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A Family of Bi-Univalent Functions Defined by (p, q)-Derivative Operator Subordinate to a Generalized Bivariate Fibonacci Polynomials

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Abstract. Making use of a generalized bivariate Fibonacci polynomials, we propose a family of normalized regular functions $\psi(\zeta) = \zeta + d_2\zeta^2 + d_3\zeta^3 + \cdots$, which are bi-univalent in the disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$ involving (p, q)-derivative operator. We find estimates on the coefficients $|d_2|$, $|d_3|$ and the Fekete-Szegö inequality for members of this family. New implications of the primary result as well as pertinent links to previously published findings are also provided.

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1. Introduction

The quantum (or q-) calculus is essential because it is applied in many different branches of mathematics, computer science, physics, and other related fields. The extension of the q-calculus to the (p, q)-calculus, was taken into consideration by the researchers. The (p, q)-calculus, which includes the (p, q)-number, is first examined around

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the same time (1991) and subsequently on its own by [12, 15, 21, 46]. Fibonacci oscillators were studied with the presentation of the (p, q)-number in [12]. The investigation of the (p, q)-number in [15] allows for the construction of a (p, q)-Harmonic oscillator. In[21], the (p, q)-number was explored to unify or generalize various forms of q-oscillator algebras. The (p, q)-numbers are investigated in [46] to calculate the (p, q)-Stirling numbers. Consequently, many mathematical, computer science, physical, chemical and other related problems require knowledge of (p, q)-calculus. Expanding upon the previously mentioned papers, numerous scientists have studied the (p, q)-calculus in a variety of research fields since 1991. A syntax for embedding the q-series into a (p, q)-series was given by the results in [31]. Additionally, they looked into (p, q)-hypergeometric series and discovered some outcomes that matched (p, q)-extensions of the well-known q-identities. The q-identities are extended correspondingly to yield the (p, q)-series (see, e.g., [11]). We provide some elementary definitions of the (p, q)-calculus concepts. The (p, q)-bracket number is given by $[j]_{p,q} = p^{j-1} + p^{j-2}q + ... + p^2q^{j-3} + pq^{j-2} + q^{j-1} = \frac{p^j - q^j}{p-q}$ $(p \neq q)$, which is an extension of q-number (see [30]), that is $[j]_q = \frac{1-q^j}{1-q}$ $(q \neq 1)$. Note that $[j]_{p,q}$ is symmetric and if p=1, then $[j]_{p,q}$=$[j]_q$.

let $\mathfrak{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, where $\mathbb{C}$ is the complex plane. Let $\mathbb{R}$ be the family of real numbers and $\mathbb{N} = \mathbb{N}_0 \backslash \{0\} := \{1, 2, 3, ...\}$.

Definition 1. *[1] Let ψ be a function defined on $\mathbb{C}$ and $0 < q < p \leq 1$. Then the (p, q)-derivative of θ is defined by*

$$D_{p,q}\psi(\zeta) = \frac{\psi(p\zeta) - \psi(q\zeta)}{(p-q)\zeta} \ (\zeta \neq 0),$$

and $D_{p,q}\psi(0) = \psi'(0)$, provided $\psi'(0)$ exists.

We note that $D_{p,q}\zeta^j = [j]_{p,q}\zeta^{j-1}$ *and* $D_{p,q}ln(\zeta) = \frac{ln(p/q)}{(p-q)\zeta}$. Also, we observe that $[j]_{p,q} \to j$, *if* $q \to 1^-$ *and* $p = 1$. Therefore, $D_{p,q}\psi(\zeta) \to \psi'(\zeta)$ as $q \to 1^-$ *and* $p = 1$. Any function's (p, q)-derivative is a linear operator. More accurately $D_{p,q}(a\psi_1(\zeta) + b\psi_2(\zeta)) = aD_{p,q}\psi_1(\zeta) + bD_{p,q}\psi_2(\zeta)$, for any constants a and b. The product rules and quotient rules are satisfied by the (p, q)-derivative (see [37]). The exponential functions are used to define the (p, q)-analogs of many functions, including sine, cosine, and tangent in the same way as their Euler expressions. Duran et al.[23] have examined the (p, q)-derivatives of these functions. To learn more about (p, q)-calculus, see [1, 8, 16, 24].

Let us take a normalized regular function ψ in $\mathfrak{D}$ given by

$$\psi(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j\zeta^j, \tag{1}$$

and let $\mathcal{A}$ be the set of all such functions. Let $\mathcal{S} = \{\psi \in \mathcal{A} : \psi \, is \, univalent \, in \, \mathfrak{D}\}$. If $\psi \in \mathcal{A}$ is of the form (1), then

$$D_{p,q}\psi(\zeta) = 1 + \sum_{j=2}^{\infty} [j]_{p,q}d_j\zeta^{j-1}, \qquad (\zeta \in \mathfrak{D}), \tag{2}$$

The renowned Koebe theorem (see[25]) states that, each function $\psi \in \mathcal{S}$ has an inverse given by

$$\psi^{-1}(\omega) = \Psi(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + ... \tag{3}$$

satisfying $\zeta = \psi^{-1}(\psi(\zeta))$ and $\omega = \psi(\psi^{-1}(\omega))$, $|\omega| < r_0(\psi)$, $r_0(\psi) \geq 1/4$, $\zeta, \omega \in \mathfrak{D}$.

The notion of bi-univalent functions was first presented by Levin in his work [33]. These are analytic functions, denoted by ψ, where both ψ and $\psi^{-1}$are univalent in $\mathfrak{D}$. The set of all bi-univalent functions of the type (1) is symbolized by Σ. $\frac{1}{2}log\left(\frac{1+\zeta}{1-\zeta}\right)$, $-log(1-\zeta)$ and $\frac{\zeta}{1-\zeta}$ are some of the functions in the Σ family. However, $\zeta - \frac{\zeta^2}{2}$, $\frac{\zeta}{1-\zeta^2}$, and the Koebe function do not belong in Σ, even though they are in $\mathcal{S}$. For a concise analysis and to discover some of the characteristics of the family Σ, see [3, 5, 13, 14, 29, 44] and the citation provided in these papers. The article by Srivastava et al.[40] gave rise to the recent momentum of studies of the bi-univalent function family. Numerous scholars have looked into several fascinating special families of Σ since this article brought the subject back to life (see [9, 10, 17, 18, 27].

