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G-filters and Generalized Complemented Distributive Lattices

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Abstract. In this work, we derive a class of filters (G-filters, normal G-filters, and co-dense filters) in a distributive lattice (with dense elements). We also verify the various algebraic properties of these filters. It is observed that the set of co-dense filters forms an uninduced distributive lattice, and the set of G-filters forms a Boolean algebra. We characterize quasi-complemented distributive lattices using G-filters and normal G-filters. Using normal G-filters, we demonstrate several necessary and sufficient requirements for a distributive lattice to become quasi-complemented. Also, we introduce generalized complementation on a distributive lattice and characterize it in terms of quasi-complemented distributive lattices.

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1. Introduction

In the order (lattice) theory, distributive lattices are foundational structures, embodying a delicate balance between order and algebraic properties. Within these lattices, filters emerge as essential constructs, offering insights into the dynamics of subsets and their interactions. In this regard, the classification of filters in a distributive lattice was studied extensively by several authors [5–7] and then introduced μ-filters, ω–filters and D-filters, etc. At the same time, within these lattices, complements emerge as fundamental

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constructs, offering profound insights into the nature of duality, negation, and complementation within the lattice framework. Thus, the class of complementations classified by many authors [[1], [8], [3]] is called ortho-complementation, pseudo-complementation, and quasi-complementation, etc. The class of maximal elements is a proper sub-collection of the class of dense elements in lattices. We start working on a distributive lattice with dense elements with this initiation.

This paper introduces G-filters, normal G-filters, and co-dense filters in a distributive lattice with dense elements. We derive some algebraic properties from them and obtain the necessary and sufficient conditions for a filter to become a G-filter (normal G-filter). Also, we characterize quasi-complemented distributive lattices using G-filters and normal G-filters. Mainly, we introduce G-complementation on a distributive lattice (which may or may not contain the zero element) and prove several algebraic properties. Every finite distributive lattice is G-complemented, as we have seen. Ultimately, we derive certain sufficient and necessary conditions under which a distributive lattice becomes G-complemented.

2. G-Filters

In this section, we introduce G-filters and co-dense filters in a distributive lattice and prove several algebraic properties of this class of filters. We prove the set of G-filters forms a Boolean algebra, and the set of co-dense filters forms a distributive lattice. Finally, we observe that every maximal filter is either G-filter or a co-dense filter.

In this article, a distributive lattice with dense elements is denoted by A. In [4], Kumar and Rao introduced a class $\{S^d \mid S \text{ is non-empty subset of } A\}$ of filters, where $S^d = \{v \in A \mid s \vee_* v \text{ is dense, for all } s \in S\}$.

Definition 1. *In a distributive lattice A, filter K is said to be a G-filter, if $K^{dd} = K$.*

Lemma 1. *For any filter K of A,*

(i) K^d and K^{dd} are G-filters.

(ii) K^{dd} is the smallest G-filter containing K.

Lemma 2. *Let K be a proper filter containing a non-dense element in A. Then K^d is proper.*

Proof. Suppose that $K^d = A$ and v is a non-dense element in K. Since $0 \in A = K^d$, $0 \vee_* f \in D$, for all $f \in K$. In particular, for $v \in K$, we have $0 \vee_* v = v \in D$. Which is contradiction to v is non-dense. Thus K^d is proper.

Let $GF(A)$ represent the set of G-filters of A..

Theorem 1. *$GF(A)$ forms a Boolean algebra with the operations $K \sqcup G = (K \vee_* G)^{dd}$ and $K \wedge_* G = K^d \cap G^d$, where $K, G \in GF(A)$ and the complement of K is K^d.*

Definition 2. *For each $v, w \in A$, $v \wedge_* w \in K$ implies $v \in K$ or $w \in K$. This indicates that K is a prime G-filter whenever K is a proper G-filter of A.*

Theorem 2. *Let I be an ideal in A such that $K \cap I = \emptyset$, where K be a G-filter. Then, U is a prime G-filter in A such that $K \subseteq U$ and $U \cap I = \emptyset$.*

