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A New Method of Generating Truncated Bivariate Families of Distributions

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Abstract. The modeling of complex data has attracted several researchers for the quest of generating new probability distributions. The joint modeling of two variables asks for some additional complexities as a bivariate distribution is needed. The field of research in developing bivariate families of distributions is somewhat new. In certain situations, the domain of data is restricted and some truncated distribution is required. Several univariate truncated families of distributions are available for modeling of a single variable but the bivariate truncated families of distributions has not been studied and in this paper, we have proposed a new bivariate truncated families of distributions. A specific sub-family has been proposed by using the bivariate Burr as a baseline distribution, resulting in a bivariate truncated Burr family of distributions. Some important statistical properties of the proposed family has been studied, which include the marginal and conditional distributions, bivariate reliability, and bivariate hazard rate functions. The maximum likelihood estimation for the parameters of the family is also carried out. The proposed bivariate truncated Burr family of distributions is studied for the Burr baseline distributions, giving rise to the bivariate truncated Burr-Burr distribution. The new bivariate truncated Burr-Burr distribution is explored in detail and several statistical properties of the new distribution are studied, which include the marginal and conditional distributions, product, ratio, and conditional moments. The maximum likelihood estimation for the parameters of the proposed distribution is done. The proposed bivariate truncated Burr-Burr distribution is used to model some real data sets. It is found that the proposed distribution performs better than the other distributions considered in this study.

2020 Mathematics Subject Classifications: 60E05, 62E15, 62G30, 62F10

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1. Introduction

Probability distributions are an important tool in statistical sciences. There are situations when complex data behavior is beyond the scope of the standard probability models and hence some extensions are needed. Several practical situations arise where one is interested in the simultaneous modeling of two or more phenomena. In these situations, the univariate distributions are not applicable, rather, some suitable bivariate or multivariate distributions are required. Investigation of the bivariate and multivariate distributions is not widespread as compared with the univariate probability distributions, but in the last few decades, various researchers have proposed a significant number of bivariate distributions from their marginals. Over the last few decades, various techniques for generating new probability distributions have been proposed. One of these techniques is the transformed-transformer approach, which was first introduced by [7], as an extensive family of univariate distributions. This family of distributions provides several families as special cases. The beta-generated family of distributions, proposed by [15], is another popular family of distributions. This family has attracted various authors to propose new probability distributions. Some of these are the beta-normal distribution, introduced by [15], the beta-exponential distribution by [23], the beta-Weibull distribution by [16], and the beta-Pareto distribution by [3], among others.

Recently, [6] have introduced a new method of generating truncated $T-X$ families of distributions including the right-truncated and left-truncated families of distributions. These families are used to construct new generalized families of continuous distributions. In comparison of the univariate distributions, the bivariate distributions have attracted a less number of researchers. The bivariate distribution is defined as the joint distribution of two random variables.

Various approaches are available to generate a bivariate distribution from the univariate marginals. A simple method has been proposed by [17] to generate a bivariate distribution from given univariate marginals. The joint cumulative distribution function of this bivariate distribution is

$$F(x_1,x_2) = G(x_1)G(x_2)\left[1+\alpha\left\{1-G(x_1)\right\}\left\{1-G(x_2)\right\}\right], \tag{1}$$

where $G(x_1)$ and $G(x_2)$ are any marginal $cdf's$ and α is some parameters. A bivariate beta family of distributions has been proposed by [28] by using the bivariate beta distribution of [26]. The bivariate Gamma distribution is a popular bivariate distribution. The joint density function is proposed by [22] as

$$f_{X_1,X_2}(x_1,x_2) = \frac{a^{b+c}}{\Gamma(b)\Gamma(c)} x_1^{b-1}(x_2-x_1)^{c-1}\mathrm{e}^{-ax_2}; 0<x_1<x_2 \tag{2}$$

where $a,b,c>0$. The marginal distributions of X_1 and X_2 are gamma distributions, with shape parameters b and $b+c$, respectively, and the scale a.

Over the past years, the truncated family of distributions has received attention from researchers. A truncated Fréchet family of distributions was proposed by [1] by using the truncated Fréchet distribution over $(0,1)$. A truncated inverted Kumaraswamy generated family was obtained by [9]. A truncated Burr-G family of distribution was obtained by [19] by using the truncated Burr distribution on (0,1). A Truncated Weibull-G (TW-G) family of distributions was obtained by [24] as an alternative to beta-G (B-G) family of distributions with more flexible hazard rate and greater reliability. The family of life-time models using a truncated negative binomial distribution was introduced by [23]. A truncated Lomax distribution was proposed by [18] by using a truncated Lomax distribution on $[0,1]$. A truncated Cauchy power-G family of distributions was proposed by [4] and some important properties of the proposed family were studied. The moment estimation for the parameters of the truncated Weibull distribution was studied by [20]. Estimation of the parameters of truncated Gamma distribution was studied by [12] . The truncated Birnbaum-Saunders distribution was studied by [2]. An Erlang-Truncated Exponential distribution was proposed by [14] as an extension of the standard one parameter exponential distribution, this distribution results from the mixture of Erlang distribution and the left-truncated one-parameter exponential distribution. The transmuted Erlang truncated exponential distribution was proposed by [25]. This new distribution extends the two-parameter Erlang truncated exponential distribution. A half-logistic inverted Topp–Leone family of distributions was proposed by [8].

In this paper, we have proposed a new method of generating bivariate truncated families of distributions. The proposed method will be used to generate a bivariate truncated Burr family of distributions. The outline for the paper follows. The bivariate truncated family of distributions is introduced in the following section. The bivariate truncated Burr family of distribution is proposed in section 3. The various statistical properties of the proposed bivariate truncated Burr family of distributions are explored in Section 4 alongside the maximum likelihood estimation for the parameters. A bivariate truncated Burr-Burr distribution is proposed in Section 5 and some useful properties of the proposed distribution are given in Section 6. In Section 7, three real datasets have been used to study the suitability of the the proposed bivariate Burr-Burr distribution. Finally, conclusions are given in Section 8.

2. Bivariate Truncated Family of Distribution

A truncated distribution represents a conditional distribution that is confined to the domain of a random variable under specific circumstances. Recently, [6] have proposed some univariate truncated families of distributions. They have proposed two right-truncated families of distributions with cumulative distribution functions

$$F_{R_1T-X}\left(x\right)=\frac{1}{R_T\left(a\right)}\int_0^{W_1[G(x)]} r\left(t\right)dt \tag{3}$$

and

$$F_{R_2T-X}\left(x\right)=\frac{1}{R_T\left(a\right)}\int_{W_2[G(x)]}^{a} r\left(t\right)dt, \tag{4}$$

where $r(t)$ is density function of some random variable defined on $\mathbb{R}_+$, $R_T(a)$ is cumulative distribution function of T at a, $W_1[G(x)]$ and $W_2[G(x)]$ are any real valued functions of $G(x)$ such that $W_1(0) \to 0, W_1(1) \to a, W_2(0) \to a$ and $W_2(1) \to 0$. It has been shown by [6] that $W_1(x) = ax$ and $W_2(x) = a(1-x)$ can be some suitable choices to propose the new families of distributions.

Recently, [5] have proposed a bivariate extension of the truncated family of distributions. The joint cdf of the proposed bivariate family of distributions is

$$F_{R_1T_1T_2-X_1X_2}(x_1, x_2) = \frac{1}{R(a_1, a_2)} \int_0^{W_{11}[G_1(x_1)]} \int_0^{W_{21}[G_2(x_2)]} r(t_1, t_2)\, dt_1 dt_2, \tag{5}$$

where $r(t_1, t_2)$ is some bivariate distribution with domain on $\mathbb{R}_+^2$, $R(a_1, a_2)$ is joint distribution function of (T_1, T_2) at (a_1, a_2), $W_{11}[G_1(x_1)]$ and $W_{21}[G_2(x_2)]$ are some functions of $G(x_1)$ and $G(x_2)$ such that $W_{11}(0) = W_{21}(0) \to 0, W_{11}(1) \to a_1$ and $W_{21}(1) \to a_2$.

A simpler version is also proposed by [5] and the joint distribution function of this version is given as

$$F_{R_1T_1T_2-X_1X_2}(x_1, x_2) = \frac{1}{R(a_1, a_2)} \int_0^{a_1G_1(x_1)} \int_0^{a_2G_2(x_2)} r(t_1, t_2)\, dt_1 dt_2. \tag{6}$$

The density function corresponding to above joint distribution function is

$$f_{R_1}(x_1, x_2) = a_1a_2g_1(x_1)\, g_2(x_2)\, \frac{r[a_1G_1(x_1), a_2G_2(x_2)]}{R(a_1, a_2)}, \tag{7}$$

where $g_1(x_1)$ and $g_2(x_2)$ are density functions corresponding to $G_1(x_1)$ and $G_2(x_2)$.

A bivariate truncated Burr family of distributions has been proposed by [5] by using

$$r(t_1, t_2) = \frac{\alpha(\alpha+1)\beta_1\beta_2 x_1^{(\beta_1-1)} x_2^{(\beta_2-1)}}{(1 + x_1^{\beta_1} + x_2^{\beta_2})^{\alpha+2}} \tag{8}$$

in 6.

In the next section, we will propose the bivariate truncated Burr family($BTBF$) of distributions by using the Burr distribution as a generating distribution in the bivariate truncated family of distributions. Various characteristics of the new family of distributions will be studied. The maximum likelihood estimation (MLE) of the new family of distributions will also be discussed.

