EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5495
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Non-Perturbative Approach in Scrutinizing Nonlinear Time-Delay of Van der Pol-Duffing Oscillator

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Abstract. The time-delayed (TD) of velocity and position are employed throughout this investigation to lessen the nonlinear vibration of an exciting Van der Pol-Duffing oscillator (VdPD). The issue encompasses multiple real-world elements such as feedback lags, signal transmission delays, and delayed responses in mechanical, electrical, or biological systems. Examining this oscillator facilitates the investigation of complex dynamics, including chaos, bifurcations, and stability alterations, making it essential for disciplines like control theory, engineering, and neuroscience. The current oscillator is analyzed using using the non-perturbative approach (NPA). This methodology is based mainly on the He's frequency formula (HFF). Simply, this approach transforms the nonlinear ordinary differential equation (ODE) into a linear one. Accordingly, the stability standards are constructed, depicted, and sketched. The analytical solution (AS) with the associated numerical data that reveals high nonlinearity, and the numerical estimation is validated via the Mathematica Software (MS). In contrast to other traditional perturbation methods, the NPA exhibits high convenience, accessibility, and great precision in analyzing the behavior of strong nonlinear oscillators. Subsequently, this technique enables the analysis of issues related to other oscillators in the dynamical systems. It is an effective and promising method for addressing similar dynamic system challenges, providing a qualitative assessment of theoretical outcomes. The study describes time histories of solutions for different natural frequencies and TD parameters and discusses the main findings based on displayed curves. It also examines how various regulatory limits impact the vibrating system. The performance is applicable in engineering and other domains owing to its flexibility in various nonlinear systems. Consequently, the NPA can be considered significant, effective, and intriguing, with potential for use in more categories within the domain of coupled dynamical systems.

2020 Mathematics Subject Classifications: 70E50 , 74H55, 34k20, 70K50, 74S99
Key Words and Phrases: Non-perturbative approach, He's frequency formula, Nonlinear oscillations, Position-velocity time-delays, Instability configuration, Time histories, Parametric resonance

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5495

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1. Introduction

Most equipment, cars, structures, buildings, and dynamical systems are fundamentally vibrating, which is a negative phenomenon because of the occurrence of undesirable vibrations. Damaging dynamical stresses may cause a breakdown of the framework or machine, energy losses, performance degradation due to vibrations, and the resultant noise. A combination of analytical, computational, and experimental methods is employed to investigate nonlinear oscillations and their applications in construction, physics, and chemistry [19, 45, 54, 58]. The nonlinear oscillators are inherently self-excited, making their dynamics potentially difficult to comprehended. Since a classical nonlinear oscillator is well-known as VdPD, several researchers examined this oscillator. It is frequently employed to confirm the effectiveness and correctness of these phenomena and has significant scientific merit and a long-standing history of application. Lately, there has been a lot of interest in studying these oscillators' structures in depth. The feedback controller is used to reduce the VdPD from vibrating when it is simultaneously exposed to fundamental parametric resonances and external parametric excitation forces [53]. The nonlinearity of an excited VdPD is prevented by using TD position and velocity. The TD is deemed appropriate as a supplemental safeguard against the nonlinearly oscillating model that is being studied. Because machinery with a TD has recently been the subject of many examinations, this topic is of particular interest. The technique of numerous scales was used to examine two various forms of different resonances [34]. The quasi-periodic movements and frequency-response curves were illustrated. It is demonstrated that the forced oscillator's differential resonance response displayed quasi-periodic movements on a three-dimensional. The influence of linear-plus-nonlinear location feedback dominance incorporation was conducted [37]. For resolving chaotic behaviors in periodically self-excited oscillators, a numerical solution (NS) in a backward difference form was put forth [47]. The focus of this research was on VdPD chaotic motions.

As common knowledge, the TDs are regularly necessary for effective control systems because of the time needed of the system status measurements, online storage processing, computing and implementing the control forces, etc. Due to the disruption in the application of the control force transported about by this TD, the controlled systems are frequently performed poorly or became unstable. Numerous studies were conducted on systems with temporal delays during stochastic excitation [29, 30, 33, 36, 59]. The fundamental and 1/3 subharmonic resonances of a harmonically induced Duffing oscillator (DO) below the state feedback control with a TD were studied analytically and numerically. The resonances' first-order approximations were obtained by applying the multiple scales method [14], and the impact of TD on the resonances was also examined. The vibration controller perspective was suggested the concept of comparable damping associated with delay feedback, and it examined the appropriate selection of feedback gains and TD. Investigations of the TD mechanism of effect in a non-autonomous system were conducted [29]. A VdPD with excitation was the first mathematical model that was considered. Including both linear and nonlinear TD position feedback in the initial structure, a delaying system was created. Much work was done on the nonlinear dynamics of a VdPD under the state feedback TD controller with linear and nonlinearity [59]. Two slow-flow equations were developed for the amplitude and phase of the principal resonance response using the averaging approach and Taylor expansion. When the TD in the feedback exceeds certain thresholds, the stability of a VdPD under linear-plus-nonlinear feedback dominance could change, potentially leading to single or double Hopf bifurcations [36]. Analytical conclusions were validated through comparison with direct numerical integration results. Exploring a Duffing oscillator with TD feedback control, we examined its asymptotic Lyapunov stabiliExploring a DO with TD feedback control, we examined its asymptotic Lyapunov stability under parametric stimulation with constrained noise [33]. The approximate solutions of several dynamical systems were obtained [14]. The solvability criteria and the adjusted equations were achieved. The stability zones for several parameters were analyzed and examined. The instability of linear TD systems was developed, where a generalized vector of several integral inequalities was outlined to interpret a variety of initial circumstances [18]. Secondly, the TD system stability criteria were developed using these multiples. A linear time-varying delay system was expected to undergo a delay-dependent stability analysis [56]. They provided support for the claim that their strategy is more effective at addressing time-varying delay structures. A numerical instance was provided to demonstrate the efficiency of the model.

