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More on the Order of Aragón Artacho–Campoy Algorithm Operators With the Help of Douglas–Rachford Operators

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Abstract. The Aragón Artacho–Campoy algorithm (AACA) is a new method for finding zeros of sums of monotone operators. In this paper we complete the analysis of their algorithm by defining their operator using Douglas Rachford operator and then study the effects of the order of the two possible Aragón Artacho–Campoy operators.

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1. Introduction

Throughout, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}, \tag{1}$$

and induced norm $\|\cdot\|: X \to \mathbb{R}: x \mapsto \sqrt{\langle x,x\rangle}$. We also assume that $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are maximally monotone operators. For more details about maximally monotone operators, we refer the reader to [3], [4], [9], [10], [11], [12], [14], [15], and the references therein. In [3], Auslender and Teboulle provide essential tools used to study monotone graphs. They focus on the behavior of a given subset of $\mathbb{R}^n$ at infinity. By using real analysis and geometric concepts, they develop a mathematical treatment to study the asymptotic behavior of sets. Moreover, the book by Bauschke and Combettes [4] is one of the best sources to learn about non-linear analysis, namely, convex analysis, monotone operators, and fixed point theory of operators. Additionally, [9] highlights the importance of maximal monotone operators and describes the progress that has been made in the field of monotone operators over the past decade. Furthermore, [10] provides a survey that discusses the developments in the theory of monotone operators. It

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is well known that a prominent example of maximal monotone operators is the subdifferential operator, which was investigated in section 5.1.6 of [11]. Moreover, Burachik and Svaiter establish new connections between maximal monotone operators and convex functions. They demonstrate that each maximal monotone operator is associated with a family of convex functions. Their study focuses on this family, determining its extremal elements using the concept of convex functions (see [12]). Following this, Patrick reviews the properties of subdifferential operators as maximally monotone operators in [14], and examines proximity operators as resolvents of these operators. Additionally, in [15], a comprehensive treatment of monotone set-valued operators is presented, utilizing mathematical programming in detail. The *resolvent* and the *reflected resolvent* associated with A are:

$$J_A = (\operatorname{Id} + A)^{-1} \text{ and } R_A = 2J_A - \operatorname{Id}, \tag{2}$$

respectively. Suppose that

$$A \text{ and } B \text{ are maximally monotone on } X,\ w \in X, \text{ and } \gamma \in \left]0,1\right[. \tag{3}$$

Fact 1. The resolvent averages between A, B and N_w are

$$A_\gamma : \mathcal{H} \rightrightarrows \mathcal{H} : x \mapsto A\Big(\gamma^{-1}\big(x - (1-\gamma)w\big)\Big) + \gamma^{-1}(1-\gamma)(x-w), \tag{4}$$

and

$$B_\gamma : \mathcal{H} \rightrightarrows \mathcal{H} : x \mapsto B\Big(\gamma^{-1}\big(x - (1-\gamma)w\big)\Big) + \gamma^{-1}(1-\gamma)(x-w). \tag{5}$$

Fact 2. A_γ and B_γ are maximally monotone and their resolvents are given by

$$J_{A_\gamma} = \gamma J_A + (1-\gamma)w \text{ and } J_{B_\gamma} = \gamma J_B + (1-\gamma)w, \tag{6}$$

respectively. Moreover, reflected resolvents are

$$R_{A_\gamma} = 2\gamma J_A + 2(1-\gamma)w - \operatorname{Id}, \text{ and } R_{B_\gamma} = 2\gamma J_B + 2(1-\gamma)w - \operatorname{Id}, \tag{7}$$

respectively. Then Aragón Artacho–Campoy operator [1] associated with the ordered pair of operators (A_γ, B_γ) is

$$T_{A_\gamma,B_\gamma} = (1-\lambda)\operatorname{Id} + \lambda R_{B_\gamma} R_{A_\gamma}. \tag{8}$$

Fact 3. *(Definition of the Douglas–Rachford splitting operator)* The Douglas–Rachford splitting operator [19] associated with the ordered pair of operators (A, B) is

$$T_{A,B} = \frac{1}{2}\big(\operatorname{Id} + R_B R_A\big) = \operatorname{Id} - J_A + J_B R_A. \tag{9}$$

Through straightforward calculations, we can determine that

$$T_{B,A} = \operatorname{Id} + J_A R_B - J_B. \tag{10}$$

In this paper, we explore the relationship between the Aragón Artacho–Campoy operators T_{A_γ,B_γ} and T_{B_γ,A_γ}. The key findings are summarized as follows:

- Key properties of J_{A_γ} and A_γ are presented in Proposition 1. These properties will be valuable for our analysis.

- We provide formulas for the Aragón Artacho–Campoy operators utilizing the Douglas–Rachford splitting operator (refer to Lemma 1). For additional details on the Douglas–Rachford splitting algorithm, see [17], [5], [8], [13], [16], and [18]. [5], [8], [13], and [18] help to understand more about the behaviour of DRS. Paper [5] studies the range of the DRS systematically. Under the assumption that the operators are 3^* monotone operators. While the second one helps to understand the behavior of the shadow sequence when the given functions have disjoint domains. The main result of this paper is proving the weak and value convergence of the shadow sequence generated by the Douglas–Rachford algorithm. Paper [13] aims to solve convex feasibility problems by using new algorithmic structures with DRS operators. Paper [18] gives a comprehensive survey about the developments of the DRS methods. Additionally, [17] shows an amazing connection between the alternating direction multiplier method (ADMM) and Douglas Rachford Splitting method (DRS) for convex problems. Finally, the paper [16] shows that the proximal point algorithm encompasses the DRS method as a specific instance, which is employed for locating a zero of the combined sum of two monotone operators.

- With the assumption A is affine relation, we prove that $R_{A_\gamma} T^n_{A_\gamma,B_\gamma} = T^n_{B_\gamma,A_\gamma} R_{A_\gamma}$ (see Theorem 1).

- We demonstrate the results by providing two examples (refer to Example 1 and Proposition 2).

- We established that the equality does not hold when substituting A_γ with B_γ in the previous result (see Proposition 2, (vii), (viii), and (ix)).

The notation employed in this paper is standard and closely aligns with that in [2], [1], and [4].

2. New Results

All the results in this section are new, highlighting the main ones, which are the relationships between the Aragón Artacho–Campoy operators T_{A_γ,B_γ} and T_{B_γ,A_γ}. Key findings include the important properties of J_{A_γ} and A_γ outlined in Proposition 1, which support our analysis. We provide formulas for the Aragón Artacho–Campoy operators using the Douglas–Rachford splitting operator, as detailed in Lemma 1. Under the assumption that A is an affine relation, we establish the equality $R_{A_\gamma} T^n_{A_\gamma,B_\gamma} = T^n_{B_\gamma,A_\gamma} R_{A_\gamma}$ (see Theorem 1). Our findings are further illustrated with two examples (refer to Example 1 and Proposition 2). Additionally, we demonstrate that replacing A_γ with B_γ in this result leads to a failure of the equality (see Proposition 2, (vii), (viii), and (ix)).