The (p, q)-calculus was used to study several subfamilies of the family $\mathcal{S}$ and the family Σ. In [41], the subordination principle is used to define the (p, q)-starlike and (p, q)-convex functions families. Novel subfamilies of the family σ associated with (p, q)-differential operators have also been presented and examined in a number of studies (refer to [6, 7, 22, 35, 45]).

Let $s(\varkappa, y)$ and $t(\varkappa, y)$ be polynomials with real coefficients. For, $j \geq 2$, the generalized bivariate Fibonacci polynomials(GBFP) are defined by the recurrence relation:

$$\mathcal{F}_j(\varkappa, y) = s(\varkappa, y)\mathcal{F}_{j-1}(\varkappa, y) + t(\varkappa, y)\mathcal{F}_{j-2}(\varkappa, y), \tag{4}$$

where $\mathcal{F}_0(\varkappa, y) = 0$, $\mathcal{F}_1(\varkappa, y) = 1$ and $s^2(\varkappa, y) + 4t(\varkappa, y) > 0$. The generating function of GBFP is (see [32])

$$\mathcal{F}(\varkappa, y, z) = \sum_{j=2}^{\infty} \mathcal{F}_j(\varkappa, y)z^j = \frac{z}{1 - s(\varkappa, y)z - t(\varkappa, y)z^2}. \tag{5}$$

For specific selections of $s(\varkappa, y)$ and $t(\varkappa, y)$, GBFP leads to various known polynomials (see [47]). Readers with an interest in GBFP can find a brief history and extensive information in [19] and its references. For members of specific subclasses of σ associated with GBFP, interesting results have been obtained in [2, 28] regarding coefficient estimates and Fekete-Szegö functional.

For brevity, we write hereafter that $s(\varkappa, y) = s$ and $t(\varkappa, y) = t$. $\mathcal{F}_2(\varkappa, y) = s$, $\mathcal{F}_3(\varkappa, y) = s^2 + t$,..., are evident from (4).

For functions θ_1, $\theta_2 \in \mathcal{A}$, we say that θ_1 is subordinate to θ_2, if there is $\kappa(\zeta)$, a Schwarz function in $\mathfrak{D}$ with $\kappa(0) = 0$ and $|\kappa(\zeta)| < 1$ $(\zeta \in \mathfrak{D})$, such that $\theta_1(\zeta) = \theta_2(\kappa(\zeta))$, $\zeta \in \mathfrak{D}$. This is indicated as $\theta_1 \prec \theta_2$ or $\theta_1(\zeta) \prec \theta_2(\zeta)$ $(\zeta \in \mathfrak{D})$. In particular, if $\theta_2 \in \mathcal{S}$, then $\theta_1(\zeta) \prec \theta_2(\zeta) \Leftrightarrow \theta_1(0) = \theta_2(0) \quad and \quad \theta_1(\mathfrak{D}) \subset \theta_2(\mathfrak{D})$.

Definition 2. *The (p,q)-analogue of Swamy differential operator for $\psi \in \mathcal{A}$ is defined as follows:*

$$\Omega_{p,q}^{\nu,\mu,0}\psi(\zeta) = \psi(\zeta),$$

$$\Omega_{p,q}^{\nu,\mu,1}\psi(\zeta) = \frac{\nu\psi(\zeta) + \mu z D_{p,q}\psi(\zeta)}{\nu+\mu},$$

$$\ldots,\qquad \Omega_{p,q}^{\nu,\mu,k}\psi(\zeta) = \Omega_{p,q}^{\nu,\mu}(\Omega_{p,q}^{\nu,\mu,k-1}\theta(\zeta)),$$

where $k \in \mathbb{N}$, $\mu \geq 0$, ν a real number with $\nu+\mu > 0$, $0 < q < p \leq 1$ and $\zeta \in \mathfrak{D}$.

Remark 1. *i). $\Omega_{p,q}^{\nu,\mu,k} : \mathcal{A} \to \mathcal{A}$ is a linear operator, as we can see, and for $\psi(\zeta)$, as provided by (1), we have*

$$\Omega_{p,q}^{\nu,\mu,k}\psi(\zeta) = \zeta + \sum_{j=2}^{\infty}\left(\frac{\nu+\mu[j]_{p,q}}{\nu+\mu}\right)^k d_j\zeta^j, \tag{6}$$

ii). If we let $\nu = 0$ and $\mu = 1$, then $\Omega_{p,q}^{\nu,\mu,k}\psi(\zeta)$ reduces to the (p,q)-analogue of Salagean operator discussed in[39].

iii). If we take $\nu = 1-\mu, \mu \geq 0$, then $A_{p,q}^{\mu,k}(= \Omega_{p,q}^{1-\mu,\mu,k}) : \mathcal{A} \to \mathcal{A}$ is a linear operator and for $\psi(\zeta)$ given by (1), we have

$$A_{p,q}^{\mu,k}\psi(\zeta) = \zeta + \sum_{j=2}^{\infty}(1+\mu([j]_{p,q}-1))^k d_j\zeta^j, \tag{7}$$

which is (p,q)-analogue of Al-Oboudi differential operator.

iv). If we put $\nu = l+1-\mu, l > -1, \mu \geq 0$, then $C_{p,q}^{l,\mu,k}(= \Omega_{p,q}^{l+1-\mu,\mu,k}) : \mathcal{A} \to \mathcal{A}$ is a linear operator and for $\psi(\zeta)$ given by (1), we have

$$= C_{p,q}^{l,\mu,k}\psi(\zeta) = \zeta + \sum_{j=2}^{\infty}\left(\frac{l+1+\mu([j]_{p,q}-1)}{l+1}\right)^k d_j\zeta^j, \tag{8}$$

which is (p,q)-analogue of Catas differential operator.

v). Swamy operator[42, 43], Al-Oboudi operator[4], and Cătaş operator [20] are obtained by taking $q \to 1^-$ and $p = 1$ in (6), (7), and (8), respectively.

