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Additional Studies on Displacement Mapping with Restrictions

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Abstract. The theory of monotone operators is fundamental in modern optimization and various areas of nonlinear analysis. Key classes of monotone operators include matrices with a positive semidefinite symmetric component and subdifferential operators. In this paper, we extend our investigation to displacement mappings. We derive formulas for set-valued and Moore-Penrose inverses. Additionally, we conduct a thorough examination of the operators (one-half times the identity plus T) and its inverse, providing a formula for the inverse of the operator. Our results are illustrated through an analysis of reflected and projection operators onto closed linear subspaces.

2020 Mathematics Subject Classifications: 47H09, 47H05, 47A06, 90C25

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1. Introduction

It is well known that one of important classes of monotone operators are Displacement mappings of nonexpansive mappings. There are many key examples that have proven how these mappings are highly useful in optimization problems. For example, in 2016 Heinz H. Bauschke, Warren Hare, and Walaa Moursi used displacement mappings in analyzing the range of the Douglas–Rachford operator to derive valuable duality results, see [5]. Additionally, the asymptotic regularity results for nonexpansive mappings were generalized in [8] to the broader context of displacement mappings. Overall, the displacement mapping framework has emerged as a powerful tool for analyzing the behavior of nonexpansive mappings, with a range of important applications in optimization and related areas. Throughout, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}, \tag{1}$$

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and induced norm $\|\cdot\|: X \to \mathbb{R}: x \mapsto \sqrt{\langle x, x\rangle}$. We also assume that $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are maximally monotone operators. The *resolvent* and the *reflected resolvent* associated with A are

$$J_A = (\operatorname{Id} + A)^{-1} \text{ and } R_A = 2J_A - \operatorname{Id}, \tag{2}$$

respectively. An operator $T : X \rightrightarrows X$ is *nonexpansive* if it is Lipschitz continuous with constant 1, i.e.,

$$\big(\forall x \in X\big)\big(\forall y \in X\big) \|Tx - Ty\| \leq \|x - y\|. \tag{3}$$

Moreover, $T : D \rightrightarrows X$ is *firmly nonexpansive* if

$$\big(\forall x \in D\big)\big(\forall y \in D\big) \; \|Tx - Ty\|^2 + \|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2 \leq \|x - y\|^2. \tag{4}$$

Fact 1. *[4, Definition 4.10] Let D be a nonempty subset of X, let $T : D \to X$, and let $\beta \in \mathbb{R}_{++}$, where $\mathbb{R}_{++}$ is the set of strictly positive real numbers $]0, +\infty[$. Then T is β-cocoercive (or β-inverse strongly monotone) if βT is firmly nonexpansive, i.e.,*

$$\big(\forall x \in D\big)\big(\forall y \in D\big) \quad \langle x - y, Tx - Ty\rangle \geq \beta\|Tx - Ty\|^2.$$

In optimization, we have seen the importance the *displacement mappings* of nonexpansive mappings:

$$\operatorname{Id} - R \tag{5}$$

because of the nice properities that have such as monotonicity which plays a central role in modern optimization (see [4, 11, 18, 20–23] for more details). A comprehensive analysis of the displacement mappings of nonexpansive mappings from the point of view of monotone operator theory under the condition of isometry of finite order of R are given in [2, Lemma] and [1, Section 3]. We refer the reder to [18, Exercise 12.16], and [4, Example 20.29], [7]. More information is in [10, 15, 17, 19].

Throughout this paper, we assume that

$$R : X \to X \text{ is linear and nonexpansive, with } D := \operatorname{Fix} R = \ker\big(\operatorname{Id} - R\big). \tag{6}$$

In this paper, we study the displacement mapping using the assumption in (6). Our results can be summarized as follows

- Proposition 1, Lemma 1, and Remark 1 collect some useful properities of the dispdisplacment mapping and its inverse, which will be useful in our study.
- Lemma 2 provides a formula and gives nice properties of the operator T.
- We derive a formula for the inverse of the displacment mapping (see Theorem 2 (i)). A formula for the Moore-Penrose inverse of the displacement mapping is given in Theorem 2(ii).

- Theorem 3 gives a comprehensive study of the the operators $\big(1/2\big)\operatorname{Id}+T$ and its inverse. Additionaly, we derive a formula of $\Big(\big(1/2\big)\operatorname{Id}+T\Big)^{-1}$ and prove that is equal to the resolvant of the operator $2T$.
- We illustrates the reults by giving four examples. The first two examples are related to the projection operator to a closed linear subspace (see Example 2 and Example 3), while the other two are related to the reflected operator to closed linear subspace (see Example 4 and Example 5).

2. Results

Important properties of the displacement mapping $(\operatorname{Id}-R)$ and its inverse are given in the next proposition.