Proposition 1. Let $\gamma \in \,]0,1[$ and assume that A is an affine relation. The following statements are true:

(i) J_{A_γ} is affine.

(ii) A_γ is an affine relation.

Proof. (i): From [7, Lemma 2.3] or [6, Theorem 2.1(xix)], it follows that J_A is affine. Utilizing (6), we can conclude that $J_{A_\gamma}$is also affine. Thus, $J_{A_\gamma}$is affine. (ii): From (i), we have that J_{A_γ} is affine if and only if $(\mathrm{Id} + A_\gamma)^{-1}$ is an affine relation, which in turn is equivalent to $(\mathrm{Id} + A_\gamma)$ being an affine relation, and this is also equivalent to A_γ being an affine relation. ■

Lemma 1. Let $\gamma \in \,]0,1[$ and $\lambda \in \,]0,1]$. We derive:

$$\begin{aligned} R_{B_\gamma} R_{A_\gamma} &= \mathrm{Id} + 2J_{B_\gamma} R_{A_\gamma} - 2J_{A_\gamma} && (11)\\ &= \mathrm{Id} + 2\gamma J_B R_{A_\gamma} - 2\gamma J_A && (12)\\ &= T_{A,B} + (1-2\gamma) J_A - J_B R_A + 2\gamma J_B R_{A_\gamma}. && (13) \end{aligned}$$

Additionally,

$$\begin{aligned} R_{A_\gamma} R_{B_\gamma} &= \mathrm{Id} + 2J_{A_\gamma} R_{B_\gamma} - 2J_{B_\gamma} && (14)\\ &= \mathrm{Id} + 2\gamma J_A R_{B_\gamma} - 2\gamma J_B && (15)\\ &= T_{B,A} + (1-2\gamma) J_B - J_A R_B + 2\gamma J_A R_{B_\gamma}. && (16) \end{aligned}$$

Moreover,

$$\begin{aligned} T_{A_\gamma,B_\gamma} &= \mathrm{Id} + 2\lambda J_{B_\gamma} R_{A_\gamma} - 2\lambda J_{A_\gamma} && (17)\\ &= \mathrm{Id} + 2\lambda\gamma J_B R_{A_\gamma} - 2\lambda\gamma J_A && (18)\\ &= \mathrm{Id} + 2\lambda\gamma \big(J_B R_{A_\gamma} - J_A\big) && (19)\\ &= T_{A,B} + (1-2\lambda\gamma) J_A - J_B R_A + 2\lambda\gamma J_B R_{A_\gamma} && (20)\\ &= T_{B,A} + J_B - J_A R_B + 2\lambda\gamma J_B R_{A_\gamma} - 2\lambda\gamma J_A. && (21) \end{aligned}$$

Furthermore,

$$\begin{aligned} T_{B_\gamma,A_\gamma} &= \mathrm{Id} + 2\lambda J_{A_\gamma} R_{B_\gamma} - 2\lambda J_{B_\gamma} && (22)\\ &= \mathrm{Id} + 2\lambda\gamma J_A R_{B_\gamma} - 2\lambda\gamma J_B && (23)\\ &= \mathrm{Id} + 2\lambda\gamma \big(J_A R_{B_\gamma} - J_B\big) && (24)\\ &= T_{B,A} + (1-2\lambda\gamma) J_B - J_A R_B + 2\lambda\gamma J_A R_{B_\gamma}. && (25) \end{aligned}$$

Proof. From (7), we can conclude:

$$R_{B_\gamma} R_{A_\gamma} = \big(2J_{B_\gamma} - \mathrm{Id}\,\big) R_{A_\gamma}$$

$$\begin{aligned}&= 2J_{B_\gamma}R_{A_\gamma} - R_{A_\gamma}\\ &= 2J_{B_\gamma}R_{A_\gamma} - \big(2J_{A_\gamma} - \operatorname{Id}\big)\\ &= \operatorname{Id} + 2J_{B_\gamma}R_{A_\gamma} - 2J_{A_\gamma},\end{aligned}$$

This establishes (11). Additionally, from (6) and (11), we have:

$$\begin{aligned}R_{B_\gamma}R_{A_\gamma} &= \operatorname{Id} + 2\big(\gamma J_B + (1-\gamma)w\big)R_{A_\gamma} - 2\big(\gamma J_A + (1-\gamma)w\big)\\ &= \operatorname{Id} + 2\gamma J_B R_{A_\gamma} + 2(1-\gamma)w - 2\gamma J_A - 2(1-\gamma)w\\ &= \operatorname{Id} + 2\gamma J_B R_{A_\gamma} - 2\gamma J_A,\end{aligned}$$

This confirms (12). Furthermore, from Fact 3, we have:

$$\begin{aligned}R_{B_\gamma}R_{A_\gamma} &= T_{A,B} + J_A - J_B R_A + 2\gamma J_B R_{A_\gamma} - 2\gamma J_A\\ &= T_{A,B} + (1-2\gamma)J_A - J_B R_A + 2\gamma J_B R_{A_\gamma}.\end{aligned}$$

This confirms (13). The proof for $R_{A_\gamma}R_{B_\gamma}$ follows similarly to that of $R_{B_\gamma}R_{A_\gamma}$. From (8), we find:

$$\begin{aligned}T_{A_\gamma,B_\gamma} &= (1-\lambda)\operatorname{Id} + \lambda R_{B_\gamma}R_{A_\gamma}\\ &= (1-\lambda)\operatorname{Id} + \lambda\big(\operatorname{Id} + 2J_{B_\gamma}R_{A_\gamma} - 2J_{A_\gamma}\big) \quad \text{(from(11))}\\ &= \operatorname{Id} + 2\lambda J_{B_\gamma}R_{A_\gamma} - 2\lambda J_{A_\gamma},\end{aligned}$$

This confirms (17). Utilizing (6) and (17), we derive:

$$\begin{aligned}T_{A_\gamma,B_\gamma} &= \operatorname{Id} + 2\lambda\big(\gamma J_B + (1-\gamma)w\big)R_{A_\gamma} - 2\lambda\big(\gamma J_A + (1-\gamma)w\big)\\ &= \operatorname{Id} + 2\lambda\gamma J_B R_{A_\gamma} + 2\lambda(1-\gamma)w - 2\lambda\gamma J_A - 2\lambda(1-\gamma)w\\ &= \operatorname{Id} + 2\lambda\gamma J_B R_{A_\gamma} - 2\lambda\gamma J_A.\\ &= \operatorname{Id} + 2\lambda\gamma\big(J_B R_{A_\gamma} - J_A\big)\end{aligned}$$