The nonlinear ODEs can be used to analyze a variety of technical problems. The small factor and the averaging technique are two examples of low nonlinear ODEs [35, 57]. Research on the dynamics of a pendulum, devoted to a stiff rotational structure through a uniform angular velocity along the vertical axis, was transient through the pendulum's pivot point [57]. The controlling equation of movement was derived analytically at this frequency. Investigation into Lyapunov exponents and their moments involved analyzing linear systems with two degrees of freedom subjected to a parametric excitation [35]. Explicit approximate equations for such exponents were computed utilizing the Homotopy perturbation method (HPM), while considering the presence of low-intensity sounds. Additionally, the solutions of vibratory systems were derived using the multiple time-scales method and the Lindstedt-Poincaré technique. These techniques, however, were dependent on a small factor, and the selection of this factor produced false results [32]. The iterative methods with the HPM were shown to be crucial lately for generating approximations of various nonlinear ODEs that are reasonably near to their solutions. These methods rely on the initial conditions of the response. Therefore, if the initial assumption does not correspond to the actual way the problem is resolved, it will be diverged, preventing the process from leading to the indicated result. One of the challenging issues that arise in the nonlinear oscillation difficulties has drawn the attention of numerous academics. Because of the nature of nonlinearity, it has been challenging for researchers and physicists to develop an exact or roughly accurate solution to various nonlinear ODEs. The HPM was used to analyze approximate analytical solution (AS) of a magnetic spherical pendulum [21]. The frequency formula was created by Prof. He, who also came up with an inventive method to approach linearity in a nonlinear ODE [22, 27]. There was a description of some recent asymptotic approach advancements that were said to be relevant to both strong and weak nonlinear ODEs. The derived approximate AS were valid across the entire solution domain. To address the limitations of conventional methods of perturbation, numerous modified techniques, along with other mathematical tools such as variation theory, HPM, and iterative approaches, were proposed. The HFF, the max-min technique, and the HPM were three proposals of the NOs in light of the simplest methods [22]. The weighted average was added to the mathematical description of his frequency construction to enhance the accuracy of frequency. Strongly nonlinear oscillators were advised to be treated straightforwardly and exclusively [27]. In a packaging system as well as an experimental micro-electro-mechanical system, the nonlinear relationship between the amplitude and frequency of a nonlinear vibration system was employed [46]. An effective approach utilizing an adaption of the conventional differential transform method was introduced [12]. The findings collected indicated that the proposed strategy was a potential method for solving the Van der Pol oscillator, providing identical information on the phase portrait and proving the system's stability efficiently. A unique two-dimensional oscillator featuring an asymmetric design and its steady vibration characteristics were presented [48]. A nonlinear oscillator equation featuring two dominant linear terms was introduced [1]. An approximate solution was derived with the power series method. Additionally, by incorporating a parameter into the original equation, the fixed points of the new nonlinear oscillator equation were identified, and a stability analysis of these fixed points was conducted.

The precise solutions of the intricate nonlinear ODEs that dictate the dynamic laws commonly observed in engineering and physics are frequently unknown. The intricate dynamics of systems can be comprehensively analyzed with these methodologies, facilitating an in-depth exploration of both quantitative and qualitative dimensions of system behavior. A substantial number of researches were undertaken on the efficacy and wide applicability of this methodology [2, 9, 10, 17]. The HFF played a significant role in deriving closed-form analytical solutions for oscillators, particularly the Duffing oscillator (DO) [11]. The HFF formula was developed into as a powerful mathematical tool for analyzing nonlinear oscillators with periodic solutions. Prof. He delivered a pioneering review article and was the inaugural presenter of it [23]. The precision and empirical validation of this HFF rendered it a preferred choice among engineers [7, 8, 13, 16, 24, 25, 49, 51]. The HFF has been meticulously refined over the years, resulting in enhanced accuracy, as previously demonstrated [6, 26, 28]. Moreover, as previously stated, the applicability of this frequency approach has broadened to encompass fractal oscillators [55]. In HFF, the correlation between frequency and amplitude of a nonlinear oscillator was derived from the residuals of two trial solutions [50]. Despite achieving a highly precise result, this method offers potential for further enhancement. The HFF originating from an ancient Chinese algorithm is an efficient method for approximating solutions of a nonlinear oscillator. A simpler formulation was presented based on HFF [52]. The basic HFF for nonlinear oscillators was presented and validated, with a proposed change [20]. A fractal vibration in a porous media was presented, and its low-frequency characteristics were clarified by the

HFF. In the examination of nonlinear oscillatory systems, the NPA distinctly contrasts with conventional perturbation methods. This approach is a robust mathematical tool capable of addressing many parameter regimes, especially those characterized by significant nonlinearity. A significant technique for deriving analytical approximations in the study of nonlinear oscillators was utilized. The principal objective of the NPA is to simplify the nonlinear model to a more tractable form, hence enabling precise specification of the solutions. This will provide a more precise estimation of the original system's behavior [3–5, 15, 31, 38–44]. The objective of this simplification is to diminish the average disparity between the two systems by converting the equation from a nonlinear ODE to a linear one. Consequently, moving away from the iterative refinement of perturbation approaches, the NPA presented a novel perspective. The objective of the NPA is to comprehend the intricacies of nonlinear systems independently, without depending on minor perturbations from a known solution.