Proof. Consider a set $\mathcal{Q} = \{G \in GF(A) \mid G$ containing K and $G \cap I = \emptyset\}$, which is a non-empty set (since $K \in \mathcal{Q}$). Suppose that $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ is an increasing chain in $\mathcal{Q}$. Take $M = \sqcup G_i$ for $i = 1, 2, 3, ...$ Then $M \in GF(A)$ (since $GF(A)$ is a distributive lattice) and $I \cap M = I \cap (\sqcup G_i) = \sqcup(I \cap (G_i) = \emptyset$ (since $I \cap G_i = \emptyset$ for all i). Therefore $M \in \mathcal{Q}$. In this case, the chain's upper bound in $\mathcal{Q}$ is M. Zorn's lemma states that $\mathcal{Q}$ has a maximal element, let's say U, such that $U \cap I = \emptyset$ and $K \subseteq U$. Choose $v, w \in A$ such that $v \notin U$ and $w \notin U$. Then $U \subseteq U \sqcup [v)^{dd} = \{U^d \wedge_* [v)^d\}^d = [U \vee_* [v)]^{dd}$ and $U \subseteq U \sqcup [w)^{dd} = \{U^d \wedge_* [w)^d\}^d = [U \vee_* [w)]^{dd}$. Since U is maximal, $[U \vee_* [v)]^{dd} \cap I \neq \emptyset$ and $[U \vee_* [w)]^{dd} \cap I \neq \emptyset$. Therefore $\{[U \vee_* [v)]^{dd} \cap I\} \cap \{U \vee_* [w)]^{dd}\} \cap I = \{U \vee_* [[v)^{dd} \cap [w)^{dd}]\} \cap I = \{U \vee_* [[v) \cap [w)]\}^{dd} \cap I = \{U \vee_* [v \vee_* w)\}^{dd} \cap I$. If $v \vee_* w \in U$, then $v \vee_* w \in U^{dd} = U$ (since U is a G-filter) and $v \vee_* w \in I$. Therefore $U \cap I \neq \emptyset$. which contradicts itself. So that $v \in U$ or $w \in U$. For this reason, U is prime.

Corollary 1. *Let $v \notin K$ and K be a G-filter in A. After that, a prime G-filter U exists such that $K \subseteq U$ and $v \notin U$.*

Theorem 3. *For every filter K of A, K^{dd} is the intersection of all prime G-filters containing K.*

Proof. Suppose there is an element v in A, $v \notin K^{dd}$, and K is a filter. Consider $\mathbf{U} = \{G \mid G$ is a G-filter of A and $v \notin G$ and $K \subseteq G\}$. Then $\mathbf{U} \neq \phi$ (since $K^{dd} \in \mathbf{U}$). Let $K_1 \subseteq K_2 \subseteq \cdots$ be a chain in $\mathbf{U}$. Then $(\sqcup K_i)^{dd} = [(\vee_* K_i)^{dd}]^{dd} = (\vee_* K_i)^{dd} = \sqcup K_i$ (since $GF(A)$ is a distributive lattice). Therefore $\sqcup K_i$ is a G-filter and $K \subseteq \sqcup K_i$ and $(\sqcup K_i) \cap (v] = \sqcup(K_i \cap (v]) = \phi$. Consequently, $\sqcup K_i \in \mathbf{U}$, and an upper bound of the chain in $\mathbf{U}$ is $\sqcup K_i$. Zorn's Lemma states that, the set $\mathbf{U}$ has a maximal element, say that the maximal, element is U. In the case where $v, w \in A$, and $v \notin U$ and $w \notin U$,

$$
\begin{aligned}
v \in \{(U \sqcup [v)^{dd}) \cap (U \sqcup [w)^{dd})\} &\Rightarrow v \in \{U \sqcup ([v)^{dd} \cap [w)^{dd})\} \\
&\Rightarrow v \in \{U \sqcup ([v) \cap [w))^{dd}\} \\
&\Rightarrow v \in \{U \sqcup ([v \vee_* w)^{dd})\} \\
&\Rightarrow v \in \{U \vee_* [v \vee_* w)^{dd}\}^{dd}.
\end{aligned}
$$

If $v \vee_* w \in U$, then $[v \vee_* w) \subseteq U$ and $[v \vee_* w)^{dd} \subseteq U^{dd} = U$ (since U is G-filter). Therefore $v \in U$. Which is a contradiction. Thus U is prime and $v \notin U$.

