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# Compositions of Resolvents: Fixed Points Sets and Set of Cycles

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**Abstract.** In this paper, we investigate the cycles and fixed point sets of compositions of resolvents using Attouch–Théra duality. We demonstrate that the cycles defined by the resolvent operators can be formulated in Hilbert space as solutions to a fixed point equation. Furthermore, we introduce the relationship between these cycles and the fixed point sets of the compositions of resolvents.

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## 1. Introduction

Throughout, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle : X \times X \to \mathbb{R},$$

and induced norm $\| \cdot \| \colon X \to \mathbb{R} \colon x \mapsto \sqrt{\langle x, x \rangle}$. For more details about Hilbert space, we refere the redear to [10] and [13]. An operator $T : X \to X$ is *nonexpansive* if it is Lipschitz continuous with constant 1, i.e.,

$$\big(\forall x \in X\big)\big(\forall y \in X\big) \quad \|Tx - Ty\| \leq \|x - y\|. \tag{1}$$

Nonexpansive operators play a major role in optimization because the set of fixed points $\operatorname{Fix} R := \{x \in X \mid x = Rx\}$ usually represents solutions to inclusion problems and optimization tasks. For more details about nonexpansive operators and the fixed point set, we refer the reader to [1]-[6], [7]-[8], [11], [16], [17], and [2, Chapters 3 and 6]. Moreover, $T : D \to X$ is *firmly nonexpansive* if

$$\big(\forall x \in D\big)\big(\forall y \in D\big) \;\; \|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2. \tag{2}$$

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Firmly nonexpansive operators are also central due to their favorable convergence properties for iterates and their correspondence with maximal monotone operators. Recall that a set-valued operator $A : X \rightrightarrows X$ with graph gra $A$ is *monotone* if

$$\big(\forall (x,u) \in \operatorname{gra} A\big)\big(\forall (y,v) \in \operatorname{gra} A\big) \quad \langle x-y, u-v\rangle \geq 0.$$

Furthermore, $A$ is *maximally monotone* if there does not exist a monotone operator $B : X \rightrightarrows X$ such that gra $B$ properly contains gra $A$, i.e., for every $(x,u) \in X \times X$,

$$(x,u) \in \operatorname{gra} A \quad \Leftrightarrow \quad \big(\forall (y,v) \in \operatorname{gra} A\big) \;\; \langle x-y, u-v\rangle \geq 0.$$

It is well known that monotone and maximally monotone operators play central roles in various areas of modern nonlinear analysis. See [10], [14], [15]-[22], and [20] for background material. Let $A : X \rightrightarrows X$ be a maximally monotone operator and denote the associated *resolvent* by

$$J_A := (\operatorname{Id} + A)^{-1}. \tag{3}$$

In [21], Minty observed that $J_A$ is a firmly nonexpansive operator from $X$ to $X$. For more information about the relationship between firmly nonexpansive mappings and maximally monotone operators, see [12]. The Hilber product space,

$$\mathbf{X} = \Big\{ \mathbf{x} = (x_i)_{i\in I} \;\Big|\; (\forall i \in I)\; x_i \in X \Big\},$$

where $m \in \{2, 3, \dots\}$ and $i = \{1, 2, \dots, m\}$. Let

$$A_i : X \rightrightarrows X \text{ be maximally monotone operators,} \tag{4}$$

with resolvents $\mathrm{J}_{A_1}, \mathrm{J}_{A_2}, \dots, \mathrm{J}_{A_m}$ which we also write more simply as $\mathrm{J}_1, \mathrm{J}_2, \dots, \mathrm{J}_m$. Set

$$\mathbf{A} = A_1 \times A_2 \times \cdots \times A_m. \tag{5}$$

Then

$$\mathrm{J}_{\mathbf{A}} : \mathbf{X} \to \mathbf{X} : (x_1, x_2, \dots, x_m) \mapsto (\mathrm{J}_1 x_1, \mathrm{J}_2 x_2, \dots, \mathrm{J}_m x_m). \tag{6}$$

Define the *circular right-shift operator*

$$\mathbf{R} : (x_1, x_2, \dots, x_m) \mapsto (x_m, x_1, x_2, \dots, x_{m-1}). \tag{7}$$

Define the *fixed point sets of the cyclic compositions of resolvants*:

$$F_1 := \operatorname{Fix}(\mathrm{J}_1 \mathrm{J}_m \dots \mathrm{J}_2), \tag{8}$$

$$F_2 := \operatorname{Fix}(\mathrm{J}_2 \mathrm{J}_1 \mathrm{J}_m \dots \mathrm{J}_3), \tag{9}$$

$$\vdots \tag{10}$$

$$F_m := \operatorname{Fix}(\mathrm{J}_m \mathrm{J}_{m-1} \dots \mathrm{J}_1). \tag{11}$$

The prospects of applying the compositions of resolvents in practical applications are broad and significant, particularly in fields such as optimization, control theory, and mathematical analysis. Here are some key areas where these applications are emerging:

1. In the area of Optimization and Control: Compositions of resolvents are crucial in optimization problems, particularly in convex optimization and monotone inclusion problems. They provide a framework for developing algorithms that can efficiently find solutions to complex optimization tasks. For instance, the resolvent composition is a monotonicity-preserving operation that can be linked to proximal compositions, which are essential in convex analysis. This relationship allows for the relaxation of monotone inclusion problems, making it easier to solve them in practical scenarios. See [18].
2. In the area of Equilibrium Problems: The compositions of resolvents encapsulate known concepts and introduce new operations that are pertinent to equilibrium problems. This is particularly relevant in economic models and game theory, where finding equilibria is essential. The properties established in the study of resolvent compositions can lead to new insights and methods for analyzing these problems [18].
3. Applications in Fluid Dynamics: In fluid dynamics, the mean resolvent operator has been used to analyze the stability of flows and predict the behavior of turbulent systems. The application of resolvent compositions in this context can enhance our understanding of flow dynamics and improve control strategies for various engineering applications [19].
4. In the area of Signal Processing and Data Analysis: Resolvent compositions can also be applied in signal processing, particularly in filtering and data reconstruction techniques. By leveraging the mathematical properties of resolvents, engineers can develop more effective algorithms for noise reduction and signal enhancement, which are critical in communications and multimedia applications [18].

The compositions of resolvents hold significant promise for practical applications across various fields, including optimization, control theory, fluid dynamics, and signal processing. As research continues to explore these compositions, we can expect to see innovative solutions and methodologies that leverage their mathematical properties to address complex real-world problems.

**Definition 1.** [2, Definition 5.1] Let $z_1 \in F_1$. Set $z_2 := \mathrm{J}_2 z_1$, $z_3 := \mathrm{J}_3 z_2$, $\cdots$, $z_{m-1} := \mathrm{J}_{m-1} z_{m-2}$, and $z_m := \mathrm{J}_m z_{m-1}$. The truple $\mathbf{z} = \big(z_1, z_2, \ldots, z_m\big) \in \mathbf{X}$ is called *a cycle*.

The notation used in the paper is standard and follows largely, e.g., [2] and [10].