Considering above-mentioned features, the evaluation of preventing nonlinear exciting VdPD with TD control is a major difficulty to given its understanding to dynamic loading, geometrical changes, and dissipating challenges. Examining this subject is the aim of this work. According to traditional mechanics, the fundamental equation of motion is controlled as an ODE with extremely nonlinear terms. The subsequent points should be highlighted in relation to the unique methodology or noteworthy outcomes:

(i) In simple terms, the distinctive approach generates a second identical linear ODE that is comparable to the current nonlinear one.

(ii) A strong matching between these two equations is achieved via the MS.

(iii) All traditional methods employ the Taylor expansion to reduce the problem's complexity. Under the current approach, this limitation has been disregarded.

(iv) The present approach enables us to examine the problem's stability analysis, which was not possible with some of the earlier traditional perturbation methodologies.

(v) In conclusion, the unusual strategy seems to be a simple, effective, and appealing tool. It can be utilized to analyze various classes of nonlinear oscillators.

It is simple to follow how the paper is set up as: A NPA that yields the equivalent linear equation is introduced in § 2. This Section displays a strong agreement between the AS and the NS. The graphical plots in § 3 are presented along with their interpretations based on the outcomes, including temporal history, stability, and polar study. Finally, the concluding remarks are presented through § 4.

2. Methodology of the Prototype

Given the importance of the above-mentioned components, the study of the VdPD has imaginable uses in manufacturing, theory of communications, and biology. It has

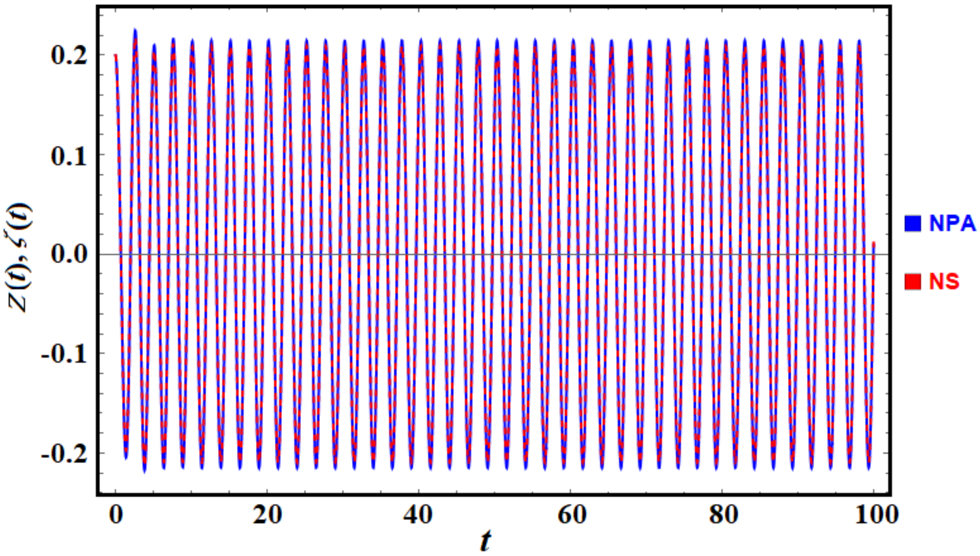

Figure 1: Sketches the model being examined.

been featured in several topics. Consequently, the current work conducts the following analysis of the excited VdPD with location and velocity delays:

$$\ddot{z} + \omega^2 z - \mu(1 - z^2)\,\dot{z} + \lambda z^3 = F\cos(\Omega t) + \alpha\, z(t-\tau) + \beta\, \dot{z}(t-\tau), \tag{1}$$

with the initial conditions (ICs):

$$z(0) = A, \quad \text{and} \quad \dot{z}(0) = 0, \tag{2}$$

where all elements considered in Eq. (1) are listed in the following:

Symbol	**Clarification**	**Symbol**	**Clarification**
Z	Displacement	F	External excited force
$\cdot$	Derivative with time	α	Coefficient of position TD
ω	Natural frequency	β	Coefficient of velocity TD
μ	Damping coefficient	A	Initial amplitude
λ	Third-order nonlinear Duffing coefficients,$\lambda > 0$,$\lambda <$ are hardening and softening spring, respectively	τ	Time-decay controller

The VdPD incorporating the TDs in position and velocity characterizes intricate dynamical systems, including mechanical oscillators and electronic circuits, where nonlinearity and memory effects are pivotal. The TDs illustrate the impact of previous states on the present behavior of the system, interpretation it valuable for analyzing phenomena such as chaotic motion, signal processing, and control systems, where delayed feedback affects stability, oscillations, and bifurcations. This equation is specifically utilized in situations necessitating the examination of delayed response effects in actual physical systems. The

VdPD has reasonable uses in various domains such as:
i. Mechanical Systems: Employed to model and analyze vibrations in mechanical systems, including beams and bridges, where nonlinear damping and stiffness are substantial.
ii. Electrical Circuits: Utilized in the construction of oscillatory circuits, such as vacuum tubes and transistor circuits, for the examination of non-linear resonance and signal modulation.
iii. Biological Systems: Simulates heart rhythms and cerebral activity, encapsulating the nonlinear dynamics of biological oscillations.
iv. Seismology: Facilitates the simulation and comprehension of the non-linear reaction of structures during seismic events.
v. Control Systems: Employed in nonlinear control analysis to devise resilient controllers for systems exhibiting intricate, nonlinear dynamics.
These applications control the oscillator's capacity to record intricate, nonlinear dynamics and forecast system reactions under diverse conditions. The usefulness of the TD in numerous fields has was emphasized in the introductory section; hence, the next evaluation will be founded on the TD of the location and velocity, i.e., the last two terms in Eq (1). Recently, this issue was approached by leveraging the properties of special functions. The following trial solution was proposed [46]:

$$\zeta = A\cos(\Delta t), \quad \dot{\zeta} = -A\Delta\sin(\Delta t), \tag{3}$$

where the initial vibration amplitude is denoted by A , and Δ is the total frequency of the TD as indicated in Eq. (1).
Appropriately, the shift of the independent time may be expressed as:

$$\begin{aligned}\zeta(t-\tau) &= A\cos\Delta(t-\tau)\\ &= A[\cos\Delta t\cos(\Delta\tau) + \sin\Delta t\sin(\Delta\tau)]\\ &= \zeta(t)\cos(\Delta\tau) - \frac{1}{\Delta}\dot{\zeta}(t)\sin(\Delta\tau).\end{aligned} \tag{4}$$