Corollary 2. *The intersection of prime G-filters is equal to D.*

Definition 3. *A filter K in A is said to be co-dense, if $K^d = D$.*

Lemma 3. *Every co-dense filter contains at least one dense element.*

Theorem 4. *With the operations $K \sqcup G = (K \vee_* G)^{dd}$ and $K \wedge_* G = K^d \cap G^d$, the set of co-dense filters forms a distributive lattice, where K and G are co-dense filters in A with the largest element A and the least element D.*

Theorem 5. *Every maximal filter is a G-filter or co-dense.*

Proof. Consider a maximal filter of A to be K. Thus, $K \subseteq K^{dd}$ is known. Thus, either $K^{dd} = A$ or $K = K^{dd}$. If $K^{dd} = A$, then $K^d = K^{ddd} = A^d = D$. Therefore K is co-dense. Otherwise $K = K^{dd}$, and then K is a G-filter.

3. Normal G-Filters

In this section, we introduce normal G-filters and obtain several algebraic properties. We prove that several necessary and sufficient conditions to become the set of normal G-filters is a Boolean algebra. Now, let us denote $(w)^d := \{v \in A \mid w \vee_* v \text{ is dense}\}$, where $w \in A$.

Lemma 4. *For any $v \in A$,*

(i) $(v)^d$ *is a filter*

(ii) $(0)^d = D$

(iii) $[v) \cap [v)^d$ *is a subset of* D

(iv) $(v)^{ddd} = (v)^d$

(v) $v \in (v)^{dd}$

(vi) $(v)^d = A \Leftrightarrow v \in D$

(vii) D *is a subset of* $(v)^d$

Lemma 5. *For any $w, x \in A$,*

(1) $w \leq x \implies (w)^d \subseteq (x)^d$

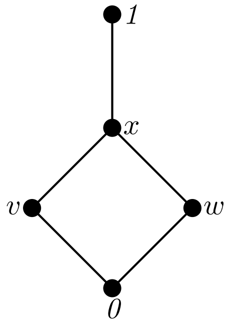
(2) $(w)^d \subseteq (x)^d \implies (x)^{dd} \subseteq (w)^{dd}$

(3) $(w)^d \cap (x)^d = (w \wedge_* x)^d$

(4) $(w \vee_* x)^d = (w)^d \sqcup (x)^d$

Remark 1. *If $v, w \in A$, then $(v)^d = (w)^d$ dose not implies $v = w$. Regarding, have a look at this instance:*

Example 1. *Given a lattice $A = \{0, v, w, x, 1\}$, its Hasse-diagram is*



Then $(1)^d = A$ *and* $(x)^d = A$*, but* $1 \neq x$*.*

Theorem 6. *The class of filters* $\mathcal{A}^d = \{(x)^d \mid x \in L\}$ *with the operations* $(x)^d \sqcup (y)^d = (x \vee_* y)^d$ *and* $(x)^d \cap (y)^d = (x \wedge_* y)^d$ *forms a distributive lattice.*

A is said to be quasi-complemented [2], if for each $v \in A$, there exists $x \in A$ such that $v \wedge_* x = 0$ and $x \vee_* v$ is dense.

Theorem 7. *If* A *is quasi-complemented, then* $(\mathcal{A}^d, \cap, \sqcup, D, A)$ *is a Boolean algebra.*

Proof. Suppose that A is quasi-complemented. Let $(v)^d \in \mathcal{A}^d$, where $v \in A$. Now, for this $v \in A$, by our assumption, there exists $x \in A$ such that $v \wedge_* x = 0$ and $v \vee_* x$ is dense. Therefore $(v)^d \cap (x)^d = (v \wedge_* x)^d = (0)^d = D$ and $(v)^d \sqcup (x)^d = (v \vee_* x)^d = A$ (since $v \vee_* x$ is dense). Thus $\mathcal{A}^d$ is a Boolean algebra.