With the generating function $\mathcal{F}(\varkappa, y, z)$ as in (5), we introduce a new family of σ subordinate to GBFP $\mathcal{F}_j(\varkappa, y)$ as in (4). The Fekete-Szegö functional[26] on some subclasses of σ associated with GBFP and the previously mentioned trends on coefficient-related problems serve as inspiration for the defined family. The inverse function $\phi^{-1}(\omega) = \psi(\omega)$ is as in (3), and $\mathcal{F}(\varkappa, y, z)$ is as in (5) are assumed throughout this paper unless otherwise noted.

Definition 3. *A function $\psi \in \Sigma$ is said to be in the family $\mathfrak{E}^{\lambda,k}_{\Sigma,p,q}(F,\nu,\mu)$, if*

$$\frac{1}{2}\left\{\frac{\zeta(\Omega^{\nu,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)}+\left(\frac{\zeta(\Omega^{\nu,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)}\right)^{\frac{1}{\lambda}}\right\}\prec F(\zeta)=\frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta},\ \zeta\in\mathfrak{D}$$

and

$$\frac{1}{2}\left\{\frac{\omega(\Omega^{\nu,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)}+\left(\frac{\omega(\Omega^{\nu,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)}\right)^{\frac{1}{\lambda}}\right\}\prec F(\omega)=\frac{\mathcal{F}(\varkappa,y,\omega)}{\omega},\ \omega\in\mathfrak{D},$$

where $0<\lambda\leq 1, \mu\geq 0, \nu$ a real number with $\nu+\mu>0,\ k\in\mathbb{N}$, and

$$F(z)=\frac{1}{1-sz-tz^2},\ s^2+4t>0. \tag{9}$$

For particular chioces of $p, q, \lambda,$ and ν, the family $\mathfrak{E}^{\lambda,k}_{\Sigma,p,q}(F,\nu,\mu)$ includes many new subfamilies of Σ as mentioned below:

Example 1.1. $\mathfrak{F}^{\lambda,k}_{\Sigma,p,q}(F,\mu)\equiv\mathfrak{E}^{\lambda,k}_{\Sigma,p,q}(F,1-\mu,\mu), 0<\lambda\leq 1, \mu\geq 0,$ *and* $k\in\mathbb{N}$ is the set of members ψ in Σ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(A^{\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)}+\left(\frac{\zeta(A^{\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)}\right)^{\frac{1}{\lambda}}\right\}\prec F(\zeta)=\frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta},\ \zeta\in\mathfrak{D},$$

and

$$\frac{1}{2}\left\{\frac{\omega(A^{\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)}+\left(\frac{\omega(A^{\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)}\right)^{\frac{1}{\lambda}}\right\}\prec F(\omega)=\frac{\mathcal{F}(\varkappa,y,\omega)}{\omega},\ \omega\in\mathfrak{D}.$$

where $F(z)$ is as mentioned in (9).

Example 1.2. $\mathfrak{G}^{\lambda,k}_{\Sigma,p,q}(F,l,\mu)\equiv\mathfrak{E}^{\lambda,k}_{\Sigma,p,q}(F,l+1-\mu,\mu), 0<\lambda\leq 1, l>-1, \mu\geq 0,$ *and* $k\in\mathbb{N}$ is the set of members $\psi\in\Sigma$ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(C^{l,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)}+\left(\frac{\zeta(C^{l,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)}\right)^{\frac{1}{\lambda}}\right\}\prec F(\zeta)=\frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta},\ \zeta\in\mathfrak{D},$$

and

$$\frac{1}{2}\left\{\frac{\omega(C^{l,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)}+\left(\frac{\omega(C^{l,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)}\right)^{\frac{1}{\lambda}}\right\}\prec F(\omega)=\frac{\mathcal{F}(\varkappa,y,\omega)}{\omega},\ \omega\in\mathfrak{D}.$$

where $F(z)$ is as mentioned in (9).

Example 1.3. $\mathfrak{H}^{k}_{\Sigma,p,q}(F,\nu,\mu) \equiv \mathfrak{E}^{1,k}_{\Sigma,p,q}(F,\nu,\mu)$ is the collection of elements $\psi \in \Sigma$ that satisfy

$$\frac{\zeta(\Omega^{\nu,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta},\ \zeta \in \mathfrak{D}$$

and

$$\frac{\omega(\Omega^{\nu,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)} \prec F(\omega) = \frac{\mathcal{F}(\varkappa,y,\omega)}{\omega},\ \omega \in \mathfrak{D},$$

where $\mu \geq 0, \nu$ a real number with $\nu+\mu>0, k \in \mathbb{N}$ and $F(z)$ is as mentioned in (9).

Example 1.4. If $q \to 1^-$ *and* $p=1$ in the set $\mathfrak{E}^{\lambda,k}_{\Sigma,p=1,q\to 1^-}(F,\nu,\mu)$, then we obtain a subset $\mathfrak{K}^{\lambda,k}_{\Sigma}(F,\nu,\mu)$, which is a collection of functions $\psi \in \Sigma$ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(\Gamma^{\nu,\mu.k}\psi(\zeta))'}{\psi(\zeta)} + \left(\frac{\zeta(\Gamma^{\nu,\mu.k}\psi(\zeta))'}{\psi(\zeta)}\right)^{\frac{1}{\lambda}}\right\} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa,y,\zeta)}{\zeta},\ \zeta \in \mathfrak{D},$$

and

$$\frac{1}{2}\left\{\frac{\omega(\Gamma^{\nu,\mu.k}\Psi(\omega))'}{\Psi(\omega)} + \left(\frac{\omega(\Gamma^{\nu,\mu.k}\Psi(\omega))'}{\Psi(\omega)}\right)^{\frac{1}{\lambda}}\right\} \prec F(\omega) = \frac{\mathcal{F}(\varkappa,y,\omega)}{\omega},\ \omega \in \mathfrak{D},$$

where $\Gamma^{\nu,\mu.k} \equiv \Omega^{\nu,\mu,k}_{p=1,q\to 1^-}$, $0<\lambda\leq 1$, $\mu \geq 0, \nu$ a real number with $\nu+\mu>0, k \in \mathbb{N}$ and $F(z)$ is as mentioned in (9)..