Consequently,

$$\dot{\zeta}(t-\tau) = \dot{\zeta}(t)\cos\Delta\tau + \Delta\zeta(t)\sin(\Delta\tau). \tag{5}$$

At this stage, Eq. (1) may be written as follows:

$$\ddot{z} + f_1(z,\, \dot{z}) + f_2(z) = F\cos(\Omega t), \tag{6}$$

where

$$\begin{aligned}f_1(z,\, \dot{z}) &= -\mu(1-z^2)\dot{z} + \frac{\alpha}{\Delta}\dot{z}\sin(\Delta\tau) - \beta\dot{z}\cos(\Delta\tau),\\ f_2(z) &= \omega^2 z + \lambda z^3 - \alpha\, z\cos(\Delta\tau) - z\beta\,\Delta\sin(\Delta\tau).\end{aligned} \tag{7}$$

Now, an equivalent frequency can be evaluated as shown previously by Moatimid et al. [45-47] in the following manner:

$$\omega_{eqv}^2 = \int_0^{2\pi/\Delta} \zeta f_2(\zeta)dt / \int_0^{2\pi/\Delta} \zeta^2 dt = \omega^2 + \frac{3}{4}\lambda A^2 - \alpha\cos(\Delta\tau) - \beta\Delta\sin(\Delta\tau). \tag{8}$$

Additionally, the equivalent damping term may be determined as shown below:

$$\Gamma = \int_0^{2\pi/\Delta} \dot{\zeta} f_1(\zeta,\dot{\zeta})dt / \int_0^{2\pi/\Delta} \dot{\zeta}^2 dt = \frac{1}{4}\mu(A^2-4) - \beta\cos(\Delta\tau) + \frac{\alpha}{\Delta}\sin(\Delta\tau). \tag{9}$$

The equivalent ODE can now be constructed as follows:

$$\ddot{\zeta} + \Gamma\dot{\zeta} + \omega_{eqv}^2\zeta = F\cos(\Omega t). \tag{10}$$

For simplicity, the stability condition will be evaluated without the external excitation force . Therefore, the nonhomogeneous differential equation as given in Eq. (10) becomes a homogeneous one. Furthermore, the standard normal form may be attained through the transformation $\zeta(t) = f(t)Exp(-\Gamma\, t/2)$. Elementary, the unknown function satisfies the following simple harmonic differential equation:

$$\ddot{f} + \Delta^2 f = 0, \tag{11}$$

where $\Delta^2 = \omega_{eqv}^2 - \Gamma^2/4$. In other words, the total frequency can be obtained by combining the results in Eqs. (8) and (9) with the previous relation to produce:

$$\Delta^2 = \omega^2 + \frac{3}{4}\lambda A^2 - \alpha\cos(\Delta\tau) - \beta\Delta\sin(\Delta\tau) - \frac{1}{4}[\frac{1}{4}\mu(A^2-4) - \beta\cos(\Delta\tau) + \frac{\alpha}{\Delta}\sin(\Delta\tau)]^2. \tag{12}$$

The stability standard requires

$$\Delta^2 > 0\,,\ \text{and}\ \Gamma > 0. \tag{13}$$

As seen, Eq. (12) is a transcendental equation in Δ . For an ease, Taylor expansion may be employed to approximate the values of the trigonometric functions in Γ as $\sin\epsilon \approx \epsilon$ and $\cos\epsilon \approx 1$. In this simplification, the total frequency can be written as

$$\Delta = \sqrt{\frac{-\alpha + \frac{3}{4}\lambda A^2 - \frac{1}{64}(-4\beta + (A^2-4)\mu + 4\alpha\tau)^2 + \omega^2}{1+\beta\tau}}. \tag{14}$$

As seen, Eq. (12) is a transcendental equation in the total frequency Δ . To obtain the value of the equivalent frequency, consider the following data sample: $\mu = -1.0,\ \tau = 0.1,\ \omega = 4,\ \alpha = 0.1,\ \beta = 0.3,\ \lambda = 0.5,\ \text{and} A = 0.2$. Using MS through the command of FindRoot, the value of the total frequency becomes $\Delta = 3.90514$. For more convenience,

along with the MS with the commend NDsolve, the graph of the original nonlinear ODE as given in Eq. (1) is graphed with the LODE as shown in Eq. (10). For this purpose, it is required to supplement the values of the external excited force F as well as its external frequency σ as $F = 2$ and $\Omega = 2.5$. The comparison reveals that the two solutions have a high degree of consistency with each other, as seen in Fig. (1), where the solutions are portrayed. Additionally, the MS reported that, up to a time of 100 units, the total variance between the AS and NS is 0.0056.
In what follows, an excellent agreement between the linear as well as the nonlinear ODEs is seen. This agreement comes from: The strong concordance between the two planar curves, one originating from a nonlinear ODE as given in Eq. (1) and the other from a linear approximation as given in Eq. (10), demonstrates that the linear model well encapsulates the fundamental dynamics of the nonlinear system under certain conditions. Despite the intrinsic complexity of the nonlinear ODE, the strong correlation between the curves indicates that, within a specific range or under particular simplifying assumptions, the linear representation serves as a robust and dependable approximation of the system's behavior, underscoring the efficacy of linearization in modelling intricate phenomena.