## 2. Aim and outline of this paper

Our main results can be summarized as follows:

- Theorem 1 and Theorem 2 sketche the relationship between the cycles and the fixed point sets of the composition of resolvants.
- The cycles that are defined by the resolvant operators can be formulated in Hilbert product space as a solution to a fixed point equation (see Lemma 2 and Lemma 4).

- We study the set of classical cycles that are defined by using resolvant operators and the set of classical gap vectors see Theorem 4.
- If one of the fixed point sets of composition of resolvents is not empty, then the individuals fixed point sets are equal and their intersection is not empty (see Lemma 5).
- In Section 5, we use Attouch–Théra duality to study the cycles and the fixed point sets of compositions of resolvents operators.

Approach of this paper is novel as it utilizes Attouch–Théra duality to conduct an in-depth investigation of the cycles and fixed point sets related to compositions of resolvents. This duality offers a powerful framework for uncovering the intricate structures and dynamics inherent in these mathematical constructs. In summary, applying Attouch–Théra duality to analyze cycles and fixed point sets in resolvent compositions represents a significant advancement in the field. More information about Attouch–Théra duality is in the next section.

## 3. Attouch–Théra duality

Let $A$ and $B$ be two maximally monotone operators on $X$. The *primal problem* associated with $\big(A, B\big)$ is to

$$\text{find } x \in X \text{ such that } 0 \in Ax + Bx. \tag{12}$$

The set of *primal solutions* associated with $\big(A, B\big)$ are the solutions to the corresponding sum problem (12) are defined as

$$\operatorname{psol}(A, B) := \operatorname{zer}(A, B) = (A, B)^{-1}(0) = \Big\{x \in X \;\Big|\; 0 \in (A + B)x\Big\}. \tag{13}$$

Now define $B^{\circledcirc} := (-\operatorname{Id}) \circ B \circ (-\operatorname{Id})$ and $B^{-\circledcirc} := (B^{-1})^{\circledcirc} = (B^{\circledcirc})^{-1}$. This allows us to define the *dual pair* of $(A, B)$:

$$(A, B)^{*} := (A^{-1}, B^{-\circledcirc}). \tag{14}$$

Then the *dual problem* associated with $(A, B)$ is defined to be the primal problem associated with the dual pair $(A^{-1}, B^{-\circledcirc})$:

$$\text{find } y \in X \text{ such that } 0 \in A^{-1}y + B^{-\circledcirc}y = A^{-1}y - B^{-1}(-y). \tag{15}$$

The set of of *dual solutions* associated with $\big(A, B\big)$ are the solutions to the corresponding sum problem (15):

$$\operatorname{dsol}(A, B) := \operatorname{psol}(A, B)^{*} = \operatorname{zer}\,(A^{-1} + B^{-\circledcirc}) = \Big\{y \in X \;\Big|\; 0 \in (A^{-1} + B^{-\circledcirc})y\Big\}. \tag{16}$$

Because $(A^{-1})^{-1} = A$, $(A^{\circledcirc})^{\circledcirc} = A$, and $(A^{-\circledcirc})^{-\circledcirc} = A$, we have

$$(A, B)^{**} = (A, B). \tag{17}$$

**Lemma 1.** Let $A$ and $B$ be maximally monotone on $X$. Let $x$ and $y$ in $X$. Then the following holds:

(i) If $\operatorname{psol}(A,B) = \{x\}$, then

$$\operatorname{dsol}(A,B) = Ax \cap (-Bx)$$

and

$$\operatorname{dsol}(A,B) = Ax \cap B^{\oslash}(-x).$$

(ii) If $\operatorname{dsol}(A,B) = \{y\}$, then

$$\operatorname{psol}(A,B) = (A^{-1}y) \cap B^{-1}(-y)$$

and

$$\operatorname{psol}(A,B) = (A^{-1}y) \cap (-B^{-\oslash}(y)).$$

(iii) If $\operatorname{psol}(A,B) = \{x\}$ and $Ax$ is a singelton, then $\operatorname{dsol}(A,B) = Ax$.

(iv) If $\operatorname{psol}(A,B) = \{x\}$ and $Bx$ and $B^{\oslash}(-x)$ are singelton, then

$$\operatorname{dsol}(A,B) = -Bx$$

and

$$\operatorname{dsol}(A,B) = B^{\oslash}(-x).$$

(v) If $\operatorname{dsol}(A,B) = \{y\}$ and $A^{-1}y$ is a singelton, then $\operatorname{psol}(A,B) = A^{-1}y$.

(vi) If $\operatorname{dsol}(A,B) = \{y\}$ and $B^{-1}(-y)$ and $(-B^{-\oslash}(y))$ are a singelton, then

$$\operatorname{psol}(A,B) = B^{-1}(-y)$$

and

$$\operatorname{psol}(A,B) = (-B^{-\oslash}(y)).$$

*Proof.* (i): From (13), it follows that

$$\begin{aligned} x \in \operatorname{psol}(A,B) &\Leftrightarrow (A+B)^{-1}(0) \neq \varnothing \\ &\Leftrightarrow \varnothing \neq Ax \cap (-Bx) \\ &\Leftrightarrow \varnothing \neq Ax \cap \Big((-\operatorname{Id}) \circ B \circ (-\operatorname{Id})(-x)\Big) \\ &\Leftrightarrow \varnothing \neq Ax \cap B^{\oslash}(-x) \\ &\Leftrightarrow \varnothing \neq Ax \cap B^{\oslash}(-x) \subseteq \operatorname{dsol}(A,B) \\ &\Leftrightarrow \varnothing \neq Ax \cap (-Bx) \subseteq \operatorname{dsol}(A,B). \end{aligned}$$

Since $\operatorname{psol}(A,B) = \{x\}$ and by using (13), it follows that $Ax \cap (-Bx) = \operatorname{dsol}(A,B)$ and $Ax \cap B^{\oslash}(-x) = \operatorname{dsol}(A,B)$.

(ii): From (16), it follows that

$$y \in \operatorname{dsol}(A,B) \Leftrightarrow (A^{-1} + B^{-\oslash})^{-1}(0) \neq \varnothing$$

$$\begin{aligned}
&\Leftrightarrow A^{-1}(y) \cap (-B^{-\ovee}(y)) \neq \varnothing \\
&\Leftrightarrow A^{-1}(y) \cap B^{-1}(-y) \neq \varnothing \\
&\Leftrightarrow \varnothing \neq A^{-1}(y) \cap B^{-1}(-y) \subseteq \mathrm{psol}(A, B).
\end{aligned}$$