Fekete-Szegö inequality[26] and estimates for $|d_2|$ and $|d_3|$ are found in Section 2 for functions $\in \mathfrak{S}^{\lambda,k}_{\Sigma,p,q}(F,\nu,\mu)$. A few intriguing ramifications of the main result as well as pertinent links to the previous results are also provided.

2. Main results

We first determine the bounds for $|d_2|$, $|d_3|$ and an inequality of Fekete-Szegö for elements in $\mathfrak{S}^{\lambda,k}_{\Sigma,p,q}(F,\nu,\mu)$.

Theorem 1. *Let* $0<\lambda\leq 1, \mu\geq 0, \nu$ *a real number such that* $\nu+\mu>0$, *and* $k\in\mathbb{N}$. *If a function* $\psi \in \mathfrak{E}^{\lambda,k}_{\Sigma,p,q}(F,\nu,\mu)$, *then*

$$i).|d_2| \leq \frac{2\lambda s\sqrt{s}}{\sqrt{|(2\lambda(\lambda+1)(\mathcal{N}-\mathcal{M})+(1-\lambda)\mathcal{M}^2)s^2-(1+\lambda)^2\mathcal{M}^2(s^2+t)|}}, \tag{10}$$

$$ii).\ |d_3| \leq \frac{2\lambda s}{(1+\lambda)\mathcal{N}} + \frac{4\lambda^2 s^2}{(1+\lambda)^2\mathcal{M}^2}, \tag{11}$$

and for $\xi \in \mathbb{R}$

$$iii).\ |d_3-\xi d_2^2| \leq \begin{cases} \frac{2\lambda s}{(1+\lambda)\mathcal{N}} & ; |1-\xi| \leq \mathcal{J} \\ \frac{4\lambda^2 s^3\,|1-\xi|}{|(2\lambda(\lambda+1)(\mathcal{N}-\mathcal{M})+(1-\lambda)\mathcal{M}^2)s^2-(1+\lambda)^2\mathcal{M}^2(s^2+t)|} & ; |1-\xi| \geq \mathcal{J}, \end{cases} \tag{12}$$

where

$$\mathcal{J} = \frac{|(2\lambda(\lambda+1)(\mathcal{N}-\mathcal{M})+(1-\lambda)\mathcal{M}^2)s^2-(1+\lambda)^2\mathcal{M}^2(s^2+t)|}{2\lambda(1+\lambda)\mathcal{N}s^2}, \tag{13}$$

$$\mathcal{M} = \left(2\left(\frac{\nu+\mu[2]_{p,q}}{\nu+\mu}\right)^k - 1\right), \tag{14}$$

and

$$\mathcal{N} = \left(3\left(\frac{\nu+\mu[3]_{p,q}}{\nu+\mu}\right)^k - 1\right). \tag{15}$$

Proof. Let $\psi \in \mathfrak{E}^{\lambda,k}_{\Sigma,p,q}(F,\nu,\mu)$. Then, based on Definition 3, we can write

$$\frac{1}{2}\left\{\frac{\zeta(\Omega^{\nu,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)} + \left(\frac{\zeta(\Omega^{\nu,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)}\right)^{\frac{1}{\lambda}}\right\} = F(\mathfrak{u}(\varsigma)),\ \varsigma \in \mathfrak{U} \tag{16}$$

and

$$\frac{1}{2}\left\{\frac{\omega(\Omega^{\nu,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)} + \left(\frac{\omega(\Omega^{\nu,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)}\right)^{\frac{1}{\lambda}}\right\} = F(\mathfrak{v}(w)),\ w \in \mathfrak{U}. \tag{17}$$

where $\mathfrak{u}(\varsigma) = \sum_{j=1}^{\infty}\mathfrak{u}_j\varsigma^j,$ *and* $\mathfrak{v}(w) = \sum_{j=1}^{\infty}\mathfrak{v}_j w^j$, $\varsigma, w \in \mathfrak{U}$ are Schwarz functions with the property (See[25])

$$|\mathfrak{u}_j| \le 1,\ \textit{and}\ |\mathfrak{v}_j| \le 1\,(j \in \mathbb{N}). \tag{18}$$

By using few fundamental mathematical technics we can write equations (16) and (17) as

$$\frac{1}{2}\left\{\frac{\zeta(\Omega^{\nu,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)} + \left(\frac{\zeta(\Omega^{\nu,\mu.k}_{p,q}\psi(\zeta))'}{\psi(\zeta)}\right)^{\frac{1}{\delta}}\right\} =$$

$$1+\left(\frac{1+\lambda}{2\lambda}\right)\mathcal{M}d_2\zeta + \left(\left(\frac{1+\lambda}{2\lambda}\right)(\mathcal{N}d_3-\mathcal{M}d_2^2)+\left(\frac{1-\lambda}{4\lambda^2}\right)\mathcal{M}^2d_2^2\right)\zeta^2+..., \tag{19}$$

$$F(\mathfrak{u}(\varsigma)) = 1+\mathcal{F}_2(\varkappa,y)\mathfrak{u}_1\varsigma + \left[\mathcal{F}_2(\varkappa,y)\mathfrak{u}_2+\mathcal{F}_3(\varkappa,y)\mathfrak{u}_1^2\right]\varsigma^2+..., \tag{20}$$