3. Discussions and Results

This Section is devoted to presenting and discussing the time histories of the obtained outcomes as well as their phase planes and the stability/instability regions in focus on the various values of the relevant factors. Returning to the TD as shown in Eq. (1), the NPA enables us to determine the solution of the basic Eq. (1) that is identical as those used by the concluding linear Eq. (10). As was previously said, every parameter affects the related frequency Δ , which is determined by Eq. (12).
The curves in Figs. (2)-(6) are calculated according to the considered data above and when the parameters $\alpha, \beta, \lambda, \mu,$ and ω have the values $(0.1, 0.4, 1.8), (0.3, 0.6, 1.1), (0.5, 4, 8), (-1, -3, -6),$ and $(1, 1.5, 4)$, respectively. As displayed in Fig. (2a), when $\alpha(= 0.1, 0.4, 1.8)$, the amplitudes of the periodic waves somewhat rise, indicating stable motion under the impact of these values. To support this conclusion, the phase plane curves for the represented solutions in Fig. (2a) are illustrated as closed curves in Fig. (2b), i.e. in a plane that combines the solutions versus their first derivatives. Segments (c), (d), and (e) in Fig. (2) show the stabilities areas of the obtained solution using the NAP at $\alpha = 0.1$, $\alpha = 0.4$ and $\alpha = 1.8$, respectively.The examination of these sections indicates that the values of α increase, the green-colored stable regions decrease, while the white-colored unstable regions expand.

The influence of the variation of the parameter β on the performance of the NPA was graphed, as indicated in Fig. (3). The drawn curves in Fig. (3a) have the forms of quasi-periodic waves. It is noted that this variation produces an increase in the wave's amplitudes. As shown in Fig. (3b), nearly symmetric closed curves in the plane $\zeta\dot{\zeta}$are observed, resembling a grid where the density of capture points increases with increasing β. The stabilities zones of the solution produced using the NAP at $\beta = 0.3$, $\beta = 0.6$, and $\beta = 1.1$ are depicted, respectively, in Figs. (2c), (2d), and (2e). An analysis of these sections shows that the white-colored unstable regions grow and the green-colored stable parts shrink as the β values.

The study of the change in various values of $\lambda(= 0.5, 4, 8)$ on the behavior of the achieved solution using NPA, phase plane plots, and stability/instability areas are explored in the portions of Fig. (4). The graphed wave in Fig. (4a) has a periodic manner over time and, therefore, the plotted phase plane curve in Fig. (4b) have closed form, which asserts the stability of the represented wave in Fig. (4a). Moreover, there is not any visible impact of λ values on the solution's behavior, at which the variation becomes very slight. The reason for the weak effect of this parameter is attributed to the minimal magnitude of its term. Based on this analysis, one can conclude that there is no variation in stability/instability area can be observed, as explored in Figs. (4c), (4d), and (4e).

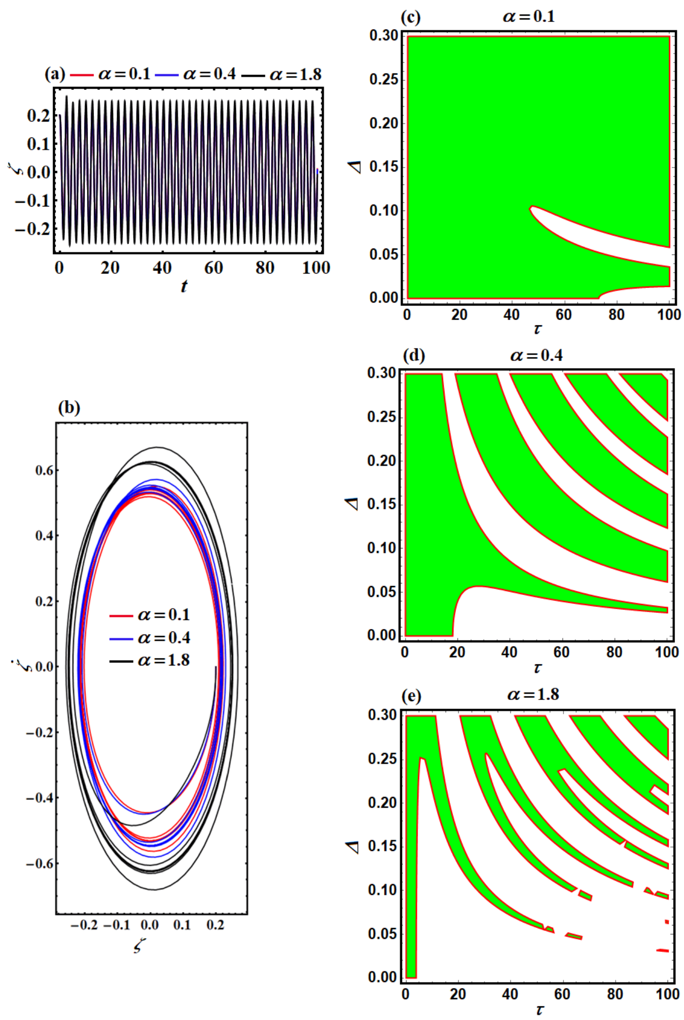

Figure 2: Explores the comparison connecting the AS and NS.

