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AI-Assisted Wearable Devices for Promoting Human Health and Strength Using Complex Interval-Valued Picture Fuzzy Soft Relations

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Abstract. Wearable technology allows users to monitor and track various fitness-related activities and data, including calories burned, distance traveled, heart rate, and sleep patterns. To choose best app for fitness, here introduced a new innovation in fuzzy algebra named as complex interval valued picture fuzzy soft relations (CIVPFSR) by defining the cartesian product (CP) of two complex interval valued picture fuzzy soft sets (CIVPFSS). Additionally, the many kinds of CIVPFS relations are described, and their outcomes are also mentioned in this article. The CIVPFSS has a complex structure that includes multidimensional variables for membership, abstinence, and non-membership degrees. As a result, this research offers modeling techniques for wearable technology that are based on CIVPFSR. The score function for best human decision making is also formulated in this process. Lastly, a comparison of current structures is done to demonstrate the viability of the suggested work with the conclusion highlighting its potential to significantly advance wearable technology. Additionally, a topological space is developed for CIVPFSR, providing a geometric interpretation of these relations and offering insights into their behavior in multi-dimensional decision spaces. The basic operations, operators, related theorems, and results are discussed. Results concerning the interiors and closures are also addressed.

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1. Introduction

Many daily decisions are characterized by uncertainty. When choosing a course of action or decision that could have multiple alternative outcomes, an ambiguous situation results. Human decisions are typically ambiguous and imprecise. It is a necessary component of life. Meanwhile, Zadeh [44] introduced the theory of fuzzy sets (FSs) in 1965, which deals with the ambiguity and uncertainty of human decision-making and successfully identifies, processes, and resolves uncertainty in mathematics. A membership function is assigned by an FS to every element whose values fall within the interval $[0, 1]$. Fuzzy sets are used to quickly and effectively answer ambiguous and uncertain real-world problems. The development of fuzzy set theory allowed for the handling and modeling of uncertainty in novel ways.

According to Klir [20], crisp sets are only defined as yes or no. On the other hand, uncertainty is incompatible with the crisp theory, which can only handle precise information. Mendal et al. [26] introduced the idea of fuzzy relations (FRs), which specify the degree, grade, and strength of positive interactions between any pair of FSs. A closer membership number to 1 indicates a positive relationship. A membership value that is closer to 0 indicates a weak relationship. To lessen ambiguity in decision-making, Zadeh [45] proposed the interval-valued fuzzy sets (IVFSs) in 1975. These sets substitute an interval of membership for a single value of membership. The interval's extremes are sub-intervals of $[0, 1]$. An extension of fuzzy relations (FRs) named interval-valued fuzzy relations (IVFRs) was introduced by Bustince and Burillo [11].

The concept of intuitionistic fuzzy sets (IFSs), which address the membership and non-membership of any element, was first put forth by Atanassov [7]. The term IFSs encompasses a broader scope than FSs. The idea of intuitionistic fuzzy relations (IFRs), which simultaneously explain a relation's effectiveness and ineffectiveness, was first presented by Burillo et al. [10]. It's a continuation of FRs.

An expansion of FSs and IFSs known as picture fuzzy sets (PFSs) is a new concept developed by Cuong et al. [22]. Three stages of an element are covered in this structure: the first is the membership level, the second is the indeterminacy, and the third is the non-membership level. Additionally, the sum of the membership, indeterminacy, and non-membership belongs to $[0, 1]$. They also presented the idea of picture fuzzy relations (PFRs).

Following the definition of FSs, a new development in fuzzy sets called the concept of complex fuzzy sets (CFSs) was first introduced by Ramot et al. [32]. CFS represents membership grades as a complex number, i.e., $r\ddot{v}^{s(\ddot{v})2\pi i}$, where r is the amplitude term and $s(\ddot{v})$ is the phase term. A CFS can model an issue with periodicity. It reduces the possibility of mistakes and misunderstandings. Ramot et al.[32] also provided a definition of complex fuzzy relations (CFRs). Rani et al. [33] established complex intuitionistic fuzzy sets (CIFSs), where membership and non-membership are represented by complex numbers. The complex intuitionistic fuzzy relations (CIFRs) were introduced by Jan et al. [18] through an investigation of cybersecurity and cybercrimes in the oil and gas industry.

1.1. Research Gap

Despite the growing use of wearable technology to track fitness data, there is a significant research gap in optimizing the selection process for the best fitness app or device. While fitness trackers provide useful data on calories burned, heart rate, distance, and sleep patterns, current methods of choosing the right app often do not fully account for the complex relationships between various factors, such as user preferences and multidimensional fitness data. Existing studies have not explored advanced mathematical models, like Complex Interval Valued Picture Fuzzy Soft Relations (CIVPFSR), that can incorporate multiple variables such as membership, abstinence, and non-membership in a user's decision-making process. A topological space can also be defined on this structure, providing a formal framework for analyzing its properties. In addition, both basic and advanced results related to topology can be integrated into this structure, allowing for a deeper exploration of its underlying characteristics and behaviors This research aims to fill this gap by applying CIVPFSR to develop a more refined model for selecting the most suitable fitness app or device, thereby enhancing the decision-making process and advancing the field of wearable technology.

1.2. Motivation

The research presented in reference by Jan et al. [17] explores Generative Adversarial Networks (GANs), powerful models that generate data samples based on statistical distributions. However, processing visual data often leads to uncertainties and misperceptions. To tackle this challenge, the authors introduce a groundbreaking mathematical framework called complex picture fuzzy soft relations (CPFSRs), which merges concepts from picture fuzzy sets and soft sets. This innovative framework provides a strong solution for managing inconsistent information and enhances decision-making by simplifying complex data structures. The study not only leverages CPFSRs to evaluate and select the optimal GAN but also conducts a comprehensive comparative analysis to demonstrate the superiority of the proposed approach. In a novel contribution, the research extends this work by incorporating R-interval valued data with complex picture fuzzy soft information, offering fresh insights and addressing intriguing applications with clear and insightful explanations.

1.3. Significance of the Study

By enabling users to watch vital parameters like heart rate, distance walked, calories burned, and sleep patterns, wearable technology has completely changed how we track and maximize our fitness journeys. These gadgets provide unmatched flexibility, enabling users to customize their workout and health regimens, whether they are independent or built into smartwatches. Fitness trackers offer a user-friendly platform for goal-setting, progress monitoring, and obtaining comprehensive insights into one's fitness journey when paired with companion smartphone apps. This work presents complex interval-valued picture fuzzy soft relations (CIVPFSR), a revolutionary development in fuzzy algebra that goes one step further in choosing the best fitness software. This novel idea provides a new way to assess fitness technology since it is defined by the Cartesian product (CP) of two complex interval-valued picture fuzzy soft sets (CIVPFSS). In order to demonstrate the complex, multifaceted character of CIVPFSS-which encompasses membership, abstention, and non-membership degrees-the paper examines a variety of CIVPFS interactions and their results. Using this sophisticated framework, the study offers state-of-the-art wearable technology modeling methods along with a recently developed score function for better decision-making. A comparison analysis with current models is shown to illustrate the practicality of this technique and highlight how this work has the potential to greatly improve wearable technology and fitness tracking in the future.

1.4. Literature Review

Complex picture fuzzy sets (CPFSs) are a novel idea developed by Akram et al. [3]. The given concept is an expansion of CIFSs and includes an additional degree of indeterminacy to convey the neutral effect of each element. Each of the three phases is represented by a complex number formed by combining the interval's left and right extremes, which belongs to $[0, 1]$. Complex interval-valued fuzzy sets (CIVFSs) were introduced by Greenfield et al. [14]. Complex interval-valued fuzzy relations (CIVFRs) are utilized to examine the interactions between two or more CIVFSs. Nasir et al. [29] proposed an application of CIVFRs to medical diagnosis in 2021. Complex interv al-valued intuitionistic fuzzy sets (CIVIFSs), which consist of the membership and non-membership degrees of complex values, were introduced by Garg et al. [13]. Even with all these advancements in decision-making, choosing the right course of action is still a challenge for people.

Molodtsov [28] developed the ideas of soft sets (SS) in 1999, which help in decision-making in unexpected ways. SSs use specific criteria to select the best options. Alkhazaleh et al. [6] defined the soft multiset theory, and Yang et al. [42] suggested a generalization of SSs. Maji et al. [25] presented an application of SSs in decision-making difficulties. Babitha and Sunil [8] established the idea of soft relations (SRs) and the study of soft sets. Park et al. [30] explored a few of the effects of equivalence SRs. Maji et al. [24] introduced the idea of the fuzzy soft set (FSS) by combining the FSs and SSs. It reduces uncertainty, which promotes better daily decision-making. Ali et al. [5] made a comment on SSs, harsh SSs, and FSSs. For FSS decision-making, Feng et al. [12] developed a flexible method. Yao et al. [43] clarified the differences between soft sets and FSSs.

FRs and SRs are combined to create fuzzy soft relations (FSRs), a novel idea created by Borah et al. [9]. While Sut et al. [37] recommended using FSRs in decision-making, Mockor and Hurtik [27] approximated FSSs using FSRs in conjunction with image processing. The concept of complex fuzzy soft sets (CFSSs) was proposed by Thirunavukarasu et al. [39]. Tamir et al. [38] provided an overview of the theory and applications of complex fuzzy logic. The idea of a sophisticated multi-fuzzy soft expert set and its applications was created by Al-Qudah and Hassan [4]. Yang et al. [41] developed the interval-valued fuzzy soft set (IVFSS), an example of an uncertainty model that is more realistic than the FS. Tripathy et al. [40] discussed the use of IVFSSs in group decision-making processes. Selvachandran et al. [34] introduced the innovative idea of CIVFSS through an application. The concept of intuitionistic fuzzy soft sets (IFSSs), an extension of fuzzy soft sets (FSSs), was introduced by Maji et al. [23]. The notion of a complex intuitionistic fuzzy soft set (CIFSS) was first presented by Kumar et al. [21]. With complex values, it is an IFSS generalized form. It handles multifaceted issues and improves the accuracy of human decision-making. Kumam et al. [19] introduced a novel advancement in fuzzy theory called the picture fuzzy soft set (PFSS). A new notion of complex picture fuzzy soft set (CPFSS) was introduced by Jan et al. [17]. Introduces by Shihadeh et al. [35] two-fold fuzzy n refined neutrosophic rings for $n \geq 3$, providing a framework for handling uncertainty in algebraic structures. Relevant for wearable devices by supporting robust data processing. Presents two-fold fuzzy algebras based on neutrosophic real numbers, offering tools for modeling imprecise data by Shihadeh et al. [36]. Enhances wearable device performance in uncertain decision-making. Rajalakshmi et al. [31] explores neutrosophic ideals in ordered ternary semigroups. Crucial for structured data analysis, such as health monitoring in wearables. Abubaker et al. [2] is an Solves neutrosophic singular boundary value problems using (LPM) polynomials. Assists wearable devices in managing variable health thresholds. Investigates threshold conversion numbers for graph products and neutrosophic graphs. Supports threshold-based decision-making in wearables by Abubaker et al. [1]. Hatamleh et al. [16] uses a complex tangent trigonometric approach with rung fuzzy sets, improving multi-sensor data fusion in wearables. Hatamleh et al. [15] by explores generalized weighted operators in trigonometric rung interval-valued fuzzy settings, aiding adaptive algorithms for health monitoring. This literature review aims to give a thorough overview of the state of CIVPFSS research today and to highlight the key findings, applications, and advancements in the field.

The concept of Complex Interval-Valued Picture Fuzzy Soft Sets (CIVPFSSs) represents a significant advancement in the area of multi-dimensional decision-making under uncertainty. This framework extends old fuzzy and soft set theories by incorporating a complex-valued membership function with interval and picture fuzzy dimensions. The presence of interval-valued grades allows the representation of uncertainty in a range rather than a fixed value, while picture fuzzy components enable the modeling of hesitancy alongside positive, neutral, and negative membership degrees. Together, these features make CIVPFSSs highly suitable for tackling problems that involve intricate and ambiguous data structures.

CIVPFSSs are particularly effective in applications requiring multi-faceted evaluation criteria. For instance, in wearable technology, selecting the most efficient device involves balancing multiple parameters such as accuracy, comfort, durability, cost, and connectivity. CIVPFSSs enable a holistic approach by incorporating interval-valued grades to accommodate varying levels of certainty in the performance of these parameters while addressing hesitancy using picture fuzzy characteristics. The amplitude term of the membership grade captures the strength or reliability of a parameter, while the phase term reflects its periodic variations over time.

This approach also supports the modeling of relations such as reflexivity, symmetry, transitivity, and equivalence, making it a versatile tool for analyzing interconnected datasets. Through its complex interval-valued representation, CIVPFSSs can effectively manage incomplete, inconsistent, and ambiguous data, outperforming traditional methodologies in both accuracy and interpretability. These features make CIVPFSSs an indispensable framework for modern decision-making problems, particularly in fields like wearable health monitoring, environmental modeling, and intelligent systems design. The score function in CIVPFSSs allows for a systematic assessment of parameters by integrating interval-valued grades to accommodate uncertainty and picture fuzzy characteristics to address hesitancy in decision-making.

The structure of the paper is outlined as follows:

2. Preliminaries / Basic Concepts

This section explains some predefined concepts related to fuzzy algebra, including fuzzy sets, complex fuzzy sets, fuzzy soft sets, complex fuzzy soft sets, complex interval-valued fuzzy soft sets, complex interval-valued intuitionistic fuzzy soft sets, picture fuzzy soft sets, complex picture fuzzy soft sets, and interval-valued fuzzy sets.

Definition 1. *[44] A fuzzy set FS A on a universal set X with a mapping $\mu(\ddot{v}) : X \to [0,1]$ is defined as:*

$$A = \{(\ddot{v}, \mu(\ddot{v})) \,|\, \ddot{v} \in X\}$$

where μ is the membership function of $\ddot{v}$.

Definition 2. *[32] Let X be a universal set; then, a CFS A with mapping $r, s : X \to [0,1]$ is expressed as:*

$$A = \{(\ddot{v}, r(\ddot{v})e^{(s(\ddot{v}))2\pi i}) : \ddot{v} \in X\}$$

where r and s are the amplitude term and phase term of the membership, respectively.

Definition 3. *[7] Let X be a universal set; then, an IFS A is expressed as:*

$$A = \{(\ddot{v}, \mu(\ddot{v}), \nu(\ddot{v})) : \ddot{v} \in X\}$$

where $0 \leq \mu(\ddot{v}) + \nu(\ddot{v}) \leq 1$. Further, the functions $\mu : X \to [0,1]$ and $\nu : X \to [0,1]$ represent the degree of membership and non-membership of the element $\ddot{v} \in X$.

Definition 4. *[22] A PFS A on a universal set X is defined as:*

$$A = \{(\ddot{v}, \mu(\ddot{v}), \phi(\ddot{v}), \nu(\ddot{v})) : \ddot{v} \in X\}$$

where $\mu(\ddot{v}), \phi(\ddot{v}), \nu(\ddot{v}) \in [0,1]$ denote the degree of membership, neutral, and non-membership, respectively. Also, $0 \leq \mu(\ddot{v}) + \phi(\ddot{v}) + \nu(\ddot{v}) \leq 1$.

Definition 5. *[13] An CIVIFS A on a universal set X is defined as:*

$$A = \{\ddot{v}, [\mu^-(\ddot{v}), \mu^+(\ddot{v})], [\nu^-(\ddot{v}), \nu^+(\ddot{v})] : \ddot{v} \in X\}$$

where $\mu^-(\ddot{v}), \mu^+(\ddot{v}) \in [0,1]$ and $\nu^-(\ddot{v}), \nu^+(\ddot{v}) \in [0,1]$, which satisfy $0 \leq \mu^+(\ddot{v}) + \nu^+(\ddot{v}) \leq 1$ and $0 \leq \mu^-(\ddot{v}) + \nu^-(\ddot{v}) \leq 1$.

The values of $\mu^-(\ddot{v}), \nu^-(\ddot{v})$ and $\mu^+(\ddot{v}), \nu^+(\ddot{v})$ denote the lower and upper terms of the degree of membership and non-membership, respectively. Also, $\mu^-(\ddot{v}), \mu^+(\ddot{v}), \nu^-(\ddot{v}), \nu^+(\ddot{v})$ have complex values, including amplitude and phase terms.

Definition 6. *[30] Let X be a universal set and E be a set of parameters. Let $P(X)$ denote the set of all possible subsets of X. Then, a FSS (A, T) with $T \subseteq E$ and a mapping $A : E \to P(X)$ is represented as:*

$$A = \{(\ddot{v}, \mu(\ddot{v})) : \ddot{v} \in E, \mu(\ddot{v}) \in P(X)\}$$

Example 1. *Let E be a set of parameters and let X be a set of refrigerator companies. Assume that each membership degree assigned by experts and the refrigerator attribute as a function of some parameter are represented by a fuzzy soft set (F, E).*

$$X = \{x_1, x_2, x_3, x_4\}, \quad i.e., \; x_1 = LG, \; x_2 = Samsung, \; x_3 = Orient, \; x_4 = Daewoo.$$

$$E = \{\ddot{v}_1, \ddot{v}_2, \ddot{v}_3\}, \quad i.e., \; \ddot{v}_1 = Design, \; \ddot{v}_2 = Beautiful, \; \ddot{v}_3 = Digital.$$

$$\begin{aligned} F(\ddot{v}_1) &= \{x_1 = 0.40, \; x_2 = 0.51, \; x_3 = 0.72, \; x_4 = 0.33\}, \\ F(\ddot{v}_2) &= \{x_1 = 0.34, \; x_2 = 0.85, \; x_3 = 0.66, \; x_4 = 0.67\}, \\ F(\ddot{v}_3) &= \{x_1 = 0.68, \; x_2 = 0.49, \; x_3 = 0.30, \; x_4 = 0.71\}. \end{aligned}$$

The fuzzy soft set (F, E) is a parameterized family of $\{F(\ddot{v}_i), i = 1, 2, 3\}$.

Definition 7. *[39] Let X be a universal set and E be a set of parameters. Let $CP(X)$ denote the set of all possible complex fuzzy subsets of X. Then, a CFSS (A, T) with $T \subseteq E$ and a mapping $A : E \to CP(X)$ is represented as:*

$$A = \{(\ddot{v}, \mu_c(\ddot{v})) : \ddot{v} \in E, \mu(\ddot{v}) \in CP(X)\} = \{(\ddot{v}, r_\mu(\ddot{v}) e^{s_\mu(\ddot{v}) 2\pi i}) : \ddot{v} \in E\},$$

where

$$\mu_c(\ddot{v}) = r_\mu(\ddot{v}) e^{s_\mu(\ddot{v}) 2\pi i},$$

and $r_\mu(\ddot{v}), s_\mu(\ddot{v}) : E \to [0, 1]$, where r_μ and s_μ are the amplitude term and phase term of the degree of membership.

Definition 8. *[23] Let X be a universal set and E be a set of parameters. Let $IP(X)$ denote the set of all possible intuitionistic fuzzy subsets of X. Then, an IFSS (A, T) with $T \subseteq E$ and a mapping $A : E \to IP(X)$ is represented as:*

$$A = \{(\ddot{v}, \mu(\ddot{v}), \nu(\ddot{v})) : \forall \ddot{v} \in E, \mu(\ddot{v}), \nu(\ddot{v}) \in IP(X)\},$$

where $\mu(\ddot{v})$ and $\nu(\ddot{v})$ are the membership and non-membership degrees, respectively.

Example 2. *From Example 1, assume an intuitionistic fuzzy soft set (F, E) describing the characteristics of refrigerators with respect to some parameters, where each membership and non-membership degree is assigned by experts.*

$$\begin{cases} F(\ddot{v}_1) = \{x_1 = (0.41, 0.53), \; x_2 = (0.50, 0.63), \; x_3 = (0.31, 0.62), \; x_4 = (0.33, 0.44)\}, \\ F(\ddot{v}_2) = \{x_1 = (0.25, 0.36), \; x_2 = (0.17, 0.58), \; x_3 = (0.39, 0.50), \; x_4 = (0.41, 0.50)\}, \\ F(\ddot{v}_3) = \{x_1 = (0.43, 0.54), \; x_2 = (0.35, 0.56), \; x_3 = (0.23, 0.46), \; x_4 = (0.30, 0.42)\}. \end{cases}$$

Then, the intuitionistic fuzzy soft set (F, E) is a parameterized family, i.e., $\{F(\ddot{v}_i), i = 1, 2, 3\}$.

Definition 9. *[21] Let X be a universal set and E be a set of parameters. Let $CIP(X)$ denote the set of all possible complex intuitionistic fuzzy subsets of X. Then, a CIFSS (A,T) with $T \subseteq E$ and a mapping $A : E \to CIP(X)$ is represented as:*

$$A = \{(\ddot{v}, \mu_c(\ddot{v}), \nu_c(\ddot{v})) : \forall \ddot{v} \in E, \mu_c(\ddot{v}), \nu_c(\ddot{v}) \in CIP(X)\} = \{(\ddot{v}, r_\mu(\ddot{v})e^{s_\mu(\ddot{v})2\pi i}, r_\nu(\ddot{v})e^{s_\nu(\ddot{v})2\pi i}) : \ddot{v} \in E\},$$

where r_μ, r_ν are the amplitude terms of the membership and non-membership degrees, respectively, and s_μ, s_ν are the phase terms of the membership and non-membership degrees, respectively.