and

$$\frac{1}{2}\left\{\frac{\omega(\Omega^{\nu,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)} + \left(\frac{\omega(\Omega^{\nu,\mu.k}_{p,q}\Psi(\omega))'}{\Psi(\omega)}\right)^{\frac{1}{\delta}}\right\} =$$

$$1+\left(\frac{1+\lambda}{2\lambda}\right)\mathcal{M}d_2\omega + \left(\left(\frac{1+\lambda}{2\lambda}\right)(\mathcal{N}(2d_2^2-d_3)-\mathcal{M}d_2^2)+\left(\frac{1-\lambda}{4\delta^2}\right)\mathcal{M}^2d_2^2\right)\omega^2+..., \tag{21}$$

$$F(\mathfrak{v}(w)) = 1+\mathcal{F}_2(\varkappa,y)\mathfrak{v}_1 w + \left[\mathcal{F}_2(\varkappa,y)\mathfrak{v}_2+\mathcal{F}_3(\varkappa,y)\mathfrak{v}_1^2\right]w^2+\dots. \tag{22}$$

where $\mathcal{M}$ and $\mathcal{N}$ are as mentioned in (14), and (15), respectively.

Comparing the terms with the same degree in (19) and (20), we conclude due to equality (16)

$$\left(\frac{1+\lambda}{2\lambda}\right)\mathcal{M}d_2 = \mathcal{F}_2(\varkappa,y)\mathfrak{u}_1, \tag{23}$$

$$\left(\frac{1+\lambda}{2\lambda}\right)(\mathcal{N}d_3-\mathcal{M}d_2^2)+\left(\frac{1-\lambda}{4\lambda^2}\right)\mathcal{M}^2d_2^2 = \mathcal{F}_2(\varkappa,y)\mathfrak{u}_2+\mathcal{F}_3(\varkappa,y)\mathfrak{m}_1^2. \tag{24}$$

Similarly, due to equality (17), we draw our conclusion by comparing the terms of the same degree in (21) and (22)

$$-\left(\frac{1+\lambda}{2\lambda}\right)\mathcal{M}d_2 = \mathcal{F}_2(\varkappa,y)\mathfrak{v}_1, \tag{25}$$

$$\left(\frac{1+\lambda}{2\lambda}\right)(\mathcal{N}(2d_2^2-d_3)-\mathcal{M}d_2^2)+\left(\frac{1-\lambda}{4\lambda^2}\right)\mathcal{M}^2d_2^2 = \mathcal{F}_2(\varkappa,y)\mathfrak{v}_2+\mathcal{F}_3(\varkappa,y)\mathfrak{v}_1^2. \tag{26}$$

From (23) and (25), we can easily obtain

$$\mathfrak{u}_1 = -\mathfrak{v}_1, \tag{27}$$

$$\left(\frac{(1+\lambda)^2}{2\lambda^2}\right)\mathcal{M}^2d_2^2 = (\mathfrak{u}_1^2+\mathfrak{v}_1^2)\mathcal{F}_2^2(\varkappa,y). \tag{28}$$

When (24) and (26) are added, we get

$$2\left[\left(\frac{1+\lambda}{\lambda}\right)(\mathcal{N}-\mathcal{M})+\left(\frac{1-\lambda}{2\lambda^2}\right)\mathcal{M}^2\right]d_2^2 = \mathcal{F}_2(\varkappa,y)(\mathfrak{u}_2+\mathfrak{v}_2)+\mathcal{F}_3(\varkappa,y)(\mathfrak{u}_1^2+\mathfrak{v}_1^2). \tag{29}$$

Substituting the value of $\mathfrak{u}_1^2+\mathfrak{v}_1^2$ from (28) in (29), we get

$$d_2^2 = \frac{2\lambda^2\mathcal{F}_2^3(\varkappa,y)(\mathfrak{u}_2+\mathfrak{v}_2)}{\left[(2\lambda(\lambda+1)(\mathcal{N}-\mathcal{M})+(1-\lambda)\mathcal{M}^2)\mathcal{F}_2^2(\varkappa,y)-(1+\lambda)^2\mathcal{M}^2\mathcal{F}_3(\varkappa,y)\right]}, \tag{30}$$

which produces (10), when applied (18).

After deducting (26) from (24) and using (27), we arrive at

$$d_3 = d_2^2+\frac{\lambda\mathcal{F}_2(\varkappa,y)(\mathfrak{u}_2-\mathfrak{v}_2)}{(1+\lambda)\mathcal{N}}. \tag{31}$$

This results in the inequality that follows:

$$|d_3| \leq |d_2|^2+\frac{|\mathcal{F}_2(\varkappa,y)||\mathfrak{u}_2-\mathfrak{v}_2|}{\left(\frac{\lambda+1}{\lambda}\right)U[3]_{p,q}}. \tag{32}$$

From (10) and (32) we obtain (11), applying (18) for $\mathfrak{u}_2$ and $\mathfrak{v}_2$.

Clearly, for $\xi\in\mathbb{R}$ we get from (30) and (31) that,

$$|d_3-\xi d_2^2| = |\mathcal{F}_2(\varkappa,y)|\left|\left(G(\xi,F)+\frac{\lambda}{(1+\lambda)\mathcal{N}}\right)u_2+\left(G(\xi,F)-\frac{\lambda}{(1+\lambda)\mathcal{N}}\right)v_2\right|,$$

where

$$G(\xi,F)=\frac{2\lambda^2(1-\xi)\mathcal{F}_2^2(\varkappa,y)}{\left[(2\lambda(\lambda+1)(\mathcal{N}-\mathcal{M})+(1-\lambda)\mathcal{M}^2)\mathcal{F}_2^2(\varkappa,y)-(1+\lambda)^2\mathcal{M}^2\mathcal{F}_3(\varkappa,y)\right]}.$$

Then, using (18) we can deduce that

$$|d_3-\xi d_2^2|\leq\begin{cases}\frac{2\lambda|\mathcal{F}_2(\varkappa,y)|}{(1+\lambda)\mathcal{N}} & ;0\leq|G(\xi,F)|\leq\frac{\lambda}{(1+\lambda)\mathcal{N}}\\ 2|\mathcal{F}_2(\varkappa,y)||G(\xi,F)| & ;|G(\xi,F)|\geq\frac{\lambda}{(1+\lambda)\mathcal{N}},\end{cases}$$

which leads us to the conclusion (12), with $\mathcal{J}$ as in (13), considering $\mathcal{F}_2(\varkappa,y)=s$, $\mathcal{F}_3(\varkappa,y)=s^2+t$. This completes the proof of Theorem 1.