It must be noted that, the values of the parameter $\mu(=-1,-3,-6)$ have a good impact on the obtained solution behavior, as seen in Fig. (5). Periodic waves are represented in Fig. (5a), where the amplitudes of the waves rise as the values of μ grow. Moreover, semi-closed curves are seen in the plane $\zeta\dot{\zeta}$ i.e. Fig. (5b). The stability/instability areas are drawn in Figs. (5c)-(5e), for the aforementioned values of this parameter.
Examining the parts in Fig. (6) reveals that the impact of the different values of a parameter $\omega(=1, 1.5, 4)$ on the solution's behavior, phase plane curves, and stability areas, as indicated in Figs. (6a), (6b), and (6c)-(6e). The amplitude of the drawn waves in portion (a) of this figure decreases with the increase of this parameter, while the curves in the plane $\zeta\dot{\zeta}$ is presented in portion (b). The associated stability/instability areas are shown in portions (c)-(e) of this figure.

Moreover, the behavior of Δ via A has been graphed in parts of Fig. (7) in view of the explicit mathematical expression of Δ , as in Eq. (14). This figure has been calculated when $\alpha=0.1, \beta=0.3, \lambda=0.5, \mu=-1, \omega=1$ and $\tau=0.1$, besides the variation of each one of these parameters when the others become fixed. An inspection of Figs (7a, b, c, d, e) shows, they are graphed when $\alpha(=0.1, 0.4, 0.7), \beta(=0.3, 0.6, 1.1), \lambda(=0.5, 0.7, 0.9), \omega(=1.0, 1.3, 1.6)$ and $\tau(=0.1, 0.4, 0.8)$, respectively. It is obvious that when α, β and τ increases, the stability regions above the drawn curves increase, while the instability ones that lie down the curves decrease, as seen in Figs (7a), (7b), and (7e), respectively. These regions must satisfy Eq. (14). On the other hand, when the values of the parameters Λ and ω are increase, the stability regions decrease while the instability ones increase, as explored in Figs. (7c) and (7d), respectively.

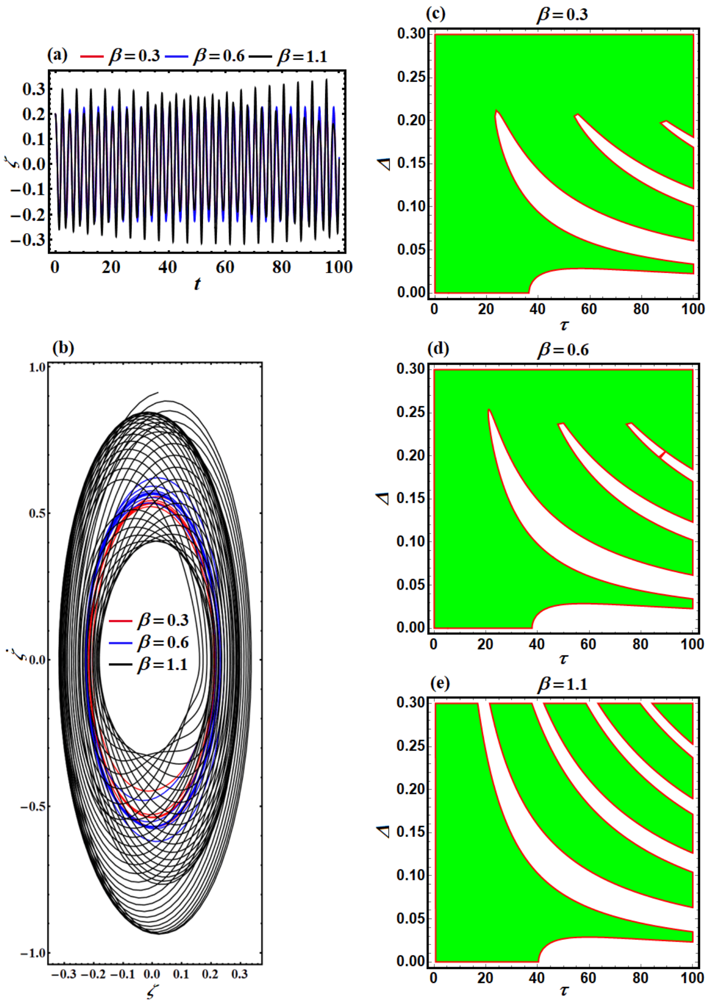

Figure 3: (a) Explores curves of at different amounts of $\theta(t)$, and (b) Reveals the phase plane trajectories in (a).