Since $\mathrm{dsol}(A, B) = \{y\}$ and by using (16), it follows that

$$A^{-1}(y) \cap B^{-1}(-y) = \mathrm{psol}(A, B).$$

(iii): From (i), we have $Ax \cap (-Bx) = \mathrm{dsol}(A, B)$, and $Ax \cap B^{\ovee}(-x) = \mathrm{dsol}(A, B)$. Since $Ax$ is a singelton then we obtain

$$\mathrm{dsol}(A, B) = Ax \cap (-Bx) = Ax,$$

and

$$\mathrm{dsol}(A, B) = Ax \cap B^{\ovee}(-x) = Ax.$$

(iv): From (i), we have $Ax \cap (-Bx) = \mathrm{dsol}(A, B)$ and $Ax \cap B^{\ovee}(-x) = \mathrm{dsol}(A, B)$. Since $Bx$ and $B^{\ovee}(-x)$ are singelton, it follows that

$$\mathrm{dsol}(A, B) = Ax \cap (-Bx) = -Bx$$

and

$$\mathrm{dsol}(A, B) = Ax \cap B^{\ovee}(-x) = B^{\ovee}(-x).$$

(v): From (ii), we have $(A^{-1}y) \cap B^{-1}(-y) = \mathrm{psol}(A, B)$ and $(A^{-1}y) \cap (-B^{-\ovee}(y)) = \mathrm{dsol}(A, B)$. Since $A^{-1}y$ is a singelton, we obtain

$$\mathrm{psol}(A, B) = (A^{-1}y) \cap B^{-1}(-y) = A^{-1}y$$

and

$$\mathrm{psol}(A, B) = (A^{-1}y) \cap (-B^{-\ovee}(y)) = A^{-1}y.$$

(vi): From (ii), we have $(A^{-1}y) \cap B^{-1}(-y) = \mathrm{psol}(A, B)$ and $(A^{-1}y) \cap (-B^{-\ovee}(y)) = \mathrm{dsol}(A, B)$. Since $B^{-1}(-y)$ and $(-B^{-\ovee}(y))$ are singelton, we obtain

$$\mathrm{psol}(A, B) = (A^{-1}y) \cap B^{-1}(-y) = B^{-1}(-y)$$

and

$$\mathrm{psol}(A, B) = (A^{-1}y) \cap (-B^{-\ovee}(y)) = -B^{-\ovee}(y).$$

■

For more information about the Attouch–Théra duality, we refer the reader to [5].

## 4. Correspondence of Properties and Results

This section presents some of our key findings regarding the cycles and fixed point sets of resolvent compositions, beginning with an exploration of their interrelationship, as demonstrated in Theorem 1 and Theorem 2.

**Theorem 1.** The following are equivalent

(i) Cycle exists.

(ii) For all $1 \le i \le m$, the fixed point sets of cyclic compositions of resolvants $F_i \neq \emptyset$.

*Proof.* " (i)⇒(ii)": Let $\mathbf{z} = \left(z_1, z_2, \dots, z_{m-1}, z_m\right)$ be a cycle. Then by Definition 1, we have $z_1 = \mathrm{J}_1 z_m$, $z_2 = \mathrm{J}_2 z_1$, $z_3 = \mathrm{J}_3 z_2$, $\cdots$, $z_{m-1} = \mathrm{J}_{m-1} z_{m-2}$, and $z_m = \mathrm{J}_m z_{m-1}$. This gives that

$$\begin{aligned} z_1 &= \mathrm{J}_1 \mathrm{J}_m \dots \mathrm{J}_3 \mathrm{J}_2 z_1 \\ z_2 &= \mathrm{J}_2 \mathrm{J}_1 \dots \mathrm{J}_4 \mathrm{J}_3 z_2 \\ &\vdots \\ z_i &= \mathrm{J}_i \mathrm{J}_{i-1} \dots \mathrm{J}_1 \mathrm{J}_m \dots \mathrm{J}_{i+1} z_i \\ &\vdots \\ z_m &= \mathrm{J}_m \mathrm{J}_{m-1} \dots \mathrm{J}_2 \mathrm{J}_1 z_m. \end{aligned}$$

Therefore, $z_1 \in F_1$, $z_2 \in F_2$, $\dots$, $z_i \in F_i$, $\dots$, $z_m \in F_m$ by (8)-(11). This implies that $F_1 \neq \emptyset$, $F_2 \neq \emptyset, \dots, F_i \neq \emptyset, \dots, F_m \neq \emptyset$.

" (ii)⇒(i)": Let $F_m \neq \emptyset$ and $z_m \in F_m$. Then

$$z_m \in \operatorname{Fix}\left(\mathrm{J}_m \mathrm{J}_{m-1} \dots \mathrm{J}_2 \mathrm{J}_1\right) z_m,$$

by (11). Therefore,

$$z_m = \mathrm{J}_m \mathrm{J}_{m-1} \dots \mathrm{J}_2 \mathrm{J}_1 z_m.$$

Next, Applying $\mathrm{J}_1$ gives

$$\mathrm{J}_1 z_m = \mathrm{J}_1 \left(\mathrm{J}_m \mathrm{J}_{m-1} \dots \mathrm{J}_2\right) \left(\mathrm{J}_1 z_m\right),$$

which is equivalent to $\mathrm{J}_1 z_m \in \operatorname{Fix}\left(\mathrm{J}_1 \mathrm{J}_m \mathrm{J}_{m-1} \dots \mathrm{J}_3 \mathrm{J}_2\right) \Leftrightarrow \mathrm{J}_1 z_m \in F_1 \neq \emptyset$. Additionaly, let $z_2 \in F_2$ such taht $z_2 = \mathrm{J}_2 z_1$. Keep doing this gives,

$$z_{m-2} = \mathrm{J}_{m-2} \mathrm{J}_{m-3} \dots \mathrm{J}_1 \mathrm{J}_m \mathrm{J}_{m-1} z_{m-2}.$$

Then,

$$\mathrm{J}_{m-1} z_{m-2} = \mathrm{J}_{m-1} \left(\mathrm{J}_{m-2} \mathrm{J}_{m-3} \dots \mathrm{J}_1 \mathrm{J}_m\right) \left(\mathrm{J}_{m-1} z_{m-2}\right),$$

and $\mathrm{J}_{m-1} z_{m-2} \in \operatorname{Fix}\left(\mathrm{J}_{m-1} \mathrm{J}_{m-2} \mathrm{J}_{m-3} \dots \mathrm{J}_1 \mathrm{J}_m\right)$, which is equivalent to $\mathrm{J}_{m-1} \mathrm{J}_{m-2} \in F_{m-1} \neq \emptyset$. Moreover, let $z_{m-1} \in F_{m-1}$ such that $z_{m-1} = \mathrm{J}_{m-1} z_{m-2}$. It follows that

$$z_{m-1} = \mathrm{J}_{m-1} \mathrm{J}_{m-2} \mathrm{J}_{m-3} \dots \mathrm{J}_1 \mathrm{J}_m z_{m-1},$$

and

$$\mathrm{J}_m z_{m-1} = \mathrm{J}_m(\mathrm{J}_{m-1}\mathrm{J}_{m-2}\mathrm{J}_{m-3}\cdots\mathrm{J}_1)\mathrm{J}_m z_{m-1}.$$

Therefore,

$$\mathrm{J}_m z_{m-1} \in \operatorname{Fix}\left(\mathrm{J}_m\mathrm{J}_{m-1}\mathrm{J}_{m-2}\mathrm{J}_{m-3}\cdots\mathrm{J}_1\right).$$

