse of part (ii) of Theorem 4 and Theorem 15, is not true in general.

Example 6. *Let* $(\mathcal{Z}_1, \tau_1) = ((0,1), \tau_{ind})$, $(\mathcal{Z}_2, \tau_2) = ((2,3), \tau_{coc})$ *and consider* $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$. *Let* $H = \{x_1\} \cup E$, *where* $x_1 \in \mathcal{Z}_1$ *and* E *is countable subset of* $\mathcal{Z}_2$. *Note that,* $\mathcal{C}l(H) = \mathcal{Z}_1 \cup E$ *and* $\mathcal{Z}_1 \subseteq \mathcal{I}nt(\mathcal{C}l(H))$. *Therefore,* $H \in \mathcal{SWD}(\mathcal{Z}, \tau_s)$ *and hence* $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau_s)$. *On the other hand* $E \notin \omega\mathcal{SWD}(\mathcal{Z}_2, \tau_2)$.

4. Almost $\omega\mathcal{SWD}$-compact spaces

In this section, we will introduce and study the notion of almost $\omega\mathcal{SWD}$-compact spaces with some of their properties.

Definition 9. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$ *and* $H \subseteq \mathcal{Z}$. *A point* $x \in \mathcal{Z}$ *is said to be* $\omega\mathcal{SWD}\theta$-*accumulation point of* H *if* $\mathcal{C}l_{\omega\mathcal{SWD}}(G) \cap H \neq \phi$ *for each* $G \in \omega\mathcal{SWD}(Z, \tau)$ *with* $x \in G$. *Furthermore, an* $\omega\mathcal{SWD}\theta$- *closure of* H *is the set off all* $\omega\mathcal{SWD}\theta$- *accumulation points of* H *and it is denoted by* $\mathcal{C}l_{\omega\mathcal{SWD}\theta}(H)$. *If* $\mathcal{C}l_{\omega\mathcal{SWD}\theta}(H) = H$, *then* H *is said to be* $\omega\mathcal{SWD}\theta$-*closed subset of* $(\mathcal{Z}, \tau)$ *and its complement is said to be* $\omega\mathcal{SWD}\theta$- *open subset of* $(\mathcal{Z}, \tau)$.

The following Proposition can be easily constructed.

Proposition 5. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$ *and* $H \subseteq \mathcal{Z}$. *Then :*

(i) H *is* $\omega\mathcal{SWD}\theta$- *open set iff for each* $x \in H$ *there is* $G \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ *with* $x \in G \subseteq \mathcal{C}l_{\omega\mathcal{SWD}}(G) \subseteq H$.

(ii) If $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau) \cap \omega\mathcal{SWDC}(\mathcal{Z}, \tau)$, *then* H *is* $\omega\mathcal{SWD}\theta$- *closed subset of* $(\mathcal{Z}, \tau)$.

(iii) $\mathcal{C}l_{\omega\mathcal{SWD}}(H) \subseteq \mathcal{C}l_{\omega\mathcal{SWD}\theta}(H)$ *and if* $H \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$, *then* $\mathcal{C}l_{\omega\mathcal{SWD}}(H) = \mathcal{C}l_{\omega\mathcal{SWD}\theta}(H)$.

Definition 10. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$ *and* $H \subseteq \mathcal{Z}$. *Then:*

(i) A family $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ *is said to be* $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-*cover(resp.,* $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-$\theta$-*cover,* τ-*cover) of* H *if* $H \subseteq \bigcup_{\alpha \in \Delta} H_\alpha$ *and* H_α *is an* $\omega\mathcal{SWD}$-*open (resp.,* $\omega\mathcal{SWD}\theta$- *open, open) subset of* $(\mathcal{Z}, \tau)$ *for each* $\alpha \in \Delta$.

(ii) $(\mathcal{Z}, \tau)$ *is said to be almost* $\omega\mathcal{SWD}$-*compact if for each* $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-*cover* $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ *of* $\mathcal{Z}$ *there is a finite subset* $\Delta_\circ \subseteq \Delta$ *with* $\mathcal{Z} = \bigcup_{\alpha \in \Delta_\circ} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)$.