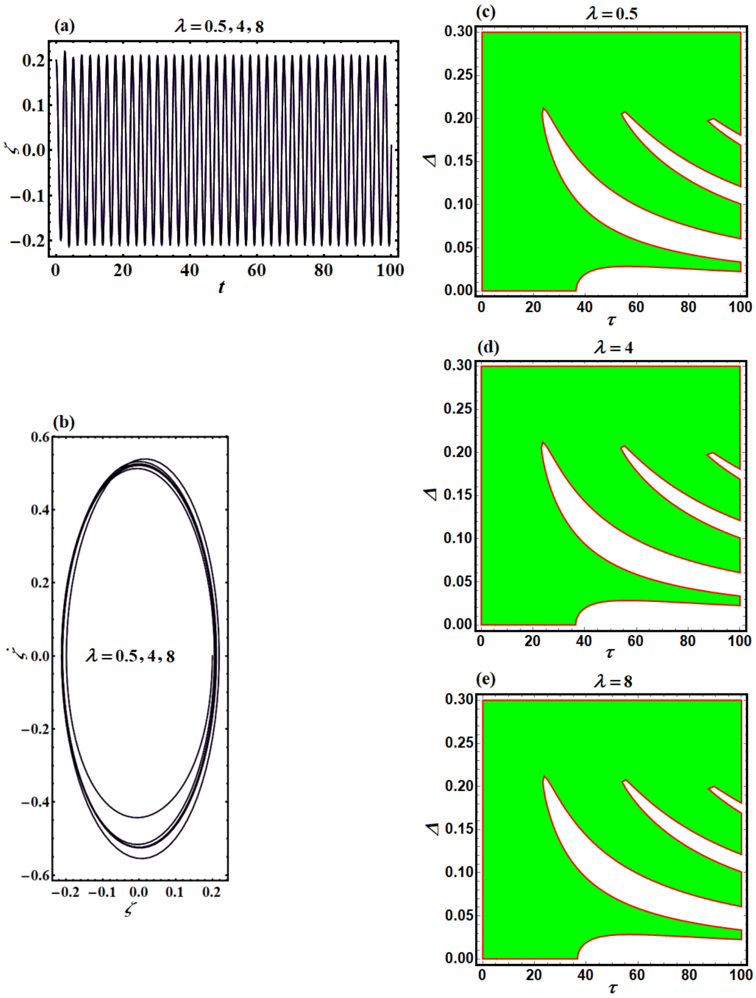

Figure 4: (a) Demonstrates curves of $\theta(t)$ at different amounts of a, and (b) Shows the phase plane diagrams in (a).

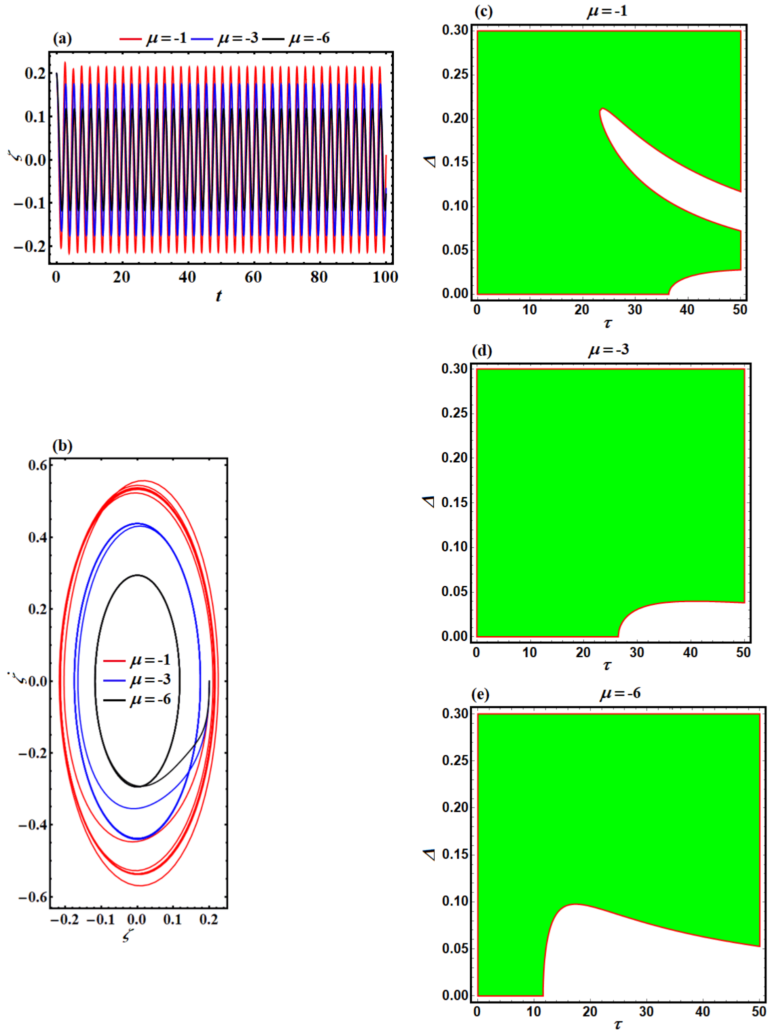

Figure 5: (a) Shows curves of $\theta(t)$ at different amounts of b, and (b) Presents the phase plane paths in (a).

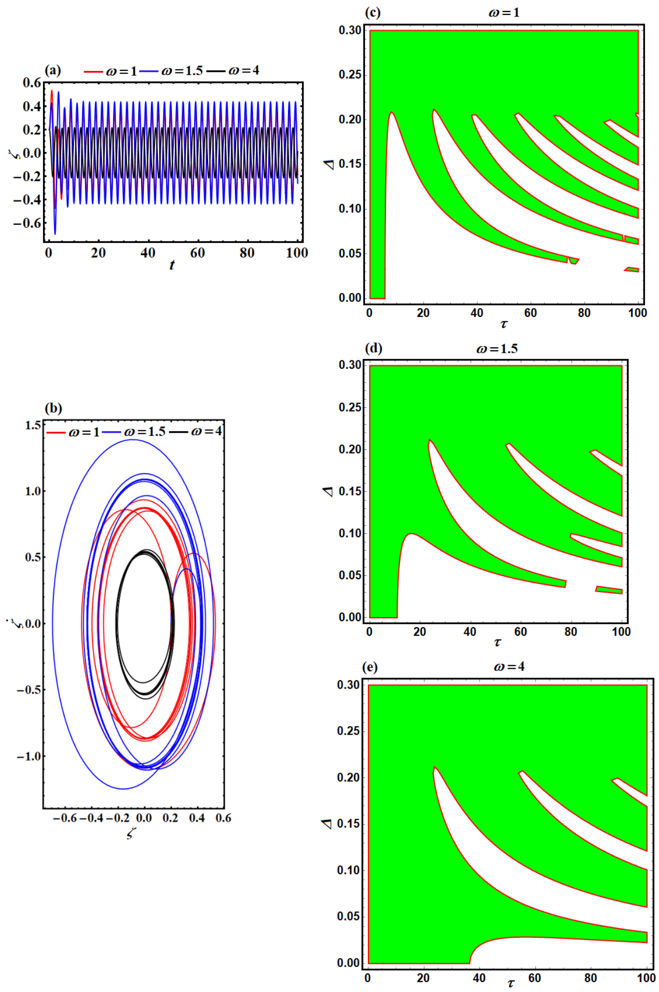

Figure 6: Show the stability/instability areas in the plane $(\tau - \Omega)$ at $a = b = 0.01$ when (a) $A = 0.4$, (b) $A = 0.7$, and (c) $A = 1$.