Theorem 8. *If* A *is a finite distributive lattice, then* $(\mathcal{A}^d, \cap, \sqcup, D, A)$ *is a Boolean algebra if and only if* A *is quasi-complemented .*

Proof. If $\mathcal{A}^d$ is a Boolean algebra, then A is quasi-complemented, as required by Theorem 7. Assume that $(v)^d, (w)^d \in \mathcal{A}^d$. Since $\mathcal{A}^d$ is a Boolean algebra, $(x)^d, (t)^d \in \mathcal{A}^d$ exist such that $D = (0)^d = (v \wedge_* x)^d = (v)^d \cap (x)^d$ and $A = (v \vee_* x)^d = (v)^d \sqcup (x)^d$. As we have $0 \in A = (v \vee_* x)^d$, $v \wedge_* x = 0$ and $v \vee_* x$ is dense. Thus A is quasi-complemented.

Definition 4. *If there is a proper filter* G *such that* $K \cap G = D$ *and* $K \vee_* G = A$*, then a filter* K *of* A *is called a G-factor.*

Theorem 9. *For any* $v \in A$*,* $(v)^d$ *is a G-factor if and only if* $(v)^d \vee_* (v)^{dd} = A$*.*

Lemma 6. *If* $v, w, \in A$*, then we have*

(i) $(v)^d = (w)^d \implies (v \wedge_* x)^d = (w \wedge_* x)^d$*, for all* $x \in A$*.*

(ii) $(v)^d = (w)^d \implies (v \vee_* x)^d = (w \vee_* x)^d$*, for all* $x \in A$*.*

Theorem 10. *The following are equivalent for any filter* K *of* A*;*

(i) K *is a G-filter*

(ii) *For any* $v \in A, v \in K$ *implies* $(v)^{dd} \subseteq K$

(iii) *Assuming* $v \in K$ *and* $(v)^d = (w)^d$, *for every* $v, w \in A$ *implies* $w \in K$

(iv) $K = \bigcup_{v \in K} [v)^{dd}$.

Definition 5. *A filter* K *of* A *is said to be a normal* G*-filter, if* $K = (v)^d$, *for some* $v \in A$.

Let us denote the class of normal G-filters of A as $N^gF(A)$. Now, we have the following;

Theorem 11. $(N^gF(A), \cap, \sqcup)$ *is a sublattice of* $GF(A)$ *in which* $(0)^d$ *is the least and* $(v)^d$ *is the greatest elements in* $N^gF(A)$, *for some* $v \in D$.

Proof. Assume that $K, G \in N^gF(A)$. $K = (v)^d$ and $G = (y)^d$ then exist a pair of $v, y \in A$. It is observe that $(x \vee_* y)^d$ is an upper bound of K, G. Let $H \in N^gF(A)$ be an upper bound of K, G. Then there exists $z \in A$ such that $H = (z)^d$ and $(v)^d, (y)^d \subseteq (z)^d$. So that $(z)^{dd} \subseteq (x)^{dd} \cap (y)^{dd} = (x \vee_* y)^{dd}$. Therefore $(x \vee_* y)^d \subseteq (z)^d$. Hence $(x \vee_* y)^d$ is the least upper bound of K and G and it is indicated by $K \sqcup G$. Thus $N^gF(A)$ is a sub-lattice of $GF(A)$. For any $v \in A$, $(v)^d \cap (0)^d = (x \wedge_* 0)^d = (0)^d$. Therefore $(0)^d$ is the least element in $N^gF(A)$. Let $w \in D$. Then $(v)^d \sqcup (y)^d = (x \vee_* y)^d$. Since $v \vee_* y \in D, (v)^d \sqcup (y)^d = A$. Therefore $(v)^d$ is the greatest element in $N^gF(A)$. Thus $N^gF(A)$ is bounded.

Denote a relation $\psi := \{(v, w) \in A \times A \mid (w)^d = (v)^d\}$. Then it is easy to prove that ψ is a congruence relation on A.

Lemma 7. *For any* $v \in A$,

(i) $v/\psi = \{0\}$ *if and only if* $v = 0$

(ii) $v/\psi = D$ *if and only if* $v \in D$.