The following results follow immediately from Definitions 5 and 10 and the fact that $\tau \subseteq \mathcal{SWD}(\mathcal{Z}, \tau) \subseteq \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ for any space $(\mathcal{Z}, \tau)$.

Theorem 16. *If a topological space* $(\mathcal{Z}, \tau)$ *is almost* $\omega\mathcal{SWD}$-*compact, then the following hold:*

(i) $(\mathcal{Z}, \tau)$ *is almost* $\mathcal{SWD}$-*compact.*

(ii) If $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ *is* τ-*cover of* $\mathcal{Z}$, *then there is a finite subset* $\Delta_\circ \subseteq \Delta$ *with* $\mathcal{Z} = \bigcup_{\alpha \in \Delta_\circ} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)$.

(iii) If $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ *is an* $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-*cover of* $\mathcal{Z}$, *then there is a finite subset* $\Delta_\circ \subseteq \Delta$ *with* $Z = \bigcup_{\alpha \in \Delta_\circ} \mathcal{C}l(H_\alpha)$.

The converses of Theorem 16 is not true in general as will see in the following examples.

Example 7. *(i) Consider* $(\mathbb{R}, \tau_{ind})$. *Then* $(\mathbb{R}, \tau_{ind})$ *satisfied (Parts ii and iii) of Theorem 16, but* $(\mathbb{R}, \tau_{ind})$ *is not almost* $\omega\mathcal{SWD}$*-compact since* $\mathcal{H} = \{\{x\} : x \in \mathbb{R}\}$ *is an* $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$*-cover of* $\mathbb{R}$ *which has no finite subset* $\Delta_\circ = \{x_1, x_2, ...x_n\} \subseteq \mathbb{R}$ *with* $\mathbb{R} = \underset{x_i \in \Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}}\{x_i\} = \underset{x_i \in \Delta_\circ}{\cup} \{x_i\}$.

(ii) Let $\mathcal{Z} = \mathbb{R}$ *with the topology* $\tau = \{\phi, \mathbb{R}, \{1\}\}$. *Then* $(\mathbb{R}, \tau)$ *is almost* $\mathcal{SWD}$*-compact but it is not almost* $\omega\mathcal{SWD}$*-compact since* $\mathcal{H} = \{\{1, x\} : x \in \mathbb{R}\}$ *is an* $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$*-cover of* $\mathbb{R}$ *which has no finite subset* $\Delta_\circ = \{x_1, x_2, ...x_n\} \subseteq \mathbb{R}$ *with* $\mathbb{R} = \underset{x \in \Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}}(\{1, x\}) = \underset{x \in \Delta_\circ}{\cup} \{1, x\}$.

Theorem 17. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$. *If* $(\mathcal{Z}, \tau)$ *is almost* $\omega\mathcal{SWD}$*-compact, then each* $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$*-*$\theta$*-cover of* $\mathcal{Z}$ *has a finite subcover.*

Proof. Let $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ be $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-$\theta$-cover of $\mathcal{Z}$. For each $x \in \mathcal{Z}$ there is $H_{\alpha(x)} \in \mathcal{H}$ with $x \in H_{\alpha(x)}$ for some $\alpha(x) \in \Delta$. By Proposition 5 there is $G_{\alpha(x)} \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ with $x \in G_{\alpha(x)} \subseteq \mathcal{C}l_{\omega\mathcal{SWD}}(G_{\alpha(x)}) \subseteq H_{\alpha(x)}$. Therefore, $\mathcal{G} = \{G_{\alpha(x)} : x \in \mathcal{Z}\}$ is an $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-cover of $\mathcal{Z}$ and hence there is a finite subset $\Delta_\circ \subseteq \Delta$ with $\mathcal{Z} = \underset{x \in \Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}}(G_{\alpha(x)}) \subseteq \underset{x \in \Delta_\circ}{\cup} H_{\alpha(x)}$.

Definition 11. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$ *and* $x \in \mathcal{Z}$. *A filter base* $\mathcal{F}$ *on* $(\mathcal{Z}, \tau)$ *is said to be:*

(i) $\omega\mathcal{SWD}\theta$*- converge to* x *if for each* $G \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ *with* $x \in G$ *there is* $F \in \mathcal{F}$ *such that* $F \subseteq \mathcal{C}l_{\omega\mathcal{SWD}}(G)$.