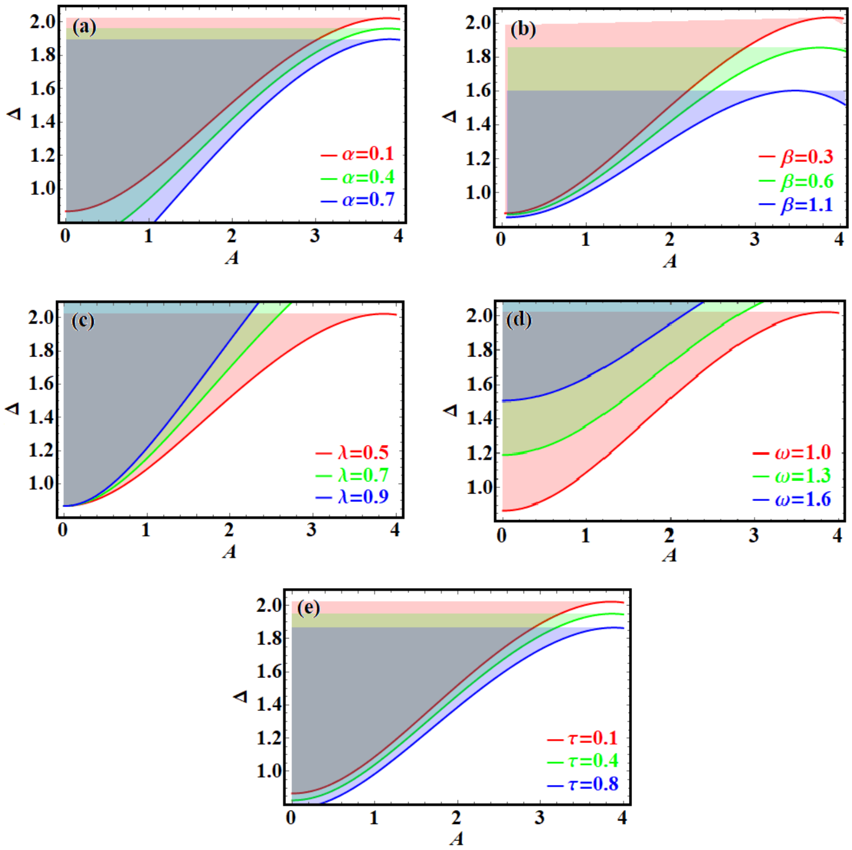

Figure 7: Presents the stability/instability zones in the plane $(\tau - \Omega)$ at $a = b = 1$ when (a) $A = 0.4$, and (b) $A = 0.7$.

4. Conclusions

In this study, the TDs in velocity and position were employed to lessen the nonlinearity vibration of an excited VdPD. Therefore, the TD idea is a protection against nonlinear oscillation of the structure under examination. The current study aimed to investigate the impact of TD, which has recently been the subject of many examinations. The NPA was used to analyze the present oscillator. The main concept of this methodology was to transform the nonlinear ODE into a linear one. Consequently, the stability constraint was built, examined, and established. The AS was numerically confirmed by comparing it to the related numerical data, which showed a remarkably high similarity. The NPA was distinguished by its convenience, accessibility, and excellent accuracy in studying the behavior of strong nonlinear oscillators in contrast to conventional perturbation methods. Accordingly, we have been able to address a variety of problems with the usage of oscillators in mechanical systems thanks to the present NPA. It was a successful, powerful, and encouraging method for examining problems related to dynamic systems. The realistic study approximation of the analytical methodology was enabled a qualitative evaluation

of the results. The temporal changes of the discovered solutions were shown for various properties of the affected parameters. A description of the outcomes was provided in view of the depicted figures through a set of curves. It has investigated the impact of various regulatory thresholds on the vibrating system. Regarding the original methodology or significant findings, the subsequent outcomes warrant emphasis:
i. The technique presented simply created an equivalent linear ODE to the current non-linear one. These two equations matched each other quite well.
ii. All conventional methods employ the Taylor expansion to make the given problem simpler in the existence of restoring forces. Under the present strategy, this weakness has been eliminated.
iii. Unlike prior traditional techniques, the current approach allows us to investigate the stability analysis of the problem.
iv. To sum up, it appears that the novel method was a simple, practical, and entertaining instrument. It applies to the analysis of numerous nonlinear oscillation categories.
v. Considering the stability criteria, and in light of the numerical calculations of the parameters, it was found that the parameters α, β, μ and ω have a destabilizing influence. By contrast, the factor λ has a slightly stabilizing effect.

As well-known, coupled dynamical systems are systems wherein two or more dynamical entities interact through a coupling mechanism, hence affecting one another's behavior. This contact may occur via physical, chemical, biological, or mathematical linkages. The state and evolution of each system are contingent not only upon its own dynamics but also on the state and dynamics of other systems. Comprehending linked dynamical systems is crucial for examining how interrelated components affect the overall behavior of complex systems. Therefore, as a future work, the novel methodology, referred to as the NPA will be used to inspect the coupled system in the field of dynamical topics.

Funding

This research was funded by Deanship of Scientific Research Project number (NBU-FFR-2024-912-01), Northern Border University, Arar, KSA.

Acknowledgements

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number "NBU-FFR-2024-912-01".

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Conflict of Interest
There are no conflicts of interest declared by the authors.

Data Availability
All data generated or analysed during this study are included in this published article.

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