Definition 10. *[19] Let X be a universal set and E be a set of parameters. Let $PP(X)$ denote the set of all possible picture fuzzy subsets of X. Then, a PFSS (A,T) with $T \subseteq E$ and a mapping $A : E \to PP(X)$ is represented as:*

$$A = \{(\ddot{v}, \mu(\ddot{v}), \phi(\ddot{v}), \nu(\ddot{v})) : \ddot{v} \in E, \mu(\ddot{v}), \phi(\ddot{v}), \nu(\ddot{v}) \in PP(X)\},$$

where $\mu(\ddot{v}), \phi(\ddot{v}), \nu(\ddot{v})$ are the membership, neutral, and non-membership degrees, respectively.

Example 3. *From Example 1, assume a PFSS (F,E) describing the characteristics of the refrigerator with respect to some parameters, where each membership, neutral, and non-membership degree is assigned by experts.*

$$\begin{cases} F(\ddot{v}_1) = \{x_1 = (0.10, 0.35, 0.50),\ x_2 = (0.20, 0.31, 0.45),\ x_3 = (0.31, 0.21, 0.50),\ x_4 = (0.23, 0.44, 0.12)\}, \\ F(\ddot{v}_2) = \{x_1 = (0.25, 0.36, 0.20),\ x_2 = (0.17, 0.38, 0.40),\ x_3 = (0.39, 0.41, 0.31),\ x_4 = (0.11, 0.30, 0.43)\}, \\ F(\ddot{v}_3) = \{x_1 = (0.13, 0.34, 0.23),\ x_2 = (0.25, 0.46, 0.12),\ x_3 = (0.23, 0.16, 0.43),\ x_4 = (0.30, 0.12, 0.50)\}. \end{cases}$$

Then, the picture fuzzy soft set (F,E) is a parameterized family, i.e., $\{F(\ddot{v}_i), i = 1, 2, 3\}$.

Definition 11. *[17] Let X be a universal set and E be a set of parameters. Let $CPP(X)$ denote the set of all possible complex picture fuzzy subsets of X. Then, a CPFSS (A,T) and $T \subseteq E$ with a mapping $A : E \to CPP(X)$ is represented as:*

$$\begin{aligned} A &= \{(\ddot{v}, \mu_c(\ddot{v}), \varphi(\ddot{v}), \psi_c(\ddot{v})) : \forall \mu(\ddot{v}), \varphi(\ddot{v}), \psi(\ddot{v}) \in CPP(X)\} \\ &= \{(\ddot{v}, r_\mu(\ddot{v})e^{s_\mu(\ddot{v})2\pi i}, \ddot{v}, r_\varphi(\ddot{v})e^{s_\varphi(\ddot{v})2\pi i}, \ddot{v}, r_\psi(\ddot{v})e^{s_\psi(\ddot{v})2\pi i}) : \ddot{v} \in E\} \end{aligned}$$

where r_μ, r_φ, r_ψ are the amplitude terms of membership, neutral, and non-membership degree, respectively, and s_μ, s_φ, s_ψ are the phase terms of membership, neutral, and non-membership degree, respectively.

3. Main Result

We clarify the idea of the CP of two complex interval-valued picture fuzzy soft sets, complex interval-valued picture fuzzy soft relations, and their types. Furthermore, we describe their examples and useful results.

Definition 12. *Let X be a universal set and E be a set of parameters. Let $CIVP(X)$ denote the set of all possible complex interval-valued picture fuzzy subsets of X. Then, a CIVPFSS (A,T) and $T \subseteq E$ with a mapping $A : E \to PPP(X)$ is represented as:*

$$F = \left\{ \ddot{v}; \begin{pmatrix} [\mu^-(\ddot{v}), \mu^+(\ddot{v})][e^{\ddot{v}2\pi i}, e^{\ddot{v}^+2\pi i}], \\ [\varphi^-(\ddot{v}), \varphi^+(\ddot{v})][e^{\varphi^-2\pi i}, e^{\varphi^+2\pi i}], \\ [\partial^-(\ddot{v}), \partial^+(\ddot{v})][e^{f^-(\ddot{v})2\pi i}, e^{f^+(\ddot{v})2\pi i}] \end{pmatrix} : \ddot{v} \in X \right\}$$

Where $\mu^+(\ddot{v}), \mu^-(\ddot{v}), \varphi^+(\ddot{v}), \varphi^-(\ddot{v}), \psi^+(\ddot{v}), \psi^-(\ddot{v})$ show the amplitude terms of the degree of the membership, neutral, and non-membership degree, respectively. $\ddot{v}^+(\ddot{v}), \ddot{v}^-(\ddot{v}), j^+(\ddot{v}), j^-(\ddot{v}), f^+(\ddot{v}), f^-(\ddot{v})$ show the phase terms of the degree of membership, neutral, and non-membership, respectively.

Example 4. *Suppose the universal set $X = \{X_1, X_2, X_3\}$ consists of three car companies, i.e., $X_1 =$ Mercedes, $X_2 =$ Porsche, $X_3 =$ McLaren, and there are three parameters $E = \{\ddot{v}_1, \ddot{v}_2, \ddot{v}_3\}$, i.e., $\ddot{v}_1 =$ Price, $\ddot{v}_2 =$ Engine, $\ddot{v}_3 =$ Model. Then,*

$$F = \left\{ \begin{array}{l} \ddot{v}_1 = \begin{pmatrix} ([0.22, 0.26]e^{2\pi i[0.12,0.21]}), & ([0.13, 0.25]e^{2\pi i[0.28,0.34]}), & ([0.12, 0.31]e^{2\pi i[0.32,0.42]}) \\ ([0.41, 0.42]e^{2\pi i[0.14,0.22]}), & ([0.32, 0.45]e^{2\pi i[0.23,0.42]}), & ([0.22, 0.45]e^{2\pi i[0.23,0.42]}) \\ ([0.31, 0.42]e^{2\pi i[0.14,0.22]}), & ([0.34, 0.44]e^{2\pi i[0.34,0.43]}), & ([0.21, 0.30]e^{2\pi i[0.34,0.44]}) \end{pmatrix} \\ \\ \ddot{v}_2 = \begin{pmatrix} ([0.23, 0.34]e^{2\pi i[0.32,0.43]}), & ([0.17, 0.42]e^{2\pi i[0.22,0.52]}), & ([0.43, 0.52]e^{2\pi i[0.23,0.43]}) \\ ([0.21, 0.41]e^{2\pi i[0.23,0.42]}), & ([0.14, 0.24]e^{2\pi i[0.23,0.42]}), & ([0.34, 0.44]e^{2\pi i[0.23,0.42]}) \\ ([0.15, 0.31]e^{2\pi i[0.12,0.42]}), & ([0.12, 0.24]e^{2\pi i[0.26,0.43]}), & ([0.13, 0.22]e^{2\pi i[0.14,0.44]}) \end{pmatrix} \\ \\ \ddot{v}_3 = \begin{pmatrix} ([0.34, 0.53]e^{2\pi i[0.33,0.43]}), & ([0.04, 0.90]e^{2\pi i[0.38,0.47]}), & ([0.25, 0.35]e^{2\pi i[0.34,0.43]}) \\ ([0.13, 0.22]e^{2\pi i[0.26,0.45]}), & ([0.12, 0.21]e^{2\pi i[0.26,0.45]}), & ([0.15, 0.25]e^{2\pi i[0.26,0.45]}) \\ ([0.12, 0.23]e^{2\pi i[0.23,0.44]}), & ([0.01, 0.05]e^{2\pi i[0.25,0.42]}), & ([0.12, 0.21]e^{2\pi i[0.26,0.45]}) \end{pmatrix} \end{array} \right\}$$

In the above observation, each value shows the membership, neutral, and non-membership degree of each company. Here, X_1, X_2, X_3 represent the first, second, and third values of the universal set. The terms in $\ddot{v}_1$ show the parameters of the universal set, where the first term shows the price, the second term shows the engine, and the third term shows the model. The same applies for $\ddot{v}_2$ and $\ddot{v}_3$.

Definition 13. *Suppose that (F, A) and (G, B) are two complex interval-valued picture fuzzy soft sets on universal sets X, Y, and $A, B \subseteq E$. Let $(F, A) \times (G, B) = (H, C)$. Then, the CP of CIVPFSSs is denoted by:*

$$F = \left\{ \ddot{v}; \begin{pmatrix} [\mu^-(\ddot{v}), \mu^+(\ddot{v})][e^{\ddot{v}^-(\ddot{v})2\pi i}, e^{\ddot{v}^+(\ddot{v})2\pi i}], \\ [\varphi^-(\ddot{v}), \varphi^+(\ddot{v})][e^{j^-(\ddot{v})2\pi i}, e^{j^+(\ddot{v})2\pi i}], \\ [\partial^-(\ddot{v}), \partial^+(\ddot{v})][e^{f^-(\ddot{v})2\pi i}, e^{f^+(\ddot{v})2\pi i}] \end{pmatrix} : \ddot{v} \in X \right\}$$

$$G = \left\{ \acute{\omega}; \begin{pmatrix} [\mu^-(\acute{\omega}), \mu^+(\acute{\omega})][e^{\ddot{v}^-(\acute{\omega})2\pi i}, e^{\ddot{v}^+(\acute{\omega})2\pi i}], \\ [\varphi^-(\acute{\omega}), \varphi^+(\acute{\omega})][e^{j^-(\acute{\omega})2\pi i}, e^{j^+(\acute{\omega})2\pi i}], \\ [\partial^-(\acute{\omega}), \partial^+(\acute{\omega})][e^{f^-(\acute{\omega})2\pi i}, e^{f^+(\acute{\omega})2\pi i}] \end{pmatrix} : \acute{\omega} \in Y \right\}$$

Define:

$$F \times G = H = \left\{ (\ddot{v}, \acute{\omega}); \begin{pmatrix} [\mu^-(\ddot{v}, \acute{\omega}), \mu^+(\ddot{v}, \acute{\omega})][e^{\ddot{v}^-(\ddot{v},\acute{\omega})2\pi i}, e^{\ddot{v}^+(\ddot{v},\acute{\omega})2\pi i}], \\ [\varphi^-(\ddot{v}, \acute{\omega}), \varphi^+(\ddot{v}, \acute{\omega})][e^{j^-(\ddot{v},\acute{\omega})2\pi i}, e^{j^+(\ddot{v},\acute{\omega})2\pi i}], \\ [\partial^-(\ddot{v}, \acute{\omega}), \partial^+(\ddot{v}, \acute{\omega})][e^{f^-(\ddot{v},\acute{\omega})2\pi i}, e^{f^+(\ddot{v},\acute{\omega})2\pi i}] \end{pmatrix} : \ddot{v} \in X, \acute{\omega} \in Y \right\}$$

Where:

$$\mu^+(\ddot{v}, \acute{\omega}) = \min[\mu^+(\ddot{v}), \mu^+(\acute{\omega})], \quad \mu^-(\ddot{v}, \acute{\omega}) = \min[\mu^-(\ddot{v}), \mu^-(\acute{\omega})]$$

$$\varphi^+(\ddot{v}, \acute{\omega}) = \min[\varphi^+(\ddot{v}), \varphi^+(\acute{\omega})], \quad \varphi^-(\ddot{v}, \acute{\omega}) = \min[\varphi^-(\ddot{v}), \varphi^-(\acute{\omega})]$$

$$\partial^+(\ddot{v}, \acute{\omega}) = \max[\partial^+(\ddot{v}), \partial^+(\acute{\omega})], \quad \partial^-(\ddot{v}, \acute{\omega}) = \max[\partial^-(\ddot{v}), \partial^-(\acute{\omega})]$$

$$\ddot{v}^+(\ddot{v}, \acute{\omega}) = \min[\ddot{v}^+(\ddot{v}), \ddot{v}^+(\acute{\omega})], \quad \ddot{v}^-(\ddot{v}, \acute{\omega}) = \min[\ddot{v}^-(\ddot{v}), \ddot{v}^-(\acute{\omega})]$$

$$j^{+}(\ddot{v},\acute{\omega}) = \min[j^{+}(\ddot{v}), j^{+}(\acute{\omega})], \quad j^{-}(\ddot{v},\acute{\omega}) = \min[j^{-}(\ddot{v}), j^{-}(\acute{\omega})]$$

$$f^{+}(\ddot{v},\acute{\omega}) = \max[f^{+}(\ddot{v}), f^{+}(\acute{\omega})], \quad f^{-}(\ddot{v},\acute{\omega}) = \max[f^{-}(\ddot{v}), f^{-}(\acute{\omega})]$$

Example 5. *Suppose the universal set $X = \{X_1, X_2, X_3\}$ consists of three car companies, i.e., $X_1 =$ Mercedes, $X_2 =$ Porsche, $X_3 =$ McLaren, and there are three parameters ($E = \{\ddot{v}_1, \ddot{v}_2, \ddot{v}_3\}$), i.e., $\ddot{v}_1 =$ Price, $\ddot{v}_2 =$ Engine, $\ddot{v}_3 =$ Model. Let $(F, A) = F$ and $(G, B) = G$ be two CIVPFSS on X, as shown below:*

$$F = \begin{cases} \ddot{v}_1 = \begin{bmatrix} ([0.22, 0.26]e^{2\pi i[0.12,0.21]}), ([0.41, 0.42]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.14,0.22]}) \\ ([0.13, 0.25]e^{2\pi i[0.28,0.34]}), ([0.32, 0.45]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.42]}) \\ ([0.12, 0.31]e^{2\pi i[0.32,0.42]}), ([0.22, 0.45]e^{2\pi i[0.23,0.42]}), ([0.21, 0.30]e^{2\pi i[0.34,0.43]}) \end{bmatrix} \\ \ddot{v}_2 = \begin{bmatrix} ([0.23, 0.34]e^{2\pi i[0.32,0.43]}), ([0.21, 0.41]e^{2\pi i[0.23,0.42]}), ([0.15, 0.31]e^{2\pi i[0.12,0.42]}) \\ ([0.17, 0.42]e^{2\pi i[0.22,0.52]}), ([0.14, 0.24]e^{2\pi i[0.23,0.42]}), ([0.12, 0.24]e^{2\pi i[0.26,0.34]}) \\ ([0.43, 0.52]e^{2\pi i[0.23,0.43]}), ([0.34, 0.44]e^{2\pi i[0.23,0.42]}), ([0.13, 0.22]e^{2\pi i[0.14,0.44]}) \end{bmatrix} \\ \ddot{v}_3 = \begin{bmatrix} ([0.34, 0.53]e^{2\pi i[0.33,0.43]}), ([0.13, 0.22]e^{2\pi i[0.26,0.45]}), ([0.12, 0.23]e^{2\pi i[0.23,0.44]}) \\ ([0.04, 0.90]e^{2\pi i[0.38,0.47]}), ([0.12, 0.21]e^{2\pi i[0.26,0.45]}), ([0.01, 0.05]e^{2\pi i[0.25,0.42]}) \\ ([0.25, 0.35]e^{2\pi i[0.34,0.43]}), ([0.15, 0.25]e^{2\pi i[0.26,0.45]}), ([0.12, 0.21]e^{2\pi i[0.26,0.45]}) \end{bmatrix} \end{cases}$$

And

$$G = \begin{cases} \ddot{v}_1 = \begin{bmatrix} ([0.12, 0.26]e^{2\pi i[0.12,0.31]}), ([0.32, 0.41]e^{2\pi i[0.26,0.45]}), ([0.21, 0.42]e^{2\pi i[0.14,0.25]}) \\ ([0.23, 0.25]e^{2\pi i[0.18,0.34]}), ([0.11, 0.21]e^{2\pi i[0.26,0.45]}), ([0.24, 0.44]e^{2\pi i[0.13,0.42]}) \\ ([0.22, 0.31]e^{2\pi i[0.22,0.42]}), ([0.22, 0.33]e^{2\pi i[0.26,0.45]}), ([0.11, 0.30]e^{2\pi i[0.24,0.43]}) \end{bmatrix} \\ \ddot{v}_2 = \begin{bmatrix} ([0.13, 0.34]e^{2\pi i[0.22,0.43]}), ([0.32, 0.41]e^{2\pi i[0.26,0.45]}), ([0.25, 0.31]e^{2\pi i[0.32,0.42]}) \\ ([0.27, 0.42]e^{2\pi i[0.12,0.52]}), ([0.22, 0.27]e^{2\pi i[0.26,0.45]}), ([0.22, 0.24]e^{2\pi i[0.16,0.34]}) \\ ([0.33, 0.52]e^{2\pi i[0.13,0.43]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.13, 0.22]e^{2\pi i[0.24,0.44]}) \end{bmatrix} \\ \ddot{v}_3 = \begin{bmatrix} ([0.24, 0.53]e^{2\pi i[0.23,0.43]}), ([0.18, 0.26]e^{2\pi i[0.26,0.45]}), ([0.12, 0.24]e^{2\pi i[0.33,0.44]}) \\ ([0.05, 0.90]e^{2\pi i[0.28,0.47]}), ([0.32, 0.41]e^{2\pi i[0.26,0.45]}), ([0.03, 0.05]e^{2\pi i[0.15,0.42]}) \\ ([0.15, 0.35]e^{2\pi i[0.24,0.43]}), ([0.32, 0.51]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \end{bmatrix} \end{cases}.$$

In the above observation, each value shows the membership, neutral, and non-membership degree of each company. Each row represents the parametric observations.

Then, their CP of (F, A) and (G, B) is defined as: The parametric observation H is given as:

$$H = \left\{ \begin{array}{l} (\ddot{\upsilon}_1, \ddot{\upsilon}_1) : \Big(([0.12, 0.26]e^{2\pi i[0.12,0.21]}, [0.32, 0.41]e^{2\pi i[0.14,0.22]}, [0.31, 0.42]e^{2\pi i[0.14,0.25]}), \\ \qquad ([0.13, 0.25]e^{2\pi i[0.18,0.34]}, [0.11, 0.21]e^{2\pi i[0.23,0.42]}, [0.34, 0.44]e^{2\pi i[0.23,0.42]}), \\ \qquad ([0.12, 0.31]e^{2\pi i[0.22,0.42]}, [0.22, 0.33]e^{2\pi i[0.26,0.45]}, [0.21, 0.30]e^{2\pi i[0.34,0.43]}) \Big), \\ \\ (\ddot{\upsilon}_1, \ddot{\upsilon}_2) : \Big(([0.13, 0.26]e^{2\pi i[0.12,0.21]}, [0.32, 0.41]e^{2\pi i[0.14,0.22]}, [0.31, 0.42]e^{2\pi i[0.32,0.42]}), \\ \qquad ([0.13, 0.25]e^{2\pi i[0.12,0.34]}, [0.22, 0.27]e^{2\pi i[0.23,0.42]}, [0.34, 0.44]e^{2\pi i[0.23,0.34]}), \\ \qquad ([0.12, 0.31]e^{2\pi i[0.13,0.42]}, [0.16, 0.29]e^{2\pi i[0.26,0.45]}, [0.21, 0.30]e^{2\pi i[0.34,0.44]}) \Big), \\ \\ (\ddot{\upsilon}_1, \ddot{\upsilon}_3) : \Big(([0.22, 0.26]e^{2\pi i[0.12,0.21]}, [0.18, 0.26]e^{2\pi i[0.14,0.22]}, [0.31, 0.42]e^{2\pi i[0.33,0.44]}), \\ \qquad ([0.05, 0.25]e^{2\pi i[0.28,0.34]}, [0.32, 0.41]e^{2\pi i[0.26,0.45]}, [0.34, 0.44]e^{2\pi i[0.23,0.42]}), \\ \qquad ([0.12, 0.31]e^{2\pi i[0.24,0.42]}, [0.22, 0.45]e^{2\pi i[0.26,0.45]}, [0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big), \\ \\ (\ddot{\upsilon}_2, \ddot{\upsilon}_1) : \Big(([0.13, 0.26]e^{2\pi i[0.12,0.21]}, [0.32, 0.41]e^{2\pi i[0.14,0.22]}, [0.31, 0.42]e^{2\pi i[0.32,0.42]}), \\ \qquad ([0.13, 0.25]e^{2\pi i[0.12,0.34]}, [0.22, 0.27]e^{2\pi i[0.23,0.42]}, [0.34, 0.44]e^{2\pi i[0.23,0.34]}), \\ \qquad ([0.12, 0.31]e^{2\pi i[0.13,0.42]}, [0.16, 0.29]e^{2\pi i[0.26,0.45]}, [0.21, 0.30]e^{2\pi i[0.34,0.44]}) \Big), \\ \\ (\ddot{\upsilon}_3, \ddot{\upsilon}_3) : \Big(([0.53, 0.24]e^{2\pi i[0.23,0.43]}, [0.13, 0.41]e^{2\pi i[0.26,0.45]}, [0.24, 0.12]e^{2\pi i[0.33,0.44]}), \\ \qquad ([0.90, 0.04]e^{2\pi i[0.28,0.47]}, [0.12, 0.21]e^{2\pi i[0.26,0.45]}, [0.05, 0.03]e^{2\pi i[0.25,0.42]}), \\ \qquad ([0.35, 0.15]e^{2\pi i[0.34,0.43]}, [0.15, 0.25]e^{2\pi i[0.26,0.45]}, [0.31, 0.22]e^{2\pi i[0.36,0.45]}) \Big). \end{array} \right\}.$$

Definition 14. *The complex interval-valued picture fuzzy soft relation R is a subset of $\mathcal{C}P$ of two CIVPFSSs.*