(ii) $\omega\mathcal{SWD}\theta$*- accumulate at* x *if* $\mathcal{C}l_{\omega\mathcal{SWD}}(G) \cap F \neq \phi$ *for each* $F \in \mathcal{F}$ *and for each* $G \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ *with* $x \in G$.

Note that, if a filter base $\mathcal{F}$ $\omega\mathcal{SWD}\theta$- converges to a point x, then $\mathcal{F}$ $\omega\mathcal{SWD}\theta$- accumulates at x. Also, it is obvious to show that a maximal filter base $\mathcal{F}$ $\omega\mathcal{SWD}\theta$- converges to a point x iff $\mathcal{F}$ $\omega\mathcal{SWD}\theta$- accumulates at x.

Theorem 18. *Let* $(\mathcal{Z}, \tau)$ *be a* $\mathcal{TS}$. *Then the following are equivalent:*

(i) $(\mathcal{Z}, \tau)$ *is almost* $\omega\mathcal{SWD}$*-compact.*

(ii) Each maximal filter base $\omega\mathcal{SWD}\theta$*- converges to some point of* $\mathcal{Z}$.

(iii) Each filter base $\omega\mathcal{SWD}\theta$*- accumulates at some point of* $\mathcal{Z}$.

(iv) For each family $\{H_\alpha : \alpha \in \Delta\}$ *of* $\omega\mathcal{SWD}$*-closed subsets of* $(\mathcal{Z}, \tau)$ *and* $\underset{\alpha \in \Delta}{\cap} H_\alpha = \phi$, *there is a finite subset* $\Delta_\circ \subseteq \Delta$ *with* $\underset{\alpha \in \Delta_\circ}{\cap} \mathcal{I}nt_{\omega\mathcal{SWD}}(H_\alpha) = \phi$.

(v) For each family $\{H_\alpha : \alpha \in \Delta\}$ *of* $\omega\mathcal{SWD}$*-closed subsets of* $(\mathcal{Z}, \tau)$ *and* $\underset{\alpha \in \Delta_\circ}{\cap} \mathcal{I}nt_{\omega\mathcal{SWD}}(H_\alpha) \neq \phi$ *for each finite subset* $\Delta_\circ \subseteq \Delta$, *then* $\underset{\alpha \in \Delta}{\cap} H_\alpha \neq \phi$.

Proof. The implication $(iv \to v)$ is obvious.

$(i \to ii)$ Let $\mathcal{F}$ be a maximal filter base on $\mathcal{Z}$ and suppose that it does not $\omega\mathcal{SWD}\theta$-converge to any point of $\mathcal{Z}$. Since $\mathcal{F}$ does not $\omega\mathcal{SWD}\theta$- accumulate at any point $x \in \mathcal{Z}$, there is $F_x \in \mathcal{F}$ and $G_x \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ with $x \in G_x$ such that $\mathcal{C}l_{\omega\mathcal{SWD}}(G_x) \cap F_x = \phi$. Therefore, $\{G_x : x \in \mathcal{Z}\}$ is an $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-cover of $\mathcal{Z}$ and so there is a finite subset

$\Delta_\circ = \{x_1, x_2, ...x_n\} \subseteq \mathcal{Z}$ with $\mathcal{Z} = \underset{x\in\Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}}(G_x)$. But $\mathcal{F}$ is a filter base on $\mathcal{Z}$ and hence there is $F_\circ \in \mathcal{F}$ with $F_\circ \subseteq \cap\{F_{x_i} : i = 1, 2, ...n\}$. Since $F_{x_i} \cap \mathcal{C}l_{\omega\mathcal{SWD}}(G_{x_i}) = \phi$, then $F_\circ = \phi$ which is a contradiction.