3. Bivariate Truncated Burr Family of Distribution

In this section, we have proposed the $BTBF$ of distributions by using the Burr distribution as a generator. Suppose X_1 and X_2 are two random variables having a bivariate Burr distribution with the joint probability density function ($pdfs$)

$$f(x_1,x_2)=\frac{\alpha(\alpha+1)\beta_1\beta_2x_1^{(\beta_1-1)}x_2^{(\beta_2-1)}}{(1+x_1^{\beta_1}+x_2^{\beta_2})^{\alpha+2}} \tag{9}$$

where $\alpha\geq 0$ is the scale parameter,$(\beta_1,\beta_2)\geq 0$ are the shape parameters. The distribution function corresponding to 9 is

$$F(x_1,x_2)=1-(1+x_1^{\beta_1})^{-\alpha}-(1+x_2^{\beta_2})^{-\alpha}+(1+x_1^{\beta_1}+x_2^{\beta_2})^{-\alpha} \tag{10}$$

The joint cumulative distribution function (cdf) of the proposed right $BTBF$ of distribution is obtained by using 9 in 6 which on simplifying becomes

$$F_R(x_1,x_2)=\frac{1-(1+(a_1G_1(x_1))^{\beta_1})^{-\alpha}-(1+(a_2G_2(x_2))^{\beta_2})^{-\alpha}+(\Delta_1(x_1,x_2))^{-\alpha}}{\gamma}, \tag{11}$$

where $\gamma=1-(1+a_1^{\beta_1})^{-\alpha}-(1+a_2^{\beta_2})^{-\alpha}+(1+a_1^{\beta_1}+a_2^{\beta_2})^{-\alpha}$ and

$\Delta_1(x_1,x_2)=1+[a_1G_1(x_{1i})]^{\beta_1}+[a_2G_2(x_{2i})]^{\beta_2}$.

The pdf of the right $BTBF$ of distribution corresponding to 11 is

$$f_R(x_1,x_2)=\frac{1}{\gamma}\alpha(\alpha+1)\beta_1\beta_2a_1^{\beta_1}a_2^{\beta_2}g_1(x_1)\,g_2(x_2)\,G_1(x_1)^{\beta_1-1}\,G_2(x_2)^{\beta_2-1}\,(\Delta_1(x_1,x_2))^{-(\alpha+2)} \tag{12}$$

In the following section, some characteristics of the $BTBF$ distribution will be presented.

4. Statistical Properties of Bivariate Truncated Burr Family of Distributions

This section discusses several statistical properties of the $BTBF$. These include, the marginal distributions, the conditional distributions, the bivariate reliability ad hazard rate functions, etc..

4.1. The Marginal Distributions

The marginal pdf of X_1 can be readily obtain from 12 as

$$f_{TB}(x_1)=\int_0^\infty f_{R(T_1T_2-X_1X_2)}(x_1,x_2)dx_2$$

$$f_{TB}(x_1) = \frac{\alpha\beta_1 a_1^{\beta_1} g_1(x_1) G_1(x_1)^{\beta_1 - 1}}{\gamma}\left[[1+(a_1G_1(x_1))^{\beta_1}]^{-(\alpha+1)} - [1+(a_1G_1(x_1))^{\beta_1} + a_2^{\beta_2}]^{-(\alpha+1)}\right]. \tag{13}$$

Similarly, the marginal *pdf* of the random variable X_2 is

$$f_{TB}(x_2) = \frac{\alpha\beta_2 a_2^{\beta_2} g_2(x_2) G_2(x_2)^{\beta_2 - 1}}{\gamma}\left[[1+(a_2G_2(x_2))^{\beta_2}]^{-(\alpha+1)} - [1+(a_2G_2(x_2))^{\beta_2} + a_1^{\beta_1}]^{-(\alpha+1)}\right], \tag{14}$$

where $g_1(x_1)$ and $g_2(x_2)$ are the density function of any baseline distribution corresponding to $G_1(x_1)$ and $G_2(x_2)$.

The marginal *cdf* of X_1 of the $BTBF$ of distributions is

$$F_{TB}(x_1) = \frac{1 - \left[1+(a_1G_1(x_1))^{\beta_1}\right]^{-\alpha} - \left[1+a_2^{\beta_2}\right]^{-\alpha} + \left[1+(a_1G_1(x_1))^{\beta_1} + a_2^{\beta_2}\right]^{-\alpha}}{\gamma}. \tag{15}$$

Similarly, the marginal *cdf* of the second random variable X_2 is

$$F_{TB}(x_2) = \frac{1 - \left[1+(a_2G_2(x_2))^{\beta_2}\right]^{-\alpha} - \left[1+a_1^{\beta_1}\right]^{-\alpha} + \left[1+(a_2G_2(x_2))^{\beta_2} + a_1^{\beta_1}\right]^{-\alpha}}{\gamma}, \tag{16}$$

where $G_1(x_1)$ and $G_2(x_2)$ are the $cdf's$ of any baseline distributions.

4.2. The Conditional Distributions

The conditional distribution of X_1 given $X_2 = x_2$ is

$$f(x_1|x_2) = \frac{(\alpha+1)\beta_1 a_1^{\beta_1} g_1(x_1) G_1(x_1)^{\beta_1-1}[\Delta_1(x_1,x_2)]^{-(\alpha+2)}}{[1+(a_2G_2(x_2))^{\beta_2}]^{-(\alpha+1)} - [1+(a_2G_2(x_2))^{\beta_2} + a_1^{\beta_1}]^{-(\alpha+1)}}. \tag{17}$$

Similarly,the conditional distribution of X_2 given $X_1 = x_1$ is obtained as

$$f(x_2|x_1) = \frac{(\alpha+1)\beta_2 a_2^{\beta_2} g_2(x_2) G_2(x_2)^{\beta_2-1}[\Delta_1(x_1,x_2)]^{-(\alpha+2)}}{[1+(a_1G_1(x_1))^{\beta_1}]^{-(\alpha+1)} - [1+(a_1G_1(x_1))^{\beta_1} + a_2^{\beta_2}]^{-(\alpha+1)}}. \tag{18}$$

The conditional distributions can be studied for any baseline distribution.

4.3. The Bivariate Reliability and Hazard Rate Function

The reliability function provides the chance that an item or system will function at any time t (for more details see [27]). The bivariate reliability function (BRF) of X_1 and X_2 is defined as

$$R(x_1,x_2) = 1 - [F_{X_1}(x_1) + F_{X_2}(x_2) - F_{X_1,X_2}(x_1,x_2)]$$

Using Eqs. 15, 16, and 11 in the above equation, the BRF for the $BTBF$ of distributions is

$$\begin{aligned} R(x_1,x_2) =& 1 - \frac{1-(1+a_1^{\beta_1})^{-\alpha}-(1+a_2^{\beta_2})^{-\alpha}+\left(1+(a_1G_1(x_1))^{\beta_1}+a_2^{\beta_2}\right)^{-\alpha}}{\gamma} \\ &+ \frac{[\Delta_1(x_1,x_2)]^{-\alpha}-[1+(a_2G_2(x_2))^{\beta_2}+a_1^{\beta_1}]^{-\alpha}}{\gamma} \end{aligned} \tag{19}$$

where $G_1(x_1)$ and $G_2(x_2)$ any baseline $cdf's$.

The hazard rate function provides the instantaneous rate at which an item or system will fail, given that it has already survived a specific length of time. The bivariate hazard rate function($BHRF$) (see [10])is

$$H(x_1,x_2) = \frac{f(x_1,x_2)}{R(x_1,x_2)}$$

By using the pdf and BRF of the $BTBF$ of distribution from Eqs.12 and 19, the $BHRF$ for the $BTBF$ of distribution is

$$\begin{aligned} H(x_1,x_2) =& \frac{1}{\gamma}\alpha(\alpha+1)\beta_1\beta_2a_1^{\beta_1}a_2^{\beta_2}g_1\left(x_1\right)g_2\left(x_2\right)G_1\left(x_1\right)^{\beta_1-1}G_2\left(x_2\right)^{\beta_2-1}[\Delta_1(x_1,x_2)]^{-(\alpha+2)} \\ &\div\Bigg[1 - \frac{1-(1+a_1^{\beta_1})^{-\alpha}-(1+a_2^{\beta_2})^{-\alpha}+\left(1+(a_1G_1(x_1))^{\beta_1}+a_2^{\beta_2}\right)^{-\alpha}}{\gamma} \\ &+ \frac{[\Delta_1(x_1,x_2)]^{-\alpha}-[1+[a_2G_2(x_2)]^{\beta_2}+a_1^{\beta_1}]^{-\alpha}}{\gamma}\Bigg] \end{aligned} \tag{20}$$

The $BHRF$ can be calculated for any baseline distribution.

4.4. Maximum Likelihood Estimation

This section discusses the maximum likelihood estimation for the parameters of the $BTBF$ of distributions. The likelihood function for the proposed $BTBF$ of distributions is

$$\begin{aligned} LF =& \alpha^n(\alpha+1)^na_1^{n\beta_1}a_2^{n\beta_2}\beta_1^n\beta_2^n\left[1-(1+a_1^{\beta_1})^{-\alpha}-(1+a_2^{\beta_2})^{-\alpha}+(1+a_1^{\beta_1}+a_2^{\beta_2})^{-\alpha}\right]^{-n} \\ &\times\prod_{i=1}^{n}g_1(x_1)\prod_{i=1}^{n}g_2(x_2)\prod_{i=1}^{n}G_1(x_1)^{\beta_1-1}\prod_{i=1}^{n}G_2(x_2)^{\beta_2-1}\prod_{i=1}^{n}\left[1+[a_1G_1(x_1)]^{\beta_1}+[a_2G_2(x_2)]^{\beta_2}\right]^{-(\alpha+2)}. \end{aligned}$$

The log of likelihood function is

$$\begin{aligned}\ell =& n\ln(\alpha) + n\ln(\alpha+1) + n\beta_1\ln(a_1) + n\beta_2\ln(a_2) + n\ln(\beta_1) + n\ln(\beta_2)\\ &+ \sum_{i=1}^{n}\ln[g_1(x_{1i})] + \sum_{i=1}^{n}\ln[g_2(x_{2i})] + (\beta_1-1)\sum_{i=1}^{n}\ln[G_1(x_{1i})] + (\beta_2-1)\sum_{i=1}^{n}\ln[G_2(x_{2i})]\\ &-(\alpha+2)\sum_{i=1}^{n}\ln(\Delta_1(x_1,x_2)) - n\ln(\gamma),\end{aligned} \tag{21}$$

where γ and $\Delta_1(x_1, x_2)$ are earlier defined.