Example 6. *From Example 5, take a subset of H which is CIVPFSR:*

$$R = \left\{ \begin{array}{l} ((\ddot{v}_1, \ddot{v}_2), \Big([0.13, 0.26]e^{2\pi i[0.12,0.21]}, [0.32, 0.41]e^{2\pi i[0.14,0.22]}, [0.31, 0.42]e^{2\pi i[0.32,0.42]}\Big), \\ \quad \Big([0.13, 0.25]e^{2\pi i[0.12,0.34]}, [0.22, 0.27]e^{2\pi i[0.23,0.42]}, [0.34, 0.44]e^{2\pi i[0.23,0.34]}\Big), \\ \quad \Big([0.12, 0.31]e^{2\pi i[0.13,0.42]}, [0.16, 0.29]e^{2\pi i[0.26,0.45]}, [0.21, 0.30]e^{2\pi i[0.34,0.44]}\Big), \\ ((\ddot{v}_1, \ddot{v}_3), \Big([0.22, 0.26]e^{2\pi i[0.12,0.21]}, [0.18, 0.26]e^{2\pi i[0.14,0.22]}, [0.31, 0.42]e^{2\pi i[0.33,0.44]}\Big), \\ \quad \Big([0.05, 0.25]e^{2\pi i[0.28,0.34]}, [0.32, 0.41]e^{2\pi i[0.26,0.45]}, [0.34, 0.44]e^{2\pi i[0.23,0.42]}\Big), \\ \quad \Big([0.12, 0.31]e^{2\pi i[0.24,0.42]}, [0.22, 0.45]e^{2\pi i[0.26,0.45]}, [0.22, 0.31]e^{2\pi i[0.36,0.45]}\Big), \\ ((\ddot{v}_2, \ddot{v}_1), \Big([0.13, 0.26]e^{2\pi i[0.12,0.21]}, [0.32, 0.41]e^{2\pi i[0.14,0.22]}, [0.31, 0.42]e^{2\pi i[0.32,0.42]}\Big), \\ \quad \Big([0.13, 0.25]e^{2\pi i[0.12,0.34]}, [0.22, 0.27]e^{2\pi i[0.23,0.42]}, [0.34, 0.44]e^{2\pi i[0.23,0.34]}\Big), \\ \quad \Big([0.12, 0.31]e^{2\pi i[0.13,0.42]}, [0.16, 0.29]e^{2\pi i[0.26,0.45]}, [0.21, 0.30]e^{2\pi i[0.34,0.44]}\Big), \\ ((\ddot{v}_3, \ddot{v}_3), \Big([0.53, 0.24]e^{2\pi i[0.23,0.43]}, [0.13, 0.41]e^{2\pi i[0.26,0.45]}, [0.24, 0.12]e^{2\pi i[0.33,0.44]}\Big), \\ \quad \Big([0.90, 0.04]e^{2\pi i[0.28,0.47]}, [0.12, 0.21]e^{2\pi i[0.26,0.45]}, [0.05, 0.03]e^{2\pi i[0.25,0.42]}\Big), \\ \quad \Big([0.35, 0.15]e^{2\pi i[0.34,0.43]}, [0.15, 0.25]e^{2\pi i[0.26,0.45]}, [0.31, 0.22]e^{2\pi i[0.36,0.45]}\Big) \end{array} \right\}.$$

Definition 15. *Suppose that (F, A) is a CIVPFSS on universal set X and the CIVPFSR is defined as:*

$$\mathcal{R} = \left\{ (\ddot{v}, \acute{\omega}) \begin{pmatrix} [\mu^-(\ddot{v}, \acute{\omega}), \mu^+(\ddot{v}, \acute{\omega})] & \left[e^{\ddot{v}^-(\ddot{v},\acute{\omega})2\pi i}, e^{\ddot{v}^+(\ddot{v},\acute{\omega})2\pi i}\right] \\ [\varphi^-(\ddot{v}, \acute{\omega}), \varphi^+(\ddot{v}, \acute{\omega})] & \left[e^{j^-(\ddot{v},\acute{\omega})2\pi i}, e^{j^+(\ddot{v},\acute{\omega})2\pi i}\right] \\ [\partial^-(\ddot{v}, \acute{\omega}), \partial^+(\ddot{v}, \acute{\omega})] & \left[e^{f^-(\ddot{v},\acute{\omega})2\pi i}, e^{f^+(\ddot{v},\acute{\omega})2\pi i}\right] \end{pmatrix} : (\ddot{v}, \acute{\omega}) \in \mathcal{R} \right\}$$

Then the inverse of the CIVPFSS relation is denoted by $\mathcal{R}^{-1}$ and is defined as:

$$\mathcal{R}^{-1} = \left\{ (\acute{\omega}, \ddot{v}) \begin{pmatrix} [\mu^-(\acute{\omega}, \ddot{v}), \mu^+(\acute{\omega}, \ddot{v})] & \left[e^{\ddot{v}^-(\acute{\omega},\ddot{v})2\pi i}, e^{\ddot{v}^+(\acute{\omega},\ddot{v})2\pi i}\right] \\ [\varphi^-(\acute{\omega}, \ddot{v}), \varphi^+(\acute{\omega}, \ddot{v})] & \left[e^{j^-(\acute{\omega},\ddot{v})2\pi i}, e^{j^+(\acute{\omega},\ddot{v})2\pi i}\right] \\ [\partial^-(\acute{\omega}, \ddot{v}), \partial^+(\acute{\omega}, \ddot{v})] & \left[e^{m^-(\acute{\omega},\ddot{v})2\pi i}, e^{m^+(\acute{\omega},\ddot{v})2\pi i}\right] \end{pmatrix} : (\acute{\omega}, \ddot{v}) \in \mathcal{R}^{-1} \right\}$$

Example 7. *Consider the CIVPFSS relation $\mathcal{R}$ from Example 6. Then, its inverse is calculated as:*

$$\mathcal{R}^{-1} = \left\{ \begin{array}{ll} (\ddot{v}_2, \ddot{v}_1), & \Big([0.13, 0.26]\, e^{2\pi i[0.12,0.21]}, [0.32, 0.41]\, e^{2\pi i[0.14,0.22]}, [0.31, 0.42]\, e^{2\pi i[0.32,0.42]}\Big), \\ & \Big([0.13, 0.25]\, e^{2\pi i[0.12,0.34]}, [0.22, 0.27]\, e^{2\pi i[0.23,0.42]}, [0.34, 0.44]\, e^{2\pi i[0.23,0.34]}\Big), \\ & \Big([0.12, 0.31]\, e^{2\pi i[0.13,0.42]}, [0.16, 0.29]\, e^{2\pi i[0.26,0.45]}, [0.21, 0.30]\, e^{2\pi i[0.34,0.44]}\Big), \end{array} \right\}.$$

$$\mathcal{R}^{-1} = \left\{ \begin{array}{ll} (\ddot{v}_3, \ddot{v}_1), & \Big([0.22, 0.26]\, e^{2\pi i[0.12,0.21]}, [0.18, 0.26]\, e^{2\pi i[0.14,0.22]}, [0.31, 0.42]\, e^{2\pi i[0.333,0.44]}\Big), \\ & \Big([0.05, 0.25]\, e^{2\pi i[0.28,0.34]}, [0.32, 0.41]\, e^{2\pi i[0.26,0.45]}, [0.34, 0.44]\, e^{2\pi i[0.23,0.42]}\Big), \\ & \Big([0.12, 0.31]\, e^{2\pi i[0.24,0.42]}, [0.22, 0.45]\, e^{2\pi i[0.26,0.45]}, [0.22, 0.31]\, e^{2\pi i[0.36,0.45]}\Big), \end{array} \right\}.$$

$$\mathcal{R}^{-1}=\left\{\begin{array}{ll}(\ddot{\upsilon}_1,\ddot{\upsilon}_2), & \left([0.13,0.26]\,e^{2\pi i[0.12,0.21]},[0.32,0.41]\,e^{2\pi i[0.14,0.22]},[0.31,0.42]\,e^{2\pi i[0.32,0.42]}\right), \\ & \left([0.13,0.25]\,e^{2\pi i[0.12,0.34]},[0.22,0.27]\,e^{2\pi i[0.23,0.42]},[0.34,0.44]\,e^{2\pi i[0.23,0.34]}\right), \\ & \left([0.12,0.31]\,e^{2\pi i[0.13,0.42]},[0.16,0.29]\,e^{2\pi i[0.26,0.45]},[0.21,0.30]\,e^{2\pi i[0.34,0.44]}\right),\end{array}\right\}.$$

$$\mathcal{R}^{-1}=\left\{\begin{array}{ll}(\ddot{\upsilon}_3,\ddot{\upsilon}_3), & \left([0.53,0.24]\,e^{2\pi i[0.23,0.43]},[0.13,0.41]\,e^{2\pi i[0.26,0.45]},[0.24,0.12]\,e^{2\pi i[0.33,0.44]}\right), \\ & \left([0.90,0.04]\,e^{2\pi i[0.28,0.47]},[0.12,0.21]\,e^{2\pi i[0.26,0.45]},[0.05,0.03]\,e^{2\pi i[0.25,0.42]}\right), \\ & \left([0.35,0.15]\,e^{2\pi i[0.34,0.43]},[0.15,0.25]\,e^{2\pi i[0.26,0.45]},[0.31,0.22]\,e^{2\pi i[0.36,0.45]}\right),\end{array}\right\}.$$

Definition 16. *For a universal set X, let a CIVPFSS (F,A) and $\mathcal{R}_1$ be a CIVPFSR on (F,A). Then:*

- $\mathcal{R}_1$ *is said to be a CIVPFS reflexive relation on (F,A) if $(\ddot{\upsilon},\ddot{\upsilon})\in\mathcal{R}_1$, for all $\ddot{\upsilon}\in(F,A)$.*
- $\mathcal{R}_1$ *is said to be a CIVPFS irreflexive relation on (F,A) if $(\ddot{\upsilon},\ddot{\upsilon})\notin\mathcal{R}_1$, for all $\ddot{\upsilon}\in(F,A)$.*
- $\mathcal{R}_1$ *is said to be a CIVPFS antisymmetric relation on (F,A) if for all $\ddot{\upsilon},\acute{\omega}\in(F,A)$, $(\ddot{\upsilon},\acute{\omega})\in\mathcal{R}_1$ and $(\acute{\omega},\ddot{\upsilon})\in\mathcal{R}_1$; then $\ddot{\upsilon}=\acute{\omega}$.*
- $\mathcal{R}_1$ *is said to be a CIVPFS complete relation on (F,A) if $(\ddot{\upsilon},\acute{\omega})\in\mathcal{R}_1$ or $(\acute{\omega},\ddot{\upsilon})\in\mathcal{R}_1$, for all $\ddot{\upsilon},\acute{\omega}\in(F,A)$.*
- $\mathcal{R}_1$ *is said to be a CIVPFS transitive relation on (F,A) if $(\ddot{\upsilon},\acute{\omega})\in\mathcal{R}_1$ and $(\acute{\omega},\gamma)\in\mathcal{R}_1$; then, $(\ddot{\upsilon},\gamma)\in\mathcal{R}_1$, for all $\ddot{\upsilon},\acute{\omega},\gamma\in(F,A)$.*
- $\mathcal{R}_1$ *is said to be a CIVPFS equivalence relation on (F,A) if $\mathcal{R}_1$ is a CIVPFS reflexive relation, CIVPFS symmetric relation, and CIVPFS transitive relation on (F,A).*
- $\mathcal{R}_1$ *is said to be a CIVPFS preorder relation on (F,A) if $\mathcal{R}_1$ is a CIVPFS reflexive relation and CIVPFS transitive relation on (F,A).*
- $\mathcal{R}_1$ *is said to be a CIVPFS strict-order relation on (F,A) if $\mathcal{R}_1$ is a CIVPFS irreflexive relation and CIVPFS transitive relation on (F,A).*
- $\mathcal{R}_1$ *is said to be a CIVPFS partial-order relation on (F,A) if $\mathcal{R}_1$ is a CIVPFS preorder relation and CIVPFS antisymmetric relation on (F,A).*
- $\mathcal{R}_1$ *is said to be a CIVPFS linear-order relation on (F,A) if $\mathcal{R}_1$ is a CIVPFS partial-order relation and CIVPFS complete relation on (F,A).*

Example 8. *From Example 5, we define the following relations:*
i. The CIVPFS reflexive relation $\mathcal{R}_1$ is

$$\mathcal{R}_1=\left\{\begin{array}{ll}(\ddot{\upsilon}_1,\ddot{\upsilon}_1), & \left(([0.12,0.26]e^{2\pi i[0.12,0.21]}),([0.32,0.41]e^{2\pi i[0.14,0.22]}),([0.31,0.42]e^{2\pi i[0.14,0.25]})\right), \\ & \left(([0.13,0.25]e^{2\pi i[0.18,0.34]}),([0.11,0.21]e^{2\pi i[0.23,0.42]}),([0.34,0.44]e^{2\pi i[0.23,0.42]})\right), \\ & \left(([0.12,0.31]e^{2\pi i[0.22,0.42]}),([0.22,0.33]e^{2\pi i[0.26,0.45]}),([0.21,0.30]e^{2\pi i[0.34,0.43]})\right)\end{array}\right\}$$

$$\mathcal{R}_1 = \left\{ \begin{array}{ll} (\ddot{v}_2, \ddot{v}_2), & \Big(([0.13, 0.34]e^{2\pi i[0.22,0.43]}), ([0.21, 0.41]e^{2\pi i[0.26,0.45]}), ([0.25, 0.31]e^{2\pi i[0.12,0.42]}) \Big), \\ & \Big(([0.17, 0.42]e^{2\pi i[0.12,0.52]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.22, 0.24]e^{2\pi i[0.26,0.34]}) \Big), \\ & \Big(([0.33, 0.52]e^{2\pi i[0.13,0.43]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.13, 0.22]e^{2\pi i[0.24,0.44]}) \Big) \end{array} \right\}$$

$$\mathcal{R}_1 = \left\{ \begin{array}{ll} (\ddot{v}_3, \ddot{v}_3), & \Big(([0.53, 0.24]e^{2\pi i[0.23,0.43]}), ([0.13, 0.41]e^{2\pi i[0.26,0.45]}), ([0.24, 0.12]e^{2\pi i[0.33,0.44]}) \Big), \\ & \Big(([0.90, 0.04]e^{2\pi i[0.28,0.47]}), ([0.12, 0.21]e^{2\pi i[0.26,0.45]}), ([0.05, 0.03]e^{2\pi i[0.25,0.42]}) \Big), \\ & \Big(([0.35, 0.15]e^{2\pi i[0.34,0.43]}), ([0.15, 0.25]e^{2\pi i[0.26,0.45]}), ([0.31, 0.22]e^{2\pi i[0.36,0.45]}) \Big) \end{array} \right\}$$

ii. The CIVPFS irreflexive relation $\mathcal{R}_2$ is

$$\mathcal{R}_2 = \left\{ \begin{array}{ll} (\ddot{v}_1, \ddot{v}_2), & \Big(([0.13, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.32,0.42]}) \Big), \\ & \Big(([0.13, 0.25]e^{2\pi i[0.12,0.34]}), ([0.22, 0.27]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.34]}) \Big), \\ & \Big(([0.12, 0.31]e^{2\pi i[0.13,0.42]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.44]}) \Big) \end{array} \right\}$$

$$\mathcal{R}_2 = \left\{ \begin{array}{ll} (\ddot{v}_2, \ddot{v}_3), & \Big(([0.23, 0.34]e^{2\pi i[0.23,0.43]}), ([0.18, 0.26]e^{2\pi i[0.26,0.45]}), ([0.15, 0.31]e^{2\pi i[0.33,0.44]}) \Big), \\ & \Big(([0.05, 0.42]e^{2\pi i[0.28,0.47]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.12, 0.024]e^{2\pi i[0.26,0.42]}) \Big), \\ & \Big(([0.15, 0.35]e^{2\pi i[0.23,0.43]}), ([0.32, 0.44]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big) \end{array} \right\}$$

$$\mathcal{R}_2 = \left\{ \begin{array}{ll} (\ddot{v}_3, \ddot{v}_1), & \Big(([0.22, 0.26]e^{2\pi i[0.12,0.21]}), ([0.18, 0.26]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.33,0.44]}) \Big), \\ & \Big(([0.05, 0.25]e^{2\pi i[0.28,0.34]}), ([0.32, 0.41]e^{2\pi i[0.26,0.45]}), ([0.34, 0.44]e^{2\pi i[0.23,0.42]}) \Big), \\ & \Big(([0.12, 0.31]e^{2\pi i[0.24,0.42]}), ([0.22, 0.45]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big) \end{array} \right\}$$

$$\mathcal{R}_2 = \left\{ \begin{array}{ll} (\ddot{v}_3, \ddot{v}_2), & \Big(([0.23, 0.34]e^{2\pi i[0.23,0.43]}), ([0.18, 0.26]e^{2\pi i[0.26,0.45]}), ([0.15, 0.31]e^{2\pi i[0.33,0.44]}) \Big), \\ & \Big(([0.05, 0.42]e^{2\pi i[0.28,0.47]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.12, 0.024]e^{2\pi i[0.26,0.42]}) \Big), \\ & \Big(([0.15, 0.35]e^{2\pi i[0.23,0.43]}), ([0.32, 0.44]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big) \end{array} \right\}$$

Example 9. *From example 5, we define the following relations:*

i. The CIVPFS symmetric relation $\mathcal{R}_1$ is:

$$\mathcal{R}_1 = \left\{ \begin{array}{l} (\ddot{v}_1, \ddot{v}_2), \Big(([0.13, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.32,0.42]}) \Big), \\ \qquad\qquad \Big(([0.13, 0.25]e^{2\pi i[0.12,0.34]}), ([0.22, 0.27]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.34]}) \Big), \\ \qquad\qquad \Big(([0.12, 0.31]e^{2\pi i[0.13,0.42]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.44]}) \Big), \\ (\ddot{v}_2, \ddot{v}_1), \Big(([0.13, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.32,0.42]}) \Big), \\ \qquad\qquad \Big(([0.13, 0.25]e^{2\pi i[0.12,0.34]}), ([0.22, 0.27]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.34]}) \Big), \\ \qquad\qquad \Big(([0.12, 0.31]e^{2\pi i[0.13,0.42]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.44]}) \Big), \\ (\ddot{v}_2, \ddot{v}_3), \Big(([0.23, 0.34]e^{2\pi i[0.23,0.43]}), ([0.18, 0.26]e^{2\pi i[0.26,0.45]}), ([0.15, 0.31]e^{2\pi i[0.33,0.44]}) \Big), \\ \qquad\qquad \Big(([0.05, 0.42]e^{2\pi i[0.28,0.47]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.12, 0.024]e^{2\pi i[0.26,0.42]}) \Big), \\ \qquad\qquad \Big(([0.15, 0.35]e^{2\pi i[0.23,0.43]}), ([0.32, 0.44]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big), \\ (\ddot{v}_3, \ddot{v}_2), \Big(([0.23, 0.34]e^{2\pi i[0.23,0.43]}), ([0.18, 0.26]e^{2\pi i[0.26,0.45]}), ([0.15, 0.31]e^{2\pi i[0.33,0.44]}) \Big), \\ \qquad\qquad \Big(([0.05, 0.42]e^{2\pi i[0.28,0.47]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.12, 0.024]e^{2\pi i[0.26,0.42]}) \Big), \\ \qquad\qquad \Big(([0.15, 0.35]e^{2\pi i[0.23,0.43]}), ([0.32, 0.44]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big) \end{array} \right\}$$

ii. The CIVPFS antisymmetric relation $\mathcal{R}_2$ is:

$$\mathcal{R}_2 = \left\{ \begin{array}{l} (\ddot{v}_1, \ddot{v}_1), \Big(([0.12, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.14,0.25]}) \Big), \\ \qquad\qquad \Big(([0.13, 0.25]e^{2\pi i[0.18,0.34]}), ([0.11, 0.21]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.42]}) \Big), \\ \qquad\qquad \Big(([0.12, 0.31]e^{2\pi i[0.22,0.42]}), ([0.22, 0.33]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.43]}) \Big), \\ (\ddot{v}_2, \ddot{v}_2), \Big(([0.13, 0.34]e^{2\pi i[0.22,0.43]}), ([0.21, 0.41]e^{2\pi i[0.26,0.45]}), ([0.25, 0.31]e^{2\pi i[0.12,0.42]}) \Big), \\ \qquad\qquad \Big(([0.17, 0.42]e^{2\pi i[0.12,0.52]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.22, 0.24]e^{2\pi i[0.26,0.34]}) \Big), \\ \qquad\qquad \Big(([0.33, 0.52]e^{2\pi i[0.13,0.43]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.13, 0.22]e^{2\pi i[0.24,0.44]}) \Big), \\ (\ddot{v}_3, \ddot{v}_3), \Big(([0.53, 0.24]e^{2\pi i[0.23,0.43]}), ([0.13, 0.41]e^{2\pi i[0.26,0.45]}), ([0.24, 0.12]e^{2\pi i[0.33,0.44]}) \Big), \\ \qquad\qquad \Big(([0.90, 0.04]e^{2\pi i[0.28,0.47]}), ([0.12, 0.21]e^{2\pi i[0.26,0.45]}), ([0.05, 0.03]e^{2\pi i[0.25,0.42]}) \Big), \\ \qquad\qquad \Big(([0.35, 0.15]e^{2\pi i[0.34,0.43]}), ([0.15, 0.25]e^{2\pi i[0.26,0.45]}), ([0.31, 0.22]e^{2\pi i[0.36,0.45]}) \Big) \end{array} \right\}$$

iii. The CIVPFS transitive relation $\mathcal{R}_3$ is:

$$\mathcal{R}_3 = \left\{ \begin{array}{l} (\ddot{v}_2, \ddot{v}_1), \Big(([0.13, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.32,0.42]}) \Big), \\ \qquad\qquad \Big(([0.13, 0.25]e^{2\pi i[0.12,0.34]}), ([0.22, 0.27]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.34]}) \Big), \\ \qquad\qquad \Big(([0.12, 0.31]e^{2\pi i[0.13,0.42]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.44]}) \Big), \\ (\ddot{v}_2, \ddot{v}_2), \Big(([0.13, 0.34]e^{2\pi i[0.22,0.43]}), ([0.21, 0.41]e^{2\pi i[0.26,0.45]}), ([0.25, 0.31]e^{2\pi i[0.12,0.42]}) \Big), \\ \qquad\qquad \Big(([0.17, 0.42]e^{2\pi i[0.12,0.52]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.22, 0.24]e^{2\pi i[0.26,0.34]}) \Big), \\ \qquad\qquad \Big(([0.33, 0.52]e^{2\pi i[0.13,0.43]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.13, 0.22]e^{2\pi i[0.24,0.44]}) \Big), \\ (\ddot{v}_2, \ddot{v}_3), \Big(([0.23, 0.34]e^{2\pi i[0.23,0.43]}), ([0.18, 0.26]e^{2\pi i[0.26,0.45]}), ([0.15, 0.31]e^{2\pi i[0.33,0.44]}) \Big), \\ \qquad\qquad \Big(([0.05, 0.42]e^{2\pi i[0.28,0.47]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.12, 0.024]e^{2\pi i[0.26,0.42]}) \Big), \\ \qquad\qquad \Big(([0.15, 0.35]e^{2\pi i[0.23,0.43]}), ([0.32, 0.44]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big), \\ (\ddot{v}_3, \ddot{v}_1), \Big(([0.22, 0.26]e^{2\pi i[0.12,0.21]}), ([0.18, 0.26]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.33,0.44]}) \Big), \\ \qquad\qquad \Big(([0.05, 0.25]e^{2\pi i[0.28,0.34]}), ([0.32, 0.41]e^{2\pi i[0.26,0.45]}), ([0.34, 0.44]e^{2\pi i[0.23,0.42]}) \Big), \\ \qquad\qquad \Big(([0.12, 0.31]e^{2\pi i[0.24,0.42]}), ([0.22, 0.45]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big), \\ (\ddot{v}_3, \ddot{v}_2), \Big(([0.23, 0.34]e^{2\pi i[0.23,0.43]}), ([0.18, 0.26]e^{2\pi i[0.26,0.45]}), ([0.15, 0.31]e^{2\pi i[0.33,0.44]}) \Big), \\ \qquad\qquad \Big(([0.05, 0.42]e^{2\pi i[0.28,0.47]}), ([0.14, 0.24]e^{2\pi i[0.26,0.45]}), ([0.12, 0.024]e^{2\pi i[0.26,0.42]}) \Big), \\ \qquad\qquad \Big(([0.15, 0.35]e^{2\pi i[0.23,0.43]}), ([0.32, 0.44]e^{2\pi i[0.26,0.45]}), ([0.22, 0.31]e^{2\pi i[0.36,0.45]}) \Big), \\ (\ddot{v}_3, \ddot{v}_3), \Big(([0.53, 0.24]e^{2\pi i[0.23,0.43]}), ([0.13, 0.41]e^{2\pi i[0.26,0.45]}), ([0.24, 0.12]e^{2\pi i[0.33,0.44]}) \Big), \\ \qquad\qquad \Big(([0.90, 0.04]e^{2\pi i[0.28,0.47]}), ([0.12, 0.21]e^{2\pi i[0.26,0.45]}), ([0.05, 0.03]e^{2\pi i[0.25,0.42]}) \Big), \\ \qquad\qquad \Big(([0.35, 0.15]e^{2\pi i[0.34,0.43]}), ([0.15, 0.25]e^{2\pi i[0.26,0.45]}), ([0.31, 0.22]e^{2\pi i[0.36,0.45]}) \Big) \end{array} \right\}$$

iv. The CIVPFS equivalence relation $\mathcal{R}_4$ is:

$$\mathcal{R}_4 = \left\{ \begin{array}{r} (\ddot{\upsilon}_1, \ddot{\upsilon}_1), \Big(\Big([0.12, 0.26] e^{2\pi i [0.12, 0.21]}, [0.32, 0.41] e^{2\pi i [0.14, 0.22]}, [0.31, 0.42] e^{2\pi i [0.14, 0.25]} \Big), \\ \Big([0.13, 0.25] e^{2\pi i [0.18, 0.34]}, [0.11, 0.21] e^{2\pi i [0.23, 0.42]}, [0.34, 0.44] e^{2\pi i [0.23, 0.42]} \Big), \\ \Big([0.12, 0.31] e^{2\pi i [0.22, 0.42]}, [0.22, 0.33] e^{2\pi i [0.26, 0.45]}, [0.21, 0.30] e^{2\pi i [0.34, 0.43]} \Big) \Big), \\ (\ddot{\upsilon}_2, \ddot{\upsilon}_2), \Big(\Big([0.13, 0.34] e^{2\pi i [0.22, 0.43]}, [0.21, 0.41] e^{2\pi i [0.26, 0.45]}, [0.25, 0.31] e^{2\pi i [0.12, 0.42]} \Big), \\ \Big([0.17, 0.42] e^{2\pi i [0.12, 0.52]}, [0.14, 0.24] e^{2\pi i [0.26, 0.45]}, [0.22, 0.24] e^{2\pi i [0.26, 0.34]} \Big), \\ \Big([0.33, 0.52] e^{2\pi i [0.13, 0.43]}, [0.16, 0.29] e^{2\pi i [0.26, 0.45]}, [0.13, 0.22] e^{2\pi i [0.24, 0.44]} \Big) \Big), \\ (\ddot{\upsilon}_3, \ddot{\upsilon}_2), \Big(\Big([0.23, 0.34] e^{2\pi i [0.23, 0.43]}, [0.18, 0.26] e^{2\pi i [0.26, 0.45]}, [0.15, 0.31] e^{2\pi i [0.33, 0.44]} \Big), \\ \Big([0.05, 0.42] e^{2\pi i [0.28, 0.47]}, [0.14, 0.24] e^{2\pi i [0.26, 0.45]}, [0.12, 0.24] e^{2\pi i [0.26, 0.42]} \Big), \\ \Big([0.15, 0.35] e^{2\pi i [0.23, 0.43]}, [0.32, 0.44] e^{2\pi i [0.26, 0.45]}, [0.22, 0.31] e^{2\pi i [0.36, 0.45]} \Big) \Big), \\ (\ddot{\upsilon}_3, \ddot{\upsilon}_3), \Big(\Big([0.53, 0.24] e^{2\pi i [0.23, 0.43]}, [0.13, 0.41] e^{2\pi i [0.26, 0.45]}, [0.24, 0.12] e^{2\pi i [0.33, 0.44]} \Big), \\ \Big([0.90, 0.04] e^{2\pi i [0.28, 0.47]}, [0.12, 0.21] e^{2\pi i [0.26, 0.45]}, [0.05, 0.03] e^{2\pi i [0.25, 0.42]} \Big), \\ \Big([0.35, 0.15] e^{2\pi i [0.34, 0.43]}, [0.15, 0.25] e^{2\pi i [0.26, 0.45]}, [0.31, 0.22] e^{2\pi i [0.36, 0.45]} \Big) \Big) \end{array} \right\}$$

v. The CIVPFS complete relation $\mathcal{R}_5$ is:

$$\mathcal{R}_5 = \left\{ \begin{array}{r} (\ddot{\upsilon}_1, \ddot{\upsilon}_2), \Big(\Big([0.13, 0.26] e^{2\pi i [0.12, 0.21]}, [0.32, 0.41] e^{2\pi i [0.14, 0.22]}, [0.31, 0.42] e^{2\pi i [0.32, 0.42]} \Big), \\ \Big([0.13, 0.25] e^{2\pi i [0.12, 0.34]}, [0.22, 0.27] e^{2\pi i [0.23, 0.42]}, [0.34, 0.44] e^{2\pi i [0.23, 0.34]} \Big), \\ \Big([0.12, 0.31] e^{2\pi i [0.13, 0.42]}, [0.16, 0.29] e^{2\pi i [0.26, 0.45]}, [0.21, 0.30] e^{2\pi i [0.34, 0.44]} \Big) \Big), \\ (\ddot{\upsilon}_2, \ddot{\upsilon}_3), \Big(\Big([0.23, 0.34] e^{2\pi i [0.23, 0.43]}, [0.18, 0.26] e^{2\pi i [0.26, 0.45]}, [0.15, 0.31] e^{2\pi i [0.33, 0.44]} \Big), \\ \Big([0.05, 0.42] e^{2\pi i [0.28, 0.47]}, [0.14, 0.24] e^{2\pi i [0.26, 0.45]}, [0.12, 0.24] e^{2\pi i [0.26, 0.42]} \Big), \\ \Big([0.15, 0.35] e^{2\pi i [0.23, 0.43]}, [0.32, 0.44] e^{2\pi i [0.26, 0.45]}, [0.22, 0.31] e^{2\pi i [0.36, 0.45]} \Big) \Big) \end{array} \right\}$$

Example 10. *Consider the CP from Example 5, we have;*
i. The CIVPFS preorder relation is expressed as:

$$\mathcal{R}_1=\left\{\begin{array}{l}((\ddot{\upsilon}_1,\ddot{\upsilon}_1),\big(([0.12,0.26]e^{2\pi i[0.12,0.21]}),([0.32,0.41]e^{2\pi i[0.14,0.22]}),([0.31,0.42]e^{2\pi i[0.14,0.25]})\big),\\ \qquad\qquad\big(([0.13,0.25]e^{2\pi i[0.18,0.34]}),([0.11,0.21]e^{2\pi i[0.23,0.42]}),([0.34,0.44]e^{2\pi i[0.23,0.42]})\big),\\ \qquad\qquad\big(([0.12,0.31]e^{2\pi i[0.22,0.42]}),([0.22,0.33]e^{2\pi i[0.26,0.45]}),([0.21,0.30]e^{2\pi i[0.34,0.43]})\big)),\\ ((\ddot{\upsilon}_1,\ddot{\upsilon}_2),\big(([0.13,0.26]e^{2\pi i[0.12,0.21]}),([0.32,0.41]e^{2\pi i[0.14,0.22]}),([0.31,0.42]e^{2\pi i[0.32,0.42]})\big),\\ \qquad\qquad\big(([0.13,0.25]e^{2\pi i[0.12,0.34]}),([0.22,0.27]e^{2\pi i[0.23,0.42]}),([0.34,0.44]e^{2\pi i[0.23,0.34]})\big),\\ \qquad\qquad\big(([0.12,0.31]e^{2\pi i[0.13,0.42]}),([0.16,0.29]e^{2\pi i[0.26,0.45]}),([0.21,0.30]e^{2\pi i[0.34,0.44]})\big)),\\ ((\ddot{\upsilon}_2,\ddot{\upsilon}_2),\big(([0.13,0.34]e^{2\pi i[0.22,0.43]}),([0.21,0.41]e^{2\pi i[0.26,0.45]}),([0.25,0.31]e^{2\pi i[0.12,0.42]})\big),\\ \qquad\qquad\big(([0.17,0.42]e^{2\pi i[0.12,0.52]}),([0.14,0.241]e^{2\pi i[0.26,0.45]}),([0.22,0.24]e^{2\pi i[0.26,0.34]})\big),\\ \qquad\qquad\big(([0.33,0.52]e^{2\pi i[0.13,0.43]}),([0.16,0.29]e^{2\pi i[0.26,0.45]}),([0.13,0.22]e^{2\pi i[0.24,0.44]})\big)),\\ ((\ddot{\upsilon}_3,\ddot{\upsilon}_3),\big(([0.53,0.24]e^{2\pi i[0.23,0.43]}),([0.13,0.41]e^{2\pi i[0.26,0.45]}),([0.24,0.12]e^{2\pi i[0.33,0.44]})\big),\\ \qquad\qquad\big(([0.90,0.04]e^{2\pi i[0.28,0.47]}),([0.12,0.21]e^{2\pi i[0.26,0.45]}),([0.05,0.03]e^{2\pi i[0.25,0.42]})\big),\\ \qquad\qquad\big(([0.35,0.15]e^{2\pi i[0.34,0.43]}),([0.15,0.25]e^{2\pi i[0.26,0.45]}),([0.31,0.22]e^{2\pi i[0.36,0.45]})\big))\end{array}\right\}.$$

ii. The CIVPFS strict-order relation is expressed as:

$$\mathcal{R}_2=\left\{\begin{array}{l}((\ddot{\upsilon}_2,\ddot{\upsilon}_1),\big(([0.13,0.26]e^{2\pi i[0.12,0.21]}),([0.32,0.41]e^{2\pi i[0.14,0.22]}),([0.31,0.42]e^{2\pi i[0.32,0.42]})\big),\\ \qquad\qquad\big(([0.13,0.25]e^{2\pi i[0.12,0.34]}),([0.22,0.27]e^{2\pi i[0.23,0.42]}),([0.34,0.44]e^{2\pi i[0.23,0.34]})\big),\\ \qquad\qquad\big(([0.12,0.31]e^{2\pi i[0.13,0.42]}),([0.16,0.29]e^{2\pi i[0.26,0.45]}),([0.21,0.30]e^{2\pi i[0.34,0.44]})\big)),\\ ((\ddot{\upsilon}_3,\ddot{\upsilon}_1),\big(([0.22,0.26]e^{2\pi i[0.12,0.21]}),([0.18,0.26]e^{2\pi i[0.14,0.22]}),([0.31,0.42]e^{2\pi i[0.333,0.44]})\big),\\ \qquad\qquad\big(([0.05,0.25]e^{2\pi i[0.28,0.34]}),([0.32,0.41]e^{2\pi i[0.26,0.45]}),([0.34,0.44]e^{2\pi i[0.23,0.42]})\big),\\ \qquad\qquad\big(([0.12,0.31]e^{2\pi i[0.24,0.42]}),([0.22,0.45]e^{2\pi i[0.26,0.45]}),([0.22,0.31]e^{2\pi i[0.36,0.45]})\big)),\\ ((\ddot{\upsilon}_3,\ddot{\upsilon}_2),\big(([0.23,0.34]e^{2\pi i[0.23,0.43]}),([0.18,0.26]e^{2\pi i[0.26,0.45]}),([0.15,0.31]e^{2\pi i[0.33,0.44]})\big),\\ \qquad\qquad\big(([0.05,0.42]e^{2\pi i[0.28,0.47]}),([0.14,0.24]e^{2\pi i[0.26,0.45]}),([0.12,0.024]e^{2\pi i[0.26,0.42]})\big),\\ \qquad\qquad\big(([0.15,0.35]e^{2\pi i[0.23,0.43]}),([0.32,0.44]e^{2\pi i[0.26,0.45]}),([0.22,0.31]e^{2\pi i[0.36,0.45]})\big))\end{array}\right\}.$$

The CIVPFS partial-order relation is expressed as:

$$\tilde{R}_3=\left\{\begin{array}{l}((\ddot{\upsilon}_1,\ddot{\upsilon}_1),\big(([0.12,0.26]e^{2\pi i[0.12,0.21]}),([0.32,0.41]e^{2\pi i[0.14,0.22]}),([0.31,0.42]e^{2\pi i[0.14,0.25]})\big),\\ \qquad\qquad\big(([0.13,0.25]e^{2\pi i[0.18,0.34]}),([0.11,0.21]e^{2\pi i[0.23,0.42]}),([0.34,0.44]e^{2\pi i[0.23,0.42]})\big),\\ \qquad\qquad\big(([0.12,0.31]e^{2\pi i[0.22,0.42]}),([0.22,0.33]e^{2\pi i[0.26,0.45]}),([0.21,0.30]e^{2\pi i[0.34,0.43]})\big)),\\ ((\ddot{\upsilon}_2,\ddot{\upsilon}_1),\big(([0.13,0.26]e^{2\pi i[0.12,0.21]}),([0.32,0.41]e^{2\pi i[0.14,0.22]}),([0.31,0.42]e^{2\pi i[0.32,0.42]})\big),\\ \qquad\qquad\big(([0.13,0.25]e^{2\pi i[0.12,0.34]}),([0.22,0.27]e^{2\pi i[0.23,0.42]}),([0.34,0.44]e^{2\pi i[0.23,0.34]})\big),\\ \qquad\qquad\big(([0.12,0.31]e^{2\pi i[0.13,0.42]}),([0.16,0.29]e^{2\pi i[0.26,0.45]}),([0.21,0.30]e^{2\pi i[0.34,0.44]})\big)),\\ ((\ddot{\upsilon}_2,\ddot{\upsilon}_2),\big(([0.13,0.34]e^{2\pi i[0.22,0.43]}),([0.21,0.41]e^{2\pi i[0.26,0.45]}),([0.25,0.31]e^{2\pi i[0.12,0.42]})\big),\\ \qquad\qquad\big(([0.17,0.42]e^{2\pi i[0.12,0.52]}),([0.14,0.241]e^{2\pi i[0.26,0.45]}),([0.22,0.24]e^{2\pi i[0.26,0.34]})\big),\\ \qquad\qquad\big(([0.33,0.52]e^{2\pi i[0.13,0.43]}),([0.16,0.29]e^{2\pi i[0.26,0.45]}),([0.13,0.22]e^{2\pi i[0.24,0.44]})\big)),\\ ((\ddot{\upsilon}_3,\ddot{\upsilon}_3),\big(([0.53,0.24]e^{2\pi i[0.23,0.43]}),([0.13,0.41]e^{2\pi i[0.26,0.45]}),([0.24,0.12]e^{2\pi i[0.33,0.44]})\big),\\ \qquad\qquad\big(([0.90,0.04]e^{2\pi i[0.28,0.47]}),([0.12,0.21]e^{2\pi i[0.26,0.45]}),([0.05,0.03]e^{2\pi i[0.25,0.42]})\big),\\ \qquad\qquad\big(([0.35,0.15]e^{2\pi i[0.34,0.43]}),([0.15,0.25]e^{2\pi i[0.26,0.45]}),([0.31,0.22]e^{2\pi i[0.36,0.45]})\big))\end{array}\right\}.$$

iii. The CIVPFS partial-order relation is expressed as:

$$\mathcal{R}_3 = \left\{ \begin{array}{l} ((\ddot{\upsilon}_1, \ddot{\upsilon}_1), (([0.12, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.14,0.25]})), \\ \qquad (([0.13, 0.25]e^{2\pi i[0.18,0.34]}), ([0.11, 0.21]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.42]})), \\ \qquad (([0.12, 0.31]e^{2\pi i[0.22,0.42]}), ([0.22, 0.33]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.43]}))), \\ ((\ddot{\upsilon}_2, \ddot{\upsilon}_1), (([0.13, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.32,0.42]})), \\ \qquad (([0.13, 0.25]e^{2\pi i[0.12,0.34]}), ([0.22, 0.27]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.34]})), \\ \qquad (([0.12, 0.31]e^{2\pi i[0.13,0.42]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.44]}))), \\ ((\ddot{\upsilon}_2, \ddot{\upsilon}_2), (([0.13, 0.34]e^{2\pi i[0.22,0.43]}), ([0.21, 0.41]e^{2\pi i[0.26,0.45]}), ([0.25, 0.31]e^{2\pi i[0.12,0.42]})), \\ \qquad (([0.17, 0.42]e^{2\pi i[0.12,0.52]}), ([0.14, 0.241]e^{2\pi i[0.26,0.45]}), ([0.22, 0.24]e^{2\pi i[0.26,0.34]})), \\ \qquad (([0.33, 0.52]e^{2\pi i[0.13,0.43]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.13, 0.22]e^{2\pi i[0.24,0.44]}))), \\ ((\ddot{\upsilon}_3, \ddot{\upsilon}_3), (([0.53, 0.24]e^{2\pi i[0.23,0.43]}), ([0.13, 0.41]e^{2\pi i[0.26,0.45]}), ([0.24, 0.12]e^{2\pi i[0.33,0.44]})), \\ \qquad (([0.90, 0.04]e^{2\pi i[0.28,0.47]}), ([0.12, 0.21]e^{2\pi i[0.26,0.45]}), ([0.05, 0.03]e^{2\pi i[0.25,0.42]})), \\ \qquad (([0.35, 0.15]e^{2\pi i[0.34,0.43]}), ([0.15, 0.25]e^{2\pi i[0.26,0.45]}), ([0.31, 0.22]e^{2\pi i[0.36,0.45]}))) \end{array} \right\}.$$

iv. The CIVPFS linear-order relation is expressed as:

$$\mathcal{R}_4 = \left\{ \begin{array}{l} ((\ddot{\upsilon}_1, \ddot{\upsilon}_1), (([0.12, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.14,0.25]})), \\ \qquad (([0.13, 0.25]e^{2\pi i[0.18,0.34]}), ([0.11, 0.21]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.42]})), \\ \qquad (([0.12, 0.31]e^{2\pi i[0.22,0.42]}), ([0.22, 0.33]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.43]}))), \\ ((\ddot{\upsilon}_2, \ddot{\upsilon}_1), (([0.13, 0.26]e^{2\pi i[0.12,0.21]}), ([0.32, 0.41]e^{2\pi i[0.14,0.22]}), ([0.31, 0.42]e^{2\pi i[0.32,0.42]})), \\ \qquad (([0.13, 0.25]e^{2\pi i[0.12,0.34]}), ([0.22, 0.27]e^{2\pi i[0.23,0.42]}), ([0.34, 0.44]e^{2\pi i[0.23,0.34]})), \\ \qquad (([0.12, 0.31]e^{2\pi i[0.13,0.42]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.21, 0.30]e^{2\pi i[0.34,0.44]}))), \\ ((\ddot{\upsilon}_2, \ddot{\upsilon}_2), (([0.13, 0.34]e^{2\pi i[0.22,0.43]}), ([0.21, 0.41]e^{2\pi i[0.26,0.45]}), ([0.25, 0.31]e^{2\pi i[0.12,0.42]})), \\ \qquad (([0.17, 0.42]e^{2\pi i[0.12,0.52]}), ([0.14, 0.241]e^{2\pi i[0.26,0.45]}), ([0.22, 0.24]e^{2\pi i[0.26,0.34]})), \\ \qquad (([0.33, 0.52]e^{2\pi i[0.13,0.43]}), ([0.16, 0.29]e^{2\pi i[0.26,0.45]}), ([0.13, 0.22]e^{2\pi i[0.24,0.44]}))), \\ ((\ddot{\upsilon}_3, \ddot{\upsilon}_3), (([0.53, 0.24]e^{2\pi i[0.23,0.43]}), ([0.13, 0.41]e^{2\pi i[0.26,0.45]}), ([0.24, 0.12]e^{2\pi i[0.33,0.44]})), \\ \qquad (([0.90, 0.04]e^{2\pi i[0.28,0.47]}), ([0.12, 0.21]e^{2\pi i[0.26,0.45]}), ([0.05, 0.03]e^{2\pi i[0.25,0.42]})), \\ \qquad (([0.35, 0.15]e^{2\pi i[0.34,0.43]}), ([0.15, 0.25]e^{2\pi i[0.26,0.45]}), ([0.31, 0.22]e^{2\pi i[0.36,0.45]}))) \end{array} \right\}.$$

Theorem 1. *A CIVPFS relation $\mathcal{R}$ is a CIVPFS symmetric relation on a CIVPFS set $\mathcal{F}$ if and only if $\mathcal{R} = \mathcal{R}^{-1}$.*

Proof: *Suppose that $\mathcal{R} = \mathcal{R}^{-1}$. Then,*

$$(\ddot{\upsilon}, \acute{\omega}) \in \mathcal{R} \implies (\acute{\omega}, \ddot{\upsilon}) \in \mathcal{R}^{-1} \implies (\acute{\omega}, \ddot{\upsilon}) \in \mathcal{R}.$$

Thus, $\mathcal{R}$ is a CIVPFS symmetric relation on a CIVPFS set $\mathcal{F}$.