Proposition 1. *Let R be nonexpansive operator, then the following holds:*

(i) $\frac{1}{2}(\operatorname{Id}-R)$ *is firmly nonexpansive.*
(ii) $\operatorname{Id}-R$ *is nonexpansive.*
(iii) $\operatorname{Id}-R$ *and* $(\operatorname{Id}-R)^{-1}$ *are maximally monotone.*
(iv) $\operatorname{Id}-R$ *is* $\frac{1}{2}$*-cocoercive.*
(v) $(\operatorname{Id}-R)^{-1}$ *is strongly monotone*with constant* $\frac{1}{2}$.
(vi) $\operatorname{Id}-R$ *is* 3^* *monotone.*
(vii) $(\operatorname{Id}-R)^{-1}$ *is* 3^* *monotone*
(viii) $\operatorname{Id}-R$ *is paramonotone.*
(ix) $\big(\operatorname{Id}-R\big)^{-1}-\frac{1}{2}\operatorname{Id}$ *is maximally monotone.*

Proof. (i): We have

$$\begin{aligned} R \text{ is nonexpansive } &\Leftrightarrow -R = 2\Big(\big(\operatorname{Id}-R\big)/2\Big) \text{ is nonexpansive}\\ &\Leftrightarrow \big(\operatorname{Id}-R\big)/2 \text{ is firmly nonexpansive,} \end{aligned}$$

by [4, Proposition 4.4]. (ii): It follows from (i) and [4, Proposition 4.2] . (iii): See [4, Example 25.20(v)] or [2, Theorem 7.1]. (iv): Combine (i) and Fact 1. (v): Take $(x,u)\in \operatorname{gra}(\operatorname{Id}-R)^{-1}$ and $(y,v)\in\operatorname{gra}(\operatorname{Id}-R)^{-1}$. Then $u\in(\operatorname{Id}-R)^{-1}x \Rightarrow x=u-Ru$ and $v\in(\operatorname{Id}-R)^{-1}y\Rightarrow y=v-Rv$.

$$\begin{aligned} &\langle u-v, x-y\rangle \geq \frac{1}{2}\|x-y\|^2\\ \Leftrightarrow\ &\langle u-v,(u-Ru)-(v-Rv)\rangle \geq \frac{1}{2}\|(u-Ru)-(v-Rv)\|^2, \end{aligned}$$

*An operator $A: X\rightrightarrows X$ is strongly monotone with constant $\beta\in\mathbb{R}_{++}$ if $A-\beta\operatorname{Id}$ is montone, i.e.,

$$\big(\forall (x,u)\in\operatorname{gra}A\big)\big(\forall(y,v)\in\operatorname{gra}A\big)\quad \langle x-y,u-v\rangle\geq\beta\|x-y\|^2.$$

which deduce from (iv) and Footnote $*$ that $(\operatorname{Id} - R)^{-1}$ is strongly monotone with constant $(1/2)$. (vi) and (vii): It follows from (iv) that $\operatorname{Id} - R$ is bounded by $(1/2)$ and its monotone by (iii). Hence, $\operatorname{Id} - R$ and $(\operatorname{Id} - R)^{-1}$ are 3^* monotone by [4, Proposition 25.16(i) & (iv)].
(viii): See [4, Example 22.9]. (ix): By (iv) and [4, Example 22.7], $(\operatorname{Id} - R)^{-1}$ is $(1/2)$-strongly monotone, i.e., $B := (\operatorname{Id} - R)^{-1} - \frac{1}{2}\operatorname{Id}$ is still monotone. If B was not maximally monotone, then neither would be $B + \frac{1}{2}\operatorname{Id} = (\operatorname{Id} - R)^{-1}$ which would contradict (iii). ■

Lemma 1. *Set* $D := \ker\big(\operatorname{Id} - R\big) = \operatorname{Fix} R$. *Then the following holds:*

(i) *D is a closed linear subspace.*
(ii) $\operatorname{Fix} R^* = D$.
(iii) $\overline{\operatorname{ran}}\big(\operatorname{Id} - R\big) = \overline{\operatorname{ran}}\big(\operatorname{Id} - R^*\big) = D^{\perp}$.

Proof. (i): Let $x, y \in D$ such that $x - Rx = 0$ and $y - Ry = 0$. Let $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}\big(\operatorname{Id} - R\big)\big(\alpha x + \beta y\big) &= \big(\operatorname{Id} - R\big)(\alpha x) + \big(\operatorname{Id} - R\big)(\beta y)\\ &= \alpha(x - Rx) + \beta(y - Ry)\\ &= 0 + 0 = 0.\end{aligned}$$

Therefore, $\alpha x + \beta y \in D$ and hence D is a linear subspace. To show that D is closed, let (x_n) be a sequence in D such that (x_n) converges to x. Then

$$\begin{aligned}\lim_{n\to\infty}(\operatorname{Id} - R)(x - x_n) &= \lim_{n\to\infty}(\operatorname{Id} - R)x - \lim_{n\to\infty}(\operatorname{Id} - R)x_n\\ &= (\operatorname{Id} - R)x - (\operatorname{Id} - R)x = 0.\end{aligned}$$

Therefore, $x \in D$ and hence D is closed.
(ii) and (iii): It follows from Proposition 1(iii) & (iv) that $\operatorname{Id} - R$ is monotone and bounded. Hence, $\operatorname{Fix} R^* = \operatorname{Fix} R = D$ and $\overline{\operatorname{ran}}\big(\operatorname{Id} - R\big) = \overline{\operatorname{ran}}\big(\operatorname{Id} - R^*\big) = D^{\perp}$ by [4, Proposition 20.17]. ■

Remark 1. *Suppose that* $X = \ell_2(\mathbb{N})$ *and that*

$$R : X \to X : (x_n)_{n\in\mathbb{N}} \mapsto \big(((1 - \varepsilon_n)x_n)\big)_{n\in\mathbb{N}}, \tag{7}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ *lies in* $]0, 1[$ *with* $\varepsilon_n \to 0$. *Then the following holds:*

(i) $\operatorname{Id} - R : (x_n)_{n\in\mathbb{N}} \mapsto \big(\varepsilon_n x_n\big)_{n\in\mathbb{N}}$ *is a compact operator.*
(ii) $D = \operatorname{Fix} R = \{0\}$.
(iii) $\operatorname{ran}(\operatorname{Id} - R)$ *is not closed.*
(iv) $\operatorname{ran} R$ *is a closed subspace.*