This is equivalent to $\mathrm{J}_m z_{m-1} \in \operatorname{Fix} F_m \neq \varnothing$. All these together give $\left(z_1, z_2, \ldots, z_{m-1}, z_m\right) \in$ $\mathbf{X}$ satisfying that

$$\left(z_1, z_2, \ldots, z_{m-1}, z_m\right) = \left(\mathrm{J}_1 z_m, \mathrm{J}_2 z_1, \ldots, \mathrm{J}_{m-1} z_{m-2}, \mathrm{J}_m z_{m-1}\right).$$

■

**Lemma 2.** Let $\mathbf{z} \in \mathbf{X}$ is a cycle. Then

$$\mathbf{z} = \mathbf{J}_{\mathbf{A}}\left(\mathbf{R}\mathbf{z}\right). \tag{18}$$

Moreover, solving (18) is equivalent to solve

$$0 \in \mathbf{A}(\mathbf{z}) + \left(\mathrm{Id} - \mathbf{R}\right)(\mathbf{z}). \tag{19}$$

*Proof*. Given that $\mathbf{z} = \left(z_1, z_2, \ldots, z_m\right) \in \mathbf{X}$ is a cycle. Then Definition 1 gives that $z_1 = \mathrm{J}_1 z_m$, $z_2 = \mathrm{J}_2 z_1$, $z_3 = \mathrm{J}_3 z_2$, $\cdots$, $z_{m-1} = \mathrm{J}_{m-1} z_{m-2}$, and $z_m = \mathrm{J}_m z_{m-1}$. Hence,

$$\begin{aligned}
\mathbf{z} = \left(z_1, z_2, \ldots, z_{m-1}, z_m\right) &= \left(\mathrm{J}_1 z_m, \mathrm{J}_2 z_1, \ldots, \mathrm{J}_{m-1} z_{m-2}, \mathrm{J}_m z_{m-1}\right) \\
&= \left(\mathrm{J}_1, \mathrm{J}_2, \ldots, \mathrm{J}_{m-1}, \mathrm{J}_m\right)\left(z_m, z_1, \ldots, z_{m-2}, z_{m-1}\right) \\
&= \mathbf{J}_{\mathbf{A}}\left(\mathbf{R}\left(z_1, z_2, \ldots, z_{m-1}, z_m\right)\right) \\
&= \mathbf{J}_{\mathbf{A}}\left(\mathbf{R}\mathbf{z}\right).
\end{aligned}$$

Note that $\mathbf{z} = \mathbf{J}_{\mathbf{A}}(\mathbf{R}\mathbf{z}) \Leftrightarrow \mathbf{z} = \left(\mathbf{Id} + \mathbf{A}\right)^{-1}(\mathbf{R}\mathbf{z})$ by (3). Therefore, we obtain

$$\mathbf{R}\mathbf{z} \in \mathbf{z} + \mathbf{A}(\mathbf{z}) \Leftrightarrow 0 \in \mathbf{A}(\mathbf{z}) + \left(\mathbf{Id} - \mathbf{R}\right)(\mathbf{z}).$$

■

Define the set of all cycles by

$$\mathbf{Z} := \operatorname{Fix}(\mathbf{J}_{\mathbf{A}}\mathbf{R}). \tag{20}$$

Define

$$F_i := \left\{z \in X \mid z = \mathrm{J}_i \ldots \mathrm{J}_1 \mathrm{J}_m \ldots \mathrm{J}_{i+1} z\right\}. \tag{21}$$

Moreover,

$$Q_i : \mathbf{X} \to X : \mathbf{z} \mapsto z_i. \tag{22}$$

The relationship between the fixed point set of composition of $m$ resolvants $F_i$'s and the set of all cycles $\mathbf{Z}$ are given in the following theorem.

**Theorem 2.** For every $1 \leq i, j \leq m$, the following hold:

(i) $F_i$ are closed and convex. Moreover,

$$F_m = (J_m J_{m-1} \ldots J_3 J_2)(F_1) = (J_m J_{m-1} \ldots J_3)(F_2) = \cdots = J_m J_{m-1}(F_{m-2}) = J_m(F_{m-1}). \tag{23}$$

$$F_{m-1} = (J_{m-1} \ldots J_3 J_2 J_1)(F_m) = (J_{m-1} \ldots J_3 J_2)(F_1) = \cdots = J_{m-1}(F_{m-2}). \tag{24}$$

$$\vdots \tag{25}$$

$$F_2 = (J_2 J_1 J_m \ldots J_4)(F_3) = (J_2 J_1 J_m \ldots J_5)(F_4) = \cdots = J_2 J_1(F_m) = J_2(F_1). \tag{26}$$

$$F_1 = (J_1 J_m \ldots J_4 J_3)(F_2) = (J_1 J_m \ldots J_4)(F_3) = \cdots = (J_1 J_m)(F_{m-1}) = J_1(F_m). \tag{27}$$

(ii) $\cap_{i=1}^m \operatorname{Fix} J_i \subseteq \cap_{i=1}^m F_i$. If $F_i = \varnothing$, then $\cap_{i=1}^m \operatorname{Fix} J_i = \varnothing$.

(iii) For $1 \leq i \leq m-1$, $J_{i+1}(F_i) = F_{i+1}$ and $J_1(F_m) = F_1$. This implies that

$$J_{\mathbf{A}} \mathbf{R}(F_1 \times F_2 \times \cdots \times F_m) = F_1 \times F_2 \times \cdots \times F_m. \tag{28}$$

(iv) $F_i \neq \varnothing$ if and only if $F_j \neq \varnothing$ if and only if $\mathbf{Z} = \varnothing$.

(v) $\mathbf{Z}$ is closed and convex, and $\mathbf{Z} \subseteq F_1 \times F_2 \times \cdots \times F_m$.

(vi) The mapping $Q_i|_{\mathbf{Z}} : \mathbf{Z} \to F_i$ is bijective and $Q_i(\mathbf{Z}) = F_i$.

*Proof.* (i): Since each $J_i$ is firmly nonexpansive, it follows that $J_i$ is nonexpansive. Therefore, by [16, Lemma 2.1.12 (ii)], the composition

$$J_i \ldots J_1 J_m \ldots J_{i+1}$$

is also nonexpansive. As a result, $F_i$ is closed and convex by [16, Proposition 2.1.11].
Let $x \in F_m \Leftrightarrow x \in \operatorname{Fix}(J_m J_{m-1} \ldots J_3 J_2 J_1) \Leftrightarrow x = J_m J_{m-1} \ldots J_3 J_2 J_1 x$. Then,

$$J_1 x = J_1(J_m J_{m-1} \ldots J_3 J_2 J_1) x = (J_1 J_m J_{m-1} \ldots J_3 J_2)(J_1 x).$$