Theorem 12. *The quotient lattice* A/ψ *forms a distributive lattice with the operations* $v/\psi \wedge_* w/\psi = (v \wedge_* w)/\psi$ *and* $v/\psi \vee_* w/\psi = (v \vee_* w)/\psi$.*Furthermore, the largest element in* A/ψ *exists only when* A *is dense.*

Theorem 13. *The subsequent algebras are equivalent;*

(1) A *is quasi complemented*

(2) $(N^gF(A), \cap, \sqcup, D, A)$ *is a Boolean algebra*

(3) $(A/\psi, \wedge_*, \vee_*, 0/\psi, d/\psi)$ *is a Boolean algebra*

(4) *All principal ideals are quasi-complemented.*

Proof. *(1)* $\Longrightarrow$ *(2)*: Suppose that A is quasi complemented. Let $v \in A$ .Then, $v \wedge_* w = 0$ and $v \vee_* w$ are dense for some $w \in A$. So that $D = (0)^d = (v \wedge_* w)^d = (v)^d \cap (w)^d$ and $A = (v \vee_* w)^d = (v)^d \sqcup (w)^d$. Therefore $N^gF(A)$ is a Boolean algebra.
(2) $\Longrightarrow$ *(3)*: Let $v \in A$. Then $(v)^d \in N^gF(A)$. For this $(v)^d \in N^gF(A)$, there exists

$w \in A$ such that $D = (0)^d = (v \wedge_* w)^d = (v)^d \cap (w)^d$ and $A = (v \vee_* w)^d = (v)^d \sqcup (w)^d$. Therefore $v \vee_* w$ is dense and $v \wedge_* w = 0$. Hence $\{0\} = 0/\psi = (v \wedge_* w)/\psi = v/\psi \wedge_* w/\psi$ and $D = (v \vee_* w)/\psi = v/\psi \vee_* w/\psi$(since $v \vee_* w$ is dense). Thus A/ψ is a Boolean algebra.
(3) $\Longrightarrow$ *(4)*: Let $v \in A$. Then, $v/\psi \wedge_* w/\psi = \{0\}$ and $v/\psi \vee_* w/\psi = D$ exist for some $w \in A$. So that $v \vee_* w$ is dense and $v \wedge_* w = 0$. So that $(v] \wedge_* (w] = (v \wedge_* w] = (0] = \{0\}$ and $(v] \vee_* (w] = (v \vee_* w]$. Now, $(v \wedge_* w] = \{0\}$ and $(v \vee_* w]$ is a principal ideal generated by a dense element $v \vee_* w$. Hence every principal ideal is quasi-complemented.
(4) $\Longrightarrow$ *(1)*: Suppose that every principal ideal is quasi-complemented. Assume $v \in A$. Then, $(v] \cap (w] = \{0\}$ and $(v] \vee_* (w]$ exist for some $w \in A$ is the principal ideals produced by $v \vee_* w$ dense elements. As a result, both $v \vee_* w$ is dense, and $v \wedge_* w = 0$. Thus A is quasi-complemented.

4. Generalized Complemented distributive lattice

This section obtains numerous algebraic characteristics and introduces a generalized complementation on a distributive lattice. For a distributive lattice to become a G-complemented one, we give both required and sufficient necessities. Furthermore, we derive sufficient and necessary conditions under which a quasi-complemented distributive lattice can be formed from a generalized complemented distributive lattice.

Definition 6. *If a unary operation g on A meets these requirements, it is called a generalized complementation.*

(i) for any $v \in A$, $v \vee_* v^g \in D$

(ii) for any $w \in A$, $v \vee_* w \in D \Leftrightarrow v^g \leq w$.

In this case, v^g is called a generalized complement of v and A is called a generalized complemented distributive lattice.