Proof. (i) and (ii): See [12, PropositionII.4.6]. (iii): It follows from [16, Proposition 3.4.6] that $\operatorname{ran}(\operatorname{Id} - R)$ is closed if and only if $\operatorname{ran}(\operatorname{Id} - R)$ is finite-dimensional. On the other hand, $X = D^{\perp} = \overline{\operatorname{ran}}(\operatorname{Id} - R)$, i.e., the range of $\operatorname{Id} - R$ is dense in the infinite-dimensional space X. Altogether,

$$\operatorname{ran}\big(\operatorname{Id} - R\big) \text{ is not closed.}$$

(iv): See [16, Lemma 3.4.20]. ■

Lemma 2. *Suppose that* ran (Id $-R$) *is closed; equivalently,*

$$\operatorname{ran}(\operatorname{Id} - R) = D^{\perp}.$$

Set

$$T := \mathrm{P}_{D^{\perp}}(\operatorname{Id} - R)^{-1}\mathrm{P}_{D^{\perp}} - \frac{1}{2}\mathrm{P}_{D^{\perp}}. \tag{8}$$

Then,

(i) $\operatorname{ran}(\operatorname{Id} - R)^* = D^{\perp}$.
(ii) T *is a linear and continuous.*
(iii) T *is monotone.*
(iv) T *is maximally monotone.*
(v) $\operatorname{ran} T \subseteq D^{\perp}$, *where* $D = \ker(\operatorname{Id} - R)$.
(vi) $\mathrm{P}_{D^{\perp}} T = T\mathrm{P}_{D^{\perp}} = T$.

Proof. (i): By using the closeness of ran (Id $-R$) and ...

$$\begin{aligned}\operatorname{ran}(\operatorname{Id} - R)^* &= \operatorname{ran}(\operatorname{Id} - R^*)\\ &= \operatorname{ran}(\operatorname{Id} - R)\\ &= D^{\perp}.\end{aligned}$$

(ii): This is clear because T is defined using $\mathrm{P}_{D^{\perp}}$, which is a linear and continuous operator. (iii): See [4, Example 20.12]. (iv): Combine (ii), (iii) and [4, Corollary 20.28]. (v): It follows directly from (8). (vi): Since $\operatorname{ran} T \subseteq D^{\perp}$ by using (v), we obtain

$$\mathrm{P}_{D^{\perp}} T = T.$$

Moreover, both T and $\mathrm{P}_{D^{\perp}}$ commute and so

$$T\mathrm{P}_{D^{\perp}} = \mathrm{P}_{D^{\perp}} T = T.$$

■

Remark 2. *It is well known that* ran (Id $-R$) *is closed if and only if there exists* $\alpha > 0$ *such that*

$$\left(\forall y \in (\ker(\operatorname{Id} - R))^{\perp} = D^{\perp}\right) \ \|y - Ry\| \geq \alpha\|y\|; \tag{9}$$

Proof. See [13, Theorem 8.18]. ■

Proposition 2. *Suppose that* (9) *holds, then the operator*

$$\mathrm{P}_{D^{\perp}}(\operatorname{Id} - R)^{-1} : D^{\perp} \to D^{\perp}, \tag{10}$$

(i) *is a linear selection of* T^{-1}.
(ii) *is continuous and its norm is bounded above by* $1/\alpha$.

Proof. (i): It follows from (8) and Lemma 2. (ii): Clear from (9). ■

Theorem 1. *Suppose that* ran $(\mathrm{Id} - R)$ *is closed. Set*

$$A := (\mathrm{Id} - R)^{-1} - \frac{1}{2}\,\mathrm{Id}, \tag{11}$$

and defined

$$Q_A : \operatorname{dom} A \to X : y \mapsto \mathrm{P}_{Ay}\, y. \tag{12}$$

Set

$$B := \mathrm{P}_{\operatorname{dom} A}\, Q_A\, \mathrm{P}_{\operatorname{dom} A}\,. \tag{13}$$

Then the following holds;

(i) $\operatorname{dom} A = D^{\perp}$ *and is closed.*

(ii) A *is linear relation.*

(iii) A *is maximally monotone.*

(iv) we have

$$(\forall y \in \operatorname{dom} A)\;\; Q_A y = \mathrm{P}_{D^{\perp}}(\mathrm{Id} - R)^{-1} y - \frac{1}{2}\,\mathrm{P}_{D^{\perp}}\, y.$$

(v) B *is maximally monotone, linear and continuous.*

(vi) $A = N_{D^{\perp}} + B$*.*

(vii) $B = T$*.*

(viii) $B|_{\operatorname{dom} A}$ *is a selection of* $A|_{\operatorname{dom} A}$*.*

Proof. (i): From (11) $\operatorname{dom} A = \operatorname{ran}(\mathrm{Id} - R) = D^{\perp}$, which is closed by the assumption.
(ii): It is clear that A is a linear relation, i.e., gra A is a linear subspace, that $A0 = D$, and by (i) the $\operatorname{dom} A = D^{\perp}$ is closed.
(iii): It follows directly from Proposition 1(ix).
(iv): By [9, Proposition 6.2], we have $\big(\forall y \in \operatorname{dom} A\big)$ $Q_A y = \mathrm{P}_{(A0)^{\perp}}(Ay) \in Ay$. Hence,

$$\begin{aligned}\big(\forall y \in D^{\perp}\big)\;\; Q_A y &= \mathrm{P}_{D^{\perp}}(Ay) = \mathrm{P}_{D^{\perp}}\Big((\mathrm{Id} - R)^{-1} y - \frac{1}{2} y\Big)\\ &= \mathrm{P}_{D^{\perp}}(\mathrm{Id} - R)^{-1} y - \frac{1}{2}\,\mathrm{P}_{D^{\perp}}\, y.\end{aligned}$$