This confirms (18) and (19). Finally, from (18) and (9), we derive:

$$\begin{aligned}T_{A_\gamma,B_\gamma} &= T_{A,B} + J_A - J_B R_A + 2\lambda\gamma J_B R_{A_\gamma} - 2\lambda\gamma J_A\\ &= T_{A,B} + (1-2\lambda\gamma)J_A - J_B R_A + 2\lambda\gamma J_B R_{A_\gamma}.\end{aligned}$$

By merging (10) and (18), we obtain (21). The proof for T_{B_γ,A_γ} follows a similar approach to that of T_{A_γ,B_γ}. ■

Example 1. Let $w \in \mathcal{H}$, U be a closed linear subspace, $\gamma \in\,]0,1[$, and $\lambda \in\,]0,1]$. Assume $A = \operatorname{Id} - v$, where $v \in U^\perp$, and $B = \mathrm{P}_{a+U}$ for some $a \in \mathcal{H}$. From (9), we have

$$T_{A,B} = \operatorname{Id} - J_A + J_B R_A,$$

and from (8), it follows that

$$T_{A_\gamma,B_\gamma} = (1-\lambda)\operatorname{Id} + \lambda\big(R_{B_\gamma}R_{A_\gamma}\big).$$

The following statements hold:

(i) $J_A = \big((\operatorname{Id} + v)/2\big)$ and $R_A = v$.

(ii) $J_{A_\gamma} = \gamma\big((\operatorname{Id} + v)/2\big) + (1-\gamma)w$. Moreover,

$$R_{A_\gamma} = \gamma v - (1-\gamma)\operatorname{Id} + 2(1-\gamma)w.$$

(iii) $J_B = \big(\operatorname{Id} - \frac{1}{2}\operatorname{P}_U\big) - \operatorname{P}_{U^\perp} a$ and $R_B = \big(\operatorname{Id} - \operatorname{P}_U\big) - 2\operatorname{P}_{U^\perp} a$.

(iv) We have

$$J_{B_\gamma} = \gamma\Big(\big(\operatorname{Id} - \frac{1}{2}\operatorname{P}_U\big) - \operatorname{P}_{U^\perp} a\Big) + (1-\gamma)w,$$

and

$$R_{B_\gamma} = (2\gamma - 1)\operatorname{Id} - \gamma\operatorname{P}_U - 2\gamma\operatorname{P}_{U^\perp} a + 2(1-\gamma)w.$$

(v) $T_{A,B} = \big((\operatorname{Id} + v)/2\big) - \operatorname{P}_{U^\perp} a$.

(vi) $T_{B,A} = \big((\operatorname{Id} + v)/2\big)$.

(vii) $J_B R_A = v - \operatorname{P}_{U^\perp} a$.

(viii) $J_A R_B = \big((\operatorname{Id} + v)/2\big) - (\operatorname{P}_U/2) - \operatorname{P}_{U^\perp} a$.

(ix) $J_B R_{A_\gamma} = \gamma v + (1-\gamma)\Big(\big(\frac{1}{2}\operatorname{P}_U - \operatorname{Id}\big) - \big(\operatorname{P}_U - 2\operatorname{Id}\big)w\Big) - \operatorname{P}_{U^\perp} a$.

(x) $J_A R_{B_\gamma} = \frac{1}{2}\Big((2\gamma - 1)\operatorname{Id} - \gamma\operatorname{P}_U - 2\gamma\operatorname{P}_{U^\perp} a + 2(1-\gamma)w + v\Big)$.

(xi) Suppose $k := \lambda\gamma\Big[(2\gamma - 1)v + 4(1-\gamma)w - 2(1-\gamma)\operatorname{P}_U w - 2\operatorname{P}_{U^\perp} a\Big]$. Then

$$T_{A_\gamma, B_\gamma}(x) = \big(1 - \lambda\gamma(3 - 2\gamma)\big)x + \lambda\gamma(1-\gamma)\operatorname{P}_U x + k. \tag{26}$$

(xii) Suppose $l := \lambda\gamma\Big[2(1-\gamma)\operatorname{P}_{U^\perp} a + v + 2(1-\gamma)w\Big]$. Then

$$T_{B_\gamma, A_\gamma}(x) = \Big(1 - \lambda\gamma(3 - 2\gamma)\Big)x + \lambda\gamma(1-\gamma)\operatorname{P}_U x + l. \tag{27}$$

Proof. (i): Let $y \in \mathcal{H}$ and define $x = J_A y$. Then, we have $y \in (\operatorname{Id} + A)x$ if and only if $y = 2x - v$, which implies $x = ((y + v)/2)$. This leads to $J_A = \big((\operatorname{Id} + v)/2\big)$. Consequently, we find that

$$R_A = 2\big((\operatorname{Id} + v)/2\big) - \operatorname{Id} \Leftrightarrow R_B = v$$

by (2).
(ii): Combine (i) and (6) yields:

$$\begin{aligned} R_{A_\gamma}(x) &= 2\gamma\big((x + v)/2\big) + 2(1-\gamma)w - x \\ &= \gamma\big(x + v\big) + 2\big(1-\gamma\big)w - x \end{aligned}$$

$$= \gamma v - \big(1-\gamma\big)x + 2\big(1-\gamma\big)w.$$

(iii): Let $y \in \mathcal{H}$ and define $x = J_B y$. Our goal is to determine x. We have:

$$\begin{aligned} y \in (\mathrm{Id} + \mathrm{P}_{a+U})x &\Leftrightarrow y = x + a + \mathrm{P}_U(x-a) \\ &\Leftrightarrow y = x + (\mathrm{Id} - \mathrm{P}_U)a + \mathrm{P}_U\, x \\ &\Leftrightarrow y = x + \mathrm{P}_{U^\perp} a + \mathrm{P}_U\, x. \end{aligned}$$

Hence,

$$y = x + \mathrm{P}_{U^\perp} a + x^*, \text{ where } x^* = \mathrm{P}_U\, x. \tag{28}$$

Applying P_U to (28) results in:

$$\mathrm{P}_U\, y = \mathrm{P}_U\, x + \mathrm{P}_U\, \mathrm{P}_{U^\perp} a + x^* \Leftrightarrow \mathrm{P}_U\, y = 2x^* \Leftrightarrow x^* = \frac{1}{2}\mathrm{P}_U\, y. \tag{29}$$

Inserting (29) back into (28) results in:

$$y = x + \mathrm{P}_{U^\perp} a + \frac{1}{2}\mathrm{P}_U\, y \Leftrightarrow x = \Big(\mathrm{Id} - \frac{1}{2}\mathrm{P}_U\Big)y - \mathrm{P}_{U^\perp} a.$$

Therefore,

$$J_B = (\mathrm{Id} - \frac{1}{2}\mathrm{P}_U) - \mathrm{P}_{U^\perp} a,$$

and

$$R_B = 2\,\big(\mathrm{Id} - \frac{1}{2}\mathrm{P}_U\big) - 2\,\mathrm{P}_{U^\perp} a - \mathrm{Id} = (\mathrm{Id} - \mathrm{P}_U) - 2\,\mathrm{P}_{U^\perp} a,$$

by (2).