Example 2. *In Example 1, let us define $0^g = x, v^g = w, w^g = v, x^g = 1^g = 0$. Then g is a generalized complementation on A.*

Example 3. *Every complemented distributive lattice is generalized complemented.*

Remark 2. *The converse of the above statement need not be true. For, in Example 1, A is a generalized complemented distributive lattice, but A is not complemented.*

Lemma 8. *If g is a generalized complementation on A and $v, w \in A$, then*

(i) $0^g \in D$

(ii) $d \in D \Longrightarrow d^g = 0$

(iii) $v \leq w \Longrightarrow w^g \leq v^g$

(iv) $v^{gg} \leq v$

(v) $v^{ggg} = v^g$

(vi) $0^{gg} = 0$

(vii) $v \in D \iff v^g = 0 \iff v^{gg} \in D$

(viii) $v^g \leq 0^g$

(ix) $v^g \leq w^g \iff w^{gg} \leq v^{gg}$

(x) $v = 0 \implies v^{gg} = 0$.

Lemma 9. *If g is a generalized complementation on A, then the following are equivalent;*

(i) $v \vee_* w \in D$, *for all* $v, w \in A$.

(ii) $v^{gg} \vee_* w \in D$, *for all* $v, w \in A$.

(iii) $v^{gg} \vee_* w^{gg} \in D$, *for all* $v, w \in A$.

(iv) $v \vee_* w^{gg} \in D$, *for all* $v, w \in A$.

Proof. Let $v, w \in A$. *(i)* $\implies$ *(ii):* Suppose that $v \vee_* w \in D$. Then $v^g \leq w$ and $v^g \vee_* w = w$. Now, $v^{gg} \vee_* w = v^{gg} \vee_* (v^g \vee_* w) = (v^{gg} \vee_* v^g) \vee_* w \in D$, since $v^{gg} \vee_* v^g \in D$.
(ii) $\implies$ *(iii):* Suppose that $v^{gg} \vee_* w \in D$. Then $w \vee_* v^{gg} \in D$ and hence $w^{gg} \vee_* v^{gg} = v^{gg} \vee_* w^{gg} \in D$.
(iii) $\implies$ *(iv):* Suppose that $v^{gg} \vee_* w^{gg} \in D$. By Lemma 8(iv), $v^{gg} \leq v$. Therefore $v \vee_* w^{gg} \in D$.
(iv) $\implies$ *(i):* Suppose that $v \vee_* w^{gg} \in D$. By Lemma 8(iv), $v^{gg} \leq v$. Therefore $v \vee_* w \in D$.

Lemma 10. *If g is a generalized complementation on A and $v, w \in A$, then*

(i) $(v \wedge_* w)^g = v^g \vee_* w^g$

(ii) $(v \vee_* w)^g \leq v^g \wedge_* w^g$

(iii) $(v \vee_* w)^{gg} = v^{gg} \vee_* w^{gg} = (v^g \wedge_* w^g)^g$

(iv) $(v \wedge_* w)^{gg} = (v^g \vee_* w^g)^g = (v^{gg} \wedge_* w^{gg})^{gg}$.

Proof. *(i):* For any $v, w \in A$, $v, w \geq v \wedge_* w$. Then $(v \wedge_* w)^g \geq v^g, w^g$. Therefore $v^g \vee_* w^g \leq (v \wedge_* w)^g$. Now, $(v \wedge_* w) \vee_* (v^g \vee_* w^g) = [v \vee_* (v^g \vee_* w^g)] \wedge_* [w \vee_* (v^g \vee_* w^g)] = [(v \vee_* v^g) \vee_* w] \wedge_* [v^g \vee_* (w \vee_* w^g)] \in D$ (since $v \vee_* v^g, w \vee_* w^g \in D$). So that $(v \wedge_* w)^g \leq v^g \vee_* w^g$. Therefore $(v \wedge_* w)^g = v^g \vee_* w^g$.
(ii): We have $v, w \leq v \vee_* w$. Then $(v \vee_* w)^g \leq v^g, w^g$. Therefore $(v \vee_* w)^g \leq v^g \wedge_* w^g$.