Therefore, $J_1 x \in \operatorname{Fix}(J_1 J_m J_{m-1} \ldots J_3 J_2) \Leftrightarrow J_1 x \in F_1$. It follows that

$$J_1(F_m) \subseteq F_1. \tag{29}$$

Moreover,

$$(J_2 J_1)(F_m) = (J_2 J_1)(\operatorname{Fix}(J_m J_{m-1} \ldots J_2 J_1)) \subseteq J_2(\operatorname{Fix}(J_1 J_m J_{m-1} \ldots J_3 J_2)) \tag{30}$$

$$\subseteq \operatorname{Fix}(J_2 J_1 J_m J_{m-1} \ldots J_4 J_3) = F_2, \tag{31}$$

hence

$$(J_3 J_2 J_1)(F_m) = (J_3 J_2 J_1)(\operatorname{Fix}(J_m J_{m-1} \ldots J_2 J_1)) \subseteq (J_3 J_2)(\operatorname{Fix}(J_1 J_m J_{m-1} \ldots J_3 J_2)) \tag{32}$$

$$\subseteq J_3(\operatorname{Fix}(J_2 J_1 J_m J_{m-1} \ldots J_4 J_3)) \tag{33}$$

$$\subseteq \operatorname{Fix}(J_3 J_2 J_1 J_m J_{m-1} \ldots J_5 J_4) = F_3, \tag{34}$$

until finally

$$F_m = \operatorname{Fix}(J_m J_{m-1} \ldots J_2 J_1 = (J_m J_{m-1} \ldots J_2 J_1)(\operatorname{Fix}(J_m J_{m-1} \ldots J_2 J_1)) \tag{35}$$

$$\subseteq \left(\mathrm{J}_m\mathrm{J}_{m-1}\dots\mathrm{J}_2\right)\left(\operatorname{Fix}\left(\mathrm{J}_1\mathrm{J}_m\dots\mathrm{J}_3\mathrm{J}_2\right)\right) \tag{36}$$

$$\vdots \tag{37}$$

$$\subseteq \mathrm{J}_m\left(\operatorname{Fix}\left(\mathrm{J}_{m-1}\mathrm{J}_{m-2}\dots\mathrm{J}_2\mathrm{J}_1\mathrm{J}_m\right)\right) = \mathrm{J}_m\left(F_{m-1}\right) \tag{38}$$

$$\subseteq \operatorname{Fix}\left(\mathrm{J}_m\mathrm{J}_{m-1}\dots\mathrm{J}_2\mathrm{J}_1\right) = F_m. \tag{39}$$

Hence, equality holds throughout (35) to (39), and we are done. The same approach will verify (24)-(27).
(ii): It is well known that $\cap_{i=1}^m \operatorname{Fix} \mathrm{J}_i \subseteq F_1$, $\cap_{i=1}^m \operatorname{Fix} \mathrm{J}_i \subseteq F_2$, $\cdots$, $\cap_{i=1}^m \operatorname{Fix} \mathrm{J}_i \subseteq F_m$. Hence

$$\bigcap_{i=1}^m \operatorname{Fix} \mathrm{J}_i \subseteq \bigcap_{i=1}^m F_i.$$

This also implies that $\cap_{i=1}^m \operatorname{Fix} \mathrm{J}_i = \varnothing$ if $F_i = \varnothing$. (iii): From (i), we have

$$\mathrm{J}_1\left(F_m\right) = F_1, \quad \mathrm{J}_2\left(F_1\right) = F_2, \quad \dots, \quad \mathrm{J}_m\left(F_{m-1}\right) = F_m. \tag{40}$$

Using (6), (7) and (40), we obtain

$$\begin{aligned}
\mathrm{J}_{\mathbf{A}}\mathbf{R}\left(F_1 \times F_2 \times \cdots \times F_{m-1} \times F_m\right) &= \mathrm{J}_{\mathbf{A}}\left(F_m \times F_1 \times F_2 \times \cdots \times F_{m-1}\right) \\
&= \left(\mathrm{J}_1, \mathrm{J}_2, \cdots, \mathrm{J}_m\right)\left(F_m \times F_1 \times F_2 \times \cdots \times F_{m-1}\right) \\
&= F_1 \times F_2 \times \cdots \times F_{m-1} \times F_m.
\end{aligned}$$

(iv): It is clear from the definitions of $F_i$, $F_j$ and $\mathbf{Z}$.
(v): Since $\mathrm{J}_{\mathbf{A}}\mathbf{R}$ is nonexpansive and $\mathbf{Z} = \operatorname{Fix} \mathrm{J}_{\mathbf{A}}\mathbf{R}$, it follows that $\mathbf{Z}$ is closed and convex by [16, Proposition 2.1.11]. Moreover, let $\mathbf{z} = \left(z_1, z_2, \dots, z_m\right) \in \operatorname{Fix} \mathrm{J}_{\mathbf{A}}\mathbf{R} \Leftrightarrow \mathbf{z} = \mathrm{J}_{\mathbf{A}}\mathbf{R}\mathbf{z}$. This implies

$$\begin{aligned}
z_1 &= \mathrm{J}_1\mathrm{J}_m\dots\mathrm{J}_2 z_1, \\
&\vdots \\
z_i &= \mathrm{J}_i\mathrm{J}_{i-1}\dots\mathrm{J}_1\mathrm{J}_m\dots\mathrm{J}_{i+1} z_i, \\
&\vdots \\
z_m &= \mathrm{J}_m\mathrm{J}_{m-1}\dots\mathrm{J}_1 z_m.
\end{aligned}$$

Hence, $\mathbf{z} = \left(z_1, z_2, \dots, z_m\right) \in F_1 \times F_2 \times \cdots \times F_{m-1} \times F_m$. Since this is true for all $\mathbf{z} \in \mathbf{Z}$, it follows that

$$\mathbf{Z} \subseteq F_1 \times F_2 \times \cdots \times F_{m-1} \times F_m.$$

(vi): It is clear from (i) that $Q_i : \mathbf{Z} \to F_i$ is surjective. To show $Q_i$ is injective, suppose $\mathbf{z} = \left(z_1, z_2, \dots, z_m\right), \widetilde{\mathbf{z}} = \left(\widetilde{z}_1, \widetilde{z}_2, \dots, \widetilde{z}_m\right) \in \mathbf{Z}$ and $Q_i(\mathbf{z}) = Q_i(\widetilde{\mathbf{z}})$. This implies that $z_i = \widetilde{z}_i$. Because $\mathbf{z}$ and $\widetilde{\mathbf{z}}$ are cycles, it follows that

$$z_{i+1} = \mathrm{J}_{i+1} z_i = \mathrm{J}_{i+1}\widetilde{z}_i = \widetilde{z}_{i+1} \tag{41}$$

$$\vdots \tag{42}$$

$$z_m = J_m z_{m-1} = J_m \widetilde{z}_{m-1} = \widetilde{z}_m \tag{43}$$

$$z_1 = J_1 z_m = J_1 \widetilde{z}_m = \widetilde{z}_1 \tag{44}$$

$$z_2 = J_2 z_1 = J_2 \widetilde{z}_1 = \widetilde{z}_2 \tag{45}$$

$$\vdots \tag{46}$$

$$z_{i-1} = J_{i-1} z_{i-2} = J_{i-1} \widetilde{z}_{i-2} = \widetilde{z}_{i-1}. \tag{47}$$

From (41)-(47), we have

$$\mathbf{z} = \widetilde{\mathbf{z}}.$$

■

**Lemma 3.** Let $\cap_{i=1}^{m} \operatorname{Fix} J_i \neq \varnothing := D$. Then the following holds:
(i) For all $i$ such that $1 \leq i \leq m$, it holds that $F_i = D$.
(ii) $\mathbf{Z} = \{(z, z, \cdots, z) \mid z \in D\} = D^m \cap \mathbf{\Delta}$.