$(ii \rightarrow iii)$ Let $\mathcal{F}$ be a filter base on $\mathcal{Z}$. Then there is $\mathcal{F}_\circ$ a maximal filter base with $\mathcal{F} \subseteq \mathcal{F}_\circ$. Since $\mathcal{F}_\circ$ $\omega\mathcal{SWD}\theta$- converges to x for some $x \in \mathcal{Z}$, then for each $G \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ with $x \in G$ there is $F_\circ \in \mathcal{F}_\circ$ such that $F_\circ \subseteq \mathcal{C}l_{\omega\mathcal{SWD}}(G)$. Therefore for each $F \in \mathcal{F}$, $\phi \neq F_\circ \cap F \subseteq \mathcal{C}l_{\omega\mathcal{SWD}}(G) \cap F$ and hence $\mathcal{F}$ $\omega\mathcal{SWD}\theta$- accumulates at x.

$(iii \rightarrow iv)$ Let $\{H_\alpha : \alpha \in \Delta\}$ be a family of $\omega\mathcal{SWDC}(\mathcal{Z}, \tau)$ with $\underset{\alpha\in\Delta}{\cap} H_\alpha = \phi$. If $\underset{\alpha\in\Delta_\circ}{\cap} \mathcal{I}nt_{\omega\mathcal{SWD}}(H_\alpha) \neq \phi$ for each finite subset $\Delta_\circ \subseteq \Delta$. Then $\mathcal{F} = \{\underset{\alpha\in\Delta_\circ}{\cap} \mathcal{I}nt_{\omega\mathcal{SWD}}(H_\alpha) : \Delta_\circ \subseteq \Delta$ and $\Delta_\circ$ is finite$\}$ is a filter base on $\mathcal{Z}$ and hence $\mathcal{F}$ $\omega\mathcal{SWD}\theta$- accumulates at x for some $x \in Z$. Since $\{\mathcal{Z} - H_\alpha : \alpha \in \Delta\}$ is an $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-cover of $\mathcal{Z}$, then $x \in \mathcal{Z} - H_{\alpha_\circ}$ for some $\alpha_\circ \in \Delta$. Therefore, $\mathcal{C}l_{\omega\mathcal{SWD}}(Z - H_{\alpha_\circ}) \cap \mathcal{I}nt_{\omega\mathcal{SWD}}(H_{\alpha_\circ}) = \phi$ which is a contradiction.

$(v \rightarrow i)$ Let $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ be an $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-cover of $\mathcal{Z}$. Then $\mathcal{Z} - H_\alpha \in \omega\mathcal{SWDC}(\mathcal{Z}, \tau)$ for each $\alpha \in \Delta$ and $\underset{\alpha\in\Delta}{\cap}(\mathcal{Z} - H_\alpha) = \phi$. Hence there is a finite subset $\Delta_\circ \subseteq \Delta$ with $\underset{\alpha\in\Delta_\circ}{\cap} \mathcal{I}nt_{\omega\mathcal{SWD}}(\mathcal{Z} - H_\alpha) = \phi$.Therefore $\mathcal{Z} = \underset{x\in\Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)$.

Theorem 19. *Let $(\mathcal{Z}, \tau)$ be almost $\omega\mathcal{SWD}$-compact $\mathcal{TS}$ and $E \subseteq \mathcal{Z}$. If $E \in \tau \cap \omega\mathcal{SWDC}(\mathcal{Z}, \tau)$, then (E, τ_E) is almost $\omega\mathcal{SWD}$-compact.*

Proof. Let $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ be a family with $E = \underset{\alpha\in\Delta}{\cup} H_\alpha$ and $H_\alpha \in \omega\mathcal{SWD}(E, \tau_E)$ for each $\alpha \in \Delta$. Then by Theorem 12, $H_\alpha \in \omega\mathcal{SWD}(\mathcal{Z}, \tau)$ for each $\alpha \in \Delta$ and hence $\{H_\alpha : \alpha \in \Delta\} \cup \{\mathcal{Z} - H\} = \mathcal{Z}$. Since $(\mathcal{Z}, \tau)$ is almost $\omega\mathcal{SWD}$-compact, then there is a finite subset $\Delta_\circ \subseteq \Delta$ with $\mathcal{Z} = [\underset{x\in\Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)] \cup \{\mathcal{Z} - H\}$.Therefore, $E = \underset{\alpha\in\Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha) \subseteq \underset{\alpha\in\Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}_E}(H_\alpha)$.