(iv): From (6) and (iii), it can be concluded that:

$$\begin{aligned} J_{B_\gamma} &= \gamma J_B + 2(1-\gamma)w \\ &= \gamma\Big(\Big(\mathrm{Id} - \frac{1}{2}\mathrm{P}_U\Big) - \mathrm{P}_{U^\perp} a\Big) + (1-\gamma)w. \end{aligned}$$

According to (7), we obtain:

$$\begin{aligned} R_{B_\gamma} &= 2\gamma\Big(\Big(\mathrm{Id} - \frac{1}{2}\mathrm{P}_U\Big) - \mathrm{P}_{U^\perp} a\Big) + 2(1-\gamma)w - \mathrm{Id} \\ &= (2\gamma - 1)\,\mathrm{Id} - \gamma\,\mathrm{P}_U - 2\gamma\,\mathrm{P}_{U^\perp} a + 2(1-\gamma)w. \end{aligned}$$

(v): Utilizing (iii), (i), and (9) yields:

$$\begin{aligned} T_{A,B}(x) &= x - \Big(\frac{x+v}{2}\Big) + \Big(\mathrm{Id} - \frac{1}{2}\mathrm{P}_U - \mathrm{P}_{U^\perp} a\Big)R_A x \\ &= \frac{x}{2} - \frac{v}{2} + R_A x - \frac{1}{2}\mathrm{P}_U\, R_A x - \mathrm{P}_{U^\perp} a \end{aligned}$$

$$= \Big(\frac{x+v}{2}\Big) - \mathrm{P}_{U^\perp} a.$$

(vi): Based on (iii), (ii), and (9), we find:

$$\begin{aligned} T_{B,A}(x) &= \frac{1}{2}x + \frac{1}{2}R_A R_B x \\ &= \frac{1}{2}x + \frac{1}{2}(v)\Big(\big(x - \mathrm{P}_U x\big) - 2\,\mathrm{P}_{U^\perp} a\Big) \\ &= \frac{1}{2}\big(x+v\big). \end{aligned}$$

(vii): By employing (iii) and (i), we derive:

$$\begin{aligned} J_B R_A x &= \big(\mathrm{Id} - \frac{1}{2}\mathrm{P}_U - \mathrm{P}_{U^\perp} a\big) R_A x \\ &= R_A x - \frac{1}{2}\mathrm{P}_U R_A x - \mathrm{P}_{U^\perp} a \\ &= v - \mathrm{P}_{U^\perp} a. \end{aligned}$$

(viii): Applying (i) and (iii) results in:

$$\begin{aligned} J_A R_B x &= \Big(\frac{\mathrm{Id}+v}{2}\Big) R_B x \\ &= \frac{1}{2}R_B x + \frac{1}{2}v \\ &= \frac{1}{2}\Big(x - \mathrm{P}_U x - 2\,\mathrm{P}_{U^\perp} a\Big) + \frac{1}{2}v \\ &= \Big(\frac{x+v}{2}\Big) - \frac{1}{2}\mathrm{P}_U x - \mathrm{P}_{U^\perp} a. \end{aligned}$$

(ix): Through the application of (ii) and (iii), we obtain:

$$\begin{aligned} J_B R_{A_\gamma} x &= \big(\mathrm{Id} - \frac{1}{2}\mathrm{P}_U - \mathrm{P}_{U^\perp} a\big) R_{A_\gamma} x \\ &= R_{A_\gamma} x - \frac{1}{2}\mathrm{P}_U R_{A_\gamma} x - \mathrm{P}_{U^\perp} a \\ &= \gamma v - (1-\gamma)x + 2(1-\gamma)w - \frac{1}{2}\mathrm{P}_U \Big(2(1-\gamma)w - (1-\gamma)x\Big) - \mathrm{P}_{U^\perp} a \\ &= \gamma v + (1-\gamma)\Big(\Big(\frac{1}{2}\mathrm{P}_U - \mathrm{Id}\Big)x - \big(\mathrm{P}_U - 2\,\mathrm{Id}\big)w\Big) - \mathrm{P}_{U^\perp} a. \end{aligned}$$

(x): Based on (i) and (iv), we derive:

$$\begin{aligned} J_A R_{B_\gamma} x &= \Big(\frac{\mathrm{Id}+v}{2}\Big)\Big((2\gamma-1)x - \gamma\,\mathrm{P}_U x - 2\gamma\,\mathrm{P}_{U^\perp} a + 2(1-\gamma)w\Big) \\ &= \frac{1}{2}\Big((2\gamma-1)x - \gamma\,\mathrm{P}_U x - 2\gamma\,\mathrm{P}_{U^\perp} a + 2(1-\gamma)w + v\Big). \end{aligned}$$

(xi): Merging (vii), (ix), (v), and (20) results in:

$$\begin{aligned}
T_{A_\gamma,B_\gamma}x &= T_{A,B} + (1-2\lambda\gamma)J_A - J_BR_A + 2\lambda\gamma J_BR_{A_\gamma} \\
&= \Big(\frac{x+v}{2}\Big) - \mathrm{P}_{U^\perp} a + (1-2\lambda\gamma)\Big(\frac{x+v}{2}\Big) - J_BR_A + 2\lambda\gamma J_BR_{A_\gamma} \\
&= (x+v) - \mathrm{P}_{U^\perp} a - \lambda\gamma(x+v) - J_BR_A + 2\lambda\gamma J_BR_{A_\gamma} \\
&= (x+v) - \lambda\gamma(x+v) - v + 2\lambda\gamma J_BR_{A_\gamma} \\
&= x - \lambda\gamma(x+v) + 2\lambda\gamma J_BR_{A_\gamma} \\
&= x - \lambda\gamma(x+v) + 2\lambda\gamma\Big[\gamma v + \frac{(1-\gamma)}{2}\mathrm{P}_U x - (1-\gamma)(x+\mathrm{P}_U w - 2w) - \mathrm{P}_{U^\perp} a\Big] \\
&= \big(1-\lambda\gamma(3-2\gamma)\big)x + \lambda\gamma(1-\gamma)\mathrm{P}_U x + \lambda\gamma\Big((2\gamma-1)v \\
&\qquad + 4(1-\gamma)w - 2(1-\gamma)\mathrm{P}_U w - 2\mathrm{P}_{U^\perp} a\Big).
\end{aligned}$$