If we take $\xi=1$ in the part *iii*) of Theorem 1, we obtain the following result:

Corollary 1. *Let* $0<\lambda\leq 1,\mu\geq 0,\nu$ *a real number such that* $\nu+\mu>0$, $k\in\mathbb{N}$ *and* $\psi(\zeta)=\zeta+\sum_{j=2}^{\infty}d_j\zeta^j$ *be in the class* $\mathfrak{E}_{\Sigma,p,q}^{\lambda,k}(F,\nu,\mu)$. *Then* $|d_3-d_2^2|\leq\frac{2\lambda s}{(1+\lambda)\mathcal{N}}$.

Corollary 2. *Let us assume that* $\nu=1-\mu$ *in Theorem 1. Then the upper bounds of* $|d_2|,|d_3|,$ *and* $|d_3-\xi d_2^2|,\xi\in\mathbb{R}$, *for a function* $\psi\in\mathfrak{F}_{\Sigma,p,q}^{\lambda,k}(F,\mu)$ *are given by (10), (11), and (12), respectively, with* $\mathcal{M}=\mathcal{M}_1=2(1+\mu([2]_{p,q}-1)^k-1)$, *and* $\mathcal{N}=\mathcal{N}_1=3(1+\mu([3]_{p,q}-1)^k-1)$. *For* $\mathcal{J}$ *in (13),* $\mathcal{M}$, *and* $\mathcal{N}$ *are to be substituted with* $\mathcal{M}_1$, *and* $\mathcal{N}_1$, *respectively.*

Corollary 3. *Let us assume that* $\nu=l+1-\mu$ *in Theorem 1. Then the upper bounds of* $|d_2|,|d_3|,$ *and* $|d_3-\xi d_2^2|,\xi\in\mathbb{R}$, *for a function* $\psi\in\mathfrak{G}_{\Sigma,p,q}^{\lambda,k}(F,l,\mu)$ *are given by (10), (11), and (12), respectively, with* $\mathcal{M}=\mathcal{M}_2=\left(2\left(\frac{l+1+\mu([2]_{p,q}-1)}{l+1}\right)^k-1\right)$, *and* $\mathcal{N}=\mathcal{N}_2=\left(3\left(\frac{l+1+\mu([3]_{p,q}-1)}{l+1}\right)^k-1\right)$. *For* $\mathcal{J}$ *in (13),* $\mathcal{M}$, *and* $\mathcal{N}$ *are to be substituted with* $\mathcal{M}_2$, *and* $\mathcal{N}_2$, *respectively.*

If $\lambda=1$ in Theorem 1, we get

Corollary 4. *Let* $\mu\geq 0,\nu$ *a real number such that* $\nu+\mu>0$, *and* $k\in\mathbb{N}$. *If a function* $\psi\in\mathfrak{H}_{\Sigma,p,q}^{k}(F,\nu,\mu)$, *then*

$$i).\ |d_2|\leq\frac{s\sqrt{s}}{\sqrt{|(\mathcal{N}-\mathcal{M})s^2-\mathcal{M}^2(s^2+t)|}},\qquad ii).\ |d_3|\leq\frac{s^2}{\mathcal{M}^2}+\frac{s}{\mathcal{N}}$$

and for $\xi\in\mathbb{R}$

$$iii).\ |d_3-\xi d_2^2|\leq\begin{cases}\frac{s}{\mathcal{N}} & ;|1-\xi|\leq\left|\frac{(\mathcal{N}-\mathcal{M})s^2-\mathcal{M}^2(s^2+t)}{\mathcal{N}s^2}\right|\\ \frac{s^3\,|1-\xi|}{|(\mathcal{N}-\mathcal{M})s^2-\mathcal{M}^2(s^2+t)|} & ;|1-\xi|\geq\left|\frac{(\mathcal{N}-\mathcal{M})s^2-\mathcal{M}^2(s^2+t)}{\mathcal{N}s^2}\right|,\end{cases}$$

where $\mathcal{M}$, *and* $\mathcal{N}$ *are given by (14) and (15), respectively.*

Remark 2. *Taking $k = 0$ in Corollary 4, we get two results of Yilmaz and Aktaş[47, Corollaries 2 and 6].*

Corollary 5. *Let us assume that $q \to 1^-$ and $p = 1$ in Theorem 1. Then the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for any function $\psi \in \mathfrak{Y}_\Sigma^{\lambda,k}(F, \nu, \mu)$, are given by (10), (11), and (12), respectively, with $\mathcal{M} = \mathcal{M}_3 = \left(2\left(\frac{\nu+2\mu}{\nu+\mu}\right)^k - 1\right)$, and $\mathcal{N} = \mathcal{N}_3 = \left(3\left(\frac{\nu+3\mu}{\nu+\mu}\right)^k - 1\right)$. For $\mathcal{J}$ in (13), $\mathcal{M}$, and $\mathcal{N}$ are to be substituted with $\mathcal{M}_3$, and $\mathcal{N}_3$, respectively.*