(v): See [9, Example 6.4(i)]. (vi): Combining (i) and [9, Example 6.4(iii)] gives

$$A = N_{\operatorname{dom} A} + B = N_{D^{\perp}} + B.$$

(vii): Using (13), (iv) and (i) gives

$$\begin{aligned}B &= \mathrm{P}_{\operatorname{dom} A}\, Q_A\, \mathrm{P}_{\operatorname{dom} A}\\ &= \mathrm{P}_{D^{\perp}}\Big(\mathrm{P}_{D^{\perp}}(\mathrm{Id} - R)^{-1} - \frac{1}{2}\,\mathrm{P}_{D^{\perp}}\Big)\mathrm{P}_{D^{\perp}}\\ &= \mathrm{P}_{D^{\perp}}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^{\perp}} - \frac{1}{2}\,\mathrm{P}_{D^{\perp}}\end{aligned}$$

$$= T \quad \text{(from (8))}.$$

(viii): Using (i) gives

$$\begin{aligned}
A|_{\operatorname{dom} A} &= (N_{D^\perp} + B)|_{D^\perp} \quad \text{(from (vi))} \\
&= (N_{D^\perp} + T)|_{D^\perp} \quad \text{(from (vii))} \\
&= \Big(N_{D^\perp} + \mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\mathrm{P}_{D^\perp} - \frac{1}{2}\mathrm{P}_{D^\perp}\Big)\Big|_{D^\perp} \quad \text{(from (8))} \\
&\equiv D + D^\perp \quad (\text{because } N_{D^\perp}|_{D^\perp} \equiv D),
\end{aligned}$$

and

$$\begin{aligned}
B|_{\operatorname{dom} A} &= T|_{D^\perp} \quad \text{(from (i) and (vii))} \\
&= \Big(\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\mathrm{P}_{D^\perp} - \frac{1}{2}\mathrm{P}_{D^\perp}\Big)\Big|_{D^\perp} \quad \text{(from (8))} \\
&= D^\perp.
\end{aligned}$$

Hence, $B|_{\operatorname{dom} A}$ is a selection of $A|_{\operatorname{dom} A}$. ■

In the next theorem we derive formulas for the inverse and Moore-Penrose inverse of the operator $(\mathrm{Id} - R)$.

Theorem 2. *Recall from* (8) *and* (11) *that*

$$T := \mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\mathrm{P}_{D^\perp} - \frac{1}{2}\mathrm{P}_{D^\perp},$$

and

$$A := (\mathrm{Id} - R)^{-1} - \frac{1}{2}\mathrm{Id},$$

respectively.Then the following holds;

(i) The set-valued inverse of $\mathrm{Id} - R$ *is*

$$(\mathrm{Id} - R)^{-1} = \frac{1}{2}\mathrm{Id} + T + N_{D^\perp}. \tag{14}$$

(ii) The Moore-Penrose inverse of $\mathrm{Id} - R$ *is*

$$(\mathrm{Id} - R)^\dagger = T + \frac{1}{2}\mathrm{P}_{D^\perp}. \tag{15}$$

Proof. (i): Combining Theorem 1(vi) & (vii) and (11) gives

$$\begin{aligned}
(\mathrm{Id} - R)^{-1} - \frac{1}{2}\mathrm{Id} &= A \\
&= N_{D^\perp} + T,
\end{aligned}$$

Hence,

$$(\operatorname{Id} - R)^{-1} = N_{D^\perp} + T + \frac{1}{2}\operatorname{Id}.$$

(ii): By using [6, Proposition 2.1] and we obtain

$$\begin{aligned}
(\operatorname{Id} - R)^{\dagger} &= \mathrm{P}_{(\operatorname{Id} - R)^*} \circ (\operatorname{Id} - R)^{-1} \circ \mathrm{P}_{\operatorname{ran}(\operatorname{Id} - R)} \\
&= \mathrm{P}_{D^\perp} \circ (\operatorname{Id} - R)^{-1} \circ \mathrm{P}_{D^\perp} \quad \text{(from Lemma 2(i))} \\
&= \mathrm{P}_{D^\perp} \circ \Big(\frac{1}{2}\operatorname{Id} + T + N_{D^\perp}\Big) \circ \mathrm{P}_{D^\perp} \quad \text{(from (i))} \\
&= \mathrm{P}_{D^\perp} \circ \Big(\frac{1}{2}\mathrm{P}_{D^\perp} + T\,\mathrm{P}_{D^\perp} + D\Big) \quad \text{(Because } N_{D^\perp}|_{D^\perp} \equiv D) \\
&= \frac{1}{2}\mathrm{P}_{D^\perp} + \mathrm{P}_{D^\perp}\, T\, \mathrm{P}_{D^\perp} + 0 \\
&= \frac{1}{2}\mathrm{P}_{D^\perp} + T \quad \text{(from Lemma 2(vi))},
\end{aligned}$$

which verified (15). ■

Proposition 3 (uniqueness of T). *Let $T_\circ : X \to X$ be such that*

$$(\operatorname{Id} - R)^{-1} = \frac{1}{2}\operatorname{Id} + T_\circ + N_{D^\perp}, \tag{16}$$

and

$$\mathrm{P}_{D^\perp}\, T_\circ\, \mathrm{P}_{D^\perp} = T_\circ. \tag{17}$$

Then $T_\circ = T$.