(xii): Using (iii), (vi), (viii), and (x), we derive:

$$\begin{aligned}
T_{B_\gamma,A_\gamma}x &= T_{B,A}x + (1-2\lambda\gamma)J_Bx - J_AR_Bx + 2\lambda\gamma J_AR_{B_\gamma}x \\
&= \Big(\frac{x+v}{2}\Big) + (1-2\lambda\gamma)J_Bx - \Big(\frac{x+v}{2}\Big) + \frac{1}{2}\mathrm{P}_U x + \mathrm{P}_{U^\perp} a + 2\lambda\gamma J_AR_{B_\gamma}x \\
&= (1-2\lambda\gamma)\Big(x - \frac{1}{2}\mathrm{P}_U x - \mathrm{P}_{U^\perp} a\Big) + \frac{1}{2}\mathrm{P}_U x + \mathrm{P}_{U^\perp} a + 2\lambda\gamma J_AR_{B_\gamma}x \\
&= (1-2\lambda\gamma)x + 2\lambda\gamma\mathrm{P}_{U^\perp} a + \lambda\gamma\mathrm{P}_U x + 2\lambda\gamma J_AR_{B_\gamma}x \\
&= \Big(1-\lambda\gamma(3-2\gamma)\Big)x + \lambda\gamma(1-\gamma)\mathrm{P}_U x + \lambda\gamma\Big[2(1-\gamma)\mathrm{P}_{U^\perp} a + v + 2(1-\gamma)w\Big],
\end{aligned}$$

which verifies (xii). ■

Lemma 2. Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone and $\gamma \in \,]0,1[$. If J_A is affine, then:

$$J_{A_\gamma}R_{A_\gamma} = R_{A_\gamma}J_{A_\gamma}. \tag{30}$$

Proof. Combine Proposition 1(i) and [7, Lemma 2.4 (i)]. ■

Lemma 3. Assuming that A is an affine relation, we can conclude:

$$R_{A_\gamma}T_{A_\gamma,B_\gamma} - T_{B_\gamma,A_\gamma}R_{A_\gamma} = 2\big(J_{A_\gamma}T_{A_\gamma,B_\gamma} - (1-\lambda)J_{A_\gamma} - \lambda J_{A_\gamma}R_{B_\gamma}R_{A_\gamma}\big). \tag{31}$$

$$= 2\gamma\big(J_AT_{A,B} - (1-\lambda)J_A - \lambda J_AR_{B_\gamma}R_{A_\gamma}\big). \tag{32}$$

Proof. From Proposition 1(ii), it follows that A_γ is an affine relation. Therefore, applying (7) and (17), we derive:

$$\begin{aligned}
R_{A_\gamma}T_{A_\gamma,B_\gamma} - T_{B_\gamma,A_\gamma}R_{A_\gamma} &= \big(2J_{A_\gamma} - \mathrm{Id}\big)T_{A_\gamma,B_\gamma} - T_{B_\gamma,A_\gamma}R_{A_\gamma} \\
&= 2J_{A_\gamma}T_{A_\gamma,B_\gamma} - T_{A_\gamma,B_\gamma} - T_{B_\gamma,A_\gamma}R_{A_\gamma} \\
&= 2J_{A_\gamma}T_{A_\gamma,B_\gamma} - T_{A_\gamma,B_\gamma} - \big(\mathrm{Id} + 2\lambda J_{A_\gamma}R_{B_\gamma} - 2\lambda J_{B_\gamma}\big)R_{A_\gamma}
\end{aligned}$$

$$
\begin{aligned}
&= 2J_{A_\gamma}T_{A_\gamma,B_\gamma} - T_{A_\gamma,B_\gamma} - R_{A_\gamma} - 2\lambda\big(J_{A_\gamma}R_{B_\gamma}R_{A_\gamma} - J_{B_\gamma}R_{A_\gamma}\big)\\
&= 2J_{A_\gamma}T_{A_\gamma,B_\gamma} - \big(\operatorname{Id} + 2\lambda J_{B_\gamma}R_{A_\gamma} - 2\lambda J_{A_\gamma}\big) - R_{A_\gamma}\\
&\quad - 2\lambda\big(J_{A_\gamma}R_{B_\gamma}R_{A_\gamma} - J_{B_\gamma}R_{A_\gamma}\big)\\
&= 2J_{A_\gamma}T_{A_\gamma,B_\gamma} - \operatorname{Id} + 2\lambda J_{A_\gamma} - R_{A_\gamma} - 2\lambda J_{A_\gamma}R_{B_\gamma}R_{A_\gamma}\\
&= 2J_{A_\gamma}T_{A_\gamma,B_\gamma} - 2(1-\lambda)J_{A_\gamma} - 2\lambda J_{A_\gamma}R_{B_\gamma}R_{A_\gamma},
\end{aligned}
$$

this confirms (31). Subsequently, utilizing (6) and (31) gives

$$
\begin{aligned}
R_{A_\gamma}T_{A_\gamma,B_\gamma} - T_{B_\gamma,A_\gamma}R_{A_\gamma} &= 2J_{A_\gamma}T_{A_\gamma,B_\gamma} - 2(1-\lambda)J_{A_\gamma} - 2\lambda J_{A_\gamma}R_{B_\gamma}R_{A_\gamma}\\
&= 2\big(\gamma J_A + (1-\gamma)w\big)T_{A_\gamma,B_\gamma} - 2(1-\lambda)\big(\gamma J_A + (1-\gamma)w\big)\\
&\quad - 2\lambda\big(\gamma J_A + (1-\gamma)w\big)R_{B_\gamma}R_{A_\gamma}\\
&= 2\gamma J_A T_{A_\gamma,B_\gamma} - 2\gamma(1-\lambda)J_A - 2\lambda\gamma J_A R_{B_\gamma}R_{A_\gamma}.
\end{aligned}
$$

■

Theorem 1. Let $\gamma \in]0,1[$ and $\lambda \in]0,1]$, and suppose that A is an affine realtion. Then:

$$R_{A_\gamma}T^n_{A_\gamma,B_\gamma} = T^n_{B_\gamma,A_\gamma}R_{A_\gamma}. \tag{33}$$

Proof. We will demonstrate by induction that $R_{A_\gamma}T^n_{A_\gamma,B_\gamma} = T^n_{B_\gamma,A_\gamma}R_{A_\gamma}$. Starting with $n=1$, we can use (32) to derive:

$$
\begin{aligned}
R_{A_\gamma}T_{A_\gamma,B_\gamma} - T_{B_\gamma,A_\gamma}R_{A_\gamma} &= 2\gamma J_A T_{A_\gamma,B_\gamma} - 2\gamma(1-\lambda)J_A - 2\lambda\gamma J_A R_{B_\gamma}R_{A_\gamma}\\
&= J_A\Big(2\gamma\big(T_{A_\gamma,B_\gamma} - (1-\lambda)\operatorname{Id}\big)\Big) - 2\lambda\gamma J_A R_{B_\gamma}R_{A_\gamma}\\
&= J_A\Big(2\gamma\big((1-\lambda)\operatorname{Id} + \lambda R_{B_\gamma}R_{A_\gamma}\big) - 2\gamma(1-\lambda)\operatorname{Id}\Big) - 2\lambda\gamma J_A R_{B_\gamma}R_{A_\gamma}\\
&= 2\lambda\gamma J_A R_{B_\gamma}R_{A_\gamma} - 2\lambda\gamma J_A R_{B_\gamma}R_{A_\gamma} = 0.
\end{aligned}
$$