Remark 3. *If $k = 0$ in the set $\mathfrak{Y}_\Sigma^{\lambda,k}(F, \nu, \mu)$, then we obtain a subset $\mathfrak{Q}_\Sigma^\lambda(F), 0 < \lambda \leq 1$, which is the collection of members of $\psi \in \Sigma$ that satisfy*

$$\frac{1}{2}\left\{\frac{\zeta\psi'(\zeta)}{\psi(\zeta)} + \left(\frac{\zeta\psi'(\zeta)}{\psi(\zeta)}\right)^{\frac{1}{\lambda}}\right\} \prec F(\zeta) = \frac{\mathcal{F}(\varkappa, y, \zeta)}{\zeta}, \zeta \in \mathfrak{D},$$

and

$$\frac{1}{2}\left\{\frac{\omega\Psi'(\omega)}{\Psi(\omega)} + \left(\frac{\omega\Psi'(\omega)}{\Psi(\omega)}\right)^{\frac{1}{\lambda}}\right\} \prec F(\omega) = \frac{\mathcal{F}(\varkappa, y, \omega)}{\omega}, \zeta \in \mathfrak{D}.$$

Corollary 6. *Let $0 < \lambda \leq 1$. If a function $\theta \in \mathfrak{Q}_\Sigma^\lambda(F)$, then*

$$i).\ |d_2| \leq \frac{2\lambda s\sqrt{s}}{\sqrt{|\lambda(\lambda-1)s^2 - (1+\lambda)^2 t|}}, \qquad ii).\ |d_3| \leq \frac{4\lambda^2 s^2}{(1+\lambda)^2} + \frac{\lambda s}{1+\lambda},$$

and for $\xi \in \mathbb{R}$

$$iii).\ |d_3 - \xi d_2^2| \leq \begin{cases} \frac{\lambda s}{1+\lambda} & ; |1-\xi| \leq \frac{|\lambda(\lambda-1)s^2 - (1+\lambda)^2 t)|}{4\lambda(1+\lambda)s^2} \\ \frac{4\lambda^2 s^3\,|1-\xi|}{|\lambda(\lambda-1)s^2 - (1+\lambda)^2 t|} & ; |1-\xi| \geq \frac{|\lambda(\lambda-1)s^2 - (1+\lambda)^2 t|}{4\lambda(1+\lambda)s^2}. \end{cases}$$

Remark 4. *We derive two results in[47, Corollaries 2 and 6] by taking $\lambda = 1$ in Corollary 6.*

3. Conclusions

This study establishes upper bounds on $|d_2|$ *and* $|d_3|$ for functions in subfamily of σ related to $(m, n)-$Lucas polynomials. Moreover, the Fekete-Szegö functional $|d_3 - \mu d_2^2|, \mu \in \mathbb{R}$ has been identified for functions in these subfamilies. Through adjusting the parameters in Theorem 1, few implications have been brought to light. Relevant connections to the current research are also discovered. Nevertheless, this paper does not address all of the significant subclasses of Σ that exist in the literature. For example, authors[34, 36, 38] have examined various subclasses involving (p, q)-operators introduced in (p, q)-calculus. It is recommended that the interested reader review these papers and the associated references.

References

[1] Tuncer Acar, Ali Aral, and Syed Abdul Mohiuddine. On kantorovich modification of (p, q)-baskakov operators. *Journal of Inequalities and Applications*, 2016(1):98, 2016.

[2] İbrahim Aktaş and Derya Hamarat. Generalized bivariate fibonacci polynomial and two new subclasses of bi-univalent functions. *Asian-European Journal of Mathematics*, 16(08):2350147, 2023.

[3] Tariq Al-Hawary, Ala Amourah, and Basem Aref Frasin. Fekete–szegö inequality for bi-univalent functions by means of horadam polynomials. *Boletín de la Sociedad Matematica Mexicana*, 27:1–12, 2021.

[4] Fatima M Al-Oboudi. On univalent functions defined by a generalized slgean operator. *International Journal of Mathematics and Mathematical Sciences*, 2004(27):1429–1436, 2004.

[5] Abdullah Alsoboh, Ala Amourah, Maslina Darus, and Carla Amoi Rudder. Investigating new subclasses of bi-univalent functions associated with q-pascal distribution series using the subordination principle. *Symmetry*, 15(5):1109, 2023.

[6] ŞAHSENE Altinkaya and S Yalçin. Certain classes of bi-univalent functions of complex order associated with quasi-subordination involving (p, q)-derivative operator. *Kragujevac Journal of Mathematics*, 44(4):639–649, 2020.

[7] Sahsene Altinkaya and Sibel Yalcin. Lucas polynomials and applications to an unified class of bi-univalent functions equipped with (p, q)-derivative operators. *TWMS Journal of Pure and Applied Mathematics*, 11(1):100–108, 2020.

[8] Ala Amourah, Heba Abdelkarim, and Anas AL-Elaumi. (p, q)- chebyshev polynomials and their applications to bi-univalent functions. 2022.

[9] Ala Amourah, Ibtisam Aldawish, Khadeejah Rasheed Alhindi, and Basem Aref Frasin. An application of rabotnov functions on certain subclasses of bi-univalent functions. *Axioms*, 11(12):680, 2022.

[10] Ala Amourah, Zabidin Salleh, BA Frasin, Muhammad Ghaffar Khan, and Bakhtiar Ahmad. Subclasses of bi-univalent functions subordinate to gegenbauer polynomials. *Afrika Matematika*, 34(3):41, 2023.

[11] Serkan Araci, Uğur Duran, Mehmet Acikgoz, and Hari M Srivastava. A certain (p, q) (p,q)-derivative operator and associated divided differences. *Journal of Inequalities and Applications*, 2016:1–8, 2016.

[12] Metin Arik, Ertugul Demircan, Teoman Turgut, Lezgin Ekinci, and Muhittin Mungan. Fibonacci oscillators. *Zeitschrift für Physik C Particles and Fields*, 55:89–95, 1992.

[13] David A Brannan, J Clunie, et al. Aspects of contemporary complex analysis. *(No Title)*, 1980.

[14] David A Brannan and TS Taha. On some classes of bi-univalent functions. In *Mathematical Analysis and Its Applications*, pages 53–60. Elsevier, 1988.

[15] G Brodimas, RP Mignani, and A Jannussis. Two-parameter quantum groups. Technical report, 1991.