(iii): By (ii), $(v^g \wedge_* w^g)^g \leq (v \vee_* w)^{gg}$. Then $v^{gg} \vee_* w^{gg} \leq (v \vee_* w)^{gg}$. On the other hand, $(v \vee_* w) \vee_* (v \vee_* w)^g \in D$. Then $v \vee_* [w \vee_* (v \vee_* w)^g] \in D$. By Lemma 9.,

$$\begin{aligned} v^{gg} \vee_* [w \vee_* (v \vee_* w)^g] \in D \ &\Rightarrow [v^{gg} \vee_* w] \vee_* [(v \vee_* w)^g] \in D \\ &\Rightarrow w \vee_* v^{gg} \vee_* (v \vee_* w)^g \in D \\ &\Rightarrow w^{gg} \vee_* v^{gg} \vee_* (v \vee_* w)^g \in D \\ &\Rightarrow (v \vee_* w)^g \vee_* v^{gg} \vee_* w^{gg} \in D \\ &\Rightarrow (v \vee_* w)^{gg} \leq v^{gg} \vee_* w^{gg}. \end{aligned}$$

Therefore $(v \vee_* w)^{gg} = v^{gg} \vee_* w^{gg}$.
(iv): $(v \wedge_* w)^{gg} = [(v \wedge_* w)^g]^g = [v^g \vee_* w^g]^g = [v^g \vee_* w^g]^{ggg} = [v^{ggg} \vee_* w^{ggg}]^g = (v^g \vee_* w^g)^g = (v^{gg} \wedge_* w^{gg})^{gg}$.

Theorem 14. *Every finite distributive lattice is generalized complemented.*

Proof. Let A be a finite distributive lattice and $D \neq \emptyset$. For any $v \in A$, define $v^g = Inf\{w \in A \mid v \vee_* w \in D\}$. Then $v \vee_* v^g = v \vee_* [\wedge_*\{w \in A \mid v \vee_* w \in D\}] = \wedge_*[\{v \vee_* w \mid v \vee_* w \in D\}] \in D$. If $v \vee_* x \in D$, then $x \in \{v \in A \mid v \vee_* v \in D\}$. Therefore $\wedge_*\{v \in A \mid v \vee_* v \in D\} \leq x$ and hence $v^g \leq x$. On other hand, suppose $v^g \leq x$, then we have $v \vee_* v^g \in D,\ v \vee_* x \in D$. Hence v^g is a generalized complementation of v. Thus A is generalized complemented.

Theorem 15. *A is generalized complemented if and only if $[0, d]$ is generalized complemented, for all dense elements $d \in A$.*

Proof. Suppose that A is a generalized complemented and g is a generalized complementation on A. Let $v \in [0, d]$. Then there exists $v^g \in A$ such that $v \vee_* v^g \in D$ and for any $w \in A, v \vee_* w \in D$ if and only if $v^g \leq w$. Since $v \vee_* d \in D$, $v^g \leq d$. Therefore $v^g \in [0, d]$. Hence $[0, d]$ is generalized complemented. Conversely suppose that $[0, d]$ is a generalized complemented distributive lattice with dense element d. Let $v \in A$. Then $v \vee_* d$ is dense in A. Therefore $v \in [0, v \vee_* d]$. Hence there exists v^g in $[0, v \vee_* d]$ such that $v \vee_* v^g$ is dense in $[0, v \vee_* d]$. Let $w \in A$, such that $v \vee_* w$ is dense in A. Then $(v \vee_* w) \wedge_* d$ is dense in $[0, v \vee_* d]$. Therefore $(v \wedge_* d) \vee_* (w \wedge_* d)$ is dense in $[0, v \vee_* d]$. So that $(v \wedge_* d)^g \leq w \wedge_* d \leq w$. Also that $v^g \leq v^g \vee_* d^g \leq w$. If $v^g \leq w$, then $v \vee_* v^g \in D$. Therefore $v \vee_* w \in D$. Hence A is generalized complemented distributive lattice with the generalized complementation g.