Hypothesis assumption: when $n = k$;

$$R_{A_\gamma}T^k_{A_\gamma,B_\gamma} - T^k_{B_\gamma,A_\gamma}R_{A_\gamma} = 0. \tag{34}$$

For $n = k+1$, and utilizing (34), we obtain:

$$
\begin{aligned}
R_{A_\gamma}T^{k+1}_{A_\gamma,B_\gamma} - T^{k+1}_{B_\gamma,A_\gamma}R_{A_\gamma} &= R_{A_\gamma}T^k_{A_\gamma,B_\gamma}T_{A_\gamma,B_\gamma} - T^k_{B_\gamma,A_\gamma}T_{B_\gamma,A_\gamma}R_{A_\gamma}\\
&= R_{A_\gamma}T^k_{A_\gamma,B_\gamma}T_{A_\gamma,B_\gamma} - T^k_{B_\gamma,A_\gamma}R_{A_\gamma}T_{A_\gamma,B_\gamma}\\
&= R_{A_\gamma}T^k_{A_\gamma,B_\gamma}T_{A_\gamma,B_\gamma} - R_{A_\gamma}T^k_{A_\gamma,B_\gamma}T_{A_\gamma,B_\gamma}\\
&= 0.
\end{aligned}
$$

Therefore, (33) has been verified. ■

Lemma 4. Suppose both A and B are affine relations. Then the following holds:

(i) The operators T_{A_γ,B_γ} and T_{B_γ,A_γ} are affine.

(ii) The equation $T_{A_\gamma,B_\gamma}R_{B_\gamma}R_{A_\gamma} = R_{B_\gamma}R_{A_\gamma}T_{A_\gamma,B_\gamma}$ is satisfied.

(iii) We have

$$\lambda^{-2}\big(T_{A_\gamma,B_\gamma}T_{B_\gamma,A_\gamma} - T_{B_\gamma,A_\gamma}T_{A_\gamma,B_\gamma}\big) = R_{B_\gamma}R^2_{A_\gamma}R_{B_\gamma} - R_{A_\gamma}R^2_{B_\gamma}R_{A_\gamma}.$$

(iv) The equality

$$T_{A_\gamma,B_\gamma}T_{B_\gamma,A_\gamma} = T_{B_\gamma,A_\gamma}T_{A_\gamma,B_\gamma}$$

holds if and only if $R_{B_\gamma}R^2_{A_\gamma}R_{B_\gamma} = R_{A_\gamma}R^2_{B_\gamma}R_{A_\gamma}$.

(v) If $R^2_{A_\gamma} = R^2_{B_\gamma}$, then it follows that $T_{A_\gamma,B_\gamma}T_{B_\gamma,A_\gamma} = T_{B_\gamma,A_\gamma}T_{A_\gamma,B_\gamma}$.

Proof. (i): Clear. (ii): From (i), we conclude that:

$$\begin{aligned} T_{A_\gamma,B_\gamma}R_{B_\gamma}R_{A_\gamma} &= T_{A_\gamma,B_\gamma}\big(\lambda^{-1}T_{A_\gamma,B_\gamma} - \lambda^{-1}(1-\lambda)\,\mathrm{Id}\,\big) \\ &= \lambda^{-1}T^2_{(A_\gamma,B_\gamma)} - \lambda^{-1}(1-\lambda)T_{A_\gamma,B_\gamma} \\ &= \big(\lambda^{-1}T_{A_\gamma,B_\gamma} - \lambda^{-1}(1-\lambda)\,\mathrm{Id}\,\big)T_{A_\gamma,B_\gamma} \\ &= R_{B_\gamma}R_{A_\gamma}T_{A_\gamma,B_\gamma}. \end{aligned}$$

(iii): Utilizing (8), we find that:

$$\lambda^{-2}\big(T_{A_\gamma,B_\gamma}T_{B_\gamma,A_\gamma}\big) = \lambda^{-2}\big((1-\lambda)\,\mathrm{Id} + \lambda R_{B_\gamma}R_{A_\gamma}\big)\big((1-\lambda)\,\mathrm{Id} + \lambda R_{A_\gamma}R_{B_\gamma}\big)$$

Hence,

$$\begin{aligned} \lambda^{-2}\big(T_{A_\gamma,B_\gamma}T_{B_\gamma,A_\gamma}\big) = \lambda^{-2}\big(&(1-\lambda)^2\,\mathrm{Id} + \lambda(1-\lambda)R_{A_\gamma}R_{B_\gamma} + \lambda(1-\lambda)R_{B_\gamma}R_{A_\gamma} \\ &+ \lambda^2 R_{B_\gamma}R^2_{A_\gamma}R_{B_\gamma}\big). \end{aligned} \tag{35}$$

Moreover,

$$\lambda^{-2}\big(T_{B_\gamma,A_\gamma}T_{A_\gamma,B_\gamma}\big) = \lambda^{-2}\big((1-\lambda)\,\mathrm{Id} + \lambda R_{A_\gamma}R_{B_\gamma}\big)\big((1-\lambda)\,\mathrm{Id} + \lambda R_{B_\gamma}R_{A_\gamma}\big)$$

Hence,

$$\begin{aligned} \lambda^{-2}\big(T_{B_\gamma,A_\gamma}T_{A_\gamma,B_\gamma}\big) = \lambda^{-2}\big(&(1-\lambda)^2\,\mathrm{Id} + \lambda(1-\lambda)R_{B_\gamma}R_{A_\gamma} + \lambda(1-\lambda)R_{A_\gamma}R_{B_\gamma} \\ &+ \lambda^2 R_{A_\gamma}R^2_{B_\gamma}R_{A_\gamma}\big). \end{aligned} \tag{36}$$

Taking the difference of (35) and (36) yields:

$$\lambda^{-2}\big(T_{A_\gamma,B_\gamma}T_{B_\gamma,A_\gamma} - T_{B_\gamma,A_\gamma}T_{A_\gamma,B_\gamma}\big) = R_{B_\gamma}R^2_{A_\gamma}R_{B_\gamma} - R_{A_\gamma}R^2_{B_\gamma}R_{A_\gamma}.$$

(iv) and (v): They are derived from (iii). ■

Proposition 2. *Let U be a closed linear subspace, and define $A = \operatorname{Id} + v$ with $v \in U^{\perp}$. Furthermore, let $B = \mathrm{P}_{a+U}$, where $a \in U^{\perp}$ and $a \neq v$. The following statements are true:*