*Proof.* (i): Since $J_i$ is firmly nonexpansive for every $1 \leq i \leq m$, then by [10, Corollary 4.51], we have

$$(\forall (1 \leq i \leq m)), \ \operatorname{Fix}\left(J_i J_{i-1} \cdots J_1 J_m \cdots J_{i+1}\right) = D.$$

(ii): Let $\mathbf{z} = (z_1, z_2, \cdots, z_m) \in \mathbf{Z}$. Then,

$$\begin{aligned}
&z_1 = J_1 J_m J_{m-1} \cdots J_2 z_1 \Leftrightarrow z_1 \in F_1 = D\\
&z_2 = J_2 J_1 J_m \cdots J_3 z_2 \Leftrightarrow z_2 \in F_2 = D\\
&\vdots\\
&z_m = J_m J_{m-1} J_{m-2} \cdots J_1 z_m \Leftrightarrow z_m \in F_m = D.
\end{aligned}$$

Therefore,

$$\mathbf{z} = (z_1, z_2, \cdots, z_m) \in F_1 \times F_2 \times \cdots \times F_m = D \times U \times D \times \cdots \times D = D^m$$

and

$$\mathbf{z} = (z_1, z_2, \cdots, z_m) = (z, z, \cdots, z) \in \mathbf{D}.$$

Therefore,

$$\mathbf{z} \in D^m \cap \mathbf{D}.$$

■

**Remark 1.** When $m = 2$, we have $J_1(F_2) = F_1$ and $J_2(F_1) = F_2$.

*Proof.* Let $z \in F_1$. This implies that $z = J_1 J_2 z$ and $J_2 z = J_2 J_1 (J_2 z)$. Therefore,

$$J_2(F_1) \subseteq F_2. \tag{48}$$

Let $\widetilde{z} \in F_2$. It follows that $\widetilde{z} = \mathrm{J}_2\mathrm{J}_1\widetilde{z}$ and $\mathrm{J}_1\widetilde{z} = \mathrm{J}_1\mathrm{J}_2(\mathrm{J}_1\widetilde{z})$. Thus,

$$\mathrm{J}_1(F_2) \subseteq F_1 \tag{49}$$

Now, applying $\mathrm{J}_1$ and $\mathrm{J}_2$ to (48) and (49), respectively, we obtain

$$F_1 \subseteq \mathrm{J}_1(F_2) \quad \text{and} \quad F_2 \subseteq \mathrm{J}_2(F_1).$$

Hence, $F_1 = \mathrm{J}_1(F_2)$ and $F_2 = \mathrm{J}_2(F_1)$. ■

**Lemma 4.** Recall from (20) that $\mathbf{Z} = \operatorname{Fix} \mathrm{J}_{\mathbf{A}}\mathbf{R}$. Then it follows that

$$\mathbf{Z} = \operatorname{Fix} \mathrm{J}_{\mathbf{A}}\mathbf{R} = \operatorname{Fix} \mathrm{J}_{\frac{1}{2}\mathbf{A}}\Big(\frac{\mathbf{Id}+\mathbf{R}}{2}\Big).$$

*Proof.* Let $\mathbf{x} \in \operatorname{Fix}\big(\mathrm{J}_A\mathbf{R}\big)$. Then

$$\begin{aligned}
\mathbf{x} = \mathrm{J}_A\mathbf{R}\mathbf{x} \Leftrightarrow \mathbf{R}\mathbf{x} \in \mathbf{x} + \mathbf{A}\mathbf{x} &\Leftrightarrow 0 \in \big(\mathbf{x} - \mathbf{R}\mathbf{x}\big) + \mathbf{A}(\mathbf{x}) \\
&\Leftrightarrow 0 \in \frac{\big(\mathbf{x} - \mathbf{R}\mathbf{x}\big)}{2} + \frac{\mathbf{A}(\mathbf{x})}{2} \\
&\Leftrightarrow 0 \in \mathbf{x} - \Big(\frac{\mathbf{Id}+\mathbf{R}}{2}\Big)\mathbf{x} + \frac{\mathbf{A}(\mathbf{x})}{2} \quad \text{adding and subtracting } \frac{\mathbf{x}}{2} \\
&\Leftrightarrow \Big(\frac{\mathbf{Id}+\mathbf{R}}{2}\Big)\mathbf{x} \in \Big(\mathbf{Id} + \frac{1}{2}\mathbf{A}\Big)(\mathbf{x}) \\
&\Leftrightarrow \mathbf{x} = \mathrm{J}_{\frac{1}{2}\mathbf{A}}\Big(\frac{\mathbf{Id}+\mathbf{R}}{2}\Big)(\mathbf{x}) \\
&\Leftrightarrow \mathbf{x} \in \operatorname{Fix} \mathrm{J}_{\frac{1}{2}\mathbf{A}}\Big(\frac{\mathbf{Id}+\mathbf{R}}{2}\Big).
\end{aligned}$$

■

**Lemma 5.** Suppose that $\operatorname{Fix} \mathrm{J}_i \neq \varnothing$ for each $1 \leq i \leq m$. Then the following are equivalent :

(i) $\cap_{i=1}^{m} \operatorname{Fix} \mathrm{J}_i \neq \varnothing$.

(ii) $F_1 = F_2 = \cdots = F_m \neq \varnothing$.

*Proof.* (i): Let $\cap_{i=1}^{m} \operatorname{Fix} \mathrm{J}_i \neq \varnothing \Rightarrow F_1 \neq \varnothing, F_2 \neq \varnothing, \cdots, F_m \neq \varnothing$ and from Lemma 3(i), it follows that

$$F_1 = F_2 = \cdots = F_m = \cap_{i=1}^{m} \operatorname{Fix} \mathrm{J}_i.$$

(ii): Let $F_1 = F_2 = \cdots = F_m \neq \varnothing$. Applying Theorem 2 (iv) gives $\mathbf{Z} \neq \varnothing$. ■

## 5. Consequences of Attouch-Théra duality

Recall (5) that

$$\mathbf{A} = A_1 \times A_2 \times \cdots \times A_m.$$

From now on, suppose that

$$\mathbf{A} \text{ is maximally monotone on } \mathbf{X}, \tag{50}$$

and

$$\mathbf{C} := \operatorname{zer} \mathbf{A} \text{ is not empty.} \tag{51}$$

Recall (20), which states that

$$\mathbf{Z} := \operatorname{Fix}(\mathbf{J_A R}).$$

**Proposition 1.** The following holds:

(i) $\mathbf{Z} = \operatorname{psol}\big(\mathbf{Id} - \mathbf{R}\big)$.

(ii) $\mathbf{A} + \mathbf{Id} - \mathbf{R}$ is maximally monotone.

(iii) $\mathbf{Z}$ is closed and convex.