Proof. By using (8), we have

$$\begin{aligned}
T &= \mathrm{P}_{D^\perp}(\operatorname{Id} - R)^{-1}\mathrm{P}_{D^\perp} - \frac{1}{2}\mathrm{P}_{D^\perp} \\
&= \mathrm{P}_{D^\perp}\Big(\frac{1}{2}\operatorname{Id} + T_\circ + N_{D^\perp}\Big)\mathrm{P}_{D^\perp} - \frac{1}{2}\mathrm{P}_{D^\perp} \quad \text{(from (16))} \\
&= \mathrm{P}_{D^\perp}\, T_\circ\, \mathrm{P}_{D^\perp} \\
&= T_\circ \quad \text{(from (17))},
\end{aligned}$$

as claimed. ■

Theorem 3. *Recall from* (8) *that*

$$T = \mathrm{P}_{D^\perp}(\operatorname{Id} - R)^{-1}\mathrm{P}_{D^\perp} - \frac{1}{2}\mathrm{P}_{D^\perp}.$$

Then the following holds;

(i) $(1/2)\operatorname{Id} + T$ *is* $\frac{1}{2}$*-strongly monotone.*

(ii) $\big((1/2)\operatorname{Id} + T\big)^{-1} = 2J_{2T}$.

(iii) $2T + \mathrm{Id} = 2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} + \mathrm{P}_D$.

(iv) $J_{2T} = \mathrm{P}_D + \frac{1}{2}(\mathrm{Id} - R)\,\mathrm{P}_{D^\perp}$.

(v) $2J_{2T} = (\mathrm{Id} - R)\,\mathrm{P}_{D^\perp} + 2\,\mathrm{P}_D$.

(vi) $(\mathrm{Id} - R)\,\mathrm{P}_{D^\perp} + 2\,\mathrm{P}_D = \mathrm{Id} - R + 2\,\mathrm{P}_D$.

(vii) *We have*

$$\Big(\frac{1}{2}\,\mathrm{Id} + T\Big)^{-1} = 2J_{2T} = (\mathrm{Id} - R)\,\mathrm{P}_{D^\perp} + 2\,\mathrm{P}_D = \mathrm{Id} - R + 2\,\mathrm{P}_D\,. \tag{18}$$

(viii) $\Big(\frac{1}{2}\,\mathrm{Id} + T\Big)^{-1}\Big|_{D^\perp} = \mathrm{Id} - R$.

Proof. (i): Showing that $\frac{1}{2}\,\mathrm{Id} + T$ is $(1/2)$-strongly monotone $\Leftrightarrow$ $\frac{1}{2}\,\mathrm{Id} + T - \frac{1}{2}\,\mathrm{Id} = T$ is montone, which is verified by Lemma 2(iii). (ii): From Lemma 2(ii) & (iv) and [14, Lemma 2], we have

$$\begin{aligned}\Big(\frac{1}{2}\,\mathrm{Id} + T\Big)^{-1} &= \Big(\frac{1}{2}(\mathrm{Id} + 2T)\Big)^{-1}\\ &= 2\big(\,\mathrm{Id} + 2T\big)^{-1}\\ &= 2J_{2T}.\end{aligned}$$

(iii): By using (8) and Lemma 2(ii), we obtain

$$\begin{aligned}2T &= 2\Big(\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} - \frac{1}{2}\,\mathrm{P}_{D^\perp}\Big)\\ &= 2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} - \mathrm{P}_{D^\perp},\end{aligned}$$

hence

$$\begin{aligned}2T + \mathrm{Id} &= 2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} - \mathrm{P}_{D^\perp} + \mathrm{Id}\\ &= 2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} + \mathrm{P}_D\,.\end{aligned}$$

(iv): From (iii), we obtain $2T + \mathrm{Id} = 2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} + \mathrm{P}_D$. Put differently,

$$2T + \mathrm{Id} : D \oplus D^\perp \to D \oplus D^\perp : d \oplus d^\perp \mapsto d + 2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1} d^\perp.$$

For two vectors $d^\perp$, $e^\perp$ in $D^\perp$, we have the equivalences,

$$e^\perp = 2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1} d^\perp \Leftrightarrow d^\perp = \Big(2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\Big)^{-1} e^\perp,$$

and therefore,

$$\begin{aligned}d^\perp &= \Big(2\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\Big)^{-1} e^\perp\\ &= \Big(2\big(\,\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\big)\Big)^{-1} e^\perp\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\Big)^{-1} e^\perp \\
&= \frac{1}{2}\big(\mathrm{Id} - R\big)\, \mathrm{P}_{D^\perp}^{-1}\, e^\perp \\
&= \frac{1}{2}\big(\mathrm{Id} - R\big) e^\perp.
\end{aligned}
$$

Hence,

$$(2T + \mathrm{Id})^{-1} : D \oplus D^\perp \to D \oplus D^\perp : d \oplus d^\perp \mapsto d + \frac{1}{2}(\mathrm{Id} - R) d^\perp;$$

equivalently,

$$J_{2T} = (2T + \mathrm{Id})^{-1} : z \mapsto \mathrm{P}_D\, z + \frac{1}{2}(\mathrm{Id} - R)\, \mathrm{P}_{D^\perp} z.$$