Conversely, assume that $\mathcal{R}$ is a CIVPFS symmetric relation on a CIVPFS set.

$$(\ddot{\upsilon}, \acute{\omega}) \in \mathcal{R} \implies (\acute{\omega}, \ddot{\upsilon}) \in \mathcal{R}.$$

However,

$$(\acute{\omega}, \ddot{\upsilon}) \in \mathcal{R} \implies (\acute{\omega}, \ddot{\upsilon}) \in \mathcal{R}^{-1} \implies \mathcal{R} = \mathcal{R}^{-1}.$$

Hence, proved.

Theorem 2. *A CIVPFS relation $\mathcal{R}$ is a CIVPFS transitive relation on a CIVPFS set $\mathcal{F}$ if and only if $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.*

Proof: *Suppose that $\mathcal{R}$ is a CIVPFS transitive relation on a CIVPFS set $\mathcal{F}$. Let $(\ddot{v}, X) \in \mathcal{R} \circ \mathcal{R}$. Then, according to the definition of a CIVPFS transitive relation,*

$$(\ddot{v}, \acute{\omega}) \in \mathcal{R} \quad \textit{and} \quad (\acute{\omega}, X) \in \mathcal{R} \implies (\ddot{v}, X) \in \mathcal{R}.$$

Thus, $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.

Conversely, assume that $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$. Then, For $(\ddot{v}, \acute{\omega}) \in \mathcal{R}$ and $(\acute{\omega}, X) \in \mathcal{R}$,

$$(\ddot{v}, X) \in \mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R} \implies (\ddot{v}, X) \in \mathcal{R}.$$

Thus, $\mathcal{R}$ is a CIVPFS transitive relation on the CIVPFS set $\mathcal{F}$.

4. Application:

This section explores an application of the proposed framework to identify the best fitness tracking app. The method evaluates multiple apps based on predefined criteria using the newly introduced concepts. The results demonstrate the practicality and effectiveness of the proposed approach in decision-making scenarios.

4.1. Fitness tracking app

Wearable technology, called fitness trackers, enables users to keep tabs on their fitness-related activities and data, like calories burned and distance traveled. Fitness trackers can be integrated into smartwatches or be used as standalone devices. Typically, a mobile app that lets users manage information and utilize app features is connected to the device.

In the middle of the 1960s, the first functional fitness tracker was created. However, wearable technology and fitness trackers really took off a decade ago. In 2012, the first Fitbit was released, and in 2014, the first Apple Watch was introduced. Since then, wearable technology and fitness trackers have grown in popularity.

According to a recent Pew Research Center study, 21% of Americans wear smartwatches or fitness trackers on a regular basis. Similarly, a 2019 Gallup poll found that over 40% of Americans track their fitness-related activities using an app or gadget.

Fitness trackers are used for a variety of purposes, but the primary goal is to improve or preserve one's health. The tool or app keeps users informed, helps them stay motivated, and lets them monitor their progress. Many people have started pursuing healthier lifestyles in recent years, and they have embraced technologies that encourage them to monitor their progress. Millennials are the generation that has embraced this trend the most; some have even dubbed them "the wellness generation."

The procedure of this application is explained in Figure (1).

First of all, we defined a universal set, which includes some fundamental fitness tracking apps. The universal set is defined as:

$$X = \{X_1, X_2, X_3, X_4\}$$

This universal set consists of four types of fitness tracking apps, where:

- $X_1 =$ Mobile application,
- $X_2 =$ Web application,
- $X_3 =$ Wearable integration, and
- $X_4 =$ Social networking integration.

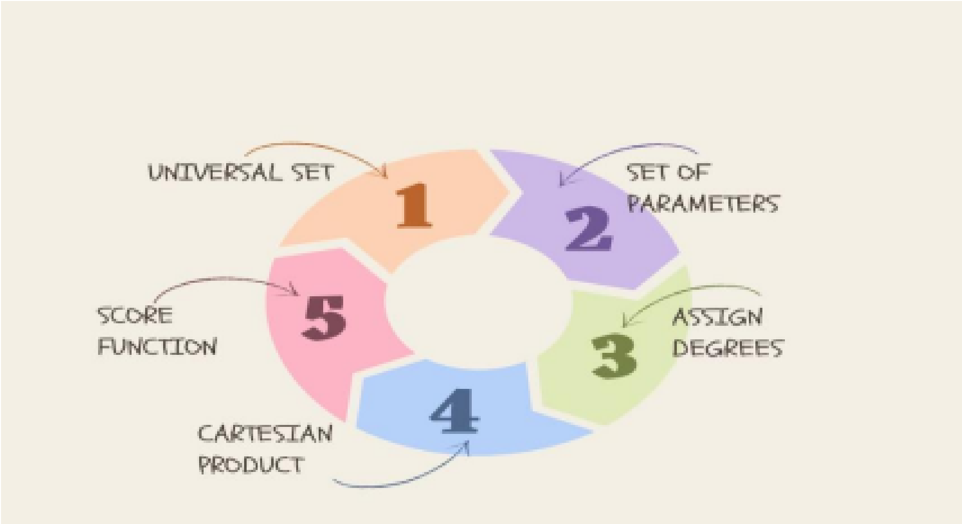

Figure 1: . Fitness tracking app application procedure.

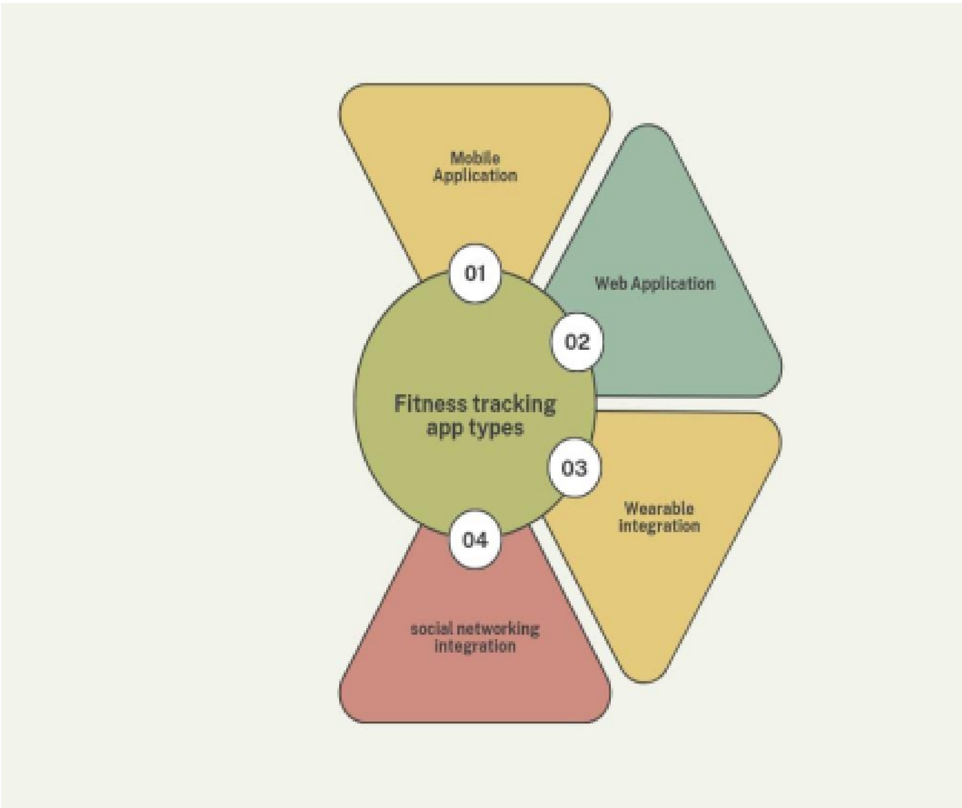

Figure 2: Summary of different fitness tracking apps.

The types of fitness tracking apps are discussed in Figure (2).

The summary of different apps is as follows:

⋇ **Mobile Application:**

A mobile application is designed to help users monitor and manage their physical activity, health metrics, and wellness goals. It typically includes features like:

- Tracking workouts,
- Counting steps,
- Monitoring heart rate,
- Providing nutrition information, and

- Setting fitness goals.

⋇ **Web Application:**

The online version of a fitness tracking app, referred to as a web application, is accessible through a web browser. Without having to download the app to your device, you can:

- Monitor your fitness activities,
- Set objectives,
- View your progress, and
- Use other features.

It functions as an online version of the app rather than a downloadable application for your computer or phone.

⋇ **Wearable Integration:**

The term "wearable integration" describes an app's capacity to link and synchronize with wearable gadgets, such as:

- Smartwatches,
- Fitness trackers, or
- Other wearable technology.

The app collects information from the wearable device, including:

- Heart rate,
- Steps taken,
- Distance traveled, and
- Sleep patterns.

This synchronization provides a more thorough and accurate picture of fitness activities and progress.

⋇ **Social Networking Integration:**

Social networking integration in fitness tracking apps allows users to connect and share their fitness activities and progress with friends or followers on social media platforms. This feature enables users to:

- Share accomplishments, challenges, and updates on platforms like Facebook, Instagram, and Twitter,
- Participate in friendly competitions with their social network,
- Discuss fitness experiences, and
- Stay motivated.

⋇ **Fitness Tracking App Parameters:**

The parameters of a fitness tracking app are defined as:

$$E = \{\ddot{v}_1, \ddot{v}_2, \ddot{v}_3, \ddot{v}_4\}$$

where:

- $\ddot{v}_1$ = Users' profiles,

- $\ddot{\upsilon}_2$ = Activity tracking,
- $\ddot{\upsilon}_3$ = Workout plans, and
- $\ddot{\upsilon}_4$ = Progress monitoring.

A summary of fitness tracking app parameters is shown in Figure (3).

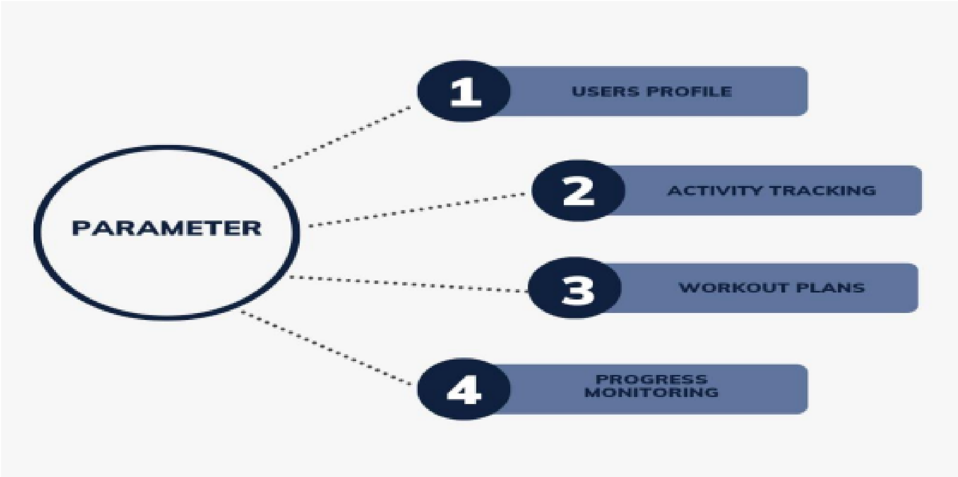

Figure 3: Summary of fitness tracking app parameters.

The summary of different parameters is as follows:

⋇ **User Profiles:**

Users create profiles with personal information, goals, and preferences, which allow the app to customize workouts and meal plans according to this input. Goal setting enables users to set specific goals, track progress, and adjust their daily routine accordingly.

The parameters related to user profiles are defined as:

$$\ddot{\upsilon}_1 = \left\{ \begin{array}{l} \Bigg(\Bigg(\big([0.27, 0.36]e^{2\pi i[0.12,0.21]}, [0.40, 0.42]e^{2\pi i[0.14,0.22]}, [0.31, 0.42]e^{2\pi i[0.14,0.22]}\big), \\ \big([0.13, 0.35]e^{2\pi i[0.28,0.34]}, [0.32, 0.40]e^{2\pi i[0.23,0.42]}, [0.34, 0.43]e^{2\pi i[0.23,0.42]}\big), \\ \big([0.12, 0.31]e^{2\pi i[0.32,0.42]}, [0.21, 0.45]e^{2\pi i[0.23,0.42]}, [0.21, 0.30]e^{2\pi i[0.34,0.43]}\big), \\ \big([0.12, 0.33]e^{2\pi i[0.32,0.42]}, [0.21, 0.40]e^{2\pi i[0.23,0.42]}, [0.21, 0.31]e^{2\pi i[0.34,0.43]}\big)\Bigg)\Bigg) \end{array} \right\}$$

⋇ **Activity Tracking:**

Activity tracking involves the practice of measuring and collecting data on an individual's physical and psychological activity to keep track of and maintain documentation regarding their health and wellness.

The parameters related to activity tracking are defined as:

$$\ddot{\upsilon}_2 = \left\{ \begin{array}{l} \Bigg(\Bigg(\big([0.23, 0.38]e^{2\pi i[0.32,0.43]}, [0.21, 0.40]e^{2\pi i[0.23,0.42]}, [0.15, 0.35]e^{2\pi i[0.12,0.42]}\big), \\ \big([0.17, 0.40]e^{2\pi i[0.22,0.52]}, [0.14, 0.24]e^{2\pi i[0.23,0.42]}, [0.12, 0.24]e^{2\pi i[0.26,0.34]}\big), \\ \big([0.40, 0.42]e^{2\pi i[0.23,0.43]}, [0.34, 0.40]e^{2\pi i[0.23,0.42]}, [0.13, 0.28]e^{2\pi i[0.14,0.44]}\big), \\ \big([0.12, 0.36]e^{2\pi i[0.32,0.42]}, [0.22, 0.45]e^{2\pi i[0.23,0.42]}, [0.21, 0.30]e^{2\pi i[0.34,0.43]}\big)\Bigg)\Bigg) \end{array} \right\}$$

⋇ **Workout Plan:**

A workout plan includes the type of exercise you need to do and how long you need to do them. This plan helps you maintain a consistent routine, so you can achieve your fitness goals. It gives you an organized structure of how your exercise should go. It can also help you make working out more creative and will help create more progress. Many fitness experts recommend creating a plan, so you can see the progress you've made and gradually achieve your goals.

The parameters related to the workout plan are defined as:

$$\ddot{\upsilon}_3 = \left\{ \begin{array}{l} \Bigg(\Bigg(\big([0.34, 0.53]e^{2\pi i[0.33,0.43]}, [0.13, 0.20]e^{2\pi i[0.26,0.45]}, [0.12, 0.20]e^{2\pi i[0.23,0.44]}\big), \\ \big([0.04, 0.80]e^{2\pi i[0.38,0.47]}, [0.12, 0.22]e^{2\pi i[0.26,0.45]}, [0.01, 0.06]e^{2\pi i[0.25,0.42]}\big), \\ \big([0.25, 0.34]e^{2\pi i[0.34,0.43]}, [0.15, 0.24]e^{2\pi i[0.26,0.45]}, [0.12, 0.24]e^{2\pi i[0.26,0.45]}\big), \\ \big([0.12, 0.37]e^{2\pi i[0.32,0.42]}, [0.20, 0.45]e^{2\pi i[0.23,0.42]}, [0.21, 0.36]e^{2\pi i[0.34,0.43]}\big)\Bigg)\Bigg) \end{array} \right\}$$

⋇ **Progress Monitoring:**

Users love to see what they have achieved through their workout sessions. When a wearable device collects the information and sends it to your fitness tracking application, users can efficiently monitor it. From how many miles they have covered during running sessions to how much time was taken, it allows them to track their progress efficiently.

The parameters related to progress monitoring are defined as:

$$\ddot{\upsilon}_4 = \left\{ \begin{array}{l} \Bigg(\Bigg(\big([0.34, 0.43]e^{2\pi i[0.33,0.43]}, [0.13, 0.20]e^{2\pi i[0.26,0.45]}, [0.12, 0.20]e^{2\pi i[0.23,0.44]}\big), \\ \big([0.06, 0.80]e^{2\pi i[0.38,0.47]}, [0.12, 0.25]e^{2\pi i[0.26,0.45]}, [0.01, 0.02]e^{2\pi i[0.25,0.42]}\big), \\ \big([0.25, 0.35]e^{2\pi i[0.34,0.43]}, [0.24, 0.25]e^{2\pi i[0.26,0.45]}, [0.12, 0.22]e^{2\pi i[0.26,0.45]}\big), \\ \big([0.12, 0.41]e^{2\pi i[0.32,0.42]}, [0.22, 0.44]e^{2\pi i[0.23,0.42]}, [0.21, 0.32]e^{2\pi i[0.34,0.43]}\big)\Bigg)\Bigg) \end{array} \right\}$$

Experts examine the fitness tracking app in which all the parameters are considered. Let $(F, A) = F$ represent observations by an expert, who assigns values of membership, neutral, and non-membership based on the parameters.

Suppose that the corresponding membership, neutral, and non-membership matrices are as follows:

$$\mathcal{F} = \left\{ \begin{array}{l} \ddot{\upsilon}_1 = \Big([0.27, 0.36]\, e^{2\pi i[0.12,0.21]}, [0.40, 0.42]\, e^{2\pi i[0.14,0.22]}, [0.31, 0.42]\, e^{2\pi i[0.14,0.22]}\Big), \\ \Big([0.13, 0.35]\, e^{2\pi i[0.28,0.34]}, [0.32, 0.40]\, e^{2\pi i[0.23,0.42]}, [0.34, 0.43]\, e^{2\pi i[0.23,0.42]}\Big), \\ \Big([0.12, 0.31]\, e^{2\pi i[0.32,0.42]}, [0.21, 0.45]\, e^{2\pi i[0.23,0.42]}, [0.21, 0.30]\, e^{2\pi i[0.34,0.43]}\Big), \\ \Big([0.12, 0.33]\, e^{2\pi i[0.32,0.42]}, [0.21, 0.40]\, e^{2\pi i[0.23,0.42]}, [0.21, 0.31]\, e^{2\pi i[0.34,0.43]}\Big) \end{array} \right\}$$

$$\mathcal{F} = \left\{ \begin{array}{l} \ddot{\upsilon}_2 = \Big([0.23, 0.38]\, e^{2\pi i[0.32,0.43]}, [0.21, 0.40]\, e^{2\pi i[0.23,0.42]}, [0.15, 0.35]\, e^{2\pi i[0.12,0.42]}\Big), \\ \Big([0.17, 0.40]\, e^{2\pi i[0.22,0.52]}, [0.14, 0.24]\, e^{2\pi i[0.23,0.42]}, [0.12, 0.24]\, e^{2\pi i[0.26,0.34]}\Big), \\ \Big([0.40, 0.42]\, e^{2\pi i[0.23,0.43]}, [0.34, 0.40]\, e^{2\pi i[0.23,0.42]}, [0.13, 0.28]\, e^{2\pi i[0.14,0.44]}\Big), \\ \Big([0.12, 0.36]\, e^{2\pi i[0.32,0.42]}, [0.22, 0.45]\, e^{2\pi i[0.23,0.42]}, [0.21, 0.30]\, e^{2\pi i[0.34,0.43]}\Big) \end{array} \right\}$$

$$\mathcal{F}=\left\{\begin{array}{l}\ddot{v}_3=\left([0.34,0.53]\,e^{2\pi i[0.33,0.43]},[0.13,0.20]\,e^{2\pi i[0.26,0.45]},[0.12,0.20]\,e^{2\pi i[0.23,0.44]}\right),\\ \left([0.04,0.80]\,e^{2\pi i[0.38,0.47]},[0.12,0.22]\,e^{2\pi i[0.26,0.45]},[0.01,0.06]\,e^{2\pi i[0.25,0.42]}\right),\\ \left([0.25,0.34]\,e^{2\pi i[0.34,0.43]},[0.15,0.24]\,e^{2\pi i[0.26,0.45]},[0.12,0.24]\,e^{2\pi i[0.26,0.45]}\right),\\ \left([0.12,0.37]\,e^{2\pi i[0.32,0.42]},[0.20,0.45]\,e^{2\pi i[0.23,0.42]},[0.21,0.36]\,e^{2\pi i[0.34,0.43]}\right)\end{array}\right\}$$

$$\mathcal{F}=\left\{\begin{array}{l}\ddot{v}_4=\left([0.34,0.43]\,e^{2\pi i[0.33,0.43]},[0.13,0.20]\,e^{2\pi i[0.26,0.45]},[0.12,0.20]\,e^{2\pi i[0.23,0.44]}\right),\\ \left([0.06,0.80]\,e^{2\pi i[0.38,0.47]},[0.12,0.25]\,e^{2\pi i[0.26,0.45]},[0.01,0.02]\,e^{2\pi i[0.25,0.42]}\right),\\ \left([0.25,0.35]\,e^{2\pi i[0.34,0.43]},[0.24,0.25]\,e^{2\pi i[0.26,0.45]},[0.12,0.22]\,e^{2\pi i[0.26,0.45]}\right),\\ \left([0.12,0.41]\,e^{2\pi i[0.32,0.42]},[0.22,0.44]\,e^{2\pi i[0.23,0.42]},[0.21,0.32]\,e^{2\pi i[0.34,0.43]}\right)\end{array}\right\}$$

Assume that the first three values of matrices $\ddot{v}_1, \ddot{v}_2, \ddot{v}_3$ of each parameter correspond to the values of membership, neutral, and non-membership assigned by an expert. The value (λ) of each parameter corresponds to the value of membership, neutral, and non-membership assigned by an expert, and the fourth indicates the general belongingness value in the fitness tracking app. Then, the self-CP of (F, A) is expressed CP of fitness tracking app as shown in Figures (4)-(7).