*Proof.* (i): Combine Lemma 2 and (13). (ii): Note that $\mathbf{Id} - \mathbf{R}$ is linear, full domain, and maximally monotone by [2, Theorem 7.1]. Moreover, $\mathbf{A}$ is maximally monotone by assumption. Therefore, the sum is maximally monotone by [10, Corollary 25.5 (i)] (iii): It follows directly from (i) and Theorem 2 (v). ■

**Theorem 3.** Recall from Lemma 2, the primal (Attouch-Théra) problem:

$$0 \in \mathbf{A}(\mathbf{z}) + (\mathbf{Id} - \mathbf{R})(\mathbf{z}),$$

for the pair $(\mathbf{A}, \mathbf{Id} - \mathbf{R})$. The Attouch-Théra dual problem is

$$0 \in \mathbf{A}^{-1}(\mathbf{y}) + (\mathbf{Id} - \mathbf{R})^{-1}(\mathbf{y}) \tag{52}$$

or

$$0 \in \big(\mathbf{A}^{-1} + N_{\mathbf{D}^\perp}\big)(\mathbf{y}) + \Big(\frac{1}{2}\mathbf{Id} + \mathbf{T}\Big)(\mathbf{y}). \tag{53}$$

Moreover,

$$\operatorname{dsol}(\mathbf{A}, \mathbf{Id} - \mathbf{R}) = \operatorname{zer}\Big(\mathbf{A}^{-1} + N_{\mathbf{D}^\perp} + \frac{1}{2}\mathbf{Id} + \mathbf{T}\Big). \tag{54}$$

*Proof.* The dual pair of $\big(\mathbf{A}, \big(\mathbf{Id} - \mathbf{R}\big)\big)$ is

$$\big(\mathbf{A}, \big(\mathbf{Id} - \mathbf{R}\big)\big)^* = \big(\mathbf{A}^{-1}, \big(\mathbf{Id} - \mathbf{R}\big)^{-\circledcirc}\big).$$

Because of the linearity of $\mathbf{R}$, it follows that

$$\big(\mathbf{Id} - \mathbf{R}\big)^{-\circledcirc} = \big(-\mathbf{Id}\big) \circ \big(\mathbf{Id} - \mathbf{R}\big)^{-1} \circ \big(-\mathbf{Id}\big) = \big(\mathbf{Id} - \mathbf{R}\big)^{-1}.$$

Hence, Attouch-Théra dual problem simplifies to

$$0 \in \mathbf{A}^{-1}\big(\mathbf{y}\big) + \big(\mathbf{Id} - \mathbf{R}\big)^{-1}\big(\mathbf{y}\big).$$

From [3, Theorem 2.8 (i)], we obtain

$$\begin{aligned}0 \in \mathbf{A}^{-1}(\mathbf{y}) + \left(\mathbf{Id} - \mathbf{R}\right)^{-1}(\mathbf{y}) &\Leftrightarrow 0 \in \mathbf{A}^{-1}(\mathbf{y}) + \left(\mathbf{N}_{\mathbf{D}^{\perp}} + \frac{1}{2}\mathbf{Id} + \mathbf{T}\right)(\mathbf{y})\\ &\Leftrightarrow 0 \in \left(\mathbf{A}^{-1} + \mathbf{N}_{\mathbf{D}^{\perp}}\right)(\mathbf{y}) + \left(\frac{1}{2}\mathbf{Id} + \mathbf{T}\right)(\mathbf{y}),\end{aligned}$$

which verifies (53). Next, applying (16), (52), and (53) yields

$$\begin{aligned}\operatorname{dsol}(\mathbf{A}, \mathbf{Id} - \mathbf{R}) &= \operatorname{zer}\left(\mathbf{A}^{-1} + \left(\mathbf{Id} - \mathbf{R}\right)^{-1}\right)\\ &= \operatorname{zer}\left(\mathbf{A}^{-1} + \mathbf{N}_{\mathbf{D}^{\perp}} + \frac{1}{2}\mathbf{Id} + \mathbf{T}\right).\end{aligned}$$

■

**Proposition 2.** The solution set of (52) is at most a sigleton and possibly empty.

*Proof.* [3, Theorem 2.8 (i)] gives $\left(\mathbf{Id} - \mathbf{R}\right)^{-1} = \mathbf{N}_{\mathbf{D}^{\perp}} + \frac{1}{2}\mathbf{Id} + \mathbf{T}$. $\mathbf{Id} - \mathbf{R}$ is $(1/2)$-cocoercive because $\mathbf{R}$ is nonexpansive by [10, Proposition 4.11]. Hence, $\left(\mathbf{Id} - \mathbf{R}\right)^{-1} = \mathbf{N}_{\mathbf{D}^{\perp}} + \frac{1}{2}\mathbf{Id} + \mathbf{T}$ is $(1/2)$-strongly monotone by [2, Lemma 7.8(iv)]. Then

$$\mathbf{A}^{-1} + \left(\mathbf{Id} - \mathbf{R}\right)^{-1} = \frac{1}{2}\mathbf{Id} + \left(\mathbf{N}_{\mathbf{D}^{\perp}} + \mathbf{T} + \mathbf{A}^{-1}\right)$$

is strongly monotone. Hence, it follows that

$$\operatorname{zer}\left(\mathbf{A}^{-1} + \left(\mathbf{Id} - \mathbf{R}\right)^{-1}\right) = \left(\mathbf{A}^{-1} + \left(\mathbf{Id} - \mathbf{R}\right)^{-1}\right)^{-1}(0)$$

is at most a singleton by [10, Proposition 23.35]. ■

**Theorem 4.** Let $\operatorname{psol}(\mathbf{A}, \mathbf{Id} - \mathbf{R}) = \mathbf{Z}$ and recall (54), which states that

$$\operatorname{dsol}(\mathbf{A}, \mathbf{Id} - \mathbf{R}) = \operatorname{zer}\left(\mathbf{A}^{-1} + \mathbf{N}_{\mathbf{D}^{\perp}} + \frac{1}{2}\mathbf{Id} + \mathbf{T}\right).$$

Then

$$\operatorname{dsol}\left(\mathbf{Id} - \mathbf{R}\right) = \left(\mathbf{R} - \mathbf{Id}\right)\mathbf{Z} = \begin{cases}\left\{\mathbf{J}_{2(\mathbf{A}^{-1} + \mathbf{N}_{\mathbf{D}^{\perp}} + \mathbf{T})}(\mathbf{0})\right\}, & if\ \ \mathbf{Z} \neq \varnothing\\ \varnothing, & if\ \ \mathbf{Z} = \varnothing.\end{cases} \tag{55}$$

Moreover, if $\mathbf{y}^* := \mathbf{J}_{2(\mathbf{A}^{-1} + \mathbf{N}_{\mathbf{D}^{\perp}} + \mathbf{T})}(\mathbf{0})$ exists, then the following holds:

(i) $\mathbf{y}^* \in \mathbf{D}^{\perp}$.

(ii) $\mathbf{y}^*$ is the only vector that makes $\mathbf{A}^{-1}\mathbf{y} \cap -(\mathbf{N}_{\mathbf{D}^{\perp}}\mathbf{y} + \frac{1}{2}\mathbf{y} + \mathbf{T}\mathbf{y})$ non-empty.