The derivatives of log of likelihood function $w.r.t$ α,a_1,a_2,β_1 and β_2 are

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} =& \frac{n}{\alpha} + \frac{n}{\alpha+1} - \sum_{i=1}^{n}\ln\left[\Delta_1(x_1,x_2)\right] - \frac{n}{\gamma}\Big[(1+a_1^{\beta_1})^{-\alpha}\ln(1+a_1^{\beta_1}) + (1+a_2^{\beta_2})^{-\alpha}\ln(1+a_2^{\beta_2})\\ &-(1+a_1^{\beta_1}+a_2^{\beta_2})^{-\alpha}\ln(1+a_1^{\beta_1}+a_2^{\beta_2})\Big]\end{aligned} \tag{22}$$

$$\frac{\partial\ell}{\partial a_1} = \frac{n\beta_1}{a_1} - (\alpha+2)\sum_{i=1}^{n}\frac{\beta_1 a_1^{\beta_1-1}G_1(x_{1i})^{\beta_1}}{\Delta_1(x_1,x_2)} - \frac{n\alpha\beta_1 a_1^{\beta_1-1}}{\gamma} - \Big[(1+a_1^{\beta_1})^{-(\alpha+1)} - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-(\alpha+1)}\Big] \tag{23}$$

$$\frac{\partial\ell}{\partial a_2} = \frac{n\beta_2}{a_2} - (\alpha+2)\sum_{i=1}^{n}\frac{\beta_2 a_2^{\beta_2-1}G_2(x_{2i})^{\beta_2}}{\Delta_1(x_1,x_2)} - \frac{n\alpha\beta_2 a_2^{\beta_2-1}}{\gamma} - \Big[(1+a_2^{\beta_2})^{-(\alpha+1)} - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-(\alpha+1)}\Big] \tag{24}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\beta_1} =& n\ln(a_1) + \frac{n}{\beta_1} + \sum_{i=1}^{n}\ln(G_1(x_{1i})) - (\alpha+2)\sum_{i=1}^{n}\frac{[a_1G_1(x_{1i})]^{\beta_1}\ln[a_1G_1(x_{1i})]}{\Delta_1(x_1,x_2)} - \frac{n\alpha a_1^{\beta_1}\ln a_1}{\gamma}\\ &\Big[(1+a_1^{\beta_1})^{-(\alpha+1)} - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-(\alpha+1)}\Big]\end{aligned} \tag{25}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\beta_2} =& n\ln(a_2) + \frac{n}{\beta_2} + \sum_{i=1}^{n}\ln(G_2(x_{2i})) - (\alpha+2)\sum_{i=1}^{n}\frac{[a_2G_2(x_{2i})]^{\beta_2}\ln[a_2G_2(x_{2i})]}{\Delta_1(x_1,x_2)} - \frac{n\alpha a_2^{\beta_2}\ln a_2}{\gamma}\\ &\Big[(1+a_2^{\beta_2})^{-(\alpha+1)} - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-(\alpha+1)}\Big]\end{aligned} \tag{26}$$

The maximum likelihood estimators of $\alpha, a_1, a_2, \beta_1$ and β_2 are obtained by equating the derivatives in (22), (23), (24), (25), and (26) to zero and solving the resulting equations.

The solution is done by using some numerical method. In the next section we will study the bivariate truncated Burr family of distributions for baseline Burr distribution. The resulting distribution is is called bivariate truncated Burr-Burr distribution.

5. The Bivariate Truncated Burr-Burr Distribution

A family of twelve cumulative distribution functions have been proposed by [11] and the Burr distribution is one of these. Several distributions are related with the Burr distribution; including the Lomax, the logistic, the log-logistic, the exponential, and Weibull distributions. In this section, we have introduced the Bivariate Truncated Burr-Burr ($BTBB$) distribution, by using the Burr distribution as a baseline distribution in the proposed $BTBF$ of distributions. We have given some desirable statistical properties of this new distribution. The maximum likelihood estimation for the parameters of the proposed $BTBB$ distribution is also discussed.
The $BTBB$ distribution is proposed by using the following *cdf* of the Burr distribution in Eq. 11.

$$F(x_1) = 1 - \frac{1}{(1+x_2^{\beta_1})^{\alpha}} \qquad \text{and} \qquad F(x_2) = 1 - \frac{1}{(1+x_2^{\beta_2})^{\alpha}}. \tag{27}$$

The *cdf* of the $BTBB$ is

$$\begin{aligned} F_{BTBB}(x_1,x_2) =& \frac{1}{\gamma}\Bigg[1 - [1 + (a_1(1-(1+x_1^{\beta_1})^{-\alpha}))^{\beta_1}]^{-\alpha} - [1 + (a_2(1-(1+x_2^{\beta_2})^{-\alpha}))^{\beta_2}]^{-\alpha} \\ & + [\Delta_2(x_1,x_2)]^{-\alpha}\Bigg], \end{aligned} \tag{28}$$

where

$$\Delta_2(x_1,x_2) = [1 + (a_1(1-(1+x_1^{\beta_1})^{-\alpha}))^{\beta_1} + (a_2(1-(1+x_2^{\beta_2})^{-\alpha}))^{\beta_2}]$$

The *pdf* for the bivariate truncated Burr-Burr distribution corresponding to Eq. 28 is

$$\begin{aligned} f_{BTBB}(x_1,x_2) =& \frac{\alpha^3(\alpha+1)a_1^{\beta_1}a_2^{\beta_2}\beta_1^2\beta_2^2 x_1^{\beta_1-1}x_2^{\beta_2-1}(1+x_1^{\beta_1})^{-(\alpha+1)}(1+x_2^{\beta_2})^{-(\alpha+1)}}{\gamma} \\ & \times\Bigg[[1-(1+x_1^{\beta_1})^{-\alpha}]^{\beta_1-1}[1-(1+x_2^{\beta_2})^{-\alpha}]^{\beta_2-1}[\Delta_2(x_1,x_2)]^{-(\alpha+2)}\Bigg], \end{aligned} \tag{29}$$

where γ and $\Delta_2(x_1,x_2)$ are defined earlier. In the following section, we have discussed some properties of the $BTBB$ distribution.

6. Properties of the Bivariate Truncated Burr-Burr Distribution

In this section, some statistical properties of the $BTBB$ distribution are discussed. The include marginal and conditional distributions, joint and ratio moments, marginal and inverse moments, conditional moments, bivariate reliability and hazard rate function and maximum likleihood estimation of the parameters.

6.1. The Marginal and Conditional Distributions

The marginal distributions of X_1 and X_2 are readily written from Eq. (28) as

$$F_{TBB}(x_1) = \frac{1 - [1 + a_1(1 - (1 + x_1^{\beta_1})^{-\alpha}))^{\beta_1}]^{-\alpha} - [1 + a_2^{\beta_2}]^{-\alpha} + [\Delta_1(x_1)]^{-\alpha}}{\gamma}, \tag{30}$$

and

$$F_{TBB}(x_2) = \frac{1 - [1 + a_2(1 - (1 + x_2^{\beta_2})^{-\alpha}))^{\beta_2}]^{-\alpha} - [1 + a_1^{\beta_1}]^{-\alpha} + [\Delta_1(x_2)]^{-\alpha}}{\gamma}, \tag{31}$$

where $\Delta_1(x_1) = 1 + (a_1(1 - (1 + x_1^{\beta_1})^{-\alpha}))^{\beta_1} + a_2^{\beta_2}$ and $\Delta_1(x_2) = 1 + (a_2(1 - (1 + x_2^{\beta_2})^{-\alpha}))^{\beta_2} + a_1^{\beta_1}$

The marginal density function for X_1 and X_2 are determined from Eq. (29) as

$$\begin{aligned} f_{TBB}(x_1) =& \frac{\alpha^2 \beta_1^2 a_1^{\beta_1} x_1^{\beta_1 - 1}(1 + x_1^{\beta_1})^{-(\alpha+1)}[1 - (1 + x_1^{\beta_1})^{-\alpha}]^{\beta_1 - 1}}{\gamma} \\ &\times \left[[1 + (a_1(1 - (1 + x_1^{\beta_1})^{-\alpha}))^{\beta_1}]^{-(\alpha+1)} - [\Delta_1(x_1)]^{-(\alpha+1)} \right] \end{aligned} \tag{32}$$

and

$$\begin{aligned} f_{TBB}(x_2) =& \frac{\alpha^2 \beta_2^2 a_2^{\beta_2} x_2^{\beta_2 - 1}(1 + x_2^{\beta_2})^{-(\alpha+1)}[1 - (1 + x_2^{\beta_2})^{-\alpha}]^{\beta_2 - 1}}{\gamma} \\ &\times \left[[1 + (a_2(1 - (1 + x_2^{\beta_2})^{-\alpha}))^{\beta_2}]^{-(\alpha+1)} - [\Delta_1(x_2)]^{-(\alpha+1)} \right] \end{aligned} \tag{33}$$

Using the density and distribution functions of the Burr distribution in Eq.17, the conditional distribution of X_1 given $X_2 = x_2$ for the $BTBB$ distribution is

$$f_{BTBB}(x_1|x_2) = \frac{\alpha(\alpha+1) a_1^{\beta_1} \beta_1^2 x_1^{\beta_1 - 1}(1 + x_1^{\beta_1})^{-(\alpha+1)}[1 - (1 + x_1^{\beta_1})^{-\alpha}]^{\beta_1 - 1} \Delta_2(x_1, x_2)^{-(\alpha+2)}}{[1 + (a_2(1 - (1 + x_2^{\beta_2})^{-\alpha}))^{\beta_2}]^{-(\alpha+1)} - [\Delta_1(x_2)]^{-(\alpha+1)}}. \tag{34}$$

Again using the density and distribution functions of Burr distribution in Eq.18, the conditional distribution of X_2 given $X_1 = x_1$ is

$$f_{BTBB}(x_2|x_1) = \frac{\alpha(\alpha+1) a_2^{\beta_2} \beta_2^2 x_2^{\beta_2 - 1}(1 + x_2^{\beta_2})^{-(\alpha+1)}[1 - (1 + x_2^{\beta_2})^{-\alpha}]^{\beta_2 - 1} \Delta_2(x_1, x_2)^{-(\alpha+2)}}{[1 + (a_1(1 - (1 + x_1^{\beta_1})^{-\alpha}))^{\beta_1}]^{-(\alpha+1)} - [\Delta_1(x_1)]^{-(\alpha+1)}} \tag{35}$$

where $\Delta_2(x_1, x_2)$ is earlier defined.