Order pair	x_1	x_2	x_3	λ
$(\ddot{v}_1,\ddot{v}_1)$	$\begin{pmatrix}([0.27,0.36]e^{2\pi i[0.12,0.21]}),\\ ([0.40,0.42]e^{2\pi i[0.14,0.22]}),\\ ([0.31,0.42]e^{2\pi i[0.14,0.22]})\end{pmatrix}$	$\begin{pmatrix}([0.13,0.35]e^{2\pi i[0.28,0.34]}),\\ ([0.32,0.40]e^{2\pi i[0.23,0.42]}),\\ ([0.34,0.43]e^{2\pi i[0.23,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.31]e^{2\pi i[0.32,0.42]}),\\ ([0.21,0.45]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.30]e^{2\pi i[0.34,0.43]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.33]e^{2\pi i[0.32,0.42]}),\\ ([0.21,0.40]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.31]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_1,\ddot{v}_2)$	$\begin{pmatrix}([0.23,0.36]e^{2\pi i[0.12,0.21]}),\\ ([0.21,0.40]e^{2\pi i[0.14,0.22]}),\\ ([0.31,0.42]e^{2\pi i[0.14,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.13,0.35]e^{2\pi i[0.22,0.34]}),\\ ([0.14,0.24]e^{2\pi i[0.23,0.42]}),\\ ([0.34,0.43]e^{2\pi i[0.26,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.31]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.40]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.30]e^{2\pi i[0.34,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.33]e^{2\pi i[0.32,0.42]}),\\ ([0.21,0.40]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.31]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_1,\ddot{v}_3)$	$\begin{pmatrix}([0.27,0.36]e^{2\pi i[0.12,0.21]}),\\ ([0.13,0.20]e^{2\pi i[0.14,0.22]}),\\ ([0.31,0.42]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.04,0.35]e^{2\pi i[0.28,0.34]}),\\ ([0.12,0.22]e^{2\pi i[0.23,0.42]}),\\ ([0.34,0.43]e^{2\pi i[0.25,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.31]e^{2\pi i[0.32,0.42]}),\\ ([0.15,0.24]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.30]e^{2\pi i[0.34,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.33]e^{2\pi i[0.32,0.42]}),\\ ([0.20,0.40]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.36]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_1,\ddot{v}_4)$	$\begin{pmatrix}([0.27,0.36]e^{2\pi i[0.12,0.21]}),\\ ([0.13,0.20]e^{2\pi i[0.14,0.22]}),\\ ([0.31,0.42]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.06,0.35]e^{2\pi i[0.28,0.34]}),\\ ([0.12,0.25]e^{2\pi i[0.23,0.42]}),\\ ([0.34,0.43]e^{2\pi i[0.23,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.31]e^{2\pi i[0.32,0.42]}),\\ ([0.21,0.25]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.30]e^{2\pi i[0.34,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.33]e^{2\pi i[0.32,0.42]}),\\ ([0.21,0.40]e^{2\pi i[0.23,0.42]}),\\ ([0.21,0.32]e^{2\pi i[0.34,0.43]})\end{pmatrix}$

Figure 4: CP of fitness tracking app for $\ddot{v}_1$

4.2. Score Function

The CP of two CIVPFSSs are shown in the above table. The complex values are converted into real values to find the score values. We use the following formula to compute the score:
$\left[\mu^+(\ddot{v})^2+\ddot{v}^+(\ddot{v})^2-\mu^-(\ddot{v})^2-\ddot{v}^-(\ddot{v})^2\right]+\left[\varphi^+(\ddot{v})^2+j^+(\ddot{v})^2-\varphi^-(\ddot{v})^2-j^-(\ddot{v})^2\right]$
$-\left[\phi^+(\ddot{v})^2+f^+(\ddot{v})^2-\phi^-(\ddot{v})^2-f^-(\ddot{v})^2\right]$

where: $\mu^+(\ddot{v})$, $\mu^-(\ddot{v})$ represent the membership values, $\ddot{v}^+(\ddot{v})$, $\ddot{v}^-(\ddot{v})$ are the neutral values, $\varphi^+(\ddot{v})$, $\varphi^-(\ddot{v})$ represent the non-membership values, $j^+(\ddot{v})$, $j^-(\ddot{v})$ are the neutral values for non-membership, $\phi^+(\ddot{v})$, $\phi^-(\ddot{v})$ represent the fuzzy sets for membership values, $f^+(\ddot{v})$, $f^-(\ddot{v})$ are the non-membership values for fuzzy sets.

The values that we obtain from the formula are given below in the Table 1.

Order pair	x_1	x_2	x_3	λ
$(\ddot{v}_2,\ddot{v}_1)$	$\begin{pmatrix}([0.23,0.36]e^{2\pi i[0.12,0.21]}),\\([0.21,0.40]e^{2\pi i[0.14,0.22]}),\\([0.31,0.42]e^{2\pi i[0.14,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.13,0.35]e^{2\pi i[0.22,0.34]}),\\([0.14,0.24]e^{2\pi i[0.23,0.42]}),\\([0.34,0.43]e^{2\pi i[0.26,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.31]e^{2\pi i[0.23,0.42]}),\\([0.21,0.40]e^{2\pi i[0.23,0.42]}),\\([0.21,0.30]e^{2\pi i[0.34,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.33]e^{2\pi i[0.32,0.42]}),\\([0.21,0.40]e^{2\pi i[0.23,0.42]}),\\([0.21,0.31]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_2,\ddot{v}_2)$	$\begin{pmatrix}([0.23,0.38]e^{2\pi i[0.32,0.43]}),\\([0.21,0.40]e^{2\pi i[0.23,0.42]}),\\([0.15,0.35]e^{2\pi i[0.12,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.17,0.40]e^{2\pi i[0.22,0.52]}),\\([0.14,0.24]e^{2\pi i[0.23,0.42]}),\\([0.12,0.24]e^{2\pi i[0.26,0.34]})\end{pmatrix}$	$\begin{pmatrix}([0.400.42]e^{2\pi i[0.23,0.43]}),\\([0.34,0.40]e^{2\pi i[0.23,0.42]}),\\([0.13,0.28]e^{2\pi i[0.14,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.36]e^{2\pi i[0.32,0.42]}),\\([0.22,0.45]e^{2\pi i[0.23,0.42]}),\\([0.21,0.30]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_2,\ddot{v}_3)$	$\begin{pmatrix}([0.23,0.38]e^{2\pi i[0.32,0.43]}),\\([0.13,0.20]e^{2\pi i[0.23,0.42]}),\\([0.15,0.35]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.04,0.40]e^{2\pi i[0.22,0.47]}),\\([0.12,0.22]e^{2\pi i[0.23,0.42]}),\\([0.12,0.24]e^{2\pi i[0.26,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.25,0.34]e^{2\pi i[0.23,0.43]}),\\([0.15,0.24]e^{2\pi i[0.23,0.42]}),\\([0.13,0.28]e^{2\pi i[0.26,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.36]e^{2\pi i[0.32,0.42]}),\\([0.20,0.45]e^{2\pi i[0.23,0.42]}),\\([0.21,0.36]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_2,\ddot{v}_4)$	$\begin{pmatrix}([0.23,0.38]e^{2\pi i[0.32,0.43]}),\\([0.13,0.20]e^{\pi i[0.23,0.42]}),\\([0.15,0.35]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.06,0.40]e^{2\pi i[0.22,0.47]}),\\([0.12,0.24]e^{2\pi i[0.23,0.42]}),\\([0.12,0.24]e^{2\pi i[0.26,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.25,0.35]e^{2\pi i[0.23,0.43]}),\\([0.24,0.25]e^{2\pi i[0.23,0.42]}),\\([0.13,0.28]e^{2\pi i[0.26,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.36]e^{2\pi i[0.32,0.42]}),\\([0.22,0.44]e^{2\pi i[0.23,0.42]}),\\([0.21,0.32]e^{2\pi i[0.34,0.43]})\end{pmatrix}$

Figure 5: CP of fitness tracking app for $\ddot{v}_2$

Order pair	x_1	x_2	x_3	λ
$(\ddot{v}_3,\ddot{v}_1)$	$\begin{pmatrix}([0.27,0.36]e^{2\pi i[0.12,0.21]}),\\([0.13,0.20]e^{2\pi i[0.14,0.22]}),\\([0.31,0.42]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.04,0.35]e^{2\pi i[0.28,0.34]}),\\([0.12,0.22]e^{2\pi i[0.23,0.42]}),\\([0.34,0.43]e^{2\pi i[0.25,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.31]e^{2\pi i[0.32,0.42]}),\\([0.15,0.24]e^{2\pi i[0.23,0.42]}),\\([0.21,0.30]e^{2\pi i[0.34,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.33]e^{2\pi i[0.32,0.42]}),\\([0.20,0.40]e^{2\pi i[0.23,0.42]}),\\([0.21,0.36]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_3,\ddot{v}_2)$	$\begin{pmatrix}([0.23,0.38]e^{2\pi i[0.32,0.43]}),\\([0.13,0.20]e^{2\pi i[0.23,0.42]}),\\([0.15,0.35]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.04,0.40]e^{2\pi i[0.22,0.47]}),\\([0.12,0.22]e^{2\pi i[0.23,0.42]}),\\([0.12,0.24]e^{2\pi i[0.26,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.25,0.34]e^{2\pi i[0.23,0.43]}),\\([0.15,0.24]e^{2\pi i[0.23,0.42]}),\\([0.13,0.28]e^{2\pi i[0.26,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.36]e^{2\pi i[0.32,0.42]}),\\([0.20,0.45]e^{2\pi i[0.23,0.42]}),\\([0.21,0.36]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_3,\ddot{v}_3)$	$\begin{pmatrix}([0.34,0.53]e^{2\pi i[0.33,0.43]}),\\([0.13,0.20]e^{2\pi i[0.26,0.45]}),\\([0.12,0.20]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.04,0.80]e^{2\pi i[0.38,0.47]}),\\([0.12,0.22]e^{2\pi i[0.26,0.45]}),\\([0.01,0.06]e^{2\pi i[0.25,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.25,0.34]e^{2\pi i[0.34,0.43]}),\\([0.15,0.24]e^{2\pi i[0.26,0.45]}),\\([0.12,0.24]e^{2\pi i[0.26,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.37]e^{2\pi i[0.32,0.42]}),\\([0.20,0.45]e^{2\pi i[0.23,0.42]}),\\([0.21,0.36]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_3,\ddot{v}_4)$	$\begin{pmatrix}([0.34,0.43]e^{2\pi i[0.33,0.43]}),\\([0.13,0.20]e^{2\pi i[0.26,0.45]}),\\([0.12,0.20]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.04,0.80]e^{2\pi i[0.38,0.47]}),\\([0.12,0.22]e^{2\pi i[0.26,0.45]}),\\([0.01,0.06]e^{2\pi i[0.25,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.25,0.34]e^{2\pi i[0.34,0.43]}),\\([0.15,0.24]e^{2\pi i[0.26,0.45]}),\\([0.12,0.24]e^{2\pi i[0.26,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.37]e^{2\pi i[0.32,0.42]}),\\([0.20,0.44]e^{2\pi i[0.23,0.42]}),\\([0.21,0.36]e^{2\pi i[0.34,0.43]})\end{pmatrix}$

Figure 6: CP of fitness tracking app for $\ddot{v}_3$

Order pair	x_1	x_2	x_3	λ
$(\ddot{v}_4,\ddot{v}_1)$	$\begin{pmatrix}([0.27,0.36]e^{2\pi i[0.12,0.21]}),\\([0.13,0.20]e^{2\pi i[0.14,0.22]}),\\([0.31,0.42]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.06,0.35]e^{2\pi i[0.28,0.34]}),\\([0.12,0.25]e^{2\pi i[0.23,0.42]}),\\([0.34,0.43]e^{2\pi i[0.23,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.31]e^{2\pi i[0.32,0.42]}),\\([0.21,0.25]e^{2\pi i[0.23,0.42]}),\\([0.21,0.30]e^{2\pi i[0.34,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.33]e^{2\pi i[0.32,0.42]}),\\([0.21,0.40]e^{2\pi i[0.23,0.42]}),\\([0.21,0.32]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_4,\ddot{v}_2)$	$\begin{pmatrix}([0.23,0.38]e^{2\pi i[0.32,0.43]}),\\([0.13,0.20]e^{\pi i[0.23,0.42]}),\\([0.15,0.35]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.06,0.40]e^{2\pi i[0.22,0.47]}),\\([0.12,0.24]e^{2\pi i[0.23,0.42]}),\\([0.12,0.24]e^{2\pi i[0.26,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.25,0.35]e^{2\pi i[0.23,0.43]}),\\([0.24,0.25]e^{2\pi i[0.23,0.42]}),\\([0.13,0.28]e^{2\pi i[0.26,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.36]e^{2\pi i[0.32,0.42]}),\\([0.22,0.44]e^{2\pi i[0.23,0.42]}),\\([0.21,0.32]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_4,\ddot{v}_3)$	$\begin{pmatrix}([0.34,0.43]e^{2\pi i[0.33,0.43]}),\\([0.13,0.20]e^{2\pi i[0.26,0.45]}),\\([0.12,0.20]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.04,0.80]e^{2\pi i[0.38,0.47]}),\\([0.12,0.22]e^{2\pi i[0.26,0.45]}),\\([0.01,0.06]e^{2\pi i[0.25,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.25,0.34]e^{2\pi i[0.34,0.43]}),\\([0.15,0.24]e^{2\pi i[0.26,0.45]}),\\([0.12,0.24]e^{2\pi i[0.26,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.37]e^{2\pi i[0.32,0.42]}),\\([0.20,0.44]e^{2\pi i[0.23,0.42]}),\\([0.21,0.36]e^{2\pi i[0.34,0.43]})\end{pmatrix}$
$(\ddot{v}_4,\ddot{v}_4)$	$\begin{pmatrix}([0.34,0.43]e^{2\pi i[0.33,0.43]}),\\([0.13,0.20]e^{2\pi i[0.26,0.45]}),\\([0.12,0.20]e^{2\pi i[0.23,0.44]})\end{pmatrix}$	$\begin{pmatrix}([0.06,0.80]e^{2\pi i[0.38,0.47]}),\\([0.12,0.25]e^{2\pi i[0.26,0.45]}),\\([0.01,0.02]e^{2\pi i[0.25,0.42]})\end{pmatrix}$	$\begin{pmatrix}([0.25,0.35]e^{2\pi i[0.34,0.43]}),\\([0.24,0.25]e^{2\pi i[0.26,0.45]}),\\([0.12,0.22]e^{2\pi i[0.26,0.45]})\end{pmatrix}$	$\begin{pmatrix}([0.12,0.41]e^{2\pi i[0.32,0.42]}),\\([0.22,0.44]e^{2\pi i[0.23,0.42]}),\\([0.21,0.32]e^{2\pi i[0.34,0.43]})\end{pmatrix}$

Figure 7: CP of fitness tracking app for $\ddot{v}_4$

To find the best fitness tracking app, we must first determine the highest numerical degree in each row while ignoring the last column. The intended value of λ is multiplied by the product of these numerical degrees to get each fitness tracking app score. We cannot compare any app with itself. So, in each same ordered pair like $(\ddot{v}_3, \ddot{v}_2)$, we write a cross $(\times)$. The fitness tracking app with the highest score is the finest. Since it is not a novel attempt to compare, we do not look at the numerical degree of the fitness tracking app of the identical parametric ordered pair.

The grade values are calculated in Table 2. Now, calculate the score function as:

$$S(X_1) = (0.426 \times 0.322) + (0.426 \times 0.322) = 0.274$$

Ordered Pair	X_1	X_2	X_3	λ
$(\ddot{v}_1, \ddot{v}_1)$	0.224	0.023	0.131	0.259
$(\ddot{v}_1, \ddot{v}_2)$	0.287	0.014	0.156	0.321
$(\ddot{v}_1, \ddot{v}_3)$	0.257	-0.083	0.132	0.182
$(\ddot{v}_1, \ddot{v}_4)$	0.280	-0.083	0.132	0.165
$(\ddot{v}_2, \ddot{v}_1)$	0.287	0.014	0.156	0.321
$(\ddot{v}_2, \ddot{v}_2)$	0.312	-0.066	0.423	0.081
$(\ddot{v}_2, \ddot{v}_3)$	0.320	0.080	0.336	0.147
$(\ddot{v}_2, \ddot{v}_4)$	0.322	0.426	0.334	0.228
$(\ddot{v}_3, \ddot{v}_1)$	0.012	-0.008	0.133	0.182
$(\ddot{v}_3, \ddot{v}_2)$	0.320	0.080	0.336	0.147
$(\ddot{v}_3, \ddot{v}_3)$	0.328	0.230	0.760	0.122
$(\ddot{v}_3, \ddot{v}_4)$	0.318	0.137	0.761	0.122
$(\ddot{v}_4, \ddot{v}_1)$	0.280	-0.083	0.032	0.165
$(\ddot{v}_4, \ddot{v}_2)$	0.322	0.426	0.334	0.228
$(\ddot{v}_4, \ddot{v}_3)$	0.318	0.137	0.761	0.122
$(\ddot{v}_4, \ddot{v}_4)$	0.367	0.147	0.413	0.124

Table 1: Score values

Table 2: Grade Table

R	$(\ddot{v}_1, \ddot{v}_1)$	$(\ddot{v}_1, \ddot{v}_2)$	$(\ddot{v}_1, \ddot{v}_3)$	$(\ddot{v}_1, \ddot{v}_4)$	$(\ddot{v}_2, \ddot{v}_1)$	$(\ddot{v}_2, \ddot{v}_2)$	$(\ddot{v}_2, \ddot{v}_3)$	$(\ddot{v}_2, \ddot{v}_4)$
h_i	h_3	h_3	h_3	h_3	h_3	h_2	h_2	h_1
Highest Degree	$\times$	0.321	0.182	0.165	0.321	$\times$	0.336	0.426
λ	$\times$	0.287	0.257	0.280	0.287	$\times$	0.320	0.322
R	$(\ddot{v}_3, \ddot{v}_1)$	$(\ddot{v}_3, \ddot{v}_2)$	$(\ddot{v}_3, \ddot{v}_3)$	$(\ddot{v}_3, \ddot{v}_4)$	$(\ddot{v}_4, \ddot{v}_1)$	$(\ddot{v}_4, \ddot{v}_2)$	$(\ddot{v}_4, \ddot{v}_3)$	$(\ddot{v}_4, \ddot{v}_4)$
h_i	h_3	h_2	h_2	h_2	h_3	h_1	h_2	h_2
Highest Degree	0.182	0.336	$\times$	0.761	0.165	0.426	0.761	$\times$
λ	0.012	0.320	$\times$	0.318	0.280	0.322	0.318	$\times$

$$S(X_2) = (0.336 \times 0.320) + (0.336 \times 0.320) + (0.761 \times 0.318) + (0.761 \times 0.318) = 0.699$$

$$\begin{aligned} S(X_3) &= (0.321 \times 0.287) + (0.182 \times 0.257) + (0.165 \times 0.280) + (0.321 \times 0.287) \\ &\quad + (0.182 \times 0.012) + (0.165 \times 0.280) = 0.326 \end{aligned}$$

Hence, X_2 is a good fitness tracking app and works effectively.