(iii) $\mathbf{Z} = \mathbf{A}^{-1}\mathbf{y}^* \cap (-\frac{1}{2}\mathbf{y}^* - \mathbf{T}\mathbf{y}^* - \mathbf{D})$.

*Proof*. By using (16), we have $\mathbf{y} \in \operatorname{dsol}\big(\mathbf{A}, \mathbf{Id} - \mathbf{R}\big)$. This implies that

$$\begin{aligned}
&\mathbf{0} \in \mathbf{A}^{-1}(\mathbf{y}) + (\mathbf{Id} - \mathbf{R})^{-1}(\mathbf{y}) \quad (\forall \mathbf{z} \in \mathbf{Z}) \\
\Leftrightarrow\ &\mathbf{z} \in \mathbf{A}^{-1}(\mathbf{y}) \text{ and } -\mathbf{z} \in (\mathbf{Id} - \mathbf{R})^{-1}(\mathbf{y}) \quad (\forall \mathbf{z} \in \mathbf{Z}) \\
\Leftrightarrow\ &\mathbf{y} \in \mathbf{A}(\mathbf{z}) \text{ and } \mathbf{y} = (\mathbf{Id} - \mathbf{R})(-\mathbf{z}) \quad (\forall \mathbf{z} \in \mathbf{Z}).
\end{aligned}$$

Hence, for all $\mathbf{z} \in \mathbf{Z}$, it follows that

$$\mathbf{y} = \mathbf{R}\mathbf{z} - \mathbf{z}$$

and

$$\begin{aligned}
\operatorname{dsol}\big(\mathbf{Id} - \mathbf{R}\big) &= \cup_{\mathbf{z} \in \mathbf{Z}}\{\mathbf{R}\mathbf{z} - \mathbf{z} \mid \mathbf{z} \in \mathbf{Z}\} \\
&= \big(\mathbf{R} - \mathbf{Id}\big)\mathbf{Z}.
\end{aligned}$$

Additionally, if $\mathbf{Z} \neq \varnothing$, then using [3, Theorem 2.4] and (3) gives

$$\begin{aligned}
&\mathbf{0} \in \mathbf{A}^{-1}(\mathbf{y}) + (\mathbf{Id} - \mathbf{R})^{-1}(\mathbf{y}) \\
\Leftrightarrow\ &\mathbf{0} \in \mathbf{A}^{-1}(\mathbf{y}) + \Big(\frac{1}{2}\mathbf{Id} + \mathbf{T} + \mathbf{N}_{\mathbf{D}^\perp}\Big)(\mathbf{y}) \\
\Leftrightarrow\ &\mathbf{0} \in \Big(\mathbf{Id} + 2\Big(\mathbf{A}^{-1} + \mathbf{T} + \mathbf{N}_{\mathbf{D}^\perp}\Big)\Big)(\mathbf{y}) \\
\Leftrightarrow\ &\mathbf{y} = \Big(\mathbf{Id} + 2\Big(\mathbf{A}^{-1} + \mathbf{T} + \mathbf{N}_{\mathbf{D}^\perp}\Big)\Big)^{-1}(\mathbf{0}) \\
\Leftrightarrow\ &\mathbf{y} = \mathrm{J}_{2(\mathbf{A}^{-1} + \mathbf{T} + \mathbf{N}_{\mathbf{D}^\perp})}(\mathbf{0}).
\end{aligned}$$

However, if $\mathbf{Z} = \varnothing$, then using [9, Proposition 2.4 (v)]

$$\varnothing = \operatorname{dsol}\big(\mathbf{A}, \mathbf{Id} - \mathbf{R}\big) = \operatorname{dsol}\big(\mathbf{Id} - \mathbf{R}\big).$$

(i): By [3, Theorem 2.7], we have $\operatorname{dom}(\mathbf{Id} - \mathbf{R})^{-1} = \mathbf{D}^\perp$. This implies that

$$\mathbf{y} \in \mathbf{D}^\perp.$$

(ii): Combine Proposition 2 and [9, Proposition 2.4].
(iii): Combine (i), (ii), and [2, Proposition 9.3 (i)] where $\mathbf{N}_{\mathbf{C}}^{-1}$ is replaced by $\mathbf{A}^{-1}$. ■

**Lemma 6.** Denote by $\mathbf{y}^* = \big(y_1, y_2, \cdots, y_m\big)$ the unique solution of (53). Then the following holds:

(i) The mapping $\mathrm{J}_1 : F_m \to F_1$ is bijective on $F_m$ and it is given by $\mathrm{J}_1\big(z\big) = z - y_1$. Moreover, for $1 \leq i \leq m - 1$ the mapping $\mathrm{J}_{i+1} : F_i \to F_{i+1}$ is bijective and is given by $\mathrm{J}_{i+1}\big(z\big) = z - y_{i+1}$.

(ii) The fixed point sets $F_1 = F_m - y_1$ and $F_{i+1} = F_i - y_{i+1}$.

*Proof.* (i): Let $z$ and $\widetilde{z}$ be in $F_m$ satisfying that $J_1 z = J_1\widetilde{z}$. Our goal is to show that $J_1$ is injective on $F_m$. Then, we have

$$z = J_m J_{m-1} \cdots J_i J_{i-1} \cdots J_1 z$$

and

$$\widetilde{z} = J_m J_{m-1} \cdots J_i J_{i-1} \cdots J_1 \widetilde{z}.$$

Since $J_1 z = J_1\widetilde{z}$, then we obtain $z = \widetilde{z}$. Thus, $J_1$ is an injective mapping on $F_m$. Moreover, Remark 1 shows that $J_1$ is a surjective mapping on $F_m$. Therefore, $J_1$ is a bijective mapping on $F_m$. For every $z \in F_m$, we have

$$z = J_m J_{m-1} \cdots J_i J_{i-1} \cdots J_1 z. \tag{56}$$

Set $z_1 = J_1 z,\ \ z_2 = J_2 z_1,\ \ \cdots,\ \ z_{m-1} = J_{m-1} z_{m-2},\ \ z_m = J_m z_{m-1}$. Therefore, using (56), we have $z = J_m z_{m-1}$ and $\mathbf{z} = \big(z_1, z_2, \cdots, z_{m-1}, z\big)$. Therefore, Theorem 4 gives

$$\big(y_1, y_2, \cdots, y_m\big) = \big(z, z_1, z_2, \cdots, z_{m-1}\big) - \big(z_1, z_2, \cdots, z_{m-1}, z_m\big)$$

and therefore, $y_1 = z - z_1 \Rightarrow z_1 = z - y_1 \Rightarrow J_1 z = z - y_1$. The proof of $J_i$ is the same as $J_1$.
(ii): It follows from (i) that for every $z \in F_m$, we obtain $J_1 z = z - y_1$. Then by Theorem 2(iii) we have $F_1 = F_m - y_1$. The proof for $F_{i+1} = F_i - y_{i+1}$ is the same as $F_1$. ∎

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## 6. Clarification

Please note that a preprint has been published on arXiv and is referenced in [4]. There is no conflict of interest, and no data were used to support this study.

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