6.2. The Joint and Ratio Moments

The $(r,s)^{th}$ joint moment of X_1 and X_2 for the $BTBB$ distribution is obtained as

$$E(X_1^r\, X_2^s) = \int_0^\infty \int_0^\infty x_1^r\, x_2^s\, f(x_1,x_2)dx_1dx_2$$

Using the joint density function of$BTBB$ distribution, the joint moment is

$$E(X_1^r\, X_2^s) = \int_0^\infty \int_0^\infty \frac{\alpha^3(\alpha+1)a_1^{\beta_1}a_2^{\beta_2}\beta_1^2\beta_2^2 x_1^{r+\beta_1-1}x_2^{s+\beta_2-1}(1+x_1^{\beta_1})^{-(\alpha+1)}(1+x_2^{\beta_2})^{-(\alpha+1)}}{\gamma}$$

$$\times\left[[1-(1+x_1^{\beta_1})^{-\alpha}]^{\beta_1-1}[1-(1+x_2^{\beta_2})^{-\alpha}]^{\beta_2-1}[\Delta_2(x_1,x_2)]^{-(\alpha+2)}\right]dx_1dx_2$$

Let $u = 1-(1+x_1^{\beta_1})^{-\alpha}$ and $v = 1-(1+x_2^{\beta_2})^{-\alpha}$, then

$$E(X_1^r\, X_2^s) = \frac{\alpha(\alpha+1)a_1^{\beta_1}a_2^{\beta_2}\beta_1\beta_2}{\gamma}\int_0^1\int_0^1 u^{\beta_1-1}\, v^{\beta_2-1}\left[(1-u)^{-(\frac{1}{\alpha\beta_1})}-1\right]^r$$
$$\left[(1-v)^{-(\frac{1}{\alpha\beta_2})}-1\right]^s\left[1+(a_1\, u)^{\beta_1}+(a_2\, v)^{\beta_2}\right]^{-(\alpha+2)}dudv$$

Simplifying above, the joint moments for the $BTBB$ distribution is

$$E(X_1^rX_2^s) = \frac{\alpha(\alpha+1)a_1^{\beta_1}a_2^{\beta_2}\beta_1\beta_2}{\gamma}\sum_{n=0}^\infty\binom{-(\alpha+2)}{n}a_2^{\beta_2[-(\alpha+2)-n]}\left[\sum_{n,k=0}^\infty(-1)^n a_1^{\beta_1 k}\binom{r}{n}\binom{n}{k}\right.$$
$$\left.B\left(\beta_1(k+1),\frac{n-r}{\alpha\beta_1}+1\right)\sum_{n=0}^\infty(-1)^n\binom{s}{n}B\left(\beta_2[-(\alpha+2)-n+1],\frac{n-s}{\alpha\beta_2}+1\right)\right] \tag{36}$$

where $B(a,b)$ is the complete Beta function. The correlation coefficient between X_1 and X_2 can be computed for the $BTBB$ distribution. The correlation coefficients are given in Table 1 below for $\alpha = 1$. It is easy to see that the correlation coefficient increases with an increase in the values of the parameters.

The $(r,s)^{th}$ ratio moment of X_1 and X_2 for the $BTBB$ distribution is

$$E\left(\frac{X_1^r}{X_2^s}\right) = \int_0^\infty \int_0^\infty \frac{\alpha^3(\alpha+1)a_1^{\beta_1}a_2^{\beta_2}\beta_1^2\beta_2^2 x_1^{r+\beta_1-1}x_2^{-s+\beta_2-1}(1+x_1^{\beta_1})^{-(\alpha+1)}(1+x_2^{\beta_2})^{-(\alpha+1)}}{\gamma}$$

$$\times\left[[1-(1+x_1^{\beta_1})^{-\alpha}]^{\beta_1-1}[1-(1+x_2^{\beta_2})^{-\alpha}]^{\beta_2-1}[\Delta_2(x_1,x2)]^{-(\alpha+2)}\right]dx_1dx_2$$

$$=\frac{\alpha(\alpha+1)a_1^{\beta_1}a_2^{\beta_2}\beta_1\beta_2}{\gamma}\int_0^1\int_0^1 u^{\beta_1-1}\,v^{\beta_2-1}\left[(1-u)^{-(\frac{1}{\alpha\beta_1})}-1\right]^r$$
$$\left[(1-v)^{-(\frac{1}{\alpha\beta_2})}-1\right]^{-s}\left[1+(a_1\,u)^{\beta_1}+(a_2\,v)^{\beta_2}\right]^{-(\alpha+2)}dudv$$

After simplify, the ratio moment for the $BTBB$ distribution is

$$E\left(\frac{X_1^r}{X_2^s}\right)=\frac{\alpha(\alpha+1)a_1^{\beta_1}a_2^{\beta_2}\beta_1\beta_2}{\gamma}\sum_{n=0}^{\infty}\binom{-(\alpha+2)}{n}a_2^{\beta_2[-(\alpha+2)-n]}\left[\sum_{n,k=0}^{\infty}(-1)^n a_1^{\beta_1 k}\binom{r}{n}\binom{n}{k}\right.$$
$$\left.\times B\left(\beta_1(k+1),\frac{n-r}{\alpha\beta_1}+1\right)\sum_{n=0}^{\infty}(-1)^n\binom{-s}{n}B\left(\beta_2[-(\alpha+2)-n+1],\frac{n+s}{\alpha\beta_2}+1\right)\right]. \tag{37}$$

The marginal and inverse moments for the $BTBB$ distribution are discussed in the next subsection.

Table 1: **Correlation Coefficient for the $BTBB$ Distribution.**

	$\beta_1=3$ $\beta_2=3$			
	a_2			
a_1	2	3	4	5
2	0.189	0.238	0.259	0.272
3	0.238	0.314	0.350	0.369
4	0.259	0.350	0.396	0.421
5	0.272	0.369	0.421	0.451
	$\beta_1=4$ $\beta_2=5$			
	a_2			
a_1	2	3	4	5
2	0.352	0.379	0.377	0.374
3	0.429	0.493	0.489	0.479
4	0.446	0.535	0.535	0.533
5	0.448	0.546	0.563	0.557
	$\beta_1=6$ $\beta_2=6$			
	a_2			
a_1	2	3	4	5
2	0.472	0.495	0.486	0.479
3	0.495	0.566	0.561	0.552
4	0.486	0.561	0.566	0.563
5	0.479	0.552	0.563	0.563

6.3. The Marginal and Inverse Moments

The r^{th} marginal moment of X_1 for the $BTBB$ distribution is obtained by using Eq.32 as

$$E(X_1^r) = \int_0^\infty x_1^r \frac{\alpha^2\beta_1^2 a_1^{\beta_1} x_1^{\beta_1-1}(1+x_1^{\beta_1})^{-(\alpha+1)}[1-(1+x_1^{\beta_1})^{-\alpha}]^{\beta_1-1}}{\gamma}$$
$$\times\left[[1+(a_1(1-(1+x_1^{\beta_1})^{-\alpha}))^{\beta_1}]^{-(\alpha+1)} - [1+(a_1(1-(1+x_1^{\beta_1})^{-\alpha}))^{\beta_1}+a_2^{\beta_2}]^{-(\alpha+1)}\right]dx_1$$
$$=\frac{\alpha\,\beta_1\,a_1^{\beta_1}}{\gamma}\int_0^1 u^{\beta_1-1}\left[(1-u)^{\frac{-1}{\alpha\beta_1}}-1\right]^r - \left[[1+(a_1u)^{\beta_1}]^{-(\alpha+1)}[1+(a_1u)^{\beta_1}+a_2^{\beta_2}]^{-(\alpha+1)}\right]du$$

Simplifying above, the r^{th} marginal moment is

$$E(X_1^r) = \frac{\alpha\,\beta_1\,a_1^{\beta_1}}{\gamma}\sum_{n=0}^{\infty}(-1)^n\,a_1^{\beta_1[-(\alpha+1)-n]}\binom{-(\alpha+1)}{n}\binom{r}{n}B\left(\beta_1[-(\alpha+2)-n+1],\frac{n-r}{\alpha\beta_1}+1\right)$$
$$\times\left[1-[1+a_2^{\beta_2}]^n\right]. \tag{38}$$

Similarly, the marginal distribution in Eq.33 can be used to obtained the s^{th} marginal moment of the random variable X_2 for the $BTBB$ distribution as

$$E(X_2^s) = \int_0^\infty x_2^s \frac{\alpha^2\beta_2^2 a_2^{\beta_2} x_2^{\beta_2-1}(1+x_2^{\beta_2})^{-(\alpha+1)}[1-(1+x_2^{\beta_2})^{-\alpha}]^{\beta_2-1}}{\gamma}$$
$$\times\left[[1+(a_2(1-(1+x_2^{\beta_2})^{-\alpha}))^{\beta_2}]^{-(\alpha+1)} - [1+(a_2(1-(1+x_2^{\beta_2})^{-\alpha}))^{\beta_2}+a_1^{\beta_1}]^{-(\alpha+1)}\right]dx_2.$$

Solving the integral, the s^{th} marginal moment is

$$E(X_2^s) = \frac{\alpha\,\beta_2\,a_2^{\beta_2}}{\gamma}\sum_{n=0}^{\infty}(-1)^n\,a_2^{\beta_2[-(\alpha+1)-n]}\binom{-(\alpha+1)}{n}\binom{s}{n}B\left(\beta_2[-(\alpha+2)-n+1],\frac{n-s}{\alpha\beta_2}+1\right)$$
$$\times\left[1-[1+a_1^{\beta_1}]^n\right]. \tag{39}$$

The r^{th} and s^{th} inverse moments can be readily written from the marginal moments by replacing r with $-r$ and s with $-s$.
The conditional moments for the $BTBB$ distribution are obtained in the following subsection.