[16] JD Bukweli-Kyemba and MN Hounkonnou. Quantum deformed algebras: coherent states and special functions. *arXiv preprint arXiv:1301.0116*, 2013.

[17] Serap Bulut. Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions. *Comptes Rendus Mathematique*, 352(6):479–484, 2014.

[18] Murat Çağlar, Erhan Deniz, and Hari Mohan Srivastava. Second hankel determinant for certain subclasses ofbi-univalent functions. *Turkish Journal of Mathematics*, 41(3):694–706, 2017.

[19] Mario Catalani. Generalized bivariate fibonacci polynomials. *arXiv preprint math/0211366*, 2002.

[20] A Catas. On certain classes of p-valent functions defined by new multiplier transformations, tc istanbul kultur university publications. In *Proceedings of the International Symposium on Geometric Function Theory and Applications (GFTA 2007), Istanbul, Turkey*, volume 91, pages 241–250, 2007.

[21] R Chakrabarti and R Jagannathan. A (p, q)-oscillator realization of two-parameter quantum algebras. *Journal of Physics A: Mathematical and General*, 24(13):L711, 1991.

[22] Luminita-Ioana Cotırla. New classes of analytic and bi-univalent functions. *AIMS Math*, 6(10):10642–10651, 2021.

[23] Ugur Duran, Mehmet Acikgoz, and Serkan Araci. A study on some new results arising from (p, q)-calculus. 2018.

[24] Ugur Duran, Mehmet Acikgoz, and Serkan Araci. A study on some new results arising from (p, q)-calculus. 2018.

[25] Peter L Duren. *Univalent functions*, volume 259. Springer Science & Business Media, 2001.

[26] M Fekete and G Szegö. Eine bemerkung über ungerade schlichte funktionen. *Journal of the london mathematical society*, 1(2):85–89, 1933.

[27] Basem A Frasin and MK Aouf. New subclasses of bi-univalent functions. *Applied Mathematics Letters*, 24(9):1569–1573, 2011.

[28] Hasan Hüseyin Güleç and İbrahim Aktaş. Coefficient estimate problems for a new subclass of bi-univalent functions linked with the generalized bivariate fibonacci-like polynomial. *Journal of Engineering Technology and Applied Sciences*, 9(2):71–84, 2024.

[29] Mohamed Illafe, Ala Amourah, and Maisarah Haji Mohd. Coefficient estimates and fekete–szegö functional inequalities for a certain subclass of analytic and bi-univalent functions. *Axioms*, 11(4):147, 2022.

[30] Frederick H Jackson. Xi.—on q-functions and a certain difference operator. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, 46(2):253–281, 1909.

[31] R Jagannathan and K Srinivasa Rao. Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series. *arXiv preprint math/0602613*, 2006.

[32] Emine Gokcen Kocer and Serife Tuncez. Bivariate fibonacci and lucas like polynomials. *Gazi University Journal of Science*, 29(1):109–113, 2016.

[33] Mordechai Lewin. On a coefficient problem for bi-univalent functions. *Proceedings of the American mathematical society*, 18(1):63–68, 1967.

[34] Duygu Malyalı. (p, q)-hahn difference operator. Master's thesis, Eastern Mediterranean University (EMU)-Doğu Akdeniz Üniversitesi (DAÜ), 2020.

[35] Susanta Kumar Mohapatra and Trailokya Panigrahi. Coefficient estimates for bi-univalent functions defined by (p, q) analogue of the salagean differential operator related to the chebyshev polynomials. *J. Math. Fund. Sci*, 53(1):49–66, 2021.

[36] Ahmad Motamednezhad and Safa Salehian. New subclass of bi-univalent functions by (p, q)-derivative operator. *Honam Mathematical Journal*, 41(2):381–390, 2019.

[37] P Njionou Sadjang. On the fundamental theorem of (p, q)-calculus and some (p, q)-taylor formulas. *arXiv e-prints*, pages arXiv–1309, 2013.

[38] S Santhiya and K Thilagavathi. Geometric properties of analytic functions defined by the (p, q) derivative operator involving the poisson distribution. *Journal of Mathematics*, 2023(1):2097976, 2023.

[39] C Selvaraj, G Thirupathi, and E Umadevi. Certain classes of analytic functions involving a family of generalized differential operators. *Transylvanian J. Math. Mechanics*, 9(1):51–61, 2017.

[40] Hari M Srivastava, Akshaya Kumar Mishra, and Priyabrat Gochhayat. Certain subclasses of analytic and bi-univalent functions. *Applied mathematics letters*, 23(10):1188–1192, 2010.

[41] HM Srivastava, Nusrat Raza, Eman SA AbuJarad, Gautam Srivastava, and Mohammed H AbuJarad. Fekete-szegö inequality for classes of (p, q)-starlike and (p, q)-convex functions. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 113(4):3563–3584, 2019.

[42] SR Swamy. Inclusion properties for certain subclasses of analytic functions defined by a generalized multiplier transformation. *Int. J. Math. Anal*, 6(32):1553–1564, 2012.

[43] SR Swamy. Inclusion properties of certain subclasses of analytic functions. In *Int. Math. Forum*, volume 7, pages 1751–1760. Citeseer, 2012.

[44] De Lin Tan. Coefficient estimates for bi-univalent functions. *Chinese Ann. Math. Ser. A*, 5(5):559–568, 1984.

[45] Sangarambadi Padmanabhan Vijayalakshmi, Thirumalai Vinjimur Sudharsan, and Teodor Bulboaca. Symmetric toeplitz determinants for classes defined by post quantum operators subordinated to the limaçon function. *MATHEMATICA*, page 299, 2024.

[46] Michelle Wachs and Dennis White. p, q-stirling numbers and set partition statistics. *Journal of Combinatorial Theory, Series A*, 56(1):27–46, 1991.

[47] Nazmiye Yilmaz and İbrahim Aktaş. On some new subclasses of bi-univalent functions defined by generalized bivariate fibonacci polynomial. *Afrika Matematika*, 33(2):59, 2022.