5. Comparative Analysis

In this section, the concept of CIVPFSS is compared with some preceding structures in the theory of FS, IFS, PFS, CFS, CIFS, CPFS, IVFS, IVIFS, IVPFS, CIVFS, CIVIFS, CIVPFS, CFSS, CIFSS, CPFSS, CIVFSS, and CIVIFSS.

5.1. CIVPFSS Comparison with FS, IFS, and PFS

FS has a single membership degree, IFS adds a non-membership degree, and PFS includes membership, neutral, and non-membership degrees. CIVPFSS combines complex intervals and picture fuzzy sets to model uncertainty in a multi-dimensional and complex way, making it suitable for complex and dynamic systems. In contrast, FS, IFS, and PFS are more suitable for simpler applications, with FS being the most basic and PFS being more advanced but still limited to three membership degrees.

5.2. CIVPFSS Comparison with CFS, CIFS, and CPFS

CFS is the simplest, with a single complex membership degree. CIFS adds a non-membership degree, increasing the uncertainty dimension. CPFS adds a neutral membership degree, further increasing the uncertainty dimension. CIVPFSS combines complex intervals and picture fuzzy sets, allowing for the most flexible and nuanced uncertainty modeling. CIVPFSS is more suitable for complex and dynamic systems, while CFS, CIFS, and CPFS are more suitable for simpler applications.

5.3. CIVPFSS Comparison with IVFS, IVIFS, and IVPFS

IVFS is the simplest, with a single interval membership degree. IVIFS adds a non-membership degree, increasing the uncertainty dimension. IVPFS adds a neutral membership degree, further increasing the uncertainty dimension. CIVPFSS associates complex intervals and picture fuzzy sets, allowing for the most flexible and nuanced uncertainty modeling, and is more suitable for complex and dynamic systems.

5.4. CIVPFSS Comparison with CIVFS, CIVIFS, and CIVPFS

CIVFS is the simplest, with a single complex interval membership degree. CIVIFS adds a non-membership degree, increasing the uncertainty dimension. CIVPFS adds a neutral membership degree, further increasing the uncertainty dimension. CIVPFSS combines complex intervals and picture fuzzy sets, allowing for the most flexible and nuanced uncertainty modeling, and is more suitable for complex and dynamic systems. The main difference between CIVPFSS and CIVPFS is that CIVPFSS uses soft sets, which allow for more flexibility in defining the membership degrees, whereas CIVPFS uses a more traditional fuzzy set approach.

5.5. CIVPFSS Comparison with CFSS, CIFSS, and CPFSS

CFSS is the simplest, with a single complex membership degree. CIFSS adds a non-membership degree, increasing the uncertainty dimension. CPFSS adds a neutral membership degree, further increasing the uncertainty dimension. CIVPFSS combines complex intervals and picture fuzzy sets, allowing for the most flexible and nuanced uncertainty modeling, and is more suitable for complex and dynamic systems. The main difference between CIVPFSS and CPFSS is that CIVPFSS uses interval-valued membership degrees, which allow for more flexibility in defining the membership degrees, whereas CPFSS uses a more traditional fuzzy set approach.

5.6. CIVPFSS Comparison with CIVFSS and CIVIFSS

CIVFSS is the simplest, with a single complex interval membership degree. CIVIFSS adds a non-membership degree, increasing the uncertainty dimension. CIVPFSS combines complex intervals and picture fuzzy sets, allowing for the most flexible and nuanced uncertainty modeling, and is more suitable for complex and dynamic systems. The main difference between CIVPFSS and CIVIFSS is that CIVPFSS uses picture fuzzy sets, which allow for more nuanced and complex uncertainty modeling, whereas CIVIFSS uses intuitionistic fuzzy sets, which are more suitable for simple and static systems. CIVPFSS is a more general and flexible framework that can handle complex and dynamic systems, while CIVFSS and CIVIFSS are more specific and suitable for simpler applications.

Table 3: Comparison Table

Structure	Membership	Neutral	Non-membership	Interval values	Multidimensional
CFS	Yes	No	No	No	Yes
CIFS	Yes	No	Yes	No	Yes
CPFS	Yes	Yes	Yes	No	Yes
CIVIFSS	Yes	No	Yes	Yes	Yes
CIVPFSS	Yes	Yes	Yes	Yes	Yes

6. Comprehensive Discussion on Complex Interval Valued Picture Fuzzy Soft Topological Spaces

In this section, the most important space is introduced, known as the complex interval valued picture fuzzy soft topological space. The related results and theorems are discussed it full detailed. The connection between interior and closure is also reflected.

Definition 17. *Let τ be the collection of complex interval valued picture fuzzy soft sets over X, then τ is said to be a complex interval valued picture fuzzy soft topology (TST) on X if:*

(i) $\ddot{\emptyset}, \ddot{X} \in \tau$,

(ii) The union of any number of members of complex interval valued picture fuzzy soft sets in τ belongs to τ,

(iii) The intersection of any two complex interval valued picture fuzzy soft sets in τ belongs to τ.

Then (X, τ, E) is called a complex interval valued picture fuzzy soft topological space over X.

Definition 18. *Let (X, τ, E) be a complex interval valued picture fuzzy soft topological space over X. Then the members of τ are said to be complex interval valued picture fuzzy soft open sets in X.*

Definition 19. *Let (X, τ, E) be a complex interval valued picture fuzzy soft topological space over X. Then the members of τ are said to be complex interval valued picture fuzzy soft closed sets in X if their relative complements $(F, E)'$ belong to τ.*

Definition 20. *Let X be initial universe sets, E be a set of parameters, and $\tau = \{\ddot{\emptyset}, \ddot{X}\}$. Then τ is called the complex interval valued picture fuzzy soft indiscrete topology on X, and (X, τ, E) is said to be a complex interval valued picture fuzzy soft indiscrete space over X.*

Definition 21. *Let X be initial universe sets, E be a set of parameters, and let τ be the collection of all complex interval valued picture fuzzy soft sets which can be defined over X. The τ is called the complex interval valued picture fuzzy soft discrete topology on X, and (X, τ, E) is said to be a complex interval valued picture fuzzy soft discrete space over X.*

Theorem 3. *Let (X, τ, E) and (X, τ', E) be two complex interval valued picture fuzzy soft topological spaces over the common initial sets X. Then $(X,\ \tau \ddot{\cap} \tau'\ , E)$ is a complex interval valued picture fuzzy soft topological space over X.*
Proof.

(i) $\ddot{\emptyset}$, $\ddot{X}$, *belongs to* $\tau \ddot{\cap} \tau'$.

(ii) Let $\{G_i, E/i \in \hat{I}\}$ be a family of complex interval valued picture fuzzy soft sets in $\tau \ddot{\cap} \tau'$. Then $(G_i, E) \in \tau$ and $(G_i, E) \in \tau'$ for all $i \in \hat{I}$, so $\ddot{\cup}_{i \in \hat{I}}(G_i, E) \in \tau$ and $\ddot{\cup}_{i \in \hat{I}}(G_i, E) \in \tau'$. Thus $\ddot{\cup}_{i \in \hat{I}}(G_i, E) \in \tau \cap \tau'$.

(iii) *Let the two complex interval valued picture fuzzy soft sets* (H,E), $(\hat{I},E) \in \ \tau \ddot{\cap} \tau'$. *Then* (H,E), $(\hat{I},E) \in \tau$ *and* (H,E), $(\hat{I},E) \in \tau'$. *Since* $(H,E)\ddot{\cap}(G,E) \in \tau$ *and* $(H,E)\ddot{\cap}(G,E) \in \tau'$, *so* $(H,E)\ddot{\cap}(G,E) \in \ \tau \ddot{\cap} \tau'$.

Definition 22. *Let* (X,τ,E) *be a complex interval valued picture fuzzy soft topological space over* X *and* (G,E) *be the complex interval valued picture fuzzy soft set over common universal sets* X. *Then the complex interval valued picture fuzzy soft closure of* (G,E), *denoted by* $\overline{\overline{(G,E)}}$, *is the intersection of all complex interval valued picture fuzzy soft closed sets of* (G,E). *Thus,* $\overline{\overline{(G,E)}}$ *is the smallest complex interval valued picture fuzzy soft closed set over* X *which contains* (G,E).

Theorem 4. *Let* (X,τ,E) *be a complex interval valued picture fuzzy soft topological space over* X *and let* $(H,E),(\hat{I},E)$ *be complex interval valued picture fuzzy soft sets over* X. *Then:*

(i) $\overline{\overline{\ddot{\emptyset}}} = \ddot{\emptyset}$ *and* $\overline{\overline{\ddot{X}}} = \ddot{X}$.

(ii) $(H,E)\ddot{\subseteq}\overline{\overline{(H,E)}}$ *implies* $\overline{\overline{(H,E)}}$ *is a complex interval valued picture fuzzy soft closed set and* $\overline{\overline{(H,E)}}$ *contains* (H,E).

(iii) (H,E) *is a complex interval valued picture fuzzy soft closed set if and only if* $(H,E)\ddot{\approx}\overline{\overline{(H,E)}}$.

(iv) $\overline{\overline{\overline{\overline{(H,E)}}}} = (H,E)$.

(v) $(H,E)\ddot{\subseteq}(\hat{I},E)$ *implies* $\overline{\overline{(H,E)}}\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}$.

(vi) $\overline{\overline{(H,E)\ddot{\cup}(\hat{I},E)}} = \overline{\overline{(H,E)}}\ddot{\cup}\overline{\overline{(\hat{I},E)}}$.

(vii) $\overline{\overline{(H,E)\ddot{\cap}(\hat{I},E)}}\ddot{\subseteq}\overline{\overline{(H,E)}}\ddot{\cap}\overline{\overline{(\hat{I},E)}}$.

Proof.

(i) *This is obvious.*

(ii) *Let* $\{(H_i,E) \mid i \in \hat{I}\}$ *be the family of all the complex interval valued picture fuzzy closed sets containing* (H,E). *Then, by definition, we know that:*

$$\overline{\overline{(H,E)\ddot{\cap}}} \in \hat{I}(H_i,E) \rightarrow (1).$$

Now, since $\{(H_i,E) \mid i \in \hat{I}\}$ *is a complex interval valued picture fuzzy soft closed set* $\forall \ i \in I \Rightarrow \ddot{\cap}_i \in \hat{I}(H_i,E)$ *is also a complex interval valued picture fuzzy soft closed set. Since an arbitrary intersection of complex interval valued picture fuzzy soft closed sets is a complex interval valued picture fuzzy soft closed set,* $\overline{\overline{(H,E)}}$ *is a complex interval valued picture fuzzy soft closed set* $(from(1))$.

$\Rightarrow$ *Thus,* $\overline{\overline{(H,E)}}$ *is a complex interval valued picture fuzzy soft closed set. Now, we prove that* $\overline{\overline{(H,E)}}\ddot{\supseteq}(H,E)$. *We know that* $\forall \ i \in I$, $\{(H_i,E) \mid i \in \hat{I}\}\ddot{\supseteq}(H,E)$.

$\Rightarrow (H,E)\ddot{\subseteq} \ \ddot{\cap}_i \in \hat{I} \ (H_i,E) \implies (H,E) \ \ddot{\subseteq} \ \overline{\overline{(H,E)}} \ [using(1)]$

$\Rightarrow \ \overline{\overline{(H,E)}}\ddot{\subseteq}(H,E)$ *Thus* $\overline{\overline{(H,E)}}$ *contains* (H,E). *Hence,* $\overline{\overline{(H,E)}}$ *is a complex interval valued picture fuzzy soft closed set and* $\overline{\overline{(H,E)}}$ *contains* (H,E).

(iii) *Let (H,E) be a complex interval valued picture fuzzy soft set. To prove $\overline{\overline{(H,E)}} = (H,E)$, suppose (H,E) is a complex interval valued picture fuzzy soft closed set. Now, we have $(H,E)\ddot{\supseteq}(H,E)$, so (H,E) is a $<< (T,S) >>$ closed set containing $(H,E) \rightarrow (1)$. But $\overline{\overline{(H,E)}}$ is the smallest complex interval valued picture fuzzy soft closed set containing $(H,E) \rightarrow (2)$. Therefore from (1) and (2), it follows that $\overline{\overline{(H,E)}}$ is smaller then (H,E) that is $\overline{\overline{(H,E)}}\ddot{\subseteq}(H,E)$. But from from (ii) of this theorem, we have $(H,E)\ddot{\subseteq}\overline{\overline{(H,E)}}$ is always true. Therefore we have $\overline{\overline{(H,E)}}\ddot{\subseteq}(H,E)$. Thus $(H,E) = \overline{\overline{(H,E)}}$. Consequently , if (H,E) is a complex interval valued picture fuzzy soft closed set then $\overline{\overline{(H,E)}} = (H,E)$.*

(iv) *Since $\overline{\overline{(H,E)}}$ is a complex interval valued picture fuzzy soft closed set, therefore by (iii), we have $\overline{\overline{\overline{\overline{(H,E}}}} = (H,E)$.*

(v) *If $(H,E)\ddot{\subseteq}(\hat{I},E)$, then $\overline{\overline{(H,E)}}\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}$. Suppose $(H,E)\ddot{\subseteq}(\hat{I},E)$. We know that $(\hat{I},E)\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}$, and we have*

$$(H,E)\ddot{\subseteq}(\hat{I},E)\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}.$$

Therefore, $(H,E)\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}$. Therefore, $\overline{\overline{(\hat{I},E)}}$ is a complex interval valued picture fuzzy soft closed set containing $(\hat{I},E) \rightarrow (1)$. But $\overline{\overline{(H,E)}}$ is the smallest complex interval valued picture fuzzy soft closed set containing $(H,E) \rightarrow (2)$ it follows that $\overline{\overline{(H,E)}}$ is smaller than $\overline{\overline{(\hat{I},E)}}$, that is $\overline{\overline{(H,E)}}\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}$. Thus if $(H,E)\ddot{\subseteq}(\hat{I},E)$. Then $\overline{\overline{(H,E)}}\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}$.

(vi) *We know $(H,E)\ddot{\subseteq}(H,E)\ddot{\cup}(\hat{I},E)$ and $(\hat{I},E)\ddot{\subseteq}(H,E)\ddot{\cup}(\hat{I},E)$. Therefore:*

$$\overline{\overline{(H,E)}}\ddot{\subseteq}(H,E)\ddot{\cup}(\hat{I},E) \quad \textit{and} \quad (\hat{I},E)\ddot{\subseteq}(H,E)\ddot{\cup}(\hat{I},E).$$

Since $(H,E)\ddot{\subseteq}(\hat{I},E)$ implies $\overline{\overline{(H,E)}}\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}$

$$\Rightarrow \{\overline{\overline{(H,E)\ddot{\cup}(\hat{I},E)}}\}\ddot{\subseteq}\{\overline{\overline{(H,E)\ddot{\cup}(\hat{I},E)}}\} \qquad \ddot{\cup}\{\overline{\overline{(H,E)}}\ddot{\cup}\overline{\overline{(\hat{I},E)}}\}.$$

$(H,E)\ddot{\cup}(\hat{I},E)\ddot{\subseteq}(H,E)\ddot{\cup}(\hat{I},E) \rightarrow (1)$.

Also, from the complex interval valued picture fuzzy soft closure property, we have $(H,E)\ddot{\subseteq}\overline{\overline{(H,E)}}$ and (
Thus$(H,E)\ddot{\cup}(\hat{I},E)\ddot{\subseteq}\overline{\overline{(H,E)}}\ddot{\cup}\overline{\overline{(\hat{I},E)}} \implies \overline{\overline{(H,E)}}\ddot{\cup}\overline{\overline{(\hat{I},E)}}$
is the complex interval valued picture fuzzy soft closed set containing $(H,E)\ddot{\cup}(\hat{I},E)$. But $(H,E)\ddot{\cup}(\hat{I},E)$ is the smallest complex interval valued picture fuzzy soft closed set containing $(H,E)\ddot{\cup}(\hat{I},E) \rightarrow (2)$

Comparing (1) and (2), we have $\overline{\overline{(H,E)\ddot{\cup}(\hat{I},E)}}$ is smaller than $\overline{\overline{(H,E)}}\ddot{\cup}\overline{\overline{(\hat{I},E)}}$. Thus, from (1) and (2), we have $\overline{\overline{(H,E)\ddot{\cup}(\hat{I},E)}} = \overline{\overline{(H,E)}}\ddot{\cup}\overline{\overline{(\hat{I},E)}}$.

(vii) *Since $(H,E)\ddot{\cap}(\hat{I},E)\ddot{\subseteq}(H,E)$,, so by part (v), $\overline{\overline{(H,E)\ddot{\cap}(\hat{I},E)}}\ddot{\subseteq}\overline{\overline{(H,E)}}$ and $\overline{\overline{(H,E)\ddot{\cap}(\hat{I},E)}}\ddot{\subseteq}\overline{\overline{(\hat{I},E)}}$. Thus, $\overline{\overline{(H,E)\ddot{\cap}(\hat{I},E)}}\ddot{\subseteq}\overline{\overline{(H,E)}}\ddot{\cap}\overline{\overline{(\hat{I},E)}}$.*

Definition 23. *Let (X,τ,A) be a complex interval valued picture fuzzy soft topological space over X and (H,A) be a complex interval valued picture fuzzy soft set. Then we associate point wise complex interval valued picture fuzzy soft closure of (F,E) over X, which is denoted by $(\overline{\overline{H}},A)$ and defined as $(\overline{\overline{H}},A)_{\ddot{\alpha}} = \overline{\overline{(H,A)}}_{\ddot{\alpha}}$ where $\overline{\overline{(H,A)}}_{\ddot{\alpha}}$ is the complex interval valued picture fuzzy soft closure of $(H,A)_{\ddot{\alpha}}$ in (X,τ,A) for each $\ddot{\alpha} \in A$.*

Theorem 5. *Let (X, τ, A) be a complex interval valued picture fuzzy soft topological space over X and (H, A) be a complex interval valued picture fuzzy soft set. Then $(\overline{\overline{H}}, A)\ddot{\subseteq}\overline{\overline{(H, A)}}$.*

Proof. *For any parameter $\ddot{\alpha} \in E$, $\overline{\overline{(H, A)}}_{\ddot{\alpha}}$ is the smallest complex interval valued picture fuzzy soft closed set in (X, τ, A) which contains $(H, A)_{\ddot{\alpha}}$. Moreover, if $(\overline{\overline{H}}, A)_{\ddot{\alpha}} = (Ł, A)$, then $(Ł, A)$ is also a complex interval valued picture fuzzy soft closed set in (X, τ, A) containing $(H, A)_{\ddot{\alpha}}$. This implies that $(\overline{\overline{H}}, A)_{\ddot{\alpha}} =, \overline{\overline{(H, A)}}_{\ddot{\alpha}}\ddot{\subseteq}(Ł, A)$. Thus, $\overline{\overline{(H, A)}}\ddot{\subseteq}\overline{\overline{(H, A)}}$.* □

Theorem 6. *Let (X, τ, A) be the complex interval valued picture fuzzy soft topological space and (F, A) be the complex interval valued picture fuzzy soft set over (X). Then, $(\overline{\overline{F}}, A)\ddot{\subseteq}\overline{\overline{(F, A)}}$.*

Proof. *Let (X, τ, A) be the complex interval valued picture fuzzy soft topological space over X. If $(\overline{\overline{F}}, A)\ddot{\subseteq}\overline{\overline{(F, A)}}$, then $(\overline{\overline{F}}, A)$ is a complex interval valued picture fuzzy soft closed set and so $(\overline{\overline{F}}, A)^{\acute{c}} \in \tau$.*

Conversely, if $(\overline{\overline{F}}, A)^{\acute{c}} \in \tau$, then complex interval valued picture fuzzy soft closed set containing (F, A). By the above theorem, $(\overline{\overline{F}}, A)\ddot{\subseteq}\overline{\overline{(F, A)}}$, and by the definition of the complex interval valued picture fuzzy soft closure of (F, A), any complex interval valued picture fuzzy closed set over X that contains (F, A) will contain $(\overline{\overline{F}}, A)$. Thus, $\overline{\overline{(F, A)}}\ddot{\subseteq}(\overline{\overline{F}}, A)$, hence $(\overline{\overline{F}}, A) = \overline{\overline{(F, A)}}$.
□

Definition 24. *Let (X, τ, A) be a complex interval valued picture fuzzy soft topological space over X and (H, A) be a complex interval valued picture fuzzy soft set. Let $e_x \in E$. Then e_x is said to be a complex interval valued picture fuzzy soft interior point of (H, A) if there exists a complex interval valued picture fuzzy soft open set $(\mathcal{K}, A)$ such that $e_x \in (H, A)\ddot{\subseteq}(\mathcal{K}, A)$.*

Definition 25. *Let (X, τ, A) be a complex interval valued picture fuzzy soft topological space over X and (F, A) be a complex interval valued picture fuzzy soft set. Let $e_x \in A$. Then (F, A) is said to be a complex interval valued picture fuzzy soft neighborhood of e_x if there exists a complex interval valued picture fuzzy soft open set $(\mathcal{K}, A)$ such that $e_x \in (F, A)\ddot{\subseteq}(\mathcal{K}, A)$.*

Theorem 7. *Let (X, τ, A) be a complex interval valued picture fuzzy soft topological space over X. Let (F, E) be a complex interval valued picture fuzzy soft set over X, and let $e_x \in E$. If e_x is a complex interval valued picture fuzzy soft interior point of (F, E), then e_x is a complex interval valued picture fuzzy soft interior point of $(F, E)_{\ddot{\alpha}}$ in (X, τ, A) for each $\ddot{\alpha} \in E$.*

The above theorem is not true in general.

Theorem 8. *Let (X, τ, A) be a complex interval valued picture fuzzy soft topological space over the initial parameter (X). Then:*

(i) Each $e_x \in E$ has a complex interval valued picture fuzzy soft neighborhood.