Definition 12. *Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$ and $H \subseteq \mathcal{Z}$. A subset H is said to be almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$ (in $\mathcal{Z}$) if whenever $\mathcal{H} = \{H_\alpha : \alpha \in \Delta\}$ is an $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$-cover of H, then there is a finite subset $\Delta_\circ$ of Δ with $H \subseteq \underset{x\in\Delta_\circ}{\cup} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)$.*

The following Theorem can be easily constructed.

Theorem 20. *Let $(\mathcal{Z}, \tau)$ be a $\mathcal{TS}$ and $H \subseteq \mathcal{Z}$. The following are equivalent:*

(i) H is almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$.

(ii) If $\mathcal{F}$ is a maximal filter base on $\mathcal{Z}$ and meets H, then it $\omega\mathcal{SWD}\theta$- converges to some point of H.

(iii) If $\mathcal{F}$ is a filter base on $\mathcal{Z}$ and meets H, then it $\omega\mathcal{SWD}\theta$- accumulates at some point of H.

(iv) If $\{H_\alpha : \alpha \in \Delta\}$ is a family of $\omega\mathcal{SWD}$-closed subsets of $(\mathcal{Z}, \tau)$ and $[\underset{\alpha\in\Delta}{\cap} H_\alpha] \cap H = \phi$, then there is a finite subset $\Delta_\circ \subseteq \Delta$ with $[\underset{\alpha\in\Delta_\circ}{\cap} \mathcal{I}nt_{\omega\mathcal{SWD}}(H_\alpha)] \cap H = \phi$.

Proposition 6. *Let $(\mathcal{Z},\tau)$ be a $\mathcal{TS}$ and $H,G \subseteq \mathcal{Z}$. If H is an $\omega\mathcal{SWD}\theta$- closed subset of $(\mathcal{Z},\tau)$ and G is almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$, then $H\cap G$ is almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$.*

Proof. Let $\mathcal{H}=\{H_\alpha : \alpha \in \Delta\}$ be $\omega\mathcal{SWD}(\mathcal{Z},\tau)$-cover of $H \cap G$. Then by Proposition 5 (Part i) for each $x \in \mathcal{Z} - H$ there is $W_x \in \omega\mathcal{SWD}(\mathcal{Z},\tau)$ with $x \in W_x \subseteq \mathcal{C}l_{\omega\mathcal{SWD}}(W_x) \subseteq \mathcal{Z} - H$. Therefore, $\mathcal{H}\cup\{W_x : x \in G - H\}$ is an $\omega\mathcal{SWD}(\mathcal{Z},\tau)$-cover of G and hence there are a finite subset $\Delta_\circ \subseteq \Delta$ and a finite subset $\{x_1, x_2, ...x_n\} \subseteq G - H$ with $G \subseteq [\bigcup_{x\in\Delta_\circ} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)] \cup [\bigcup_{i=1}^{n}\mathcal{C}l_{\omega\mathcal{SWD}}(W_{x_i})]$. Since $\mathcal{C}l_{\omega\mathcal{SWD}}(W_x) \subseteq \mathcal{Z} - H$, then $H \cap G \subseteq \bigcup_{x\in\Delta_\circ} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)$.

Corollary 6. *If $(\mathcal{Z},\tau)$ is an almost $\omega\mathcal{SWD}$-compact and $H \subseteq \mathcal{Z}$ is an $\omega\mathcal{SWD}\theta$- closed subset of $(\mathcal{Z},\tau)$, then H is almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$.*

Theorem 21. *Let $(\mathcal{Z},\tau)$ be a $\mathcal{TS}$. If there is a non-empty proper subset $E \in \omega\mathcal{SWD}(\mathcal{Z},\tau)\cap \omega\mathcal{SWDC}(\mathcal{Z},\tau)$ of $\mathcal{Z}$, then $(\mathcal{Z},\tau)$ is almost $\omega\mathcal{SWD}$-compact iff each $H \in \omega\mathcal{SWD}(\mathcal{Z},\tau) \cap \omega\mathcal{SWDC}(\mathcal{Z},\tau)$ is an almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$.*

Proof. Let $H \in \omega\mathcal{SWD}(\mathcal{Z},\tau) \cap \omega\mathcal{SWDC}(\mathcal{Z},\tau)$. Then by Proposition 5 (Part ii) H is an $\omega\mathcal{SWD}\theta$- closed subset of $(\mathcal{Z},\tau)$ and hence by Corollary 6, H is almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$. Conversely, Let $\mathcal{H}=\{H_\alpha : \alpha \in \Delta\}$ be $\omega\mathcal{SWD}(\mathcal{Z},\tau)$-cover of $\mathcal{Z}$. Then E and $(\mathcal{Z} - E)$ are almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$ and hence there are finite subset $\Delta_\circ \subseteq \Delta$ with $\mathcal{Z} = \bigcup_{x\in\Delta_\circ} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)$.