(v): It follows directly from (iv). (vi): Because $\ker(\mathrm{Id} - R) = D$, we have $(\mathrm{Id} - R)\,\mathrm{P}_D \equiv 0$. Therefore,

$$(\mathrm{Id} - R)\, \mathrm{P}_{D^\perp} + 2\,\mathrm{P}_D = \mathrm{Id} - R + 2\,\mathrm{P}_D\,.$$

(vii): Combine (ii), (v), and (vi). (viii): From (v), we obtain

$$\Big(\frac{1}{2}\,\mathrm{Id} + T\Big)^{-1}\Big|_{D^\perp} = \big(\mathrm{Id} - R + 2\,\mathrm{P}_D\,\big)\big|_{D^\perp} = \mathrm{Id} - R.$$

■

Proposition 4. *Let $m \in \{2, 3, \dots\}$ and assume that $R^m = \mathrm{Id}$, i.e., R is an isometry of finite rank m. Assume that $X = R^m$ and recall from [2, Lemma] that*

$$\mathrm{P}_D = \frac{1}{m}\sum_{k=0}^{m-1} R^k \quad \text{and} \quad \mathrm{P}_{D^\perp} = \mathrm{Id} - \frac{1}{m}\sum_{k=0}^{m-1} R^k, \tag{19}$$

where $D = \mathrm{Fix}\, R$. Then

$$\frac{1}{2}\,\mathrm{P}_{D^\perp}\,(R + R^*)\,\mathrm{P}_{D^\perp} = \frac{1}{m}\Big(-\mathrm{Id} - \sum_{k=2}^{m-2} R^k + \frac{\max\{1, m-2\}}{2}\big(R + R^{m-1}\big)\Big). \tag{20}$$

Proof. Noted that R is an isometry $\Rightarrow R^*R = RR^* = \mathrm{Id}$, so $R^{-1} = R^*$. But also R has rank m, hence $R^{m-1} = R^{-1} = R^*$. By using these facts, we obtain

$$
\begin{aligned}
\mathrm{P}_{D^\perp}(R + R^*)\,\mathrm{P}_{D^\perp} &= \mathrm{P}_{D^\perp}\,\big(R + R^{-1}\big)\,\mathrm{P}_{D^\perp} \\
&= \mathrm{P}_{D^\perp}\,\big(R + R^{-1}\big)\Big(\mathrm{Id} - \frac{1}{m}\sum_{k=0}^{m-1} R^k\Big) \quad \text{(from (19))} \\
&= \mathrm{P}_{D^\perp}\,\Big(\big(R + R^{-1}\big) - \frac{1}{m}\sum_{k=0}^{m-1}\big(R + R^{-1}\big)R^k\Big) \\
&= \mathrm{P}_{D^\perp}\,\Big(\big(R + R^{-1}\big) - \frac{1}{m}\sum_{k=0}^{m-1}\big(R^{k+1} + R^{k-1}\big)\Big).
\end{aligned}
$$

Since R has rank m, the following holds:

$$\sum_{k=0}^{m-1} R^{k+1} = \sum_{k=0}^{m-1} R^{k-1} = \sum_{k=0}^{m-1} R^{k}. \tag{21}$$

Moreover,

$$R^{l}\sum_{k=0}^{m-1} R^{k} = \sum_{k=0}^{m-1} R^{l+k} = \sum_{k=0}^{m-1} R^{k} \tag{22}$$

Thus,

$$\left(R+R^{-1}\right)\Big(\frac{1}{m}\sum_{k=0}^{m-1} R^{k}\Big) = \frac{1}{m}\sum_{k=0}^{m-1}\left(R^{k+1}+R^{k-1}\right) = \frac{2}{m}\sum_{k=0}^{m-1} R^{k}. \tag{23}$$

Therefore,

$$\begin{aligned}
\mathrm{P}_{D^{\perp}}\left(R+R^{*}\right)\mathrm{P}_{D^{\perp}} &= \mathrm{P}_{D^{\perp}}\left(\left(R+R^{-1}\right) - \frac{1}{m}\sum_{k=0}^{m-1}\left(R^{k+1}+R^{k-1}\right)\right) \\
&= \mathrm{P}_{D^{\perp}}\left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) \\
&= \left(\mathrm{Id} - \frac{1}{m}\sum_{k=0}^{m-1} R^{k}\right)\left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) \\
&= \left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) - \left(\frac{1}{m}\sum_{k=0}^{m-1} R^{k}\right)\left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) \\
&= \left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) - \left(\frac{2}{m}\sum_{k=0}^{m-1} R^{k} - \frac{2}{m^{2}}\sum_{l=0}^{m-1} R^{l}\sum_{k=0}^{m-1} R^{k}\right) \\
&= \left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) - \left(\frac{2}{m}\sum_{k=0}^{m-1} R^{k} - \frac{2}{m^{2}}\sum_{l=0}^{m-1}\sum_{k=0}^{m-1} R^{k}\right) \\
&= \left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) - \left(\frac{2}{m}\sum_{k=0}^{m-1} R^{k} - \frac{2m}{m^{2}}\sum_{k=0}^{m-1} R^{k}\right) \\
&= \left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) - \left(\frac{2}{m}\sum_{k=0}^{m-1} R^{k} - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right) \\
&= \left(\left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}\right).
\end{aligned}$$