(ii) The intersection of any two complex interval valued picture fuzzy soft neighborhoods of a complex interval valued picture fuzzy soft point e_x is again a complex interval valued picture fuzzy soft neighborhood.

(iii) Every complex interval valued picture fuzzy soft superset of a complex interval valued picture fuzzy soft neighborhood of a point e_x is again a complex interval valued picture fuzzy soft neighborhood of the point e_x.

Proof. ***(i)***. *For any $e_x \in \ddot{X}$, we have $e_x \in \ddot{X}\ddot{\subseteq}\ddot{X}$. Thus, $\ddot{X}$ is a complex interval valued picture fuzzy soft neighborhood of e_x.*

(ii). *Let (X, τ, A) be a complex interval valued picture fuzzy soft topological space, and let $e_x \in E$ be any complex interval valued picture fuzzy soft point. Let (F, E) and (G, E) be any two complex interval valued picture fuzzy soft neighborhoods of e_x. To prove that $(F, E)\ddot{\cap}(G, E)$ is also a complex interval valued picture fuzzy soft neighborhood of e_x, we note that (F, E) being* *a complex interval valued picture fuzzy soft neighborhood of e_x implies there exists a complex interval valued picture fuzzy soft open set $(\mathcal{K}, E)$ such that $e_x \in (\mathcal{K}, E)\ddot{\subseteq}(F, E)$. Similarly, (G, E) is a complex interval valued picture fuzzy soft neighborhood of e_x, implying there exists a complex interval valued picture fuzzy soft open set $(Ł, E)$ such that $e_x \in (Ł, E)\ddot{\subseteq}(G, E)$.*

Now, $(\mathcal{K}, E)\ddot{\cap}(Ł, E)$ is a complex interval valued picture fuzzy soft open set, and from the previous conditions, we have $e_x \in [(\mathcal{K}, E)\ddot{\cap}(Ł, E)]\ddot{\subseteq}[(F, E)\ddot{\cap}(G, E)]$. Thus, there exists a complex interval valued picture fuzzy soft open set $[(\mathcal{K}, E)\ddot{\cap}(Ł, E)]$ such that $e_x \in [(\mathcal{K}, E)\ddot{\cap}(Ł, E)]\ddot{\subseteq}[(F, E)\ddot{\cap}(G, E)]$. From the definition of a complex interval valued picture fuzzy soft neighborhood, it follows that $(F, E)\ddot{\cap}(G, E)$ is a complex interval valued picture fuzzy soft neighborhood of e_x. Hence, the intersection of any two complex interval valued picture fuzzy soft neighborhoods is again a complex interval valued picture fuzzy soft neighborhood.

***(iii)**. Let (X, τ, E) be a complex interval valued picture fuzzy soft topological space, and let $e_x \in E$ be any complex interval valued picture fuzzy soft point. Let (F, E) be a complex interval valued picture fuzzy soft neighborhood of e_x, and let (G, E) be any complex interval valued picture fuzzy soft superset of (F, E). Since (G, E) is also a complex interval valued picture fuzzy soft neighborhood of e_x, there exists a complex interval valued picture fuzzy soft open set (H, E) such that $e_x \in (H, E)\ddot{\subseteq}(F, E)$.*

Now, (F, E) being a complex interval valued picture fuzzy soft subset of (G, E) implies $(G, E)\ddot{\supseteq}(F, E)$, which gives $(F, E)\ddot{\subseteq}(F, E)$. From the previous results, we have $e_x \in (H, E)\ddot{\subseteq}(F, E)\ddot{\subseteq}(G, E)$, which implies $e_x \in (H, E)\ddot{\subseteq}(G, E)$. Therefore, there exists a complex interval valued picture fuzzy soft open set (H, E) such that $e_x \in (H, E)\ddot{\subseteq}(G, E)$. Hence, (G, E) is a complex interval valued picture fuzzy soft neighborhood of e_x. Thus, every complex interval valued picture fuzzy soft superset of a complex interval valued picture fuzzy soft neighborhood is again a complex interval valued picture fuzzy soft neighborhood of that point.

Theorem 9. *Let (X, τ, E) be a complex interval valued picture fuzzy soft topological space. Let (F, E) be any complex interval valued picture fuzzy soft subset over X. Then the following hold true:*

(i) $(F, E)^\circ$ is a complex interval valued picture fuzzy soft open set contained in (F, E), i.e. $(F, E)^\circ$ is a complex interval valued picture fuzzy soft open set and $(F, E)^\circ\ddot{\subseteq}(F, E)$.

(ii) $(F, E)^\circ$ is the largest complex interval valued picture fuzzy soft open set contained in (F, E).

(iii) (F, E) is complex interval valued picture fuzzy soft open if and only if $(F, E)\ddot{=}(F, E)^\circ$.

***Proof**. By the definition of complex interval valued picture fuzzy soft interior, we have $(F, E)^\circ = \ddot{\cup}_{\lambda \in A}(H, E)_\lambda$, where $\{(H, E)_\lambda\} : \lambda \in A$ is the family of all complex interval valued picture fuzzy soft open sets contained in (F, E). $(H, E)_\lambda\ddot{\subseteq}(F, E)\ \forall\ \lambda \in \Lambda$ implies the union of all complex interval valued picture fuzzy soft open sets implies that open sets by the definition of complex interval valued picture fuzzy soft topological space. Also, we have $(H, E)_\lambda\ddot{\subseteq}(F, E)\ \forall\ \lambda \in \Lambda \Rightarrow \ddot{\cup}_{\lambda \in A}(H, E)_\lambda\ddot{\subseteq}(F, E)$*
$\Rightarrow\ (F, E)^\circ\ddot{\subseteq}(F, E)$. Hence, $(F, E)^\circ$ is a complex interval valued picture fuzzy soft open set and $(F, E)^\circ\ddot{\subseteq}(F, E)$.

(ii) from (i), we have that $(F, E)^\circ$ is a complex interval valued picture fuzzy soft open set contained in (F, E). Let (H, E) be any complex interval valued picture fuzzy soft open set contained in (F, E). Let (H, E) be any complex interval valued picture fuzzy soft open set contained in (F, E). This implies that the family $\{(H, E)_\lambda : \lambda \in \Lambda\} =$, the family of all complex interval valued picture fuzzy soft open sets contained in (F, E) implies $(H, E)\ddot{\subseteq}\ddot{\cup}_{\lambda \in A}(H, E)_\lambda \Rightarrow (H, E)\ddot{\subseteq}(F, E)^\circ$. $\Rightarrow (F, E)^\circ\ddot{\supseteq}(H, E)$. $\Rightarrow (F, E)^\circ$ is larger than every complex interval valued picture fuzzy soft open set contained in (F, E). Thus, $(F, E)^\circ$ is the largest complex interval valued picture fuzzy soft open set contained in (F, E).

***(ii)**. Suppose (F, E) is complex interval valued picture fuzzy soft open. Therefore, (F, E) is a complex interval valued picture fuzzy soft open set contained in (F, E), (i.e. $(F, E)\ddot{\subseteq}(F, E)$)*

$\rightarrow (1)$. *But* $(F,E)^\circ$ *is the largest complex interval valued picture fuzzy soft open set contained in* $(F,E) \rightarrow (2)$. *Therefore, from* (1) *and* (2), *it follows that* $(F,E)^\circ$ *must be larger than* (F,E), *that is,* $(F,E)^\circ \ddot{\supseteq} (F,E)$ *or* (F,E) *is smaller than* $(F,E)^\circ$, *that is* $(F,E) \ddot{\subseteq} (F,E)^\circ \rightarrow (3)$. *But* $(F,E)^\circ \ddot{\supseteq} (F,E)$ *is always true* $\rightarrow (4)$. *From* (3) *and* (4), *we have* $(F,E) \ddot{=} (F,E)^\circ$. *Note that the right-hand side result, i.e.,* $(F,E)^\circ$ *is a complex interval valued picture fuzzy soft open set, implies that the left-hand side, i.e.,* (F,E), *must also be a complex interval valued picture fuzzy soft open set. Consequently,* (F,E) *is a complex interval valued picture fuzzy soft open set. If* $(F,E) \ddot{=} (F,E)^\circ$, *then* (F,E) *is a complex interval valued picture fuzzy soft open set. Hence,* (F,E) *is complex interval valued picture fuzzy soft open if and only if* $(F,E) \ddot{=} (F,E)^\circ$.

Theorem 10. *Let* (X,τ,E) *be the complex interval valued picture fuzzy soft topological space. Let* (F,E) *and* (G,E) *be any two complex interval valued picture fuzzy soft subsets over* X. *Then the following properties hold true:*

(i) $\ddot{X}^\circ \ddot{=} \ddot{X}$.

(ii) $\ddot{\emptyset}^\circ = \ddot{\emptyset}$.

(iii) If $(F,E) \ddot{\subseteq} (G,E)$, *then* $(F,E)^\circ \ddot{\subseteq} (G,E)^\circ$.

(iv) $[(F,E) \ddot{\cap} (G,E)]^\circ \ddot{=} (F,E)^\circ \ddot{\cap} (G,E)^\circ$.

(v) $[(F,E)^\circ]^\circ = (F,E)^\circ$.

(vi) $(F,E)^\circ \ddot{\cup} (G,E)^\circ \ddot{\subseteq} [(F,E) \ddot{\cup} (G,E)]^\circ$.

Proof.

(i) We know that $\ddot{X}$ *is a complex interval valued picture fuzzy soft open set. This implies* $\ddot{X}^\circ \ddot{=} \ddot{X}$. *(Since* (F,E) *is open if and only if* $(F,E) \ddot{=} (F,E)^\circ$*). Therefore,* $\ddot{X}^\circ \ddot{=} \ddot{X}$.

(ii) The result follows from (i).

(iii) Suppose $(F,E) \ddot{\subseteq} (G,E)$. *Then we know that* $(F,E)^\circ \ddot{\subseteq} (F,E)$ *and* $(F,E) \ddot{\subseteq} (G,E)$, *therefore* $(F,E)^\circ \ddot{\subseteq} (G,E)$. *Therefore,* $(F,E)^\circ$ *is a* $\ll (T,S) \gg$ *complex interval valued picture fuzzy soft open set contained in* $(G,E) \rightarrow (1)$. *But* $(G,E)^\circ$ *is the largest complex interval valued picture fuzzy soft open set contained in* $(G,E) \rightarrow (2)$. *From* (1) *and* (2), *we have that* $(G,E)^\circ$ *is larger than* $(F,E)^\circ$, *which implies* $(F,E)^\circ \ddot{\subseteq} (G,E)^\circ$. *Thus,* $(F,E) \ddot{\subseteq} (G,E)$ *implies* $(F,E)^\circ \ddot{\subseteq} (G,E)^\circ$.

(iv) Let (X,τ,E) *be the complex interval valued picture fuzzy soft topological space. To prove*

$$[(F,E) \ddot{\cap} (G,E)]^\circ \ddot{=} (F,E)^\circ \ddot{\cap} (G,E)^\circ,$$

we know that $(F,E) \ddot{\cap} (G,E) \widehat{\ddot{\subseteq}} (F,E)$ *and* $(F,E) \ddot{\cap} (G,E) \ddot{\subseteq} (G,E)$, *which implies*

$$[(F,E) \ddot{\cap} (G,E)]^\circ \ddot{\subseteq} (F,E)^\circ \quad \textit{and} \quad [(F,E) \ddot{\cap} (G,E)]^\circ \ddot{\subseteq} (G,E)^\circ (by (iii)).$$

This implies

$$[(F,E) \ddot{\cap} (G,E)]^\circ \ddot{\subseteq} (F,E)^\circ \ddot{\cap} (G,E)^\circ.$$

Also, we have $(F,E)^\circ \ddot{\subseteq} (F,E)$ *and* $(G,E)^\circ \ddot{\subseteq} (G,E)$, *which implies*

$$(F,E)^\circ \ddot{\cap} (G,E)^\circ \ddot{\subseteq} (F,E) \ddot{\cap} (G,E),$$

which implies that $(F,E)^\circ \ddot{\cap} (G,E)^\circ$ *is a complex interval valued picture fuzzy soft open set contained in* $(F,E) \ddot{\cap} (G,E) \Rightarrow (2)$. *But* $[(F,E) \ddot{\cap} (G,E)]^\circ$ *is the largest complex interval valued picture fuzzy soft open set contained in* $(F,E) \ddot{\cap} (G,E) \Rightarrow (3)$. *Therefore, from* (2) *and* (3), *it follows that* $[(F,E) \ddot{\cap} (G,E)]^\circ$ *is larger than* $(F,E)^\circ \ddot{\cap} (G,E)^\circ$, *that is* $(F,E)^\circ \ddot{\cap} (G,E)^\circ$ *is smaller than* $[(F,E) \ddot{\cap} (G,E)]^\circ$ *this leads to* $(F,E)^\circ \ddot{\cap} (G,E)^\circ \ddot{\subseteq} [(F,E) \ddot{\cap} (G,E)]^\circ \rightarrow (4)$. *From* (1) *and* (4) *it follows that* $[(F,E) \ddot{\cap} (G,E)]^\circ \ddot{=} (F,E)^\circ \ddot{\cap} (G,E)^\circ$.

(v) *We know that* $[(F,E)^\circ]^\circ$ *is a complex interval valued picture fuzzy soft open set. Let us assume* $(F,E)^\circ \ddot{=} (H,E)$. *Therefore,* (H,E) *is a complex interval valued picture fuzzy soft open set, which implies* $(H,E) \ddot{=} (H,E)^\circ$. *Therefore,* $(F,E)^\circ \ddot{=} [(F,E)^\circ]^\circ$, *hence the result.*

(vi) *Since* $(F,E) \ddot{\subseteq} (F,E) \ddot{\cup} (G,E)$ *and* $(G,E) \ddot{\subseteq} (F,E) \ddot{\cup} (G,E)$, *by (iii), we have*

$$(F,E)^\circ \ddot{\subseteq} [(F,E) \ddot{\cup} (G,E)]^\circ \quad \text{and} \quad (F,E)^\circ \ddot{\subseteq} [(F,E) \ddot{\cup} (G,E)]^\circ.$$

So that

$$(F,E)^\circ \ddot{\cup} (G,E)^\circ \ddot{\subseteq} [(F,E) \ddot{\cup} (G,E)]^\circ,$$

since $(F,E)^\circ \ddot{\cup} (G,E)^\circ$ *is a complex interval valued picture fuzzy soft open set.*

Remark 1. *Let* (X, τ, E) *be the complex interval valued picture fuzzy soft topological space over* X, *and let* (F,E) *be any complex interval valued picture fuzzy soft open set. Then we always have*

$$(H,E)^\circ \ddot{\subseteq} (F,E) \ddot{\subseteq} \overline{\overline{(F,E)}}.$$

Remark 2. $\underline{\underline{(H,E)}} \ddot{=} [\underline{\underline{\ddot{X} - (H,E)}}]$ *and* $bd\ (H,E) = bd\ [\ddot{X} - (H,E)]$.
To prove;

$$bd(H,E) = bd\overline{\overline{(H,E)}} \ \ddot{\cap} \ [\overline{\overline{\ddot{X} - (H,E)}}] \quad \Rightarrow \quad (1)$$

$$bd\ [\ddot{X} - (H,E)] = [\overline{\overline{\ddot{X} - (H,E)}}] \ddot{\cap} \ [\overline{\overline{\ddot{\mathcal{X}} - (\ddot{X} - (H,E))}}].$$

Hence,

$$[\overline{\overline{\ddot{X} - (H,E)}}] \ddot{\cap} \ \overline{\overline{(H,E)}} \quad \Rightarrow \quad \overline{\overline{(H,E)}} \ddot{\cap} \ [\overline{\overline{\ddot{X} - (H,E)}}] \ddot{\cap} \ = bd(H,E).$$

Thus,

$$bd\ (H,E) = bd\ [\ddot{X} - (H,E)].$$

Theorem 11. *Let* (X, τ, E) *be the complex interval valued picture fuzzy soft topological space. Let* (H,E) *be any complex interval valued picture fuzzy soft subset of* $\ddot{X}$. *Then the following properties are true:*

(i) $bd(H,E) = \overline{\overline{(H,E)}} - (H,E)^\circ$.

(ii) $(H,E)^\circ = (H,E) - bd(H,E)$.

(iii) $\ddot{X} \ddot{=} (H,E)^\circ \ddot{\cap} bd(H,E) \ddot{\cup} [\ddot{X} - (H,E)^\circ]$.

Proof.

(i) *We know that* $bd(H,E) = \underline{\underline{(H.E)}} \ \ddot{=} \ \overline{\overline{(H,E)}} \ \ddot{\cap} \ \overline{\overline{(H,E)'}}$

$$\Rightarrow \quad \overline{\overline{(H,E)}} = \overline{\overline{(H,E)}} - [\ddot{X} - \overline{\overline{(\ddot{X} - (H,E))}}]^\circ = \overline{\overline{(H,E)}} - [\ddot{X}(\ddot{X} - (H,E))]^\circ$$

.
Thus, $= \overline{\overline{(H,E)}} - (H,E)^\circ = bd\ (H,E) = \overline{\overline{(H,E)}} - (H,E)^\circ$.

(ii) *Consider*

$$(H,E) - bd\ (H,E) = (H,E) - [\overline{\overline{(H,E)}} - \ \overline{\overline{(\ddot{X} - (H,E))}}]$$

$$= (H,E) \ \ddot{\cap} \ [\ddot{X} - \overline{\overline{(H,E)}} \ \ddot{\cap} \ \overline{\overline{(\ddot{X} - (H,E))}}]$$

$$= \ (H,E) \ \ddot{\cap} \ [(\ddot{X} - \overline{\overline{(H,E)}}) \ \ddot{\cup} \ (\ddot{X} - [\overline{\overline{\ddot{X} - (H,E)}}])]$$

.

This simplifies to

$$\Rightarrow \ (H,E) \ddot{\cap} [(\ddot{X} - \overline{\overline{(H,E)}}) \ddot{\cup} (\ddot{X} - [(H,E)^{\circ}])]$$

$$= \ (H,E) \ddot{\cap} [(\ddot{X} - \overline{\overline{(H,E)}}) \ddot{\cup} [(H,E) \ddot{\cap} (H,E)^{\circ}]]$$

. *Therefore,* $(H,E)^{\circ} = (H,E) - bd\ (H,E)$.

(iii) Consider the right-hand side

$(H,E)^{\circ} \ddot{\cup} bd\ (H,E) \ddot{\cup}\ [\ddot{X} - (H,E)^{\circ}]$

$$= (H,E)^{\circ} \ \ddot{\cup} \ bd\ (H,E) \ \ddot{\cup} \ (\ddot{X} - \overline{\overline{(H,E)}})$$

$$= (H,E) \ \ddot{\cup} \ \overline{\overline{(\ddot{X} - (H,E))}} = \ddot{X}.$$

Further, we know that $bd\ (H,E) = \overline{\overline{(H,E)}} - (H,E)^{\circ}$, *which implies that* $bd(H,E)$ *and* $(H,E)^{\circ}$ *are disjoint* $\rightarrow$ (1). *Replacing* (H,E) *with* $(\ddot{X} - (H,E)^{\circ})$, *are disjoint sets. Hence,* $\ddot{X} = \overline{\overline{(H,E)}} \ddot{\cup} \overline{\overline{(\ddot{X} - (H,E))}}$, *which is a complex interval valued picture fuzzy soft disjoint union.*

7. Conclusion and Future Work

This research introduces the CP of two CIVPFSSs, a novel framework aimed at enhancing the applicability of fuzzy soft set theory in addressing multi-criteria decision-making problems. The study presents groundbreaking concepts such as CIVPFSS symmetric relation, CIVPFSS antisymmetric relation, CIVPFSS asymmetric relation, CIVPFSS complete relation, CIVPFSS transitive relation, CIVPFSS equivalence relation, CIVPFSS partial-order relation, CIVPFSS strict-order relation, CIVPFSS preorder relation, and CIVPFSS equivalence classes. A practical application of the proposed CIVPFSS framework is demonstrated in evaluating fitness tracking applications. This application incorporates diverse criteria and attributes, leveraging expert opinions expressed through neutral, membership, and non-membership values. The innovative score function associated with CIVPFSS relations was utilized during the decision-making process, ensuring an objective and comprehensive evaluation. The findings reveal that CIVPFSS relations exhibit superior performance compared to existing fuzzy soft set structures, particularly in addressing periodicity challenges inherent in decision-making scenarios. The robustness and flexibility of CIVPFSS relations make them a promising tool for modeling complex and uncertain systems. This study opens new avenues for extending the generalization of fuzzy soft sets, paving the way for the development of advanced mathematical models with wide-ranging applications in domains such as artificial intelligence, data science, and operations research. Future research could focus on refining these concepts to create even more sophisticated structures capable of addressing emerging challenges across various scientific and industrial fields. The creation of a topological space for CIVPFSR builds a geometrical framework which clarifies how decision factors connect while examining choice behaviors operating in multi-dimensional domains. This work identifies the key operations and operators with their relevant theorems and major results. The text explores results that pertain to both interiors and closures among specific concepts then details their theoretical applications. Future developments in wearable technology might concentrate on improving personalization through the use of real-time AI-driven data processing for instant insights, linking wearables with larger health management systems, and incorporating a variety of data kinds. The CIVPFSR model may become more scalable with more development, allowing it to be used to bigger datasets and more intricate health variables. The capabilities of wearable technology could also be further increased, making them more essential to total health management, by investigating cross-domain applications and long-term health tracking, as well as by creating sophisticated user interfaces for improved accessibility.

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