Theorem 16. *A is generalized complemented if and only if $PI(A)$ is generalized complemented.*

Proof. Suppose that A is a generalized complemented distributive lattice and g is the generalized complementation on A. Let $(v] \in PI(A)$, for some $v \in A$. Then there exists $v^g \in A$ such that $v \vee_* v^g \in D$ and, $v \vee_* w \in D$ if and only if $v^g \leq w$, for any $w \in A$. Define

$(v]^g = (v^g]$. For $(t] \in PI(A)$,

$$\begin{aligned}
(t] \in [(v] \vee_* (v]^g]^* &\implies (t] \in [(v \vee_* v^g]]^* \\
&\implies (t] \wedge_* (v \vee_* v^g] = (0] \\
&\implies (t \wedge_* (v \vee_* v^g)] = (0] \\
&\implies t \wedge_* (v \vee_* v^g) = 0 \\
&\implies t = 0 \qquad\qquad (\text{since } v \vee_* v^g = 0) \\
&\implies (t] = (0].
\end{aligned}$$

Therefore $(v] \vee_* (v]^g$ is dense in $PI(A)$. If $(v] \vee_* (w]$ is dense in $PI(A)$, for some $w \in A$. Then $(v \vee_* w]$ is dense in $PI(A)$ if and only if $v \vee_* w$ is dense in A. Therefore $v^g \le w$. So that $(v^g] \subseteq (w]$. Hence $(v]^g \subseteq (w]$. If $(v]^g \subseteq (w]$. Since $(v] \vee_* (v^g]$ is dense in $PI(A)$, $(v^g] \vee_* (w]$ is dense in $PI(A)$. Hence $(v] \vee_* (w]$ is dense in $PI(A)$. Conversely suppose that $PI(A)$ is generalized complemented. Let $v \in A$. Then $(v] \in PI(A)$. Then there exists $(v]^g$ in $PI(A)$ such that $(v] \vee_* (v]^g$ is dense in $PI(A)$ and $(v] \vee_* (w]$ is dense if and only if $(v]^g \subseteq (w]$. Now, we can write $(v]^g = (v^+]$ for some $v^+ \in A$. Then $(v] \vee_* (v]^g = (v] \vee_* (v^+] = (v \vee_* v^+]$ is dense in $PI(A)$ if and only if $v \vee_* v^+$ is dense in A. If $v \vee_* w \in D$ for some $w \in A$. Then $(v] \vee_* (w]$ is dense in $PI(A)$. Therefore $(v]^g \subseteq (w]$. Hence $(v^+] \subseteq (w]$. So that $(v^+] \cap (w] = (v^+]$ and also $(v^+ \wedge_* w] = (v^+]$. Now $v^+ \wedge_* w = v^+$. Hence $v^+ \le w$. If $v^+ \le w$. Then $(v^+] \subseteq (w]$. Therefore $(v^+] \subseteq (w]$. So that $(v]^g \subseteq (w]$ if and only if $(v] \vee_* (w] = (v \vee_* w]$ is dense in $PI(A)$. Hence $v \vee_* w$ is dense in A. Thus $+$ is a generalized complementation on A and A is generalized complemented.

Definition 7. *A distributive lattice A with a generalized complementation g is said to be p-complemented if for any $v \in A$, there exists $v^g \in A$ such that $v \wedge_* v^g = 0$.*

Theorem 17. *Every G-complemented distributive lattice is quasi complemented if and only if it is p-complemented.*

Proof. Let g be a generalized complementation on A. Suppose A is quasi complemented. Let $v \in A$. Then there exists $w \in A$ such that $v \wedge_* w = 0$ and $v \vee_* w$ is dense. Since g is a generalized complementation, $v^g \le w$. Now $0 = v \wedge_* w = v \wedge_* (v^g \vee_* w) = (v \wedge_* v^g) \vee_* (v \wedge_* w)$. Therefore $v \wedge_* v^g = 0$. Hence A is p-complemented. Conversely suppose that A is p-complemented. Let $v \in A$. Then there exists $v^g \in A$ such that $v \wedge_* v^g = 0$. By definition of G-complementation, $v \vee_* v^g$ is dense. Thus A is quasi complemented.

5. Conclusions

This paper extensively studied on G-filters, normal G-filters, co-dense filters and generalized complementations in a distributive lattice with dense elements. The class of quasi complemented distributive lattices and the class of generalized complemented distributive lattices are characterized. Further we can co relate the class of relatively complemented distributive lattices and the class of ortho-complemented distributive lattices, using the class of G-filters and the class of generalized complemented distributive lattices.

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Conflicts of interest or competing interests

The authors declare that they have no conflicts of interest.

Informed Consent

The authors are fully aware and satisfied with the contents of the article.

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