First: assume that $m > 2$. Therefore, $\max\{1, m-2\} = m-2$. Then

$$\mathrm{P}_{D^{\perp}}\left(R+R^{*}\right)\mathrm{P}_{D^{\perp}} = \left(R+R^{-1}\right) - \frac{2}{m}\sum_{k=0}^{m-1} R^{k}$$

$$
\begin{aligned}
&= \frac{2}{m}\left(\frac{m}{2}\left(R+R^{-1}\right)-\sum_{k=0}^{m-1} R^k\right)\\
&= \frac{2}{m}\left(\left(\frac{m}{2}-1\right)\left(R+R^{-1}\right)-\mathrm{Id}-\sum_{k=2}^{m-2} R^k\right)\\
&= \frac{2}{m}\left(-\mathrm{Id}+\frac{m-2}{2}\left(R+R^{m-1}\right)-\sum_{k=2}^{m-2} R^k\right),
\end{aligned}
$$

which prove (20) when $m > 2$.
Next, assume that $m = 2$. Then $\max\{1, m-1\} = 1$ and $R^{-1} = R^{2-1} = R$. Therefore,

$$
\begin{aligned}
\mathrm{P}_{D^\perp}\left(R+R^*\right)\mathrm{P}_{D^\perp} &= \left(R+R^{-1}\right)-\frac{2}{m}\sum_{k=0}^{m-1} R^k\\
&= 2R-\frac{2}{2}\left(\mathrm{Id}+R\right)\\
&= 2R-\mathrm{Id}-R\\
&= R-\mathrm{Id}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\frac{2}{m}\left(-\mathrm{Id}+\frac{\max\{1,m-2\}}{2}\left(R+R^{m-1}\right)-\sum_{k=2}^{m-2} R^k\right) &= \frac{2}{2}\left(-\mathrm{Id}+\frac{1}{2}\left(R+R\right)-\sum_{k=2}^{0} R^k\right)\\
&= -\mathrm{Id}+\frac{1}{2}(2R)-0\\
&= -\mathrm{Id}+R,
\end{aligned}
$$

so equality holds when $m = 2$. ■

3. Examples

Example 1 (isometry of finite rank). *Let $m \in \{2, 3, \dots\}$ and assume that*

$$R^m = \mathrm{Id}. \tag{24}$$

Then the results in Section 2 were derived already in [1]. Moreover, the work there based on exploiting (24) *yielded to* (19) *and*

$$T = \frac{1}{2m}\sum_{k=1}^{m-1}\left(m-2k\right)R^k = -T^*, \tag{25}$$

which is always skew right-shift operator, T is symmetric only when $m = 2$.

Example 2. *Let U be a closed subspace of X and suppose that*

$$R = \mathrm{P}_U\,. \tag{26}$$

Then

(i) $D = U$.
(ii) $\mathrm{Id} - R = \mathrm{P}_{U^\perp}$.
(iii) $\operatorname{ran}(\mathrm{Id} - R) = D^\perp$ *is closed.*
(iv) $\big(\mathrm{Id} - R\big)^{-1} = \mathrm{Id} + N_U$.
(v) $T = \frac{1}{2}\mathrm{P}_{U^\perp} = T^*$.
(vi) T *is always symmetric, but skew only when* $U = X$.

Proof. (i): $D = \operatorname{Fix} R = \operatorname{Fix} \mathrm{P}_U = \{x \in X \mid x = \mathrm{P}_U x\} = U$. (ii): $\mathrm{Id} - R = \mathrm{Id} - \mathrm{P}_U = \mathrm{P}_{U^\perp}$. (iii): By using (ii), we obtain $\operatorname{ran}(\mathrm{Id} - R) = \operatorname{ran}(\mathrm{Id} - \mathrm{P}_U) = U^\perp = D^\perp$. (iv): From [4, Example 1], we have $\big(\mathrm{Id} - R\big)^{-1} = \big(\mathrm{Id} - \mathrm{P}_U\big)^{-1} = \mathrm{P}_{U^\perp}^{-1} = \mathrm{Id} + N_{U^\perp}$.
(v): By using (8), we have

$$\begin{aligned} T &= \mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} - \frac{1}{2}\,\mathrm{P}_{D^\perp} \\ &= \mathrm{P}_{U^\perp}(\mathrm{Id} + N_{U^\perp})\,\mathrm{P}_{U^\perp} - \frac{1}{2}\,\mathrm{P}_{U^\perp} \\ &= \frac{1}{2}\,\mathrm{P}_{U^\perp} \\ &= T^*. \end{aligned}$$

(vi): Follows from (v). ■

Example 3. *Let U be a closed subspace of X and suppose that*

$$R = -\,\mathrm{P}_U\,. \tag{27}$$

Then

(i) $D = \{0\}$.
(ii) $\mathrm{Id} - R = \mathrm{Id} + \mathrm{P}_U$.
(iii) $\operatorname{ran}(\mathrm{Id} - R) = X$.
(iv) $\big(\mathrm{Id} - R\big)^{-1} = \frac{1}{2}\mathrm{Id} + \frac{1}{2}\mathrm{P}_{U^\perp}$.
(v) $T = \frac{1}{2}\mathrm{P}_U$.