Proposition 7. *A finite union of almost $\omega\mathcal{SWD}$-compact subsets relative to $\mathcal{Z}$ is almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$.*

Proof. Let $\bigcup_{i=1}^{n} G_i$ be a finite union of almost $\omega\mathcal{SWD}$-compact subsets relative to $\mathcal{Z}$ and $\mathcal{H}=\{H_\alpha : \alpha \in \Delta\}$ be $\omega\mathcal{SWD}(\mathcal{Z},\tau)$-cover of $\bigcup_{i=1}^{n} G_n$. Then for each $i \in \{1, 2, 3...n\}$ there is a finite subset $\Delta_i \subseteq \Delta$ with $G_i \subseteq \bigcup_{\alpha\in\Delta_i} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)$. It is clear that $\bigcup_{i=1}^{n} \Delta_i$ is finite set. Therefore, $\bigcup_{i=1}^{n} G_i \subseteq \bigcup_{\alpha\in \bigcup_{i=1}^{n}\Delta_i} \mathcal{C}l_{\omega\mathcal{SWD}}(H_\alpha)$.

The proofs of the following results are obvious and hence they are omitted.

Theorem 22. *Let $\Gamma : (\mathcal{Z},\tau) \to (\mathcal{K},\sigma)$ be $\omega\mathcal{SWD}$-irresolute. If H is almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{Z}$, then $\Gamma(H)$ is almost $\omega\mathcal{SWD}$-compact relative to $\mathcal{K}$.*

Corollary 7. *Let $\Gamma : (\mathcal{Z},\tau) \to (\mathcal{K},\sigma)$ be surjective $\omega\mathcal{SWD}$-irresolute. If $(\mathcal{Z},\tau)$ is almost $\omega\mathcal{SWD}$-compact, then $(\mathcal{K},\sigma)$ is almost $\omega\mathcal{SWD}$-compact.*

Corollary 8. *If $\Pi Z_{\alpha\in\Delta}$ is almost $\omega\mathcal{SWD}$-compact, then $\mathcal{Z}_\alpha$ is almost $\omega\mathcal{SWD}$-compact for each $\alpha \in \Delta$.*

5. Conclusions

The study of different types of generalized open sets has been one of the main areas of research in general topology during the last several decades. Mathematicians investigate the properties of various broad topological concepts using generalized open sets. To continue this line of research, this manuscript has been written.

The main achievements of this work are:

(i) We present a generalization for Theorem 2 which was introduced in [1] and provide additional features of $\mathcal{SWD}(\mathcal{Z}, \tau)$.

(ii) We introduce the notion of $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$ which is a new generalization for somewhere dense subsets of a topological space $(\mathcal{Z}, \tau)$ and hence it is a new generalization for open subsets of a topological space $(\mathcal{Z}, \tau)$.

(iii) We verify some fundamental features of $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$ and study the requirements for the equivalence between the classes $\mathcal{SWD}(\mathcal{Z}, \tau)$, $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$ and $\omega\mathcal{SWD}(\mathcal{Z}, \tau_\omega)$.

(iv) We study the notions of the interior, closure, $\omega\mathcal{SWD}$-continuous and $\omega\mathcal{SWD}$-irresolute via $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$.

(v) We study the notion of almost $\omega\mathcal{SWD}$-compact spaces with some of their properties.

This work can be considered as a starting point for many topics and studies in topology since $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$ forms a generalization of open sets. Therefore, in upcoming papers, we plan to study the notion of connected, separation axioms and other types of covering such as paracompact spaces via the class $\omega\mathcal{SWD}(\mathcal{Z}, \tau)$.

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