(i) We have

$$T_{A_\gamma,B_\gamma}(x) = \big(1 - \lambda\gamma(3-2\gamma)\big)x + \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x + k,$$

where

$$k = \lambda\gamma\big((2\gamma-1)v + 4(1-\gamma)w - 2(1-\gamma)\,\mathrm{P}_U\,w - 2a\big).$$

(ii) We have

$$T_{B_\gamma,A_\gamma}(x) = \big(1 - \lambda\gamma(3-2\gamma)\big)x + \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x + l,$$

where

$$l = \lambda\gamma\big(2(1-\gamma)a + v + 2(1-\gamma)w\big).$$

(iii) We have

$$\begin{aligned} R_{A_\gamma}T_{A_\gamma,B_\gamma}(x) &= T_{B_\gamma,A_\gamma}R_{A_\gamma}(x) \\ &= (1-\gamma)\Big(\big(\lambda\gamma(3-2\gamma)-1\big)x - \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x\Big) + h, \end{aligned}$$

where

$$\begin{aligned} h = {} & \gamma\big[\lambda\gamma(2\gamma-3)+1+\lambda\big]v + 2(1-\gamma)\big[(1-2\lambda\gamma(1-\gamma))w \\ & + \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,w\big] + 2\lambda\gamma(1-\gamma)a. \end{aligned}$$

(iv) We have

$$R_{B_\gamma}T_{A_\gamma,B_\gamma} = (1-2\gamma)\Big(\big(\lambda\gamma(3-2\gamma)-1\big)x - \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x\Big) + m,$$

where

$$\begin{aligned} m = {} & (1-2\gamma)\lambda\gamma\big[(1-2\gamma)v + 2(1-\gamma)\,\mathrm{P}_U\,w - 4(1-\gamma)w\big] \\ & + 2\big[(1-\gamma)w + \gamma(\lambda-1-2\lambda\gamma)a\big]. \end{aligned}$$

(v) We have

$$T_{B_\gamma,A_\gamma}R_{B_\gamma} = (1-2\gamma)\Big(\big(\lambda\gamma(3-2\gamma)-1\big)x - \gamma\big(1+\lambda-\lambda\gamma(5-3\gamma)\big)\,\mathrm{P}_U\,x + s,$$

where

$$\begin{aligned} s = {} & \lambda\gamma v + 2\Big(\lambda\gamma^2 - (2\lambda+1)\gamma + 1\Big)w \\ & - 2\gamma\Big(\lambda(3\gamma+4)+1\Big)a + 2\lambda\gamma(1-\gamma)^2\,\mathrm{P}_U\,w. \end{aligned}$$

(vi) $R_{B_\gamma} T_{A_\gamma,B_\gamma} \neq T_{B_\gamma,A_\gamma} R_{B_\gamma}$.

(vii) *We have*

$$R_{B_\gamma} T_{B_\gamma,A_\gamma} = (2\gamma-1)\Big(1-\lambda\gamma(3-2\gamma)\Big)x + \gamma\Big(\lambda\gamma(5-3\gamma)-\lambda-1\Big)\,\mathrm{P}_U\,x + b,$$

where

$$b = -2\gamma\Big(\lambda\gamma(2\gamma-3)+\lambda+1\Big)a - 2\Big(\gamma\Big(\lambda\gamma(2\gamma-3)+1\Big)-1\Big)w + \lambda\gamma(2\gamma-1)v \\ -2\lambda\gamma^2(1-\gamma)\,\mathrm{P}_U\,w.$$

(viii) *We have*

$$T_{A_\gamma,B_\gamma} R_{B_\gamma} = (2\gamma-1)\Big(1-\lambda\gamma(3-2\gamma)\Big)x + \gamma\Big(\lambda\gamma(5-3\gamma)-\lambda-1\Big)\,\mathrm{P}_U\,x + c,$$

where

$$c = -2\gamma\Big(\lambda\gamma(2\gamma-3)+\lambda+1\Big)a - 2\Big(\gamma\Big(\lambda\gamma(2\gamma-3)+\lambda+1\Big)-1\Big)w + \lambda\gamma(2\gamma-1)v \\ -2\lambda\gamma^2(1-\gamma)\,\mathrm{P}_U\,w.$$

(ix) $R_{B_\gamma} T_{B_\gamma,A_\gamma} \neq T_{A_\gamma,B_\gamma} R_{B_\gamma}$.

Proof. (i): This is derived from Example 1 (xi). (ii) : This is derived from Example 1 (xii). (iii) : Utilizing (i), (ii), and Example 1(ii), we find that:

$$\begin{aligned} R_{A_\gamma} T_{A_\gamma,B_\gamma}(x) &= \Big(\gamma v - (1-\gamma)\,\mathrm{Id} + 2(1-\gamma)w\Big) T_{A_\gamma,B_\gamma}(x) \\ &= \gamma v + 2(1-\gamma)w - (1-\gamma) T_{A_\gamma,B_\gamma}(x) \\ &= \gamma v - (1-\gamma)\lambda\gamma(2\gamma-1)v + 2(1-\gamma)w - 4\lambda\gamma(1-\gamma)^2 w \\ &\quad -(1-\gamma)\Big(\Big(1-\lambda\gamma(3-2\gamma)\Big)x + \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x + \lambda\gamma\Big(-2(1-\gamma)\,\mathrm{P}_U\,w - 2a\Big)\Big) \\ &= (1-\gamma)\Big((\lambda\gamma(3-2\gamma)-1)x - \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x\Big) \\ &\quad + \gamma[\lambda\gamma(2\gamma-3)+1+\lambda]v + 2(1-\gamma)\big[(1-2\lambda\gamma(1-\gamma))w \\ &\quad + \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,w\big] + 2\lambda\gamma(1-\gamma)a. \end{aligned}$$

Moreover,

$$\begin{aligned} T_{B_\gamma,A_\gamma} R_{A_\gamma}(x) &= (1-\lambda\gamma(3-2\gamma))R_{A_\gamma}(x) + \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,R_{A_\gamma}(x) \\ &\quad + \lambda\gamma(2(1-\gamma)a + v + 2(1-\gamma)w). \\ &= (1-\gamma)\Big((\lambda\gamma(3-2\gamma)-1)x - \lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x\Big) \\ &\quad + \gamma[\lambda\gamma(2\gamma-3)+1+\lambda]v + 2(1-\gamma)\big[(1-2\lambda\gamma(1-\gamma))w \end{aligned}$$

$$+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,w\big]+2\lambda\gamma(1-\gamma)a.$$