Proof. (i): $D = \operatorname{Fix} R = \operatorname{Fix}(-\mathrm{P}_U) = \{x \in X \mid x = -\mathrm{P}_U x\} = \{0\}$. (ii): $\mathrm{Id} - R = \mathrm{Id} + \mathrm{P}_U$. (iii): By [4, Minty Theorem], $Id + \mathrm{P}_U$ has full range $D = X$. (iv): $\big(\mathrm{Id} - R\big)^{-1} = J_{\mathrm{P}_U} = \frac{1}{2}\mathrm{P}_U + \mathrm{P}_{U^\perp} = \frac{1}{2}\mathrm{Id} + \frac{1}{2}\mathrm{P}_{U^\perp}$. (v): We have

$$\begin{aligned} T &= \mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} - \frac{1}{2}\,\mathrm{P}_{D^\perp} \\ &= \frac{1}{2}\,\mathrm{Id} + \frac{1}{2}\,\mathrm{Id} - \frac{1}{2}\,\mathrm{P}_{U^\perp} - \frac{1}{2}\,\mathrm{Id} \end{aligned}$$

$$= \frac{1}{2} \mathrm{P}_U .$$

■

Example 4. *Let U be a closed subspace of X and suppose that*

$$R = R_U. \tag{28}$$

Then

(i) $D = U$.
(ii) $\mathrm{Id} - R = 2\,\mathrm{P}_{U^\perp}$.
(iii) $\mathrm{ran}\,(\mathrm{Id} - R) = D^\perp$ *is closed.*
(iv) $\big(\mathrm{Id} - R\big)^{-1} = \frac{1}{2}\,\mathrm{Id} + N_U$.
(v) $T = 0$.

Proof. (i): $D = \mathrm{Fix}\,R = \mathrm{Fix}(R_U) = \{x \in X \mid x = R_U x\} = \{x \in X \mid 2x = 2\,\mathrm{P}_U\} = U$. (ii): $\mathrm{Id} - R = \mathrm{Id} - R_U = \big(\mathrm{P}_U + \mathrm{P}_{U^\perp}\big) - \big(\mathrm{P}_U - \mathrm{P}_{U^\perp}\big) = 2\,\mathrm{P}_{U^\perp}$. (iii): $\mathrm{ran}\,(\mathrm{Id} - R) = \mathrm{ran}\,(2\,\mathrm{P}_{U^\perp}) = D^\perp$ is closed. (iv): $\big(\mathrm{Id} - R\big)^{-1} = \big(2(\mathrm{Id} - \mathrm{P}_U)\big)^{-1} = \frac{1}{2}\,\mathrm{Id} + N_{U^\perp}$. (v): We have

$$\begin{aligned}
T &= \mathrm{P}_{D^\perp}(\mathrm{Id} - R)^{-1}\,\mathrm{P}_{D^\perp} - \frac{1}{2}\,\mathrm{P}_{D^\perp} \\
&= \mathrm{P}_{D^\perp}\left(\frac{1}{2}\,\mathrm{Id} + N_{U^\perp}\right)\mathrm{P}_{U^\perp} - \frac{1}{2}\,\mathrm{P}_{U^\perp} \\
&= \frac{1}{2}\,\mathrm{P}_{U^\perp} - \frac{1}{2}\,\mathrm{P}_{U^\perp} \\
&= 0.
\end{aligned}$$

■

Example 5. *Let U be a closed subspace of X and suppose that*

$$R = -R_U. \tag{29}$$

Then

(i) $D = \mathrm{Fix}\,\big(-R_U\big) = U^\perp$.
(ii) $\mathrm{Id} - R = 2\,\mathrm{P}_U$.
(iii) $\mathrm{ran}\,(\mathrm{Id} - R) = U$ *is closed.*
(iv) $\big(\mathrm{Id} - R\big)^{-1} = \frac{1}{2}\,\mathrm{Id} + N_U$.
(v) $T = 0$.

Proof. (i): Note that $-R_U = R_{U^\perp}$ and we learn from Example 4 that $D = \mathrm{Fix}\,R = U^\perp$. (ii): $\mathrm{Id} - R = \mathrm{Id} - R_{U^\perp} = \big(\mathrm{P}_U + \mathrm{P}_{U^\perp}\big) - \big(2\,\mathrm{P}_{U^\perp} - \mathrm{Id}\big) = \big(\mathrm{P}_U + \mathrm{P}_{U^\perp}\big) - \big(\mathrm{P}_{U^\perp} - \mathrm{P}_U\big) = 2\,\mathrm{P}_U$. (iii): By using (ii), we have $\mathrm{ran}\,\big(\mathrm{Id} - R\big) = \mathrm{ran}\,\big(2\,\mathrm{P}_U\big) = D = U$. (iv): $\big(\mathrm{Id} - R\big)^{-1} =$

$\left(\operatorname{Id}-(R_{U^\perp})\right)^{-1} = \left(\operatorname{Id}-(2\operatorname{P}_{U^\perp}-\operatorname{Id})\right)^{-1} = \left(2(\operatorname{Id}-\operatorname{P}_{U^\perp})\right)^{-1} = \frac{1}{2}\operatorname{Id}+N_U$ by [4, Example]. (v): By using (8), we have

$$\begin{aligned}
T &= \operatorname{P}_{D^\perp}(\operatorname{Id}-R)^{-1}\operatorname{P}_{D^\perp} - \frac{1}{2}\operatorname{P}_{D^\perp} \\
&= \operatorname{P}_{D^\perp}\left(\frac{1}{2}\operatorname{Id}+N_U\right)\operatorname{P}_{U^\perp} - \frac{1}{2}\operatorname{P}_{U^\perp} \\
&= \frac{1}{2}\operatorname{P}_{U^\perp} - \frac{1}{2}\operatorname{P}_{U^\perp} \\
&= 0.
\end{aligned}$$

■

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Clarification

Please note that a preprint has previously been published in arXiv and available in [3]. There is no conflict of interest and there is no data were used to support this study.

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