6.4. The Conditional Moments

The r^{th} conditional moment of X_1 given $X_2 = x_2$ for the $BTBB$ distribution is obtained by using Eq.34 as

$$E(X_1^r|X_2) = \int_0^\infty \frac{\alpha(\alpha+1)a_1^{\beta_1}\beta_1^2 x_1^{r+\beta_1-1}(1+x_1^{\beta_1})^{-(\alpha+1)}[1-(1+x_1^{\beta_1})^{-\alpha}]^{\beta_1-1}}{[1+(a_2(1-(1+x_2^{\beta_2})^{-\alpha}))^{\beta_2}]^{-(\alpha+1)} - [\Delta_1(x_2)]^{-(\alpha+1)}}\Big[\Delta_2(x_1,x_2)\Big]^{-(\alpha+2)} dx_1$$

$$= \frac{(\alpha+1)a_1^{\beta_1}\beta_1}{[1+(a_2(1-(1+x_2^{\beta_2})^{-\alpha}))^{\beta_2}]^{-(\alpha+1)} - [\Delta_1(x_2)]^{-(\alpha+1)}} \times \int_0^1 u^{\beta_1-1}[(1-u)^{\frac{-1}{\alpha\beta_1}}-1]^r \Big[1+(a_1u)^{\beta_1}+[a_2(1-(1+x_2^{\beta_2})^{-\alpha})]^{\beta_2}\Big]^{-(\alpha+2)} du$$

After simplification, we have

$$E(X_1^r|X_2) = \frac{(\alpha+1)\, a_1^{\beta_1}\, \beta_1}{[1+(a_2(1-(1+x_2^{\beta_2})^{-\alpha}))^{\beta_2}]^{-(\alpha+1)} - [\Delta_1(x_2)]^{-(\alpha+1)}} \sum_{n=0}^{\infty}(-1)^n a_1^{\beta_1[-(\alpha+2)-n]} \Big[1+[a_2(1-(1+x_2^{\beta_2})^{-\alpha})]^{\beta_2}\Big]^n \binom{-(\alpha+2)}{n}\binom{r}{n} B\Big(\beta_1[-(\alpha+2)-n+1], \frac{n-r}{\alpha\beta_1}+1\Big). \tag{40}$$

Again, the s^{th} conditional moment X_2 given $X_1 = x_1$ for the $BTBB$ distribution is obtained by using Eq.35 as

$$E(X_2^s|X_1) = \int_0^\infty \frac{\alpha(\alpha+1)a_2^{\beta_2}\beta_2^2 x_2^{s+\beta_2-1}(1+x_2^{\beta_2})^{-(\alpha+1)}[1-(1+x_2^{\beta_2})^{-\alpha}]^{\beta_2-1}}{[1+(a_1(1-(1+x_1^{\beta_1})^{-\alpha}))^{\beta_1}]^{-(\alpha+1)} - [\Delta_1(x_1)]^{-(\alpha+1)}}\Big[\Delta_2(x_1,x_2)\Big]^{-(\alpha+2)} dx_2$$

$$= \frac{(\alpha+1)a_2^{\beta_2}\beta_2}{[1+(a_1(1-(1+x_1^{\beta_1})^{-\alpha}))^{\beta_1}]^{-(\alpha+1)} - [\Delta_1(x_1)]^{-(\alpha+1)}} \int_0^1 v^{\beta_2-1}[(1-v)^{\frac{-1}{\alpha\beta_2}}-1]^s \Big[1+(a_2v)^{\beta_2}+[a_1(1-(1+x_1^{\beta_1})^{-\alpha})]^{\beta_1}\Big]^{-(\alpha+2)} dv$$

Simplifying, the s^{th} conditional moment of X_2 given $X_1 = x_1$ is

$$E(X_2^s|X_1) = \frac{(\alpha+1)\, a_2^{\beta_2}\, \beta_2}{[1+(a_1(1-(1+x_1^{\beta_1})^{-\alpha}))^{\beta_1}]^{-(\alpha+1)} - [\Delta_1(x_1)]^{-(\alpha+1)}} \sum_{n=0}^{\infty}(-1)^n a_2^{\beta_2[-(\alpha+2)-n]} \Big[1+[a_1(1-(1+x_1^{\beta_1})^{-\alpha})]^{\beta_1}\Big]^n \binom{-(\alpha+2)}{n}\binom{s}{n} B\Big(\beta_2[-(\alpha+2)-n+1], \frac{n-s}{\alpha\beta_2}+1\Big). \tag{41}$$

The conditional moments are functions of random variables..

6.5. The Bivariate Reliability Characteristics and Hazard Rate Function

Using Eq.19, the BRF of $BTBB$ distribution is

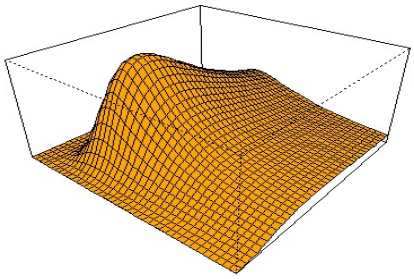
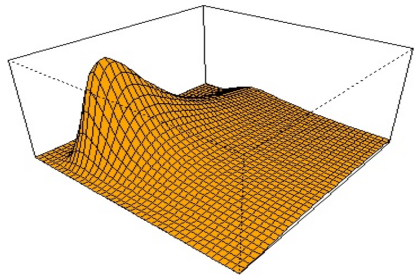
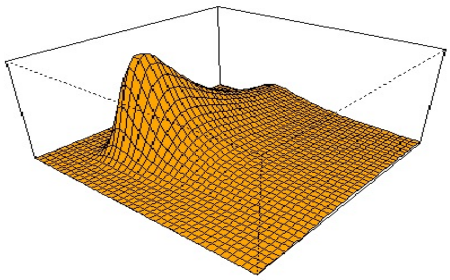
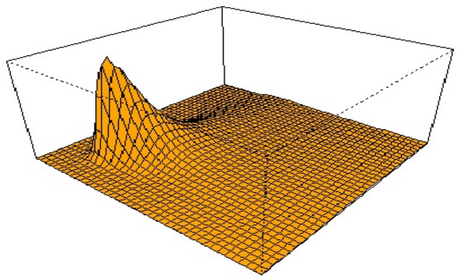
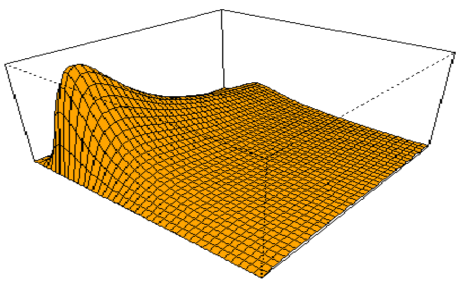
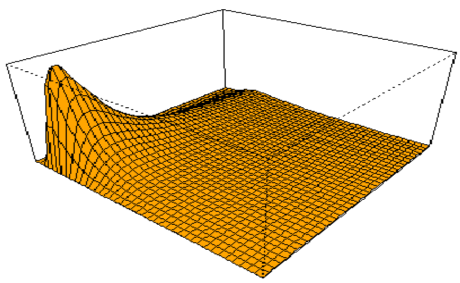
$$R_{BTBB}(x_1,x_2) = 1-\left[\frac{1-(1+a_1^{\beta_1})^{-\alpha}-(1+a_2^{\beta_2})^{-\alpha}}{\gamma}\right]-\left[\frac{[\Delta_1(x_1)]^{-\alpha}+[\Delta_1(x_2)]^{-\alpha}-[\Delta_2(x_1,x_2)]^{-\alpha}}{\gamma}\right] \tag{42}$$

Now using Eqs.(29) and 42 in Eq.20, the $BHRF$ for the $BTBB$ distribution is obtained as

$$\begin{aligned} H_{BTBB}(x_1,x_2) = &\left[\frac{\alpha^3(\alpha+1)a_1^{\beta_1}a_2^{\beta_2}\beta_1^2\beta_2^2x_1^{\beta_1-1}x_2^{\beta_2-1}(1+x_1^{\beta_1})^{-(\alpha+1)}(1+x_2^{\beta_2})^{-(\alpha+1)}}{\gamma}\right]\\ &\times\left[1-(1+x_1^{\beta_1})^{-\alpha}]^{\beta_1-1}[1-(1+x_2^{\beta_2})^{-\alpha}]^{\beta_2-1}\Delta_2(x_1,x_2)^{-(\alpha+2)}\right]\\ &\times\left[1-\left[\frac{1-(1+a_1^{\beta_1})^{-\alpha}-(1+a_2^{\beta_2})^{-\alpha}}{\gamma}\right]-\left[\frac{[\Delta_1(x_1)]^{-\alpha}+[\Delta_1(x_2)]^{-\alpha}-[\Delta_2(x_1,x_2)]^{-\alpha}}{\gamma}\right]\right]^{-1}. \end{aligned} \tag{43}$$

where γ , $\Delta_1(x_1)$,$\Delta_1(x_2)$ and $\Delta_2(x_1,x_2)$ are defined earlier. Table 2 contains graphs of the hazard rate function for different values of the parameters. The graph indicate that the HRF diminishes with decreases in α, a_1, and a_2. Conversely, increasing these parameters results in an elevation of the HRF.