Therefore,

$$R_{A_\gamma}T_{A_\gamma,B_\gamma}(x)=T_{B_\gamma,A_\gamma}R_{A_\gamma}(x).$$

and (iii) is verified.
(iv): Using Example 1(iv) and (i) gives

$$\begin{aligned}
R_{B_\gamma}T_{A_\gamma,B_\gamma}(x)&=\Big((2\gamma-1)\,\mathrm{Id}-\gamma\,\mathrm{P}_U-2\gamma a+2(1-\gamma)w\Big)T_{A_\gamma,B_\gamma}(x)\\
&=2(1-\gamma)w-2\gamma a-(1-2\gamma)T_{A_\gamma,B_\gamma}(x)-\gamma\,\mathrm{P}_U\,T_{A_\gamma,B_\gamma}(x)\\
&=2(1-\gamma)w-2\gamma a-(1-2\gamma)\Big[\Big(1-\lambda\gamma(3-2\gamma)\Big)x+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x+k\Big]\\
&=(1-2\gamma)\Big[\Big(\lambda\gamma(3-2\gamma)-1\Big)x-\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x\Big]\\
&\quad+\lambda\gamma(1-2\gamma)\Big[(1-2\gamma)v+2(1-\gamma)\,\mathrm{P}_U\,w-4(1-\gamma)w\Big]\\
&\quad+2\Big[(1-\gamma)w+\gamma(\lambda-1-2\lambda\gamma)a\Big].
\end{aligned}$$

(v): Utilizing (ii) and Example 1(iv), we obtain

$$\begin{aligned}
T_{B_\gamma,A_\gamma}R_{B_\gamma}&=\Big(\Big(1-\lambda\gamma(3-2\gamma)\,\mathrm{Id}\Big)+\lambda\gamma(1-\gamma)\,\mathrm{P}_U+l\Big)R_{B_\gamma}(x)\\
&=\Big(1-\lambda\gamma(3-2\gamma)R_{B_\gamma}(x)+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,(R_{B_\gamma}(x))+l(R_{B_\gamma}(x))\\
&=\Big(1-\lambda\gamma(3-2\gamma)\Big)(2\gamma-1)x-\gamma\Big(1+\lambda-\lambda\gamma(5-3\gamma)\Big)\,\mathrm{P}_U\,x\\
&\quad+2\lambda\gamma(1-\gamma)(a+w)+\Big(1-\lambda\gamma(3-2\gamma)\Big)\Big(2(1-\gamma)w-2\gamma a\Big)\\
&\quad+2\lambda\gamma(1-\gamma)^2\,\mathrm{P}_U\,w+\lambda\gamma v\\
&=(1-2\gamma)\Big((\lambda\gamma(3-2\gamma)-1)x-\gamma(1+\lambda-\lambda\gamma(5-3\gamma))\,\mathrm{P}_U\,x\\
&\quad+\lambda\gamma v+2\Big(\lambda\gamma^2-(2\lambda+1)\gamma+1\Big)w\\
&\quad-2\gamma\Big(\lambda(3\gamma+4)+1\Big)a+2\lambda\gamma(1-\gamma)^2\,\mathrm{P}_U\,w.
\end{aligned}$$

(vi): This is derived from (iv) and (v).
(vii): By using Example 1(iv) and (ii) we have

$$\begin{aligned}
R_{B_\gamma}T_{B_\gamma,A_\gamma}(x)&=\Big((2\gamma-1)\,\mathrm{Id}-\gamma\,\mathrm{P}_U-2\gamma a+2(1-\gamma)w\Big)T_{B_\gamma,A_\gamma}(x)\\
&=(2\gamma-1)T_{B_\gamma,A_\gamma}(x)-\gamma\,\mathrm{P}_U\,(T_{B_\gamma,A_\gamma}(x))-2\gamma a+2(1-\gamma)w\\
&=(2\gamma-1)\Big[\Big(1-\lambda\gamma(3-2\gamma)\Big)x+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x+l\Big]\\
&\quad-\gamma\,\mathrm{P}_U\Big[\Big(1-\lambda\gamma(3-2\gamma)\Big)x+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x+l\Big]-2\gamma a+2(1-\gamma)w
\end{aligned}$$

$$
\begin{aligned}
&= (2\gamma-1)\Big[\Big(1-\lambda\gamma(3-2\gamma)\Big)x+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x\Big]\\
&\quad -\gamma\Big[\Big(1-\lambda\gamma(3-2\gamma)\Big)\,\mathrm{P}_U\,x+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,x\Big]\\
&\quad +(2\gamma-1)l-\gamma\,\mathrm{P}_U\,l-2\gamma a+2(1-\gamma)w\\
&= (2\gamma-1)\Big(1-\lambda\gamma(3-2\gamma)\Big)x+\gamma\Big(\lambda\gamma(5-3\gamma)-\lambda-1\Big)\,\mathrm{P}_U\,x\\
&\quad -2\gamma\Big(\lambda\gamma(2\gamma-3)+\lambda+1\Big)a-2\Big(\gamma\Big(\lambda\gamma(2\gamma-3)+1\Big)-1\Big)w+\lambda\gamma(2\gamma-1)v\\
&\quad -2\lambda\gamma^2(1-\gamma)\,\mathrm{P}_U\,w.
\end{aligned}
$$

(viii): Utilizing Example 1(iv) and (ii) yields

$$
\begin{aligned}
T_{A_\gamma,B_\gamma}R_{B_\gamma}(x) &= \Big(1-\lambda\gamma(3-2\gamma)\,\mathrm{Id}+\lambda\gamma(1-\gamma)\,\mathrm{P}_U+k\Big)R_{B_\gamma}(x)\\
&= \Big(1-\lambda\gamma(3-2\gamma)\Big)R_{B_\gamma}(x)+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\,R_{B_\gamma}(x)+k\\
&= \Big(1-\lambda\gamma(3-2\gamma)\Big)\Big((2\gamma-1)x-\gamma\,\mathrm{P}_U\,x\Big)+\Big(1-\lambda\gamma(3-2\gamma)\Big)\Big(2(1-\gamma)w-2\gamma a\Big)\\
&\quad +\lambda\gamma(1-\gamma)\,\mathrm{P}_U\Big((2\gamma-1)x-\gamma\,\mathrm{P}_U\,x\Big)+\lambda\gamma(1-\gamma)\,\mathrm{P}_U\Big(2(1-\gamma)w-2\gamma a\Big)+k\\
&= (2\gamma-1)\Big(1-\lambda\gamma(3-2\gamma)\Big)x+\gamma\Big(\lambda\gamma(5-3\gamma)-\lambda-1\Big)\,\mathrm{P}_U\,x\\
&\quad -2\gamma\Big(\lambda\gamma(2\gamma-3)+\lambda+1\Big)a-2\Big(\gamma\Big(\lambda\gamma(2\gamma-3)+\lambda+1\Big)-1\Big)w\\
&\quad +\lambda\gamma(2\gamma-1)v-2\lambda\gamma^2(1-\gamma)\,\mathrm{P}_U\,w.
\end{aligned}
$$

(ix): It follows from (vii) and (viii). ■

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