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Chen's Inequality for CR-Warped Products in Locally Metallic Riemannian Manifolds

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Abstract. In this paper, we study CR-warped product submanifolds in locally metallic Riemannian manifolds. We provide several non-trivial examples of such submanifolds. We establish a sharp inequality known as Chen's inequality for the squared norm of the second fundamental form. We also discuss the equality case of Chen's inequality.

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Key Words and Phrases: Warped product, CR warped product, Metallic Riemannian structure, Golden Structure, CR submanifold

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1. Introduction

Warped products are considered a generalization of Cartesian products. The study of warped product manifolds was developed by Bishop and O'Neill in [1], who obtained fundamental properties of warped product manifolds and constructed a class of complete manifolds with negative curvature. Subsequently, B. Y. Chen studied CR-submanifolds of Kähler manifolds, that are warped products of complex and totally real submanifolds, and published his findings in a series of papers [6–8]. Also, He presented a multitude of properties for warped product manifolds and submanifolds in [9] and discussed the applications of these properties to differential geometry and geometric analysis.

Hretcanu and Crasmareanu introduced the notion of a Golden structure on Riemannian manifolds in [15]. They showed that a Golden structure is a generalization of an almost product structure. The properties of submanifolds in Golden Riemannian manifolds were then studied in [10, 16] using the correspondences between a Golden structure and an almost product structure. The metallic structure, defined in [17], is a further generalization of the Golden structure. Different types of submanifolds in metallic Riemannian manifolds

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were studied in [3, 13], which involved obtaining different integrability conditions for the distributions related to these submanifolds. The metallic warped product Riemannian manifold was studied in [2, 14]. After that, Hretcanu and Blaga worked on the existence problem of proper warped product bi-slant submanifolds in locally metallic Riemannian manifolds [14]. They provided a brief overview of metallic Riemannian manifolds and their submanifolds, and then discussed slant and bi-slant submanifolds (including semi-slant and hemi-slant submanifolds) in locally metallic Riemannian manifolds. They also studied the properties of warped product bi-slant submanifolds in metallic Riemannian manifolds and investigated the existence of various types of warped products, including warped product CR submanifolds in locally metallic Riemannian manifolds [14] where they proved that there is no proper CR warped product of the form $M_T \times_f M_\perp$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds, respectively, in a locally metallic Riemannian manifold. A study related to this field using other mathematical methods that may be relevant from another point of view in [18].

In this paper, we continue the research on warped product CR-submanifolds of the form $M_\perp \times_f M_T$ in locally metallic Riemannian manifolds. We provide some examples of warped product CR-submanifolds in a metallic Riemannian manifold. Also, we obtain some useful lemmas that will be used to prove our main theorem. We derive a relation for the squared norm of the second fundamental form in terms of the components of the gradient of the warping function, and consider the equality case.

2. Preliminaries

Let $\tilde{M}$ be a smooth manifold of dimension m. The metallic structure J is a (1,1) tensor field defined by the equation

$$J^2 = pJ + qI, \tag{1}$$

where $p, q \in \mathbb{N}$ and I is the identity operator on the space of all vector fields on $\tilde{M}$, denoted by $\Gamma(T\tilde{M})$, [17].

A metallic Riemannian manifold is a Riemannian manifold $(\tilde{M}, \tilde{g})$ where the Riemannian metric $\tilde{g}$ is J-compatible. This means that

$$\tilde{g}(JX, Y) = \tilde{g}(X, JY) \tag{2}$$

holds for all vector fields X and Y in $\Gamma(T\tilde{M})$[17]. Specifically, a Golden structure is a special type of a metallic structure. It is defined by the equation $J^2 = J + I$, where $p = q = 1$ [11]. On a metallic Riemannian manifold $\tilde{M}$, the Riemannian metric $\tilde{g}$ satisfies the equation

$$\tilde{g}(JX, JY) = p\tilde{g}(JX, Y) + q\tilde{g}(X, Y), \tag{3}$$

for all $X, Y \in \Gamma(T\tilde{M})$. This equation can be derived from equations (1) and (2).

Now, let M be a submanifold embedded in a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$. Let TX and NX be the tangential and normal components of JX, respectively, for any

$X \in \Gamma(TM)$. Similarly, let tV and nV be the tangential and normal components of JV, respectively, for any $V \in \Gamma(T^{\perp}M)$. That is

$$JX = TX + NX, \tag{4}$$

$$JV = tV + nV. \tag{5}$$

This implies that for any X, Y in $\Gamma(TM)$ and U, V in $\Gamma(T^{\perp}M)$, we have [3]

$$\tilde{g}(TX, Y) = \tilde{g}(X, TY), \tag{6}$$

$$\tilde{g}(nU, V) = \tilde{g}(U, nV), \tag{7}$$

$$\tilde{g}(NX, V) = \tilde{g}(X, tV). \tag{8}$$

Clearly, the maps T and n are $\tilde{g}$ -symmetric. Consequently, the following equations hold for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, [13]

$$T^2X = pTX + qX - tNX, \qquad pNX = NTX + nNX, \tag{9}$$

$$n^2V = pnV + qV - NtV, \qquad ptV = TtV + tnV. \tag{10}$$

Suppose that $\tilde{\nabla}$ and ∇ be the Levi-Civita connections on Riemannian manifolds $\tilde{M})$ and M, respectively. Then, for any $X.Y \in \Gamma(M)$, $V \in \Gamma(T^{\perp}M)$, the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{11}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{12}$$

where h and A_V are the second fundamental form and the shape operator on M, respectively [5]. They are related by

$$\tilde{g}(h(X, Y), V) = \tilde{g}(A_V X, Y). \tag{13}$$

A locally metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ is a manifold that has a metallic Riemannian structure such that J is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$, that is $\tilde{\nabla}J = 0$, [12].

Hence, from equation (1), one can see that the following equation holds for any $X, Y, Z \in \Gamma(TM)$, [4]

$$\tilde{g}((\tilde{\nabla}_X J)Y, Z) = \tilde{g}(Y, (\tilde{\nabla}_X J)Z). \tag{14}$$

Now, let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, then the warped product $M_1 \times_f M_2$ is a Riemannian manifold with Riemannian metric $g = g_1 + f^2 g_2$, where f is a positive smooth function on M_1, called the warping function [1]. Note that if M_1 and M_2 have dimension n_1 and n_2, respectively, then the dimension of the warped product $M_1 \times_f M_2$ is $n = n_1 + n_2$.

On the warped product $M = M_1 \times_f M_2$, if $X, Y \in \Gamma(TM_1)$, and $Z, W \in \Gamma(TM_2)$, then [9],

$$\nabla_X Y \in \Gamma(TM_1), \tag{15}$$

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{16}$$

$$\nabla_Z W = \acute{\nabla}_Z W - g(Z, W)\vec{\nabla} \ln f. \tag{17}$$

where ∇ is the Levi-Civita connection on M and $\vec{\nabla} \ln f$ is the gradient of $\ln f$ which is defined for any $X \in \Gamma(TM)$ as

$$g(\vec{\nabla} f, X) = X(f). \tag{18}$$

This implies that

$$\| \vec{\nabla} f \|^2 = \sum_{i=1}^{n} (e_i(f))^2 \tag{19}$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field on M. Moreover, on any warped product $M = M_1 \times_f M_2$, M_1 and M_2 are totally geodesic and totally umbilical submanifolds of M [1].

A submanifold M of a metallic Riemannian manifold $\tilde{M}$ is called invariant under the metallic structure J if $J(T_xM) \subset T_xM$, for any $x \in M$. It follows that the orthogonal complement of the tangent space of M is also invariant under J, $J(T_xM^{\perp}) \subset T_xM^{\perp}$, for any $x \in M$. This is because $g(X, JU) = g(JX, U) = 0$, for any $U \in \Gamma(TM^{\perp})$ and $X \in \Gamma(TM)$. An anti-invariant submanifold M of $\tilde{M}$ is a submanifold such that $J(T_xM) \subset T_xM^{\perp}$, for any $x \in M$ [4].

The existence of warped product CR submanifolds in a locally metallic Riemannian manifold $\tilde{M}$ has been studied in[14], and proved the following:

Theorem 1. *([14]) Let $M = M_T \times_f M_{\perp}$ be a warped product CR-submanifold in a locally metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, where M_T and $M_{\perp}$ are invariant and anti-invariant submanifolds of $\tilde{M}$, respectively. Then, $M = M_T \times_f M_{\perp}$ is a non-proper warped product submanifold in $\tilde{M}$, that is, the warping function f is constant on M_T.*

3. Basic Lemmas and Examples

In this section, we consider warped product CR-submanifolds in the form $M = M_{\perp} \times_f M_T$ such that $M_{\perp}$ is an anti-invariant submanifold and M_T is an invariant submanifold of a locally metallic Riemannian manifold $\tilde{M}$.

Lemma 1. *Let $M = M_{\perp} \times_f M_T$ be a warped product CR-submanifold in a locally metallic Riemannian manifold $\tilde{M}$, then for any $X, Y \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_{\perp})$, we have*

$$g(h(X, Y), JZ) = -Z(\ln f)\, g(X, JY), \tag{20}$$

$$g(h(X, Z), JW) = 0, \tag{21}$$

$$g(h(JX, Y), JZ) = -pZ(\ln f)\, g(JX, Y) + qZ(\ln f)\, g(X, Y). \tag{22}$$

Proof. Since $\tilde{M}$ is a locally metallic Riemannian manifold, then by using (3), for any $Z \in \Gamma(TM_\perp)$ and $X, Y \in \Gamma(TM_T)$, we have

$$g(h(X,Y), JZ) = \frac{1}{p} g(\tilde{\nabla}_X JY, JZ) - \frac{q}{p} g(\tilde{\nabla}_X Y, Z) \tag{23}$$

Also, by Gauss formula (11) and (16), we have

$$g(h(X,Y), JZ) = \frac{1}{p} g(h(X, JY), JZ) + \frac{q}{p} Z(\ln f)\, g(X,Y). \tag{24}$$

By interchanging Y by JY in (24), we have

$$g(h(X,JY), JZ) = g(h(X,JY), JZ) + \frac{q}{p} g(h(X,Y), JZ) - \frac{q}{p} Z(\ln f)\, g(X, JY). \tag{25}$$

Then, we get

$$g(h(X,Y), JZ) = -Z(\ln f)\, g(X, JY), \tag{26}$$

which proves (20). Now, by using the same strategies,

$$g(h(X,Z), JW) = \frac{1}{p} g(\tilde{\nabla}_Z JX, JW) - \frac{q}{p} g(\tilde{\nabla}_Z X, W). \tag{27}$$

On the other hand, we derive

$$g(h(X,Z), JW) = \frac{1}{p} g(h(Z, JX), JW). \tag{28}$$

By replacing X with JX, and applying (3), we get

$$g(h(JX,Z), JW) = g(h(Z,JX), JW) + \frac{q}{p} g(h(Z,X), JW). \tag{29}$$

This implies that

$$g(h(X,Z), JW) = 0.$$

By rewriting the equation (20) with JX instead of X, and by using (3), we can see that (22) holds. This completes the proof of the lemma.

Corollary 1. *Let $M = M_\perp \times_f M_T$ be a warped product CR-submanifold M in a locally metallic Riemannian manifold $\tilde{M}$, then*

$$g(h(JX,JY), JZ) = qg(h(X,Y), JZ) + pg(h(JX,Y), JZ). \tag{30}$$

for any $X, Y \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$,

Proof. After interchanging Y by JY in (22), and applying (3), we obtain

$$g(h(JX,JY), JZ) = -(p^2+q) Z(\ln f)\, g(JX,Y) - pqZ(\ln f)\, g(X,Y). \tag{31}$$

Using equation (31)and (20) along with (22), we get (30).

Here, we provide some examples of a CR-warped product manifold $M = M_\perp \times_f M_T$ in a locally metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$.

Example 1. Let M be a submanifold of $\tilde{M}$ defined by the immersion i as follows:

$$i(f,\alpha,\theta) = (f\cos\theta\sin\alpha, f\cos\theta\cos\alpha, f\sin\theta\sin\alpha, f\sin\theta\cos\alpha, f, \tilde{\sigma}q^{\frac{-1}{2}}f, \sigma q^{\frac{-1}{2}}f),$$

where $f > 0$, $\alpha, \theta \in (0, \frac{\pi}{2})$, σ is a metallic number, defined by $\sigma = \frac{p+\sqrt{p^2+4q}}{2}$, and $\tilde{\sigma} = p - \sigma$, for some positive integers p and q.

It is straightforward to compute that the tangent bundle of M is spanned by the vectors $\{Z_1, Z_2, Z_3\}$, where

$$\begin{aligned}
Z_1 &= \cos\theta\sin\alpha\frac{\partial}{\partial x_1} + \cos\theta\cos\alpha\frac{\partial}{\partial x_2} + \sin\theta\sin\alpha\frac{\partial}{\partial x_3} + \sin\theta\cos\alpha\frac{\partial}{\partial x_4} \\
&\quad + \frac{\partial}{\partial x_5} + \frac{\tilde{\sigma}}{\sqrt{q}}\frac{\partial}{\partial x_6} + \frac{\sigma}{\sqrt{q}}\frac{\partial}{\partial x_7}, \\
Z_2 &= f\cos\theta\cos\alpha\frac{\partial}{\partial x_1} - f\cos\theta\sin\alpha\frac{\partial}{\partial x_2} + f\sin\theta\cos\alpha\frac{\partial}{\partial x_3} - f\sin\theta\sin\alpha\frac{\partial}{\partial x_4}, \\
Z_3 &= -f\sin\theta\sin\alpha\frac{\partial}{\partial x_1} - f\sin\theta\cos\alpha\frac{\partial}{\partial x_2} + f\cos\theta\sin\alpha\frac{\partial}{\partial\partial x_3} + f\cos\theta\cos\alpha\frac{\partial}{\partial x_4}.
\end{aligned}$$

By using the metallic structure J of $\tilde{M}$ which is

$$J(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (\sigma x_1, \sigma x_2, \sigma x_3, \sigma x_4, \bar{\sigma}x_5, \sigma x_6, \bar{\sigma}x_7),$$

Then, we find that

$$\begin{aligned}
JZ_1 &= \sigma\cos\theta\sin\alpha\frac{\partial}{\partial x_1} + \sigma\cos\theta\cos\alpha\frac{\partial}{\partial x_2} + \sigma\sin\theta\sin\alpha\frac{\partial}{\partial x_3} + \sigma\sin\theta\cos\alpha\frac{\partial}{\partial x_4} \\
&\quad + \tilde{\sigma}\frac{\partial}{\partial x_5} + \frac{\sigma\tilde{\sigma}}{\sqrt{q}}\frac{\partial}{\partial x_6} + \frac{\tilde{\sigma}\sigma}{\sqrt{q}}\frac{\partial}{\partial x_7}, \\
JZ_2 &= f\sigma\cos\theta\cos\alpha\frac{\partial}{\partial x_1} - f\sigma\cos\theta\sin\alpha\frac{\partial}{\partial x_2} + f\sigma\sin\theta\cos\alpha\frac{\partial}{\partial x_3} - f\sigma\sin\theta\sin\alpha\frac{\partial}{\partial x_4}, \\
JZ_3 &= -f\sigma\sin\theta\sin\alpha\frac{\partial}{\partial x_1} - f\sigma\sin\theta\cos\alpha\frac{\partial}{\partial x_2} + f\sigma\cos\theta\sin\alpha\frac{\partial}{\partial x_3} + f\sigma\cos\theta\cos\alpha\frac{\partial}{\partial x_4},
\end{aligned}$$

Now, we define two vector spaces D_T and $D_\perp$, where $D_T = Span\{Z_2, Z_3\}$ is the invariant distribution and $D_\perp = Span\{Z_1\}$ is the anti-invariant distribution, which are preserved by the action of J. Hence, the Riemannian metric of the warped product CR-submanifold M is given by the following:

$$g = (2 + \tilde{\sigma}^2 q^{-1} + \sigma^2 q^{-1})d^2 f + f^2(d^2\theta + d^2\alpha).$$

Then, $M = M_\perp \times fM_T$ is a warped product CR-submanifold in the metallic Riemannian manifold $\tilde{M}$.

Example 2. Let M be a submanifold of $\tilde{M}$ defined by the immersion i as follows:

$$\begin{aligned} i(u,v,\theta,\phi) =&(u\cos\theta, u\sin\theta, v\cos\theta, v\sin\theta, u\cos\phi, u\sin\phi, v\cos\phi, v\sin\phi, \frac{1}{\sqrt{q}}\sigma u,\\ &\frac{1}{\sqrt{q}}\tilde{\sigma}v, \frac{1}{\sqrt{q}}\tilde{\sigma}u, \frac{1}{\sqrt{q}}\sigma v), \end{aligned}$$

where $u, v > 0$, $\theta, \phi \in (0, \frac{\pi}{2})$, σ is a metallic number, defined by $\sigma = \frac{p+\sqrt{p^2+4q}}{2}$, and $\tilde{\sigma} = p - \sigma$, for some positive integers p and q.
By some computation, it is easy to find that the tangent bundle of M is spanned by $\{Z_1, Z_2, Z_3, Z_4\}$, where

$$\begin{aligned} Z_1 &= \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos\phi\frac{\partial}{\partial x_5} + \sin\phi\frac{\partial}{\partial x_6} + \frac{\sigma}{\sqrt{q}}\frac{\partial}{\partial x_9} + \frac{\tilde{\sigma}}{\sqrt{q}}\frac{\partial}{\partial x_{11}},\\ Z_2 &= \cos\theta\frac{\partial}{\partial x_3} + \sin\theta\frac{\partial}{\partial x_4} + \cos\phi\frac{\partial}{\partial x_7} + \sin\phi\frac{\partial}{\partial x_8} + \frac{\tilde{\sigma}}{\sqrt{q}}\frac{\partial}{\partial x_{10}} + \frac{\sigma}{\sqrt{q}}\frac{\partial}{\partial x_{12}},\\ Z_3 &= -u\sin\theta\frac{\partial}{\partial x_1} + u\cos\theta\frac{\partial}{\partial x_2} - v\sin\theta\frac{\partial}{\partial x_3} + v\cos\theta\frac{\partial}{\partial x_4},\\ Z_4 &= -u\sin\phi\frac{\partial}{\partial x_5} + u\cos\phi\frac{\partial}{\partial x_6} - v\sin\phi\frac{\partial}{\partial\partial x_7} + v\cos\phi\frac{\partial}{\partial x_8}, \end{aligned}$$

By using the metallic structure J of $\tilde{M}$ which defines as

$$\begin{aligned} &J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) =\\ &\qquad(\sigma x_1, \sigma x_2, \sigma x_3, \sigma x_4, \bar{\sigma}x_5, \bar{\sigma}x_6, \bar{\sigma}x_7, \bar{\sigma}x_8, \bar{\sigma}x_9, \sigma x_{10}, \sigma x_{11}, \bar{\sigma}x_{12}). \end{aligned}$$

Following this, we obtain that

$$\begin{aligned} JZ_1 &= \sigma\cos\theta\frac{\partial}{\partial x_1} + \sigma\sin\theta\frac{\partial}{\partial x_2} + \tilde{\sigma}\cos\phi\frac{\partial}{\partial x_5} + \tilde{\sigma}\sin\phi\frac{\partial}{\partial x_6} + \frac{\tilde{\sigma}\sigma}{\sqrt{q}}\frac{\partial}{\partial x_9} + \frac{\tilde{\sigma}\sigma}{\sqrt{q}}\frac{\partial}{\partial x_{11}},\\ JZ_2 &= \sigma\cos\theta\frac{\partial}{\partial x_3} + \sigma\sin\theta\frac{\partial}{\partial x_4} + \tilde{\sigma}\cos\phi\frac{\partial}{\partial x_7} + \tilde{\sigma}\sin\phi\frac{\partial}{\partial x_8} + \frac{\tilde{\sigma}\sigma}{\sqrt{q}}\frac{\partial}{\partial x_{10}} + \frac{\tilde{\sigma}\sigma}{\sqrt{q}}\frac{\partial}{\partial x_{12}},\\ JZ_3 &= -u\sigma\sin\theta\frac{\partial}{\partial x_1} + u\sigma\cos\theta\frac{\partial}{\partial x_2} - v\sigma\sin\theta\frac{\partial}{\partial\partial x_3} + v\sigma\cos\theta\frac{\partial}{\partial x_4},\\ JZ_4 &= -u\tilde{\sigma}\sin\phi\frac{\partial}{\partial x_5} + u\tilde{\sigma}\cos\phi\frac{\partial}{\partial x_6} - v\tilde{\sigma}\sin\phi\frac{\partial}{\partial\partial x_7} + v\tilde{\sigma}\cos\phi\frac{\partial}{\partial x_8}, \end{aligned}$$

Now, we define two vector spaces D_T and $D_\perp$ such that the invariant distribution D_T is spanned by $\{Z_3, Z_4\}$ and the anti-invariant distribution $D_\perp$ is spanned by $\{Z_1, Z_2\}$. These distributions are preserved by the action of J. The Riemannian metric of the warped product CR-submanifold M is given by the following

$$g = (2 + \frac{1}{q}\tilde{\sigma}^2 + \frac{1}{q}\sigma^2)(d^2u + d^2v) + (u^2 + v^2)(d^2\theta + d^2\phi).$$

In other words, $M = M_\perp \times_f M_T$ is a warped product CR-submanifold in the metallic Riemannian manifold $\tilde{M}$.

4. Main Theorem

In this section, we prove the main result, which is based on Lemma 1. Now, we can define a canonical frame field for an n-dimensional warped product CR-submanifold $M = M_\perp \times fM_T$ of an m-dimensional locally metallic Riemannian manifold $\tilde{M}$. Let $\dim(M_T) = n_1$ and $\dim(M_\perp) = n_2$, and so $n = n_1 + n_2$. Also, let $\mathfrak{D}$ and $\mathfrak{D}^\perp$ be the tangent bundles of M_T and $M_\perp$, respectively. The canonical frame field of $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are given by the orthonormal vectors

$$\{e_1, e_2, \cdots, e_t, e_{t+1} = \frac{Je_1}{\sigma}, e_{t+2} = \frac{Je_2}{\sigma}, \cdots, e_{2t} = e_{n_1} = \frac{Je_t}{\sigma}\}, \tag{32}$$

$$\{e_{n_1+1}, e_{n_1+2}, \cdots, e_{n=n_1+n_2}\}, \tag{33}$$

respectively, where σ is a metallic number. On the other hand, the orthonormal frame fields of the normal subbundles of $JD^\perp$ and μ are respectively

$$\{e_1^* = Je_{n_1+1}, e_2^* = Je_{n_1+2}, \cdots, e_{n_2}^* = Je_{n=n_1+n_2}\} \tag{34}$$

$$\{e_{n_2+1}^* = e_{n+n_2+1}, e_{n_2+2}^* = e_{n+n_2+2}, ..., e_{m-n}^* = e_m\} \tag{35}$$

Theorem 2. *Let M be a warped product CR-submanifold of a locally metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\tilde{M}$, respectively. In accordance with this, we have*

(i) The squared norm of the second fundamental form of M satisfies the following inequality

$$\|h\|^2 \geq (\frac{p^2}{2} + \frac{q^2}{\sigma^2})n_1\|\vec{\nabla}^\perp \ln f\|^2, \tag{36}$$

where $\dim(M_T) = n_1$, $\dim(M_\perp) = n_2$, and $\vec{\nabla}^\perp \ln f$ is gradient of $\ln f$ in the normal direction to M.

(ii) If equality sign, in the above inequality, holds identically, then $M_\perp$ and M_T are totally geodesic and totally umbilical submanifolds of $\tilde{M}$, respectively .

Proof. As can be inferred from the definition of h, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=1}^{m-n}\sum_{i,j=1}^{n} g(h(e_i, e_j), e_r^*)^2. \tag{37}$$

In view of the above equation and the definitions of the frame fields of $\mathfrak{D}$, $\mathfrak{D}^{\perp}$, $J\mathfrak{D}^{\perp}$ and μ, we can derive the following:

$$\begin{aligned}\|h\|^2 &= \sum_{r=1}^{n_2}\sum_{i,j=1}^{n_1} g(h(e_i,e_j),e_r^*)^2 + \sum_{r=n_2+1}^{m-n}\sum_{i,j=1}^{n_1} g(h(e_i,e_j),e_r^*)^2 \\ &+ 2\sum_{r=1}^{n_2}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} g(h(e_i,e_j),e_r^*)^2 + 2\sum_{r=n_2+1}^{m-n}\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n} g(h(e_i,e_j),e_r^*)^2 + \\ &\sum_{r=1}^{n_2}\sum_{i,j=n_1+1}^{n} g(h(e_i,e_j),e_r^*)^2 + \sum_{r=n_2+1}^{m-n}\sum_{i,j=n_1+1}^{n} g(h(e_i,e_j),e_r^*)^2. \end{aligned} \tag{38}$$

Leaving the μ-components in (38), and using (21), we obtain that

$$\|h\|^2 \geq \sum_{r=1}^{n_2}\sum_{i,j=1}^{n_1} g(h(e_i,e_j),e_r^*)^2.$$

With the help of equations (32) then (34), we can find

$$\begin{aligned}\|h\|^2 &\geq \sum_{r=n_1+1}^{n}\sum_{i,j=1}^{t} g(h(e_i,e_j),Je_r)^2 + 2\sum_{r=n_1+1}^{n}\sum_{i=1}^{t}\sum_{j=t+1}^{n_1} g(h(e_i,e_j),Je_r)^2 + \\ &\sum_{r=n_1+1}^{n}\sum_{i,j=t+1}^{n_1} g(h(e_i,e_j),Je_r)^2. \end{aligned} \tag{39}$$

Using (20) with the help the canonical frame field of $\mathfrak{D}$ for all $r = 1, \cdots, n_1$, we can simplify the last inequality as follows

$$\begin{aligned}\|h\|^2 \geq &\sum_{r=n_1+1}^{n}\sum_{j=t+1}^{n_1}\sum_{i=1}^{t}(\sigma(e_r \ln f)g(e_i,e_j))^2 + 2\sum_{r=n_1+1}^{n}\sum_{i,j=1}^{t}((e_r \ln f)g(e_i,\frac{J^2e_j}{\sigma}))^2 + \\ &\sum_{r=n_1+1}^{n}\sum_{i,j=1}^{t}((e_r \ln f)g(\frac{Je_i}{\sigma},\frac{J^2e_j}{\sigma}))^2. \end{aligned}$$

From (1), we obtain the following result

$$\|h\|^2 \geq (\frac{p^2}{2}+\frac{q^2}{\sigma^2})2t\|\vec{\nabla}^{\perp}\ln f\|^2 = (\frac{p^2}{2}+\frac{q^2}{\sigma^2})n_1\|\vec{\nabla}^{\perp}\ln f\|^2.$$

Focusing on the fifth and sixth terms of (38), we get

$$h(D^{\perp},D^{\perp}) = 0. \tag{40}$$

Similarly, we obtain the following from leaving the second term of (34)

$$h(\mathfrak{D},\mathfrak{D}) \subseteq J\mathfrak{D}^{\perp}. \tag{41}$$

While the other terms in (34) vanish, the fourth term gives the following:

$$h(\mathfrak{D}, \mathfrak{D}^{\perp}) \subseteq J\mathfrak{D}^{\perp}. \tag{42}$$

Then we can find that $M_{\perp}$ is totally geodesic in $\tilde{M}$ due to its totally geodesic in M and (40). Similarly, equations (41) and (42) imply that M_T is totally umbilical in $\tilde{M}$ due to M_T being totally umbilical in M, which ends the proof.

Clearly, Theorem 2 is true for the Golden Riemannian manifolds i.e. $p = q = 1$, and locally product Riemannian manifolds $p = 0,\ q = 1$.

5. Conclusion

The exploration of warped product submanifolds in locally metallic Riemannian manifolds opens several intriguing avenues for future research. In particular, focusing on warped products formed by the product of a proper slant submanifold with an invariant submanifold, called warped product semi-slant, or with an anti-invariant submanifold, called warped product hemi-slant, opens significant aspects for upcoming studies. Further investigation into the geometric properties and characteristics of these submanifolds can deepen our understanding of their structure and the implications of the local metallicity of the ambient manifold. Additionally, establishing a comprehensive classification of warped product submanifolds, alongside concrete examples, will enhance the literature and provide benchmarks for further studies. Obtaining Chen's inequality for semi-slant and hemi-slant warped product submanifolds in locally metallic Riemannian manifolds reveals significant insights about the relationship between the second fundamental form and the warping function of such submanifolds. This study also raises the question of whether warped product pointwise semi-slant and hemi-slant submanifolds can be discussed in the context of metallic Riemannian manifolds.

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