Table 2: Bivariate Hazard Rate Function for $BTBB$ Distribution

α=0.25 ,β_1=2, β_2=2.5, a_1=2, a_2=3 	α=0.25 , β_1=2, β_2=2.5 ,a_1=5, a_2=6 
α=0.5 ,β_1=3,β_2=3,a_1=2, a_2=3 	α=0.5 ,β_1=3,β_2=3,a_1=5, a_2=6 
α=0.75 ,β_1=1.5,β_2=2,a_1=2, a_2=3 	α=0.75 ,β_1=1.5,β_2=2,a_1=5, a_2=6 

6.6. Maximum Likelihood Estimation

This subsection contains the maximum likelihood estimation for the parameters of the $BTBB$ distribution. Let $X_1, X_2, ..., X_n$ represent a size-n random sample from the $BTBB$ distribution. The likelihood function is

$$\begin{aligned} LH =& \alpha^n(\alpha+1)^n a_1^{n\beta_1} a_2^{n\beta_2} \beta_1^n \beta_2^n \left[1-(1+a_1^{\beta_1})^{-\alpha}-(1+a_2^{\beta_2})^{-\alpha}+(1+a_1^{\beta_1}+a_2^{\beta_2})^{-\alpha}\right]^{-n} \\ &\times \prod_{i=1}^{n}\left[\alpha\beta_1 x_1^{(\beta_1-1)}(1+x_1^{\beta_1})^{-(\alpha+1)}\right] \prod_{i=1}^{n}\left[\alpha\beta_2 x_2^{(\beta_2-1)}(1+x_2^{\beta_2})^{-(\alpha+1)}\right] \\ &\times \prod_{i=1}^{n}\left[1-(1+x_1^{\beta_1})^{-\alpha}\right]^{\beta_1-1} \prod_{i=1}^{n}\left[1-(1+x_2^{\beta_2})^{-\alpha}\right]^{\beta_2-1} \\ &\times \prod_{i=1}^{n}\left[1+(a_1(1-(1+x_1^{\beta_1})^{-\alpha})^{\beta_1}+(a_2(1-(1+x_2^{\beta_2})^{-\alpha})^{\beta_2}\right]^{-(\alpha+2)}. \end{aligned} \tag{44}$$

The log of likelihood function is

$$\begin{aligned} l =& 3n\ln(\alpha)+n\ln(\alpha+1)+n\beta_1\ln(a_1)+n\beta_2\ln(a_2)+2n\ln(\beta_1)+2n\ln(\beta_2) \\ &-n\ln\left[1-(1+a_1^{\beta_1})^{-\alpha}-(1+a_2^{\beta_2})^{-\alpha}+(1+a_1^{\beta_1}+a_2^{\beta_2})^{-\alpha}\right] \\ &+(\beta_1-1)\sum_{i=1}^{n}\ln(x_{1i})+(\beta_2-1)\sum_{i=1}^{n}\ln(x_{2i})-(\alpha+1)\sum_{i=1}^{n}\ln(1+x_{1i}^{\beta_1}) \\ &-(\alpha+1)\sum_{i=1}^{n}\ln(1+x_{2i}^{\beta_2})+(\beta_1-1)\sum_{i=1}^{n}\ln(1-(1+x_{1i}^{\beta_1})^{-\alpha}) \\ &+(\beta_2-1)\sum_{i=1}^{n}\ln(1-(1+x_{2i}^{\beta_2})^{-\alpha})-(\alpha+2)\sum_{i=1}^{n}\ln\left[\Delta(x_1,x_2)\right] \end{aligned} \tag{45}$$

By maximizing the log-likelihood function in Eq. (45), the MLEs of $\alpha, a_1, a_2, \beta_1$, and β_2 are found. The derivatives of the log-likelihood function with respect to the unknown parameters are

$$\begin{aligned} \frac{\partial l}{\partial\alpha} =& \frac{3n}{\alpha}+\frac{n}{\alpha+1}-\sum_{i=1}^{n}\ln(1+x_{1i}^{\beta_1})-\sum_{i=1}^{n}\ln(1+x_{2i}^{\beta_2})-\sum_{i=1}^{n}\ln(\Delta_2(x_1,x_2))-n\sum_{i=1}^{n}\frac{\lambda}{\gamma} \\ &+(\beta_1-1)\sum_{i=1}^{n}\frac{\ln(1+x_{1i}^{\beta_1})(1+x_{1i}^{\beta_1})^{-\alpha}}{1-(1+x_{1i}^{\beta_1})^{-\alpha}}+(\beta_2-1)\sum_{i=1}^{n}\frac{\ln(1+x_{2i}^{\beta_2})(1+x_{2i}^{\beta_2})^{-\alpha}}{1-(1+x_{2i}^{\beta_2})^{-\alpha}} \\ &+(\alpha+2)\beta_1\sum_{i=1}^{n}\frac{\ln(1+x_{1i}^{\beta_1})(1+x_{1i}^{\beta_1})^{-\alpha}(a_1(1-(1+x_{1i}^{\beta_1})^{-\alpha}))^{\beta_1-1}}{\Delta_2(x_1,x_2)} \\ &+(\alpha+2)\beta_2\sum_{i=1}^{n}\frac{\ln(1+x_{2i}^{\beta_2})(1+x_{2i}^{\beta_2})^{-\alpha}(a_2(1-(1+x_{2i}^{\beta_2})^{-\alpha}))^{\beta_2-1}}{\Delta_2(x_1,x_2)} \end{aligned} \tag{46}$$

where $\Delta_2(x_1,x_2)$ and γ earlier defined and

$$\lambda = (1+a_1^{\beta_1})^{-\alpha}\ln(1+a_1^{\beta_1}) + (1+a_2^{\beta_2})^{-\alpha}\ln(1+a_2^{\beta_2}) - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-\alpha}\ln(1+a_1^{\beta_1}+a_2^{\beta_2})$$

$$\begin{aligned}\frac{\partial l}{\partial a_1} =&\frac{n\beta_1}{a_1} - (\alpha+2)\sum_{i=1}^{n}\frac{\beta_1(a_1(1-(1+x_{1i}^{\beta_1})^{-\alpha}))^{\beta_1-1}(1-(1+x_{1i}^{\beta_1})^{-\alpha})}{\Delta_2(x_1,x_2)}\\ &-\frac{n\,\alpha\,a_1^{\beta_1-1}\,\beta_1}{\gamma}\left[(1+a_1^{\beta_1})^{-(\alpha+1)} - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-(\alpha+1)}\right]\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{\partial l}{\partial a_2} =&\frac{n\beta_2}{a_2} - (\alpha+2)\sum_{i=1}^{n}\frac{\beta_2(a_2(1-(1+x_{2i}^{\beta_2})^{-\alpha}))^{\beta_2-1}(1-(1+x_{2i}^{\beta_2})^{-\alpha})}{\Delta_2(x_1,x_2)}\\ &-\frac{n\,\alpha\,a_2^{\beta_2-1}\,\beta_2}{\gamma}\left[(1+a_2^{\beta_2})^{-(\alpha+1)} - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-(\alpha+1)}\right]\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial l}{\partial \beta_1} =&n\ln(a_1) + \frac{2n}{\beta_1} + \sum_{i=1}^{n}\ln(x_{1i}) - (\alpha+1)\sum_{i=1}^{n}\frac{x_{1i}^{\beta_1}\ln(x_{1i}))}{1+x_{1i}^{\beta_1}}\\ &+\sum_{i=1}^{n}\ln(1-(1+x_{1i}^{\beta_1})^{-\alpha}) + \alpha(\beta_1-1)\sum_{i=1}^{n}\frac{(1+x_{1i}^{\beta_1})\ln(x_{1i})}{1-(1+x_{1i}^{\beta_1})^{-\alpha}}\\ &-(\alpha+2)\,a_1^{\beta_1}\sum_{i=1}^{n}\frac{\left[1-(1+x_{1i}^{\beta_1})^{-\alpha})^{\beta_1-1}\right][\Delta_3(x_1)+\Delta_2(x_1)]}{\Delta_2(x_1,x_2)}\\ &-\frac{n\,\alpha\,a_1^{\beta_1}\,\ln(a_1)}{\gamma}\left[(1+a_1^{\beta_1})^{-(\alpha+1)} - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-(\alpha+1)}\right]\end{aligned} \tag{49}$$

where $\Delta_2(x_1) = a_1^{\beta_1}(1-(1+x_{1i}^{\beta_1})^{-\alpha})\ln(a_1(1-(1+x_{1i}^{\beta_1})^{-\alpha}))$ and

$\Delta_3(x_1) = \alpha\beta_1 x_{1i}^{\beta_1}\ln(x_{1i})(1+x_{1i}^{\beta_1})^{-(\alpha+1)}$.

$$\begin{aligned}\frac{\partial l}{\partial \beta_2} =&n\ln(a_2) + \frac{2n}{\beta_2} + \sum_{i=1}^{n}\ln(x_{2i}) - (\alpha+1)\sum_{i=1}^{n}\frac{x_{2i}^{\beta_2}\ln(x_{2i}))}{1+x_{2i}^{\beta_2}}\\ &+\sum_{i=1}^{n}\ln(1-(1+x_{2i}^{\beta_2})^{-\alpha}) + \alpha(\beta_2-1)\sum_{i=1}^{n}\frac{(1+x_{2i}^{\beta_2})\ln(x_{2i})}{1-(1+x_{2i}^{\beta_2})^{-\alpha}}\\ &-(\alpha+2)\,a_2^{\beta_2}\sum_{i=1}^{n}\frac{\left[1-(1+x_{2i}^{\beta_2})^{-\alpha})^{\beta_2-1}\right][\Delta_3(x_2)+\Delta_2(x_2)]}{\Delta_2(x_1,x_2)}\\ &-\frac{n\,\alpha\,a_2^{\beta_2}\,\ln(a_2)}{\gamma}\left[(1+a_2^{\beta_2})^{-(\alpha+1)} - (1+a_1^{\beta_1}+a_2^{\beta_2})^{-(\alpha+1)}\right]\end{aligned} \tag{50}$$

where $\Delta_2(x_2) = a_2^{\beta_2}(1-(1+x_{2i}^{\beta_2})^{-\alpha})\ln(a_2(1-(1+x_{2i}^{\beta_2})^{-\alpha}))$ and

$$\Delta_3(x_2) = \alpha\beta_2 x_{2i}^{\beta_2} \ln(x_{2i})(1 + x_{2i}^{\beta_2})^{-(\alpha+1)}.$$

The MLEs for $\alpha, a_1, a_2, \beta_1$, and β_2 are obtained by equating the above derivatives to zero and numerically solving the resulting equations.

7. Real Data Application

In this section, we have checked the suitability of the proposed $BTBB$ distribution by using three data sets. These data sets have been modeled by using the $BTBB$ distribution alongside five other distributions. The competing distributions that we have used in the study are bivariate Burr (BB) distribution by Durling [13],bivariate Lomax (BL), bivariate log logistic (BL-L), bivariate transmuted Burr (BTB), and bivariate transmuted Weibull (BTW) distribution. The distributions are fitted by computing the $MLEs$ of the parameters. To compare the performance of the proposed $BTBB$ distribution with the competing distributions, we calculated Akaike's information criteria (AIC) and Bayesian information criteria (BIC) for each model. The best model has the smallest AIC and BIC. The application on the real data sets are given in the following.

7.1. Patients Data

This data set refers to 30 patients set from [21]. The first recurrence time is represented by X_1, and the asecond recurrence time is represented by X_2. Table 3 contains the MLE of the parameters while Table 4 provides the computed values of AIC and BIC. It is easy to see, from Table 4, that The $BTBB$ distribution has the smallest values of AIC and BIC, and hence is the best fit for this data.

7.2. Gross National Income Data

This data represents the Gross National Income of all countries of the World having 191 observations. In the data set, X_1 represents the Gross National Income for year 2016 and X_2 represents the Gross National Income for 2017. The $MLEs$ of various distributions are given in Table 5 and the computed values of AIC and BIC are given in Table 6. From Table 6, it is clear that the $BTBB$ distribution is the best fitted distribution as it has the smallest values of AIC and BIC.

7.3. Breaking Strength of Fluid Data

The third data set is Breaking Strength of Fluid, and its size is 70. In this data, X_1 denotes the force while X_2 denotes the fluid's breaking strength. The $MLE's$ of different distributions are given in Table 7 while Table 8 contains the computed values of AIC and BIC. From Table8, it is obvious that the $BTBB$ distribution is the best fit for the third data also since it has the smallest values of AIC and BIC values.

In order to assess whether the $BTBB$ is an appropriate model, we have given plots of original data and the fitted $BTBB$ distribution in Table 9. From these plots, We conclude that the $BTBB$ distribution provides a good fit to the three data sets.

Table 3: Maximum Likelihood Estimation and Standard Error for Given Distributions.

Distribution	Parameter	Estimate	Standard Error
$BTBB$	a_1	6.099633	3.757314
	a_2	1.196423	0.079680
	α	0.035319	0.004062
	β_1	2.942372	1.055427
	β_2	17.900089	1.115601
BB	α	0.24009	0.05489
	β_1	1.01058	0.14462
	β_1	1.00785	0.14352
BL	α	0.2437	0.04343
BL-L	β_1	0.48789	0.05733
	β_1	0.48784	0.05713
BTB	c	0.61384	0.13757
	k	0.57343	0.16657
	λ_1	-0.10532	4.61707
	λ_2	-0.66142	4.19430
	λ_3	-0.06115	4.32286
BTW	α_1	0.14116	0.01599
	α_2	0.08136	0.01173
	θ_1	0.09075	0.04044
	θ_2	0.02337	0.01651
	λ_1	-0.07125	2.46306
	λ_2	-0.02464	2.47251
	λ_3	-0.09762	3.21034

Table 4: Akaike's and Bayesian Information Criteria for Given Distributions

Distribution	Log-Lik	AIC	BIC
$BTBB$	-352.2676	714.5352	721.5412
BB	-367.7602	741.5204	745.724
BL	-367.7632	737.5263	738.9276
BL-L	-391.7709	787.5418	790.3442
BTB	-381.0651	772.1303	779.1362
BTW	-455.7447	925.4893	935.2978

Table 5: Maximum Likelihood Estimation and Standard Error for Given Distributions

Distribution	Parameter	Estimate	Standard Error
$BTBB$	a_1	4.451e+03	1.024
	a_2	7.187e+03	1.274
	α	3.767e-02	3.447e-03
	β_1	2.855	5.850e-03
	β_1	2.718	5.607e-02
BB	α	1.02615	0.07057
	β_1	1.73827	0.09080
	β_1	1.72734	0.09005
BL	α	1.26400	0.07984
BL-L	β_1	1.74835	0.08739
	β_1	1.73769	0.08659
BTB	c	1.39380	0.06815
	k	1.06489	0.07529
	λ_1	-0.61765	2.97112
	λ_2	-0.80989	2.97112
	λ_3	0.65903	2.09854
BTW	α_1	0.63299	0.03652
	α_2	0.41041	0.02767
	θ_1	0.52677	0.04261
	θ_2	0.32809	0.03480
	λ_1	-0.92304	2.89542
	λ_2	-0.28514	2.96639
	λ_3	0.39789	2.95426

Table 6: Akaike's and Bayesian Information Criteria for Given Distributions

Distribution	Log-Lik	AIC	BIC
$BTBB$	-427.0594	864.1188	870.2132
BB	-505.2134	1016.427	1020.083
BL	-567.7754	1137.551	1138.77
BL-L	-505.2836	1014.567	1017.005
BTB	-610.8848	1231.77	1248.031
BTW	-805.9473	1625.895	1634.427

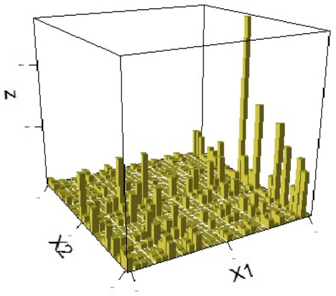
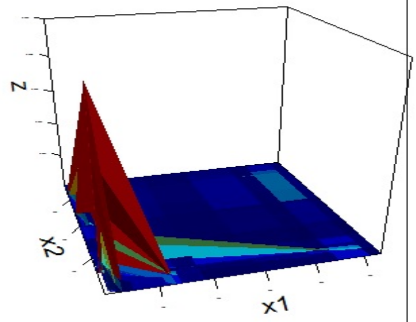
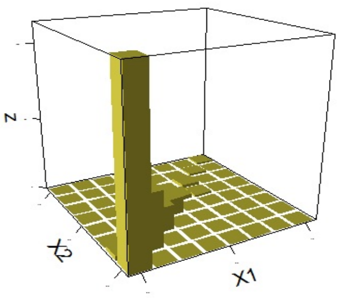
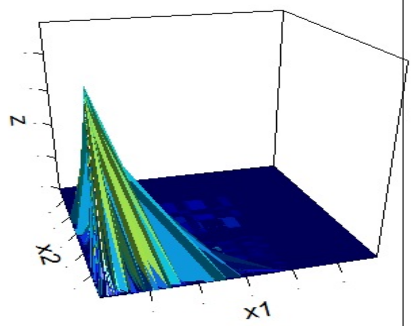
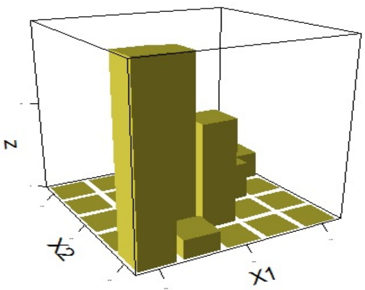
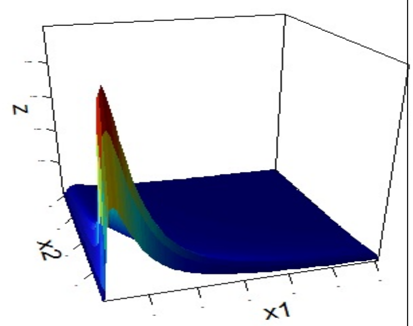
Table 7: Maximum Likelihood Estimation and Standard Error for Given Distributions

Distribution	Parameter	Estimate	Standard Error
$BTBB$	a_1	0.88335	0.20265
	a_2	1.14359	0.33553
	α	0.13561	0.02215
	β_1	5.64257	0.73521
	β_1	3.11552	0.48151
BB	α	0.22861	0.03475
	β_1	1.71784	0.15556
	β_1	1.41722	0.16461
BL	α	0.36525	0.04217
BL-L	β_1	0.78856	0.06185
	β_1	0.91949	0.07211
BTB	c	0.75154	0.08629
	k	0.70208	0.10160
	λ_1	-0.12896	33.69200
	λ_2	-0.80981	27.06451
	λ_3	-0.07487	30.5993
BTW	α_1	0.116303	0.012094
	α_2	0.129937	0.012523
	θ_1	0.025941	0.009192
	θ_2	0.021107	0.006740
	λ_1	-0.115249	1.896561
	λ_2	0.017064	1.896561
	λ_3	-0.060330	2.389490

Table 8: Akaike's and Bayesian Information Criteria for Given Distributions

Distribution	Log-Lik	AIC	BIC
$BTBB$	-616.712	1243.424	1254.666
BB	-626.6695	1259.339	1266.084
BL	-623.3973	1248.795	1251.043
BL-L	-672.6314	1349.263	1353.76
BTB	–672.4948	1354.99	1366.232
BTW	-873.6064	1761.213	1776.952

Table 9: Bivariate Histograms and Fitted Distribution for Three Data Sets.

Data	Observed Histogram	Fitted Distribution
First Data		
Second Data		
Third Data		

8. Conclusion

In this paper, we have presented a new truncated bivariate families of distributions. The new families of distributions are explored by using the Burr distribution as a generator, giving rise to the bivariate truncated Burr family of distributions. The bivariate truncated Burr family of distributions is further investigated by using the Burr distribution as a baseline distribution, resulting in a bivariate truncated Burr-Burr distributions. We have investigated various statistical characteristics of the proposed bivariate Burr-Burr distribution. Additionally, we have used three real data sets with the bivariate truncated Burr-Burr distributions. We have found that the proposed bivariate truncated Burr-Burr distribution performs well for modeling the given data. It is possible to explore the new $BTBF$ of distributions for various baseline distributions.

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Appendix

Real Data Sets

1- Patients Data

X_1

8, 23, 22, 447, 30, 24, 7, 511, 53, 15, 7, 141, 96, 149, 536, 17, 185, 292, 22, 15, 152, 402, 13, 39, 12, 113,132, 34, 2, 130.

X_2

16, 13, 28, 318, 12, 245, 9, 30, 196, 154, 333, 8, 38, 70,25, 4, 117, 114, 159, 108, 362, 24, 66, 46, 40, 201, 156, 30, 25, 26.

2- Gross National Income Data

X_1

0.1822, 1.1512, 1.3809, 4.6252, 0.5956, 2.0302, 1.7857, 0.8350, 4.3637, 4.4443, 1.5751, 2.6632, 4.1918, 0.3509, 1.5622, 1.5765, 4.1588, 0.7419, 0.2010, 0.7574, 0.6621, 1.1353, 1.5455, 1.3730, 7.6870, 1.7759, 0.1600, 0.0721, 0.5829, 0.3246, 0.3280, 4.2664, 0.0644, 0.1850, 2.1768, 1.4354, 1.3050, 0.1396, 0.6630, 0.0792, 1.4490, 2.1088, 0.7487, 3.0955, 2.9400, 0.3323, 4.7209, 0.3268, 0.8756, 1.3282, 1.0234, 1.0185, 0.7663, 2.1316, 0.1700, 2.7645, 0.7702, 0.1603, 0.8080, 4.0066, 3.8702, 1.6623, 0.1510, 0.8785, 4.5203, 0.3889, 2.4284, 1.2460, 0.7191, 0.1779,

0.1540, 0.7278, 0.1681, 0.4096, 5.5809, 2.4337, 4.4971, 0.6026, 1.0437, 1.8544, 1.8446, 5.0475, 3.2273, 3.4733, 0.7832, 3.8267, 0.8320, 2.2054, 0.2898, 0.3002, 3.5122, 7.4109, 0.3113, 0.5822, 2.3685, 1.3011, 0.3124, 0.0667, 0.8876, 9.7355, 2.6884, 6.5460, 0.1339, 0.1053, 2.4968, 1.2717, 0.1901, 3.3025, 0.5006, 0.3520, 1.9468, 1.6623, 0.3789, 0.5311, 1.0618, 1.5961, 0.7149, 0.1098, 0.5282, 0.9582, 1.8652, 0.2334, 4.6711, 3.3679, 0.5145, 0.0898, 0.5326, 6.7340, 3.8129, 0.5155, 1.3466, 0.5410, 1.8494, 0.3398, 0.8424, 1.1635, 0.8729, 2.4983, 2.6521, 11.8088, 2.1060, 2.3843, 0.1744, 2.3792, 1.1441, 1.0358, 0.5804, 0.2894, 5.1329, 0.2297, 1.2877, 2.5334, 0.1216, 7.8427, 2.8546, 2.9161, 0.1850, 1.1948, 0.1115, 3.3307, 1.1118, 0.4015, 1.3413, 4.7378, 5.7636, 0.2432, 0.3164, 0.2557, 1.4971, 1.2557, 0.8045, 0.1407, 0.5447, 2.9396, 1.0192, 2.3500, 1.4890, 0.5752, 0.1654, 0.7593, 6.8121, 3.8680, 5.4104, 1.9502, 0.6135, 0.2928, 1.2570, 0.5589, 0.1480, 0.3522, 0.1677.

X_2

0.1824, 1.1886, 1.3802, 4.7574, 0.5790, 2.0764, 1.8461, 0.9144, 4.3560, 4.5415, 1.5600, 2.6681, 4.1580, 0.3677, 1.5843, 1.6323, 4.2156, 0.7166, 0.2061, 0.8065, 0.6714, 1.1716, 1.5534, 1.3755, 7.6427, 1.8740, 0.1650, 0.0702, 0.5983, 0.3413, 0.3315, 4.3433, 0.0663, 0.1750, 2.1910, 1.5270, 1.2938, 0.1399, 0.5694, 0.0796, 1.4636, 2.2162, 0.7524, 3.1568, 3.0588, 0.3481, 4.7918, 0.3392, 0.8344, 1.3921, 1.0347, 1.0355, 0.6868, 1.9513, 0.1750, 2.8993, 0.7620, 0.1719, 0.8324, 4.1002, 3.9254, 1.6431, 0.1516, 0.9186, 4.6136, 0.4096, 2.4648, 1.2864, 0.7278, 0.2067, 0.1552, 0.7447, 0.1665, 0.4215, 5.8420, 2.5393, 4.5810, 0.6353, 1.0846, 1.9130, 1.7789, 5.3754, 3.2711, 3.5299, 0.7846, 3.8986, 0.8288, 2.2626, 0.2961, 0.3042, 3.5945, 7.0524, 0.3255, 0.6070, 2.5002, 1.3378, 0.3255, 0.0667, 1.1100, 9.7336, 2.8314, 6.5016, 0.1358, 0.1064, 2.6107, 1.3567, 0.1953, 3.4396, 0.5125, 0.3592, 2.0189, 1.6944, 0.3843, 0.5554, 1.0103, 1.6779, 0.7340, 0.1093, 0.5567, 0.9387, 1.8573, 0.2471, 4.7900, 3.3970, 0.5157, 0.0906, 0.5231, 6.8012, 3.6290, 0.5311, 1.2831, 0.5055, 1.9178, 0.3403, 0.8380, 1.1789, 0.9154, 2.6150, 2.7315, 11.6818, 2.2646, 2.4233, 0.1811, 2.3978, 1.1695, 1.0499, 0.5909, 0.2941, 4.9680, 0.2384, 1.3019, 2.6077, 0.1240, 8.2503, 2.9467, 3.0594, 0.1872, 1.1923, 0.0963, 3.4258, 1.1326, 0.4119, 1.3306, 4.7766, 5.7625, 0.2337, 0.3317, 0.2655, 1.5516, 1.2505, 0.6846, 0.1453, 0.5547, 2.8622, 1.0275, 2.4804, 1.5594, 0.5888, 0.1658, 0.8130, 6.7805, 3.9116, 5.4941, 1.9930, 0.6470, 0.2995, 1.0672, 0.5859, 0.1239, 0.3557, 0.1683.

3- Breaking Strength of Fluid Data

X_1

0.66552, 1.32920, 1.99020, 2.64780, 3.30510, 3.96520, 4.61540, 5.26260, 5.91770, 6.55930, 7.20630, 7.85490, 8.48920, 9.12330, 9.75220, 10.39500, 11.01200, 11.65400, 12.27500, 12.90500, 13.51400, 14.14600, 14.77300, 15.39700, 16.00400, 16.62100, 17.23300, 17.84900, 18.46800, 19.04800, 19.68000, 20.24500, 20.84200, 21.45700, 22.04300, 22.65200, 23.21200, 23.82100, 24.43700, 25.02500, 25.58700, 26.19300, 26.74800, 27.31500, 27.88500, 28.51000, 29.07300, 29.61700, 30.21500, 30.74700, 31.33800, 33.04800, 34.72200, 36.38200, 38.03300, 39.60300, 41.23000, 42.84500, 44.44800, 45.98700, 47.54100, 49.14000, 50.65300, 52.27000, 53.63200, 55.23800, 56.51000, 58.14300, 59.55000, 60.92400.

X_2

1.433193, 1.921320, 2.669056, 3.279862, 3.703085, 4.175902, 5.180407, 5.311075, 5.480077, 5.944436, 6.644375, 6.870545, 7.567605, 7.363139, 7.708288, 7.861320, 8.041136, 8.567537, 8.816040, 8.680506, 9.233999, 9.491558, 9.536301, 9.699463, 9.991789, 10.379586, 10.684267, 11.283192, 11.077667, 11.295088, 11.256025, 12.181274, 12.205218, 11.998618, 12.517958, 12.407828, 13.409593, 14.055641, 13.446431, 14.335534, 14.528933, 15.351420, 15.256486, 15.450883, 15.000152, 16.024559, 16.133539, 16.790022, 17.621796, 17.252878, 17.397621, 18.850729, 18.921950, 20.292019, 21.790637, 22.214351, 23.902455, 24.655257, 26.207445, 27.813584, 27.866845, 29.506678, 30.753878, 32.763854, 34.007137, 35.547121, 37.161858,38.220492, 39.666